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LOCAL DIMENSIONS OF THE BRANCHING MEASURE ON A GALTON–WATSON TREE

Quansheng LIU [a,b]

[a] *IRMAR, Université de Rennes I, Campus de Beaulieu, 35042 Rennes, France*
[b] *LMAM, Université de Bretagne-Sud, Campus de Tohannic, BP573, 56017 Vannes, France*

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ABSTRACT. – Let $\mu = \mu_\omega$ be the branching measure on the boundary $\partial\mathbf{T}$ of a supercritical Galton–Watson tree $\mathbf{T} = \mathbf{T}(\omega)$. Denote by $\underline{d}(\mu, u)$ and $\overline{d}(\mu, u)$ the lower and upper local dimensions of μ at $u \in \partial\mathbf{T}$. It is well known that almost surely, $\underline{d}(\mu, u) = \overline{d}(\mu, u) = \log m$ for μ-almost all $u \in \partial\mathbf{T}$, where m is the expected value of the offspring distribution. Here we find exactly when the result holds for *all* $u \in \partial\mathbf{T}$, and obtain some limit theorems about the uniform local dimensions of μ. We also find the exact local dimension of μ at $u \in \partial\mathbf{T}$ for μ-almost all u. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Soit $\mu = \mu_\omega$ la mesure de branchement sur le bord $\partial\mathbf{T}$ d'un arbre super-critique de Galton–Watson $\mathbf{T} = \mathbf{T}(\omega)$. Notons $\underline{d}(\mu, u)$ et $\overline{d}(\mu, u)$ les dimensions locales inférieures et supérieures de μ en $u \in \partial\mathbf{T}$. Il est bien connu que presque sûrement, $\underline{d}(\mu, u) = \overline{d}(\mu, u) = \log m$ pour μ-presque tout $u \in \partial\mathbf{T}$, où m est la moyenne de la loi de reproduction. Ici nous trouvons exactement quand le résultat vaut pour *tout* $u \in \partial\mathbf{T}$, tout en établissant des théorèmes limites pour les dimensions locales uniformes de μ. Nous trouvons aussi la dimension locale exacte de μ en $u \in \partial\mathbf{T}$ pour μ-presque tout u. © 2001 Éditions scientifiques et médicales Elsevier SAS

0. Introduction

Set $\mathbb{N}^* = \{1, 2, \ldots\}$ and $\mathbb{N} = \{0\} \cup \mathbb{N}^*$, and write $\mathbf{U} = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} (\mathbb{N}^*)^n$ for the set of all finite sequences $u = u_1 \ldots u_n = (u_1, \ldots, u_n)$ including the null sequence $\emptyset$. If $u = u_1 \ldots u_n$ ($u_k \in \mathbb{N}^*$), we write $|u| = n$ and $u|k = u_1 \ldots u_k, k \leqslant n$; by convention $|\emptyset| = 0$ and $u|0 = \emptyset$. For two sequences $u = u_1 \ldots u_m$ and $v = v_1 \ldots v_n$, we write $uv = u_1 \ldots u_m v_1 \ldots v_n$ for the juxtaposition; by convention $u\emptyset = \emptyset u = u$. If $uu' = v$ for some sequence u', we write $u < v$ or $v > u$; otherwise we write $u \not< v$ or $v \not> u$. The notations are extended to infinite sequences in an evident manner.

E-mail address: liu@univ-rennes1.fr, Quansheng.Liu@univ-ubs.fr (Q. Liu).

Let $(\Omega, \mathbb{F}, P)$ be a probability space, $\{p_n\colon n \in \mathbb{N}\}$ be a probability distribution on $\mathbb{N}$, and $\{N_u\colon u \in \mathbf{U}\}$ be a family of independent random variables defined on Ω, each distributed according to the law $\{p_n\}$. Let $\mathbf{T} = \mathbf{T}(\omega)$ be the corresponding *Galton–Watson tree* [19] with defining elements $\{N_u\colon u \in \mathbf{T}\}$: we have $\emptyset \in \mathbf{T}$ and, if $u \in \mathbf{T}$ and $i \in \mathbb{N}^*$, then $ui \in \mathbf{T}$ if and only if $1 \leqslant i \leqslant N_u$. We shall write

$$z_n = \{u \in \mathbf{T}\colon |u| = n\}$$

for the set of individuals in nth generation, and Z_n for its cardinality. Let

$$\partial\mathbf{T} = \{u_1 u_2 \ldots\colon \forall n \geqslant 0, u_1 \ldots u_n \in \mathbf{T}\}$$

be the *boundary* of $\mathbf{T}$ endowed with the ultra-metric

$$d(u, v) = \mathrm{e}^{-n}, \quad \text{where } n = \max\{k \in \mathbb{N}\colon u|k = v|k\},\ u, v \in \partial\mathbf{T}.$$

We always assume that $p_0 = 0$, that $N = N_\emptyset$ is not almost surely (a.s.) constant, and that

$$EN \log N < \infty, \tag{0.1}$$

unless otherwise specified. Write

$$m = EN \quad \text{and} \quad \alpha = \log m. \tag{0.2}$$

It is well known that the limit

$$W = \lim_{n\to\infty} Z_n / m^n$$

exists a.s. with $EW = 1$ and $P(W > 0) = 1$.

For all $u \in \mathbf{U}$, let $\mathbf{T}_u$ be the *shifted tree* of $\mathbf{T}$ at u: this is the tree with defining elements $\{N_{uv}\colon v \in \mathbf{U}\}$: we have $\emptyset \in \mathbf{T}_u$ and, if $v \in \mathbf{T}_u$, then for all $i \in \mathbb{N}^*$, $vi \in \mathbf{T}_u$ if and only if $1 \leqslant i \leqslant N_{uv}$. Let $\partial\mathbf{T}_u = \{v_1 v_2 \ldots\colon \forall n \geqslant 0, v_1 \ldots v_n \in \mathbf{T}_u\}$ be the boundary of $\mathbf{T}_u$, and let $B_u = \{uv\colon v \in \partial\mathbf{T}_u\}$ be the set of infinite descendants of u. Therefore $\mathbf{T} = \mathbf{T}_\emptyset$, $\partial\mathbf{T} = \partial\mathbf{T}_\emptyset$ and if $u \in \mathbf{T}$, then $B_u = \{v \in \partial\mathbf{T}\colon u < v\}$ is a ball in $\partial\mathbf{T}$ with center $u \in \mathbf{T}$ and diameter $|B_u| = \mathrm{e}^{-|u|}$. Let $\mu = \mu_\omega$ be the *branching measure* on $\partial\mathbf{T}$: it is the unique Borel measure such that for all $u \in \mathbf{T}$,

$$\mu(B_u) = W \lim_{n\to\infty} \frac{{}^{\#}\{v \in \mathbf{T}_u\colon |v| = n\}}{{}^{\#}\{v \in \mathbf{T}\colon |v| = n\}}, \tag{0.3}$$

where ${}^{\#}\{.\}$ denotes the cardinality of the set $\{.\}$. Equivalently, μ is the unique Borel measure on $\partial\mathbf{T}(\omega)$ such that for all $u \in \mathbf{T}$,

$$\mu(B_u) = m^{-|u|} W_u, \tag{0.4}$$

where

$$W_u = \lim_{n\to\infty} \#\{v \in \mathbf{T}_u\colon |v| = n\}/m^n \quad \text{if } u \in \mathbf{U}. \tag{0.5}$$

It proves convenient to define $\mu(B_u)$ by (0.4) for all $u \in \mathbf{U}$, and it will be useful to remark that $W = W_\emptyset$, that W_u and W_v are independent of each other if neither $u < v$ nor $v < u$, and that each of them follows the law of W.

The branching measure plays an essential role in the study of branching processes, and has been studied by many authors: see for example [9,11,13,16,18,20,15] and [17].

For each $u \in \partial\mathbf{T}$, let $\underline{d}(\mu, u)$ and $\overline{d}(\mu, u)$ be the lower and upper local dimensions of μ at u:

$$\underline{d}(\mu, u) = \liminf_{n\to\infty} \frac{-\log \mu(B_{u|n})}{n}, \qquad \overline{d}(\mu, u) = \limsup_{n\to\infty} \frac{-\log \mu(B_{u|n})}{n}. \tag{0.6}$$

When $\underline{d}(\mu, u) = \overline{d}(\mu, u)$, we write $d(\mu, u)$ for the common value. It is well-known (see [9] and [18]) that a.s.

$$d(\mu, u) = \alpha \tag{0.7}$$

for μ-almost all $u \in \partial\mathbf{T}$. A natural question is to know when (0.7) holds for *all* $u \in \partial\mathbf{T}$. We shall answer this question in Theorem 4.1, where we give a necessary and sufficient condition, and where we also establish a similar result for $\underline{d}(\mu, u)$ instead of $d(\mu, u)$.

Our approach to Theorem 4.1 is divided into two steps.

First, we establish some limit theorems about the uniform local dimensions of μ; in other words we obtain asymptotic properties of

$$m_n = \min_{u\in z_n} \mu(B_u) = \min_{u\in\partial\mathbf{T}} \mu(B_{u|n}) \quad \text{and} \quad M_n = \max_{u\in z_n} \mu(B_u) = \max_{u\in\partial\mathbf{T}} \mu(B_{u|n}) \tag{0.8}$$

as $n \to \infty$. In fact, we shall prove that there are some constants $\alpha_- \geqslant \alpha$ and $\alpha_+ \leqslant \alpha$, explicitly determined by the given distribution $\{p_n\}$, such that a.s.

$$\lim_{n\to\infty} \frac{-\log m_n}{n} = \alpha_- \quad \text{and} \quad \lim_{n\to\infty} \frac{-\log M_n}{n} = \alpha_+$$

(Theorems 2.1 and 3.1).[1] Since $m_n \leqslant \underline{d}(\mu, u) \leqslant \overline{d}(\mu, u) \leqslant M_n$ for all u, α_+ is a uniform lower bound of $\underline{d}(\mu, u)$ while α_- is a uniform upper bound of $\overline{d}(\mu, u)$ (Lemma 4.1). The condition $\alpha_- = \alpha_+$ is then sufficient for (0.7) to hold for *all* u. Our proof of the asymptotic properties uses two basic tools given in Section 1: one is an interesting convergence result about the convergence of iterations of a probability generating function (Proposition 1.1), the other is the "first moment method" (Proposition 1.2).

Secondly, we prove that there are exceptional points if $\alpha_+ < \alpha_-$ (Lemma 4.3). The main idea of the proof is to construct a non-homogeneous branching process by choosing "good" generations and "good" individuals of the initial branching process, and to prove that the new process does not terminate (cf. the proof of Lemma 4.3). In the proof, we need the fact that the martingale $\{Z_n/m^n\}_n$ converges in L^p $(p > 1)$ at a geometric rate, which is shown in Section 1.

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[1] We use the symbols $\alpha_-, \beta_-, \ldots$ (respectively $\alpha_+, \beta_+, \ldots$) to stand for numbers which are related to some exponents of the left (respectively right) tail of W.

Since the study of asymptotic properties of m_n and M_n is interesting by its own, we shall also find exact equivalents of m_n and M_n in the case where the limit variable W has exponential left or right tails (Theorems 5.1 and 6.1). These results give exact uniform local dimensions of μ, and lead to exact uniform bounds of the local dimensions (Theorem 7.1).

Our final result concerns the exact local dimension of μ at typical $u \in \partial\mathbf{T}$ (Theorem 8.1): it gives a precise estimation of the large values of $\mu_\omega(B_{u|n})$ for P-almost all $\omega \in \Omega$ and μ_ω-almost all $u \in \partial\mathbf{T}(\omega)$, and solves a conjecture of Hawkes [9, p. 382].

An interesting phenomenon revealed by our results is that, in some cases, the branching measure behaves like the occupation measure of a stable subordinator or a Brownian motion: for example, our Theorems 4.1(a)(ii), 5.1 and 6.1 correspond to Theorem 3.1 of Hu and Taylor [10], Theorems 1 and 2 of Hawkes [8] and Théorème 52.2 of Lévy [12]; but in other cases the branching measure has some properties which the occupation measure does not share: cf. parts (b)(i) and (b)(iii) of Theorem 4.1.

1. Iteration of a probability generating function and the first moment method. Exponential convergence rate in L^p of Z_n/m^n

The following three propositions will be used several times in the paper. The first is an interesting result about the convergence of the n-fold composition of a probability generating function, evaluated at a point a_n which converges to 1 at a geometric rate; the second concerns the "first moment method"; the third says that the sequence $\{Z_n/m^n\}$ converges in L^p $(p > 1)$ *at a geometric rate*, if $EN^p < \infty$.

Throughout the paper, f denotes the probability generating function of N: $f(x)= \sum_{n\geqslant 0} p_n x^n$, and f_n is its n-fold composition. In the following proposition we do not need the condition (0.1).

PROPOSITION 1.1. – *Assume only* $p_0 = 0$ *and* $m = f'(1) < \infty$, *and let* ρ, c *be two numbers in* $(0, 1]$. *Then the following assertions hold*:

- (i) *if* $1/m < \rho$, *then there are some constants* $\lambda < 1$ *and* $0 < K < \infty$ *such that for all* $n \geqslant 1$ *large enough*, $f_n(1 - c\rho^n) \leqslant K\lambda^n$;
- (ii) *if* $1/m = \rho$, *then* $\liminf_{n\to\infty} f_n(1 - c\rho^n) \geqslant \mathrm{e}^{-c}$;
- (iii) *if* $1/m > \rho$, *then* $\lim_{n\to\infty} f_n(1 - c\rho^n) = 1$.

In particular, $\sum_{n\geqslant 1} f_n(1 - c\rho^n) < \infty$ *if and only if* $1/m < \rho$.

Remark. – If $\rho < 1$, the conclusions also hold for each $c > 1$, and so for all $0 < c < \infty$; of course in this case in the series we should change "$n \geqslant 1$" to "$n \geqslant n_0$", where $n_0 > 0$ is large enough such that $1 - c\rho^n \geqslant 0$ for all $n \geqslant n_0$. This will be easily seen by the proof. If $\rho = 1$, we naturally need the condition $c \leqslant 1$ to ensure that $1 - c\rho^n \geqslant 0$.

Proof. – (a) We first prove that for all $c \in (0, 1]$ and all $\rho \in (1/m, 1]$,

$$\lim_{n\to\infty} f_n\bigl(1 - c\rho^n\bigr) = 0.$$

By the famous Seneta–Heyde theorem, there is a sequence (C_n) of positive numbers which converges to ∞ with n, such that $C_{n+1}/C_n \to m$ and that Z_n/C_n converges a.s.

to a strictly positive random variable (recall that $p_0 = 0$). Therefore $\lim Z_n^{1/n} = m$ a.s. Consequently,

$$\lim Z_n \log\bigl(1 - c\rho^n\bigr) = -\infty \quad \text{a.s. if } 1/m < \rho \leqslant 1.$$

The conclusion then follows by the dominated convergence theorem and the fact that $f_n(1 - c\rho^n) = E\mathrm{e}^{Z_n \log(1-c\rho^n)}$.

(b) We next prove that if $c \in (0, 1]$ and $\rho \in (1/m, 1]$, then there are some constants $\lambda < 1$ and $0 < K < \infty$ such that $f_n(1 - c\rho^n) \leqslant K\lambda^n$ for all $n \geqslant 1$ large enough. Let $\delta \in (0, 1)$ be sufficiently close to 1 such that $\rho_1 = \rho^{1/\delta} > 1/m$. Denote by $\{\delta n\}$ the least integer $\geqslant \delta n$. Then

$$\begin{aligned} f_n\bigl(1 - c\rho^n\bigr) &= f_{n-\{\delta n\}}\bigl(f_{\{\delta n\}}\bigl(1 - c\rho^n\bigr)\bigr) \\ &\leqslant f_{n-\{\delta n\}}\bigl(f_{\{\delta n\}}\bigl(1 - c\rho_1^{\{\delta n\}}\bigr)\bigr). \end{aligned}$$

Because $\lim_{k\to\infty} f_k(1 - c\rho^k) = 0$, there is $n_0 \in \mathbb{N}$ large enough such that for all $n \geqslant n_0$, $f_{\{\delta n\}}(1 - c\rho_1^{\{\delta n\}}) \leqslant 1/2$. It follows that

$$f_n\bigl(1 - c\rho^n\bigr) \leqslant f_{n-\{\delta n\}}(1/2), \quad n \geqslant n_0.$$

Now since $f(x) \leqslant x$, $f_k(1/2)$ decreases to a limit < 1; this limit is equal to 0 because it is a fixed point of f. Let $\varepsilon > 0$ be small enough such that $f'(\varepsilon) < 1$, and let k_ε be large enough such that $f_k(1/2) \leqslant \varepsilon$ for all $k \geqslant k_\varepsilon$. Therefore, using $f(x) \leqslant xf'(x)$ gives $f_{k+1}(1/2) \leqslant f'(f_k(1/2)) \leqslant f'(\varepsilon) f_k(1/2), k \geqslant k_\varepsilon$. It follows that for some $K_0 > 0$ and all $k \in \mathbb{N}$,

$$f_k(1/2) \leqslant K_0 f'(\varepsilon)^k.$$

Using this for $k = n - \{\delta n\}$ and the preceding inequality for $f_n(1 - c\rho^n)$, we see that for all $n \geqslant n_0$,

$$f_n\bigl(1 - c\rho^n\bigr) \leqslant K_0 f'(\varepsilon)^{n-\{\delta n\}} \leqslant K\lambda^n,$$

where $K = K_0/f'(\varepsilon)$ and $\lambda = f'(\varepsilon)^{1-\delta} < 1$.

(c) Finally by Jensen's inequality, we have

$$f_n\bigl(1 - c\rho^n\bigr) = E\mathrm{e}^{Z_n \log(1-c\rho^n)} \geqslant \mathrm{e}^{m^n \log(1-c\rho^n)},$$

from which $\liminf f_n(1 - c\rho^n) \geqslant 1$ if $\rho < 1/m$, and $\geqslant \mathrm{e}^{-c}$ if $\rho = 1/m$. The proof of the proposition is then finished, remarking that we have always $f_n(1 - c\rho^n) \leqslant 1$. □

PROPOSITION 1.2. – *Let B be a Borel set on the real line and set $A = \{W \in B\}$. Define, for $n \geqslant 0$,*

$$A_n = \bigl\{\exists u \in z_n, m^n \mu(B_u) \in B\bigr\}, \quad \textit{and} \quad A_n' = \bigl\{\forall u \in z_n, m^n \mu(B_u) \in B\bigr\}.$$

Then $A_0 = A_0' = A$, and for all $n \geqslant 1$, $P(A_n) \leqslant m^n P(A)$ and $P(A_n') = f_n(P(A))$.

Proof. – If $\{.\}$ is a set or a statement, we write $\mathbf{1}_{\{.\}}$ or $\mathbf{1}\{.\}$ for its indicator function. It is easily seen that

$$\mathbf{1}_{A_n} \leqslant \sum_{u \in z_n} \mathbf{1}\{m^n \mu(B_u) \in B\} = \sum_{u \in z_n} \mathbf{1}\{W_u \in B\}$$

and

$$\mathbf{1}_{A'_n} = \prod_{u \in z_n} \mathbf{1}\{m^n \mu(B_u) \in B\} = \prod_{u \in z_n} \mathbf{1}\{W_u \in B\}.$$

The conclusion then follows by taking expectations on each side of the above displays, using the fact that for each fixed n, the random variables $W_u, |u| = n$, are independent of each other and have the same distribution as W. □

PROPOSITION 1.3. – *Fix $p > 1$ and write $W_{(k)} = Z_k/m^k$ $(k \geqslant 1)$. If $EN^p < \infty$, then for some constant $c > 0$ and all $k \geqslant 1$,*

$$E\,|W_{(k)} - W|^p \leqslant \begin{cases} cm^{-(p-1)k} & \text{if } 1 < p \leqslant 2, \\ cm^{-pk/2} & \text{if } p > 2. \end{cases}$$

The result is well-known for $p = 2$ (cf. [7, p.13]), and seems to be unknown for $p \neq 2$. The proof uses the following very useful inequality.

LEMMA 1.4. – *If $\{X_i\colon\, i \geqslant 1\}$ are independent and integrable real random variables with $EX_i = 0$ $(\forall i)$, then for all $n \geqslant 1$ and all $p > 1$,*

$$E\left(\left|\sum_{i=1}^n X_i\right|^p\right) \leqslant \begin{cases} (B_p)^p E(\sum_{i=1}^n |X_i|^p) & \text{if } 1 < p \leqslant 2, \\ (B_p)^p E(\sum_{i=1}^n |X_i|^p) n^{(p/2)-1} & \text{if } p > 2, \end{cases}$$

where $B_p = 2\min\{k^{1/2}\colon\, k \in \mathbb{N}, k \geqslant p/2\}$ (so that $B_p = 2$ if $1 < p \leqslant 2$).

It is a direct consequence of the Marcinkiewicz–Zigmund inequality [3, p. 356]: $E(|\sum_{i=1}^n X_i|^p) \leqslant B_p^p E(|\sum_{i=1}^n X_i^2|^{p/2})$, remarking that $(\sum_{i=1}^n X_i^2)^{p/2} \leqslant \sum_{i=1}^n |X_i|^p$ if $1 < p \leqslant 2$ (sub-additivity), and $(\frac{1}{n}\sum_{i=1}^n X_i^2)^{1/2} \leqslant (\frac{1}{n}\sum_{i=1}^n |X_i|^p)^{1/p}$ if $p > 2$ (Hölder).

Proof of Proposition 1.3. – By the construction of the Galton–Watson process, we can write

$$W_{(k+n)} - W_{(k)} = m^{-k} \sum_{i=1}^{Z_k} (W_{n,i} - 1), \quad k, n \geqslant 1,$$

where $\{W_{n,i}\}_{i \geqslant 1}$ are independent of each other and independent of Z_k, and have the same distribution as $W_{(n)}$. So by the preceding lemma,

$$E\left[|W_{(k+n)} - W_{(k)}|^p | Z_k\right] \leqslant \begin{cases} m^{-kp} 2^p Z_k E|W_{(n)} - 1|^p & \text{if } 1 < p \leqslant 2, \\ m^{-kp} (B_p)^p (Z_k)^{p/2} E|W_{(n)} - 1|^p & \text{if } p > 2. \end{cases}$$

Therefore

$$E\left[|W_{(k+n)} - W_{(k)}|^p\right] \leqslant \begin{cases} m^{-k(p-1)} 2^p E|W_{(n)} - 1|^p & \text{if } 1 < p \leqslant 2, \\ m^{-kp/2} (B_p)^p E[W_{(k)}]^{p/2} E|W_{(n)} - 1|^p & \text{if } p > 2. \end{cases}$$

Using the inequality for $n=1$ and an easy argument of induction on $[p]$ (the integral part of p), we obtain the following classical result: for each fixed $p>1$, $EN^p<\infty$ implies $\sup_k E[W_{(k)}]^p<\infty$, so that $W_{(k)}\to W$ in L^p; therefore letting $n\to\infty$ in the preceding inequality, we obtain the desired result. □

2. An equivalent of $\log m_n$

In this section, we prove that without any condition other than (0.1), almost surely $(\log m_n)/n$ has a constant limit that we determine explicitly.

Let $p_->0$ be defined by

$$p_-=-\frac{\log p_1}{\log m}\quad \text{if } p_1>0,\qquad \text{and}\quad p_-=\infty\quad \text{if } p_1=0. \tag{2.1}$$

It is known that: (a) if $p_1>0$, then for some constants $c_1,c_2>0$ and all $x>0$ small enough,

$$c_1x^{p_-}\leqslant P(W\leqslant x)\leqslant c_2x^{p_-} \tag{2.2}$$

(see, for example, [1], p. 217); (b) whether $p_1>0$ or not,

$$p_-=\sup\{b>0\colon\ EW^{-b}<\infty\}=\lim_{x\to 0}\frac{\log P(W\leqslant x)}{\log x}. \tag{2.3}$$

In the following theorem and in all this paper, we shall write $1/\infty=0$ by convention.

THEOREM 2.1. – *With probability* 1,

$$\lim_{n\to\infty}\frac{-\log m_n}{n}=\left(1+\frac{1}{p_-}\right)\alpha. \tag{2.4}$$

We need two lemmas for the proof.

LEMMA 2.1. – *If there exist some constants $b>0$ and $c>0$ such that $P[W\leqslant x]\leqslant cx^b$ for all $x>0$ small enough, then for all $\eta>(1+1/b)\alpha$,*

$$P\left[m_n\geqslant \mathrm{e}^{-n\eta}\ \textit{for all } n\in\mathbb{N}\ \textit{large enough}\right]=1. \tag{2.5}$$

Proof. – Notice that $m_n\geqslant \mathrm{e}^{-n\eta}$ if and only if $\mu(B_u)<\mathrm{e}^{-n\eta}$ for some $u\in z_n$. So by Proposition 1.2, we have, for all $n\in\mathbb{N}$,

$$P\left[m_n<\mathrm{e}^{-n\eta}\right]\leqslant \mathrm{e}^{n\alpha}P\left[W<\mathrm{e}^{-n(\eta-\alpha)}\right].$$

By our condition, there is a constant $C>0$ large enough such that for all $x>0$, $P[W\leqslant x]\leqslant Cx^b$. Hence by the preceding inequality, $P[m_n<\mathrm{e}^{-n\eta}]\leqslant C\mathrm{e}^{-n[b(\eta-\alpha)-\alpha]}$. Therefore $\sum_{n=1}^\infty P[m_n<\mathrm{e}^{-n\eta}]<\infty$ whenever $\eta>(1+1/b)\alpha$, and the desired conclusion follows by Borel–Cantelli's lemma. □

LEMMA 2.2. –
(i) *With probability* 1, $m_n < \mathrm{e}^{-n\alpha}$ *for all* $n \in \mathbb{N}$ *large enough*;
(ii) *if* $P[W \leqslant x] \geqslant cx^b$ *for some constants* $b, c > 0$ *and all* $x > 0$ *small enough, then for all* $\eta < (1 + 1/b)\alpha$,

$$P\left[m_n < \mathrm{e}^{-n\eta} \text{ for all } n \in \mathbb{N} \text{ large enough}\right] = 1. \tag{2.6}$$

Proof. – By Borel–Cantelli's lemma, it suffices to prove that the series $\sum_{n=1}^{\infty} P[m_n \geqslant \mathrm{e}^{-n\eta}]$ converges in each of the following cases: (a) $\eta = \alpha$, (b) the condition of (ii) is satisfied and $\alpha < \eta < (1 + 1/b)\alpha$. Notice that $m_n \geqslant \mathrm{e}^{-n\eta}$ if and only if $\mu(B_u) \geqslant \mathrm{e}^{-n\eta}$ for all $u \in z_n$; so by Proposition 1.2, for all $n \geqslant 1$,

$$P\left[m_n \geqslant \mathrm{e}^{-n\eta}\right] = f_n\left(P\left(W \geqslant \mathrm{e}^{-n(\eta-\alpha)}\right)\right).$$

Therefore by Proposition 1.1 (with $\rho = 1$), the series converges in case (a). In case (b), there is a constant $c_1 \in (0, 1)$ small enough such that $P(W < x) \geqslant c_1 x^b$ for all $x \in (0, 1]$, so that $P(W \geqslant \mathrm{e}^{-n(\eta-\alpha)}) \leqslant 1 - c_1\rho^n$, where $\rho = \mathrm{e}^{-b(\eta-\alpha)} > \mathrm{e}^{-\alpha} = 1/m$; hence the series also converges, again by Proposition 1.1. □

Proof of Theorem 2.1. – If $p_1 > 0$, then (2.2) holds, so that the conclusion follows from Lemmas 2.1 and 2.2(ii). If $p_1 = 0$, then for each $b > 0$, there is a constant $c > 0$ such that $P(W \leqslant x) \leqslant cx^b$ for all $x > 0$ small enough, so that the conclusion follows from Lemmas 2.1 and 2.2(i). □

3. An equivalent of $\log M_n$

In this section we find an equivalent of $\log M_n$ which is similar to that of $\log m_n$ obtained in the last section.

Let $p_+ \in [1, \infty]$ be defined by

$$p_+ = \sup\{a \geqslant 1\colon\ EN^a < \infty\}. \tag{3.1}$$

Therefore $p_+ = \infty$ if and only if $EN^a < \infty$ for all $a > 1$. Recall that for all fixed $a > 1$, $EN^a < \infty$ if and only if $EW^a < \infty$ (cf. [2]). So we can replace N by W in the definition of p_+. Consequently by Theorem 3.1 of Ramachandran [21],

$$p_+ = \liminf_{x\to\infty} \frac{-\log P(N > x)}{\log x} = \liminf_{x\to\infty} \frac{-\log P(W > x)}{\log x}. \tag{3.2}$$

We shall sometimes need the condition that

$$p_+ = \lim_{x\to\infty} \frac{-\log P(W > x)}{\log x}. \tag{3.3}$$

Notice that by (3.2), condition (3.3) holds automatically if $p_+ = \infty$; when $p_+ < \infty$, it is equivalent to the condition that for all $a > p_+$, there is a constant $c > 0$ such that

$$P(W > x) \geqslant cx^{-a} \tag{3.4}$$

for all $x > 0$ large enough. Standard results from [2] and [5] show that (3.4) holds if $p_+ > 1$ and if the function $x \mapsto P(N > x)x^{p_+}$ slowly varies at ∞.

The following result is the counter part of Theorem 2.1. Recall that $1/\infty = 0$ by our convention.

THEOREM 3.1. – *Let p_+ be defined by* (3.1)*, then*

$$\liminf_{n\to\infty} \frac{-\log M_n}{n} = \left(1 - \frac{1}{p_+}\right)\alpha \quad a.s. \tag{3.5}$$

If, furthermore, condition (3.3) *holds, then the* lim inf *above is in fact a* lim: *we have*

$$\lim_{n\to\infty} \frac{-\log M_n}{n} = \left(1 - \frac{1}{p_+}\right)\alpha \quad a.s. \tag{3.6}$$

For the proof, just as in the proof of Theorem 2.1, we first establish two lemmas.

LEMMA 3.1. – *If $P(W > x) \leqslant cx^{-a}$ for some constants $a, c > 0$ and all $x > 0$ large enough, then for all $\eta < (1 - 1/a)\alpha$,*

$$P\left[M_n \leqslant \mathrm{e}^{-n\eta} \text{ for all } n \in \mathbb{N} \text{ large enough}\right] = 1. \tag{3.7}$$

Proof. – Notice that $M_n > \mathrm{e}^{-n\eta}$ if and only if there is $u \in z_n$ such that $\mu(B_u) > \mathrm{e}^{-n\eta}$. Therefore by Proposition 1.2,

$$P\left[M_n > \mathrm{e}^{-n\eta}\right] \leqslant \mathrm{e}^{n\alpha} P\left[W > \mathrm{e}^{-n(\eta-\alpha)}\right].$$

By the condition we can choose a constant $K > 0$ large enough such that $P[W > x] \leqslant Kx^{-a}$ for all $x > 0$, so that $P[W > \mathrm{e}^{-n(\eta-\alpha)}] \leqslant K\mathrm{e}^{n(\eta-\alpha)b}$. Therefore $\sum_{n=1}^{\infty} P[M_n > \mathrm{e}^{-n\eta}] < \infty$ whenever $\eta < (1 - 1/a)\alpha$. So the desired conclusion follows by Borel–Cantelli's lemma. □

LEMMA 3.2. –

(i) *With probability* 1*, $M_n > \mathrm{e}^{-n\alpha}$ for all $n \in \mathbb{N}$ large enough*;

(ii) *if $P(W > x) \geqslant cx^{-a}$ for some constants $a, c > 0$ and all $x > 0$ large enough, then for all $\eta > (1 - 1/a)\alpha$,*

$$P\left[M_n > \mathrm{e}^{-n\eta} \text{ for all } n \in \mathbb{N} \text{ large enough}\right] = 1; \tag{3.8}$$

(iii) *if $P(W > x) \geqslant cx^{-a}$ for some constants $a, c > 0$ and a non-bounded set of values of $x > 0$, then for all $\eta > (1 - 1/a)\alpha$,*

$$P\left[M_n > \mathrm{e}^{-n\eta} \text{ for infinitely many } n \in \mathbb{N}\right] = 1. \tag{3.9}$$

Proof. – Since $M_n \leqslant \mathrm{e}^{-n\eta}$ if and only if $\mu(B_u) \leqslant \mathrm{e}^{-n\eta}$ for all $u \in z_n$, by Proposition 1.2, we have

$$P\left[M_n \leqslant \mathrm{e}^{-n\eta}\right] = f_n\left(P\left[W \leqslant \mathrm{e}^{n(\alpha-\eta)}\right]\right) = f_n\left(1 - P\left[W > \mathrm{e}^{n(\alpha-\eta)}\right]\right).$$

Under the condition of (ii), there is a constant $c_1 \in (0,1)$ such that $P(W > x) \geqslant c_1 x^{-a}$ for all $x \geqslant 1$. Therefore $P[W > \mathrm{e}^{n(\alpha-\eta)}] \geqslant c_1 \mathrm{e}^{-na(\alpha-\eta)}$ if $\alpha > \eta$, so that by Proposition (1.1), the series $\sum_{n=1}^{\infty} P[M_n \leqslant \mathrm{e}^{-n\eta}]$ converges if either (a) $\eta = \alpha$, or (b) the condition of (ii) is satisfied and $\alpha > \eta > (1-1/a)\alpha$. Hence the conclusions in parts (i) and (ii) follow from Borel–Cantelli's lemma.

For part (iii), notice that if (3.9) holds for some $\eta = \eta_0$, then it also holds for all $\eta > \eta_0$; therefore we need only prove the result for $\alpha > \eta > (1-1/a)\alpha$. By the monotonicity of $P(W \geqslant x)$, it is easily seen that

$$\liminf_{x\to\infty} \frac{-\log P[W > x]}{\log x} = \liminf_{n\to\infty} \frac{-\log P[W > \mathrm{e}^{n(\alpha-\eta)}]}{\log \mathrm{e}^{n(\alpha-\eta)}}.$$

By the condition, their common value is bounded by a. Therefore, for all $\varepsilon > 0$, there are infinitely many $n \in \mathbb{N}^*$ such that

$$P\left[W > \mathrm{e}^{n(\alpha-\eta)}\right] \geqslant \mathrm{e}^{-n(\alpha-\eta)(a+\varepsilon)},$$

so that by the preceding argument, for all these n,

$$P\left[M_n \leqslant \mathrm{e}^{-n\eta}\right] \leqslant f_n\left(1 - \mathrm{e}^{-n(a+\varepsilon)(\alpha-\eta)}\right).$$

Notice that by Proposition (1.1) , the term on the right hand side tends to 0 if $\rho := \mathrm{e}^{-(a+\varepsilon)(\alpha-\eta)} > \mathrm{e}^{-\alpha} = 1/m$. Therefore for all $\eta > \alpha[1 - 1/(a+\varepsilon)]$,

$$P\left(\liminf\left[M_n \leqslant \mathrm{e}^{-n\eta}\right]\right) \leqslant \liminf P\left[M_n \leqslant \mathrm{e}^{-n\eta}\right] \leqslant \lim f_n\left(1 - \mathrm{e}^{-n(a+\varepsilon)(\alpha-\eta)}\right) = 0.$$

This implies that (3.9) holds for all $\alpha > \eta > \alpha[1 - 1/(a+\varepsilon)]$, and hence for all $\alpha > \eta > \alpha(1-1/a)$ since $\varepsilon > 0$ is arbitrary. □

Proof of Theorem 3.1. – Notice that by (3.2), for each fixed $0 < a < p_+$ ($\leqslant \infty$), $P(W > x) \leqslant x^{-a}$ for all $x > 0$ large enough, so that by Lemma 3.1, $\liminf_{n\to\infty}(-\log M_n/n) \geqslant (1-\frac{1}{p_+})\alpha$ a.s. By Lemma 3.2(i), $\limsup_{n\to\infty}(-\log M_n/n) \leqslant \alpha$ a.s. Hence the proof is finished if $p_+ = \infty$. Assume $p_+ < \infty$ and let $a' > p_+$ be arbitrarily fixed. Then $P(W > x) \geqslant x^{a'}$ for a non-bounded set of $x > 0$ by (3.2), and for all $x > 0$ large enough if (3.3) is satisfied. So by Lemma 3.2(iii), $\liminf_{n\to\infty}(-\log M_n/n) \leqslant (1-\frac{1}{a'})\alpha$ a.s., and by Lemma 3.2(ii), $\limsup_{n\to\infty}(-\log M_n/n) \leqslant (1-\frac{1}{a'})\alpha$ a.s. if (3.3) is satisfied. The proof is then finished by letting $a' \to p_+$. □

4. A necessary and sufficient condition for no exceptional point, and uniform bounds of local dimensions

The main result of the present section is the following theorem.

Recall that p_- and p_+ are defined by (2.1) and (3.1), that $p_- = \infty$ if and only if $p_1 = 0$, and that $p_+ = \infty$ if and only if $EN^a < \infty$ for all $a > 1$; recall also that the condition (3.3) automatically holds if $p_+ = \infty$.

THEOREM 4.1. – *If* $EN^{1+\delta} < \infty$ *for some* $\delta > 0$ *and if* (3.3) *holds, then*:

(a) *The following assertions hold*:
   (i) *a.s.* $\underline{d}(\mu,u)=\overline{d}(\mu,u)=\alpha$ *for all* $u\in\partial\mathbf{T}$ *if and only if* $p_+=p_-=\infty$;
   (ii) *a.s.* $\underline{d}(\mu,u)=\alpha$ *for all* $u\in\partial\mathbf{T}$ *if and only if* $p_+=\infty$.
(b) *More precisely, we have*:
   (i) *if* $p_+=p_-=\infty$, *then a.s.* $\underline{d}(\mu,u)=\overline{d}(\mu,u)=\alpha$ *for all* $u\in\partial\mathbf{T}$;
   (ii) *if* $p_+=\infty$ *and* $p_-<\infty$, *then a.s.* $\underline{d}(\mu,u)=\alpha$ *for all* $u\in\partial\mathbf{T}$ *but* $\overline{d}(\mu,u)>\alpha$ *for some* $u\in\partial\mathbf{T}$;
   (iii) *if* $p_+<\infty$, *then a.s.* $\underline{d}(\mu,u)<\alpha$ *for some* $u\in\partial\mathbf{T}$.
(c) *Moreover, a.s.* $\sup_{u\in\partial\mathbf{T}}\underline{d}(\mu,u)=\alpha$ *and* $\inf_{u\in\partial\mathbf{T}}\underline{d}(\mu,u)=(1-\frac{1}{p_+})\alpha$.

Remark. – As we shall see in the proof, Part (b)(i), the conclusions for $\underline{d}(\mu,u)$ in parts (b)(ii) and (c), and therefore the "if" parts of (a)(i) and (a)(ii), all hold without the conditions of the theorem.

Part (a)(i) gives a necessary and sufficient condition under which there is *no exceptional point* u in (0.7), for almost all ω. Similarly, part (a)(ii) gives a criterion for $\{u\in\partial\mathbf{T}\colon \underline{d}(\mu,u)\neq\alpha\}=\emptyset$ a.s. We conjecture that a similar result would also hold for the upper local dimension: the condition $p_-=\infty$ would be necessary and sufficient for $\{u\in\partial\mathbf{T}\colon \overline{d}(\mu,u)\neq\alpha\}=\emptyset$ a.s.

Part (b)(ii) shows that, when $p_+=\infty$ and $p_-<\infty$, the branching measure and the occupation measure of a stable process [10] have the same property that a.s. the lower local dimension is constant but the upper local dimension is not so. Parts (b)(i) and (b)(iii) show that in the other cases, a new phenomenon occurs for the branching measure compared with the stable occupation measure.

Part (c) gives the exact uniform bounds of the lower local dimension. We presume that the following similar result for the upper local dimension would also hold: a.s.

$$\inf_{u\in\partial\mathbf{T}}\overline{d}(\mu,u)=\alpha \quad\text{and}\quad \sup_{u\in\partial\mathbf{T}}\overline{d}(\mu,u)=\left(1+\frac{1}{p_-}\right)\alpha.$$

(This conjecture is of course sharper than the preceding one about a necessary and sufficient condition for $\{u\in\partial\mathbf{T}\colon \overline{d}(\mu,u)\neq\alpha\}=\emptyset$.) Therefore, since a.s. $\sup_{u\in\partial\mathbf{T}}\underline{d}(\mu,u)=\alpha$, in the case where $p_-<\infty$ or $p_+<\infty$, a.s. there would be no point $u\in\partial\mathbf{T}$ for which $\underline{d}(\mu,u)=\overline{d}(\mu,u)\neq\alpha$; in other words, a.s. the limit in (0.7) would not exist at every point where (0.7) is false.

One would be able to calculate explicitly the Hausdorff dimensions of some sets of exceptional points u where (0.7) fails; in some special cases this has been done very recently by Shieh and Taylor [23], using Theorem 4.1. [2]

We need three lemmas for the proof of our Theorem.

LEMMA 4.1. – *With probability* 1, *for all* $u\in\partial\mathbf{T}$, $(1-\frac{1}{p_+})\alpha\leqslant\underline{d}(\mu,u)\leqslant\overline{d}(\mu,u)\leqslant(1+\frac{1}{p_-})\alpha$.

[2] Note added in Proof. – A more complete description of the multifractal spectra of the branching measure is recently given by Quansheng Liu and Zhiying Wen: *Analyse multifractale de la mesure de branchement* (in preparation).

Proof. – The conclusion comes directly from Theorems 2.1 and 3.1, remarking that for all $u \in \partial\mathbf{T}$, $m_n \leqslant \mu(B_{u|n}) \leqslant M_n$, $n \in \mathbb{N}$, so that $\limsup_{n\to\infty}(-\log M_n/n) \leqslant \overline{d}(\mu, u) \leqslant \limsup_{n\to\infty}(-\log m_n/n)$ and $\liminf_{n\to\infty}(-\log M_n/n) \leqslant \underline{d}(\mu, u) \leqslant \liminf_{n\to\infty}(-\log m_n/n)$. □

LEMMA 4.2. – *With probability* 1, *for all* $u \in \partial\mathbf{T}$, $\underline{d}(\mu, u) \leqslant \alpha$.

Proof. – Let $\eta > \alpha$ be arbitrarily fixed. We need to prove that $P[\sup_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \leqslant \eta] = 1$. Notice that $\underline{d}(\mu, u) \leqslant \eta$ if $\mu(B_{u|n}) > \mathrm{e}^{-n\eta}$ for infinitely many $n \in \mathbb{N}$. Hence

$$\begin{aligned}
\Big[\omega\colon \sup_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \leqslant \eta\Big] &= \bigcap_{u\in\partial\mathbf{T}} [\,\underline{d}(\mu, u) \leqslant \eta] \\
&\supset \bigcap_{u\in\partial\mathbf{T}} \big[\mu(B_{u|n}) > \mathrm{e}^{-n\eta} \text{ for infinitely many } n \in \mathbb{N}\big] \\
&= \bigcap_{u\in\partial\mathbf{T}} \bigcap_{k\geqslant 1} \bigcup_{n\geqslant k} \big[\mu(B_{u|n}) > \mathrm{e}^{-n\eta}\big] = \bigcap_{k\geqslant 1} \bigcap_{u\in\partial\mathbf{T}} \bigcup_{n\geqslant k} \big[\mu(B_{u|n}) > \mathrm{e}^{-n\eta}\big].
\end{aligned}$$

Therefore we need only to prove that for all $k \geqslant 1$, $P(\bigcap_{u\in\partial\mathbf{T}} \bigcup_{n\geqslant k} [\mu(B_{u|n}) > \mathrm{e}^{-n\eta}]) = 1$, or, equivalently,

$$P\bigg(\bigcup_{u\in\partial\mathbf{T}} \bigcap_{n\geqslant k} \big[\mu(B_{u|n}) \leqslant \mathrm{e}^{-n\eta}\big]\bigg) = 0. \tag{4.1}$$

Denote by A_k the event in the left hand side of (4.1). For all $k \geqslant 1$ and all $l \geqslant k$, we have

$$\begin{aligned}
A_k &\subset \bigcup_{u\in\partial\mathbf{T}} \bigcap_{k\leqslant n\leqslant l} \big[\mu(B_{u|n}) \leqslant \mathrm{e}^{-n\eta}\big] = \bigcup_{u\in z_l} \bigcap_{k\leqslant n\leqslant l} \big[\mu(B_{u|n}) \leqslant \mathrm{e}^{-n\eta}\big] \\
&\subset \bigcup_{u\in z_l} \bigcap_{k\leqslant n< l} \Big\{\big\{[N_{u|n} > 1] \cap [\mu(B_{u|n}) \leqslant \mathrm{e}^{-n\eta}]\big\} \cup [N_{u|n} = 1]\Big\}.
\end{aligned} \tag{4.2}$$

Now for each $n \geqslant 0$ and each $u = (u_1, \ldots, u_{n+1}) = (u|n, u_{n+1}) \in \mathbb{N}^{*(n+1)}$, given $\{N_{u|(n)}\}$, we define $u_* = (u|n, u_{n+1} + 1)$ if either $1 \leqslant u_{n+1} < N_{u|n}$, or $N_{u|n} = 1$, or $u_{n+1} > N_{u|n}$, and $u_* = (u|n, 1)$ if $u_{n+1} = N_{u|n} > 1$. Then a.s. for each $u \in \mathbb{N}^{*(n+1)}$, $\mu(B_{u|n}) \geqslant \mu(B_{(u|(n+1))_*})$ if $N_{u|n} > 1$ and $u_{n+1} \leqslant N_{u|n}$. If $N_{u|n} = 1$ or $u_{n+1} > N_{u|n}$, the sequence $(u|(n+1))_*$ will play no role for our purpose; we have defined it as well only for the sake of convenience. By (4.2), for all $k \geqslant 1$ and all $l > k$,

$$\begin{aligned}
P(A_k) &\leqslant E \sum_{u\in z_l} \prod_{k\leqslant n<l} \big[\mathbf{1}\{N_{u|n} > 1\}\mathbf{1}\big\{\mu(B_{u|n}) \leqslant \mathrm{e}^{-n\eta}\big\} + \mathbf{1}\{N_{u|n} = 1\}\big] \\
&\leqslant E \sum_{u\in z_l} \prod_{k\leqslant n<l} \big[\mathbf{1}\{N_{u|n} > 1\}\mathbf{1}\big\{\mu(B_{u|(n+1)_*}) \leqslant \mathrm{e}^{-n\eta}\big\} + \mathbf{1}\{N_{u|n} = 1\}\big].
\end{aligned} \tag{4.3}$$

Denote by $I_k(l)$ the last expectation. We shall prove that $\lim_{l\to\infty} I_k(l) = 0$ for all $k \geqslant 1$. For convenience, let us only consider the case where $k = 1$, the general case being very similar. By the definition of z_l, we have

$$\begin{aligned}
I_1(l) = E &\sum_{u_1\ldots u_l \in \mathbb{N}^{*l}} \mathbf{1}\{u_1 \leqslant N\}\mathbf{1}\{u_2 \leqslant N_{u_1}\} \\
&\times \big[\mathbf{1}\{N_{u_1} > 1\}1\big\{\mu(B_{u_1u_{2*}}) \leqslant \mathrm{e}^{-\eta}\big\} + \mathbf{1}\{N_{u_1} = 1\}\big]\mathbf{1}\{u_3 \leqslant N_{u_1u_2}\}
\end{aligned}$$

$$\times\big[\mathbf{1}\{N_{u_1u_2}>1\}\mathbf{1}\big\{\mu(B_{u_1u_2u_{3*}})\leqslant \mathrm{e}^{-2\eta}\big\}+\mathbf{1}\{N_{u_1u_2}=1\}\big]\cdots\mathbf{1}\{u_l\leqslant N_{u_1\ldots u_{l-1}}\}$$
$$\times\big[\mathbf{1}\{N_{u_1\ldots u_{l-1}}>1\}\mathbf{1}\big\{\mu(B_{u_1\ldots u_{l*}})\leqslant \mathrm{e}^{-(l-1)\eta}\big\}+\mathbf{1}\{N_{u_1\ldots u_{l-1}}=1\}\big]. \tag{4.4}$$

Notice that for each fixed $u_1\ldots u_l\in\mathbb{N}^{*l}$ and for given $\{N,N_{u_1},\ldots,N_{u_1\ldots u_{l-1}}\}$, the random variables

$$\mu(B_{u_1u_{2*}}),\mu(B_{u_1u_2u_{3*}}),\ldots,\mu(B_{u_1\ldots u_{l*}})$$

are (conditionally) independent each other, and their conditional distributions are the same as

$$\mathrm{e}^{-2\alpha}W,\mathrm{e}^{-3\alpha}W,\ldots,\mathrm{e}^{-l\alpha}W,$$

respectively. Therefore by exchanging the order of the expectation E and the sum $\sum$ in (4.4) and by calculating the conditional expectation of each general term conditional on the family of random variables $\{N,N_{u_1},\ldots,N_{u_1\ldots u_{l-1}}\}$, we obtain

$$I_1(l)=\sum_{u_1\ldots u_l\in\mathbb{N}^{*l}}E\mathbf{1}\{u_1\leqslant N\}\mathbf{1}\{u_2\leqslant N_{u_1}\}\big[\mathbf{1}\{N_{u_1}>1\}P\big\{W\leqslant \mathrm{e}^{\alpha}\mathrm{e}^{-(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1}=1\}\big]$$
$$\times\mathbf{1}\{u_3\leqslant N_{u_1u_2}\}\big[\mathbf{1}\{N_{u_1u_2}>1\}P\big\{W\leqslant \mathrm{e}^{\alpha}\mathrm{e}^{-2(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1u_2}=1\}\big]\times\cdots$$
$$\times\mathbf{1}\{u_l\leqslant N_{u_1\ldots u_{l-1}}\}\big[\mathbf{1}\{N_{u_1\ldots u_{l-1}}>1\}P\big\{W\leqslant \mathrm{e}^{\alpha}\mathrm{e}^{-(l-1)(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1\ldots u_{l-1}}=1\}\big].$$

That is

$$I_1(l)=E\sum_{u_1\ldots u_l\in z_l}\prod_{n=1}^{l-1}\big[\mathbf{1}\{N_{u_1\ldots u_n}>1\}P\big\{W\leqslant m\mathrm{e}^{-n(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1\ldots u_n}=1\}\big]. \tag{4.5}$$

Now for each fixed $u_1\ldots u_{l-1}\in\mathbb{N}^{*(l-1)}$,

$$\begin{aligned}x_l:=E&\sum_{1\leqslant u_l\leqslant N_{u_1\ldots u_{l-1}}}\big[\mathbf{1}\{N_{u_1\ldots u_{l-1}}>1\}P\big\{W\leqslant m\mathrm{e}^{-(l-1)(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1\ldots u_{l-1}}=1\}\big]\\&=EN_{u_1\ldots u_{l-1}}\big[\mathbf{1}\{N_{u_1\ldots u_{l-1}}>1\}P\big\{W\leqslant m\mathrm{e}^{-(l-1)(\eta-\alpha)}\big\}+\mathbf{1}\{N_{u_1\ldots u_{l-1}}=1\}\big]\\&=EN\big[\mathbf{1}\{N>1\}P\big\{W\leqslant m\mathrm{e}^{-(l-1)(\eta-\alpha)}\big\}+\mathbf{1}\{N=1\}\big]\\&=(m-p_1)P\big\{W\leqslant m\mathrm{e}^{-(l-1)(\eta-\alpha)}\big\}+p_1.\end{aligned}$$

Therefore by calculating the conditional expectation of $I_1(l)$ given $\{N_v\colon |v|<l-1\}$, we see that

$$\frac{I_1(l)}{I_1(l-1)}=x_l\to p_1,\quad \text{as } l\to\infty.$$

Since $p_1<1$, this implies $\lim_{l\to\infty}I_1(l)=0$. A similar argument implies that $\lim_{l\to\infty}I_k(l)=0$ for all $k\geqslant 1$. Because $P(A_k)\leqslant I_l(k)$ for all $l>k$, we see that (4.1) holds, so that the proof is finished. □

LEMMA 4.3. – *The following assertions hold*:

(i) *if* $EN^{1+\delta}<\infty$ *for some* $\delta>0$ *and if* (3.3) *holds, then a.s.* $\{u\in\partial\mathbf{T}\colon \underline{d}(\mu,u)\leqslant a\}\neq\emptyset$ *for all* $a>\alpha(1-1/p_+)$;

(ii) *if* $EN^{1+\delta} < \infty$ *for some* $\delta > 0$, *then a.s.* $\{u \in \partial\mathbf{T}\colon\ \overline{d}(\mu,u) \geqslant a\} \neq \emptyset$ *for all* $a < \bar{\alpha}_0 := \alpha\Big[1 + \Big(\frac{p_+-1}{p_+}\Big)\frac{1}{p_-+1}\Big]$, *where one sets* $\frac{p_+-1}{p_+} = 1$ *if* $p_+ = \infty$, *and* $\frac{1}{p_-+1} = 0$ *if* $p_- = \infty$.

Notice that (ii) implies that a.s. $\sup_{u\in\partial\mathbf{T}} \overline{d}(\mu,u) \geqslant \bar{\alpha}_0$, so that $\sup_{u\in\partial\mathbf{T}} \overline{d}(\mu,u) > \alpha$ if $p_- < \infty$, and $\sup_{u\in\partial\mathbf{T}} \overline{d}(\mu,u) = \alpha$ if $p_- = \infty$, using Lemma 4.1.

The conclusion in part (i) may seem to be a direct consequence of Theorem 3.1. But a difficulty occurs when we use the standard argument by compactness: by Theorem 3.1 a.s. for each $\varepsilon > 0$, there is a sequence $(u_n)_n \subset \partial\mathbf{T}$ such that, for all n, $-n^{-1}\log\mu(B_{u_n|n}) \leqslant (1-1/p_+)\alpha + \varepsilon$; by the compactness of $\partial\mathbf{T}$, we can assume that $u_n \to u$ for some $u \in \partial\mathbf{T}$; however all these implies nothing for the sequence $\mu(B_{u|n})$. We therefore present a new approach; the main idea is to construct a non-homogeneous branching process whose infinite descendants satisfy the desired property.

Proof. – (i) We shall prove the following slightly more general result: whether (3.3) holds or not, we have, with probability 1,

$$\{u \in \partial\mathbf{T}\colon\ \underline{d}(\mu,u) \leqslant a\} \neq \emptyset \quad \text{for all } a > \alpha_0 := \alpha\left[1 - \left(\frac{p_+-1}{p_+}\right)\frac{1}{\bar{p}_+-1}\right], \tag{4.6}$$

where $\bar{p}_+ = \limsup_{x\to\infty} -\log P(W > x)/\log x$, and one sets $\alpha_0 = \alpha$ if $\bar{p}_+ = \infty$. The result is evident if $\bar{p}_+ = \infty$, since a.s. $d(\mu,u) = \alpha$ for μ-a.e. u. So we assume $\bar{p}_+ < \infty$. By (3.2) and the definition of $\bar{p}_+$, if $0 < \underline{b} < p_+ \leqslant \bar{p}_+ < b < \infty$, then there is some $x_0 > 0$ large enough such that for all $x > x_0$,

$$x^{-b} \leqslant P(W > x) \leqslant x^{-\underline{b}}. \tag{4.7}$$

Since $EN^{1+\delta} < \infty$, $p_+ > 1$. Fix $\alpha > a > 0$, $b > \bar{p}_+$ and $1 < \underline{b} < p_+$. Set $n_k = \lambda^k$ for $k \geqslant 1$, where $\lambda \in \mathbb{N}^*$ will be chosen large enough. Write $\delta_1 = n_1$ and $\delta_k = n_k - n_{k-1}$ if $k > 1$. For simplicity, let us assume that for *all* $\omega \in \Omega$, $\mu(B_u)$ is well-defined for all finite sequence $u \in \mathbf{U}$ with $\lim_{k\to\infty} \mu B_{u|k} = \mu(\{u\}) = 0$ for all $u \in \partial\mathbf{T}$ (recall that μ has no atom a.s. [16]); otherwise we can restrict ourselves to a subset of Ω with probability 1. Define a sub-tree $D = \bigcup_{k\geqslant 0} D_k$ of $\mathbf{T}$ as follows (D_k represents the nodes in kth level): $D_0 = \{\emptyset\}$, $D_1 = \{u_1 \in \mathbf{T}\colon\ |u_1| = n_1\}$, and for $k \geqslant 1$,

$$D_{k+1} = \big\{u_1\ldots u_{k+1} \in \mathbf{T}\colon\ u_1\ldots u_k \in D_k,\ |u_{k+1}| = \delta_{k+1},\\ \mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\big\}.$$

Then for all $k \geqslant 1$,

$$\begin{aligned} {}^{\#}D_{k+1} &= \sum_{\substack{u_1\ldots u_{k+1}\in\mathbf{T},\\ \forall i\ |u_i|=\delta_i}} \prod_{i=1}^{k} \mathbf{1}\big\{\mu(B_{u_1\ldots u_i}) - \mu(B_{u_1\ldots u_{i+1}}) > \mathrm{e}^{-an_i} - \mathrm{e}^{-an_{i+1}}\big\} \\ &= \sum_{u_1\ldots u_k\in D_k} X_{u_1\ldots u_k}, \end{aligned}$$

where

$$X_{u_1\ldots u_k} = \sum_{\substack{u_{k+1}\in \mathbf{T}_{u_1\ldots u_k},\\ |u_{k+1}|=\delta_{k+1}}} \mathbf{1}\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\}.$$

Notice that for each fixed $u_1\ldots u_{k+1} \in \mathbf{U}$ with $|u_i| = \delta_i$, the random variable

$$\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) = \sum_{v\in \mathbf{T}_{u_1\ldots u_k}, |v|=\delta_{k+1}, v\neq u_{k+1}} \mu(B_{u_1\ldots u_k v})$$

is independent of $\mu(B_{u_1\ldots u_{k+1}})$; similarly, $\{\mu(B_{u_1\ldots u_i}) - \mu(B_{u_1\ldots u_{i+1}})\}$ $(1 \leqslant i \leqslant k)$ is a sequence of independent random variables. This is the reason why we consider the events $\{\mu(B_{u_1\ldots u_i}) - \mu(B_{u_1\ldots u_{i+1}}) > \mathrm{e}^{-an_i} - \mathrm{e}^{-an_{i+1}}\}_i$ rather than $\{\mu(B_{u_1\ldots u_i}) > \mathrm{e}^{-an_i}\}_i$. It is easily seen that for each fixed $u_1\ldots u_{k+1} \in \mathbf{U}$ with $|u_i| = \delta_i$, the random variable $\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}})$ (which depends only on $\{N_v\colon v > u_1\ldots u_k\}$) is independent of each of the following three families:

(a) $\{\mu(B_{u_1\ldots u_i}) - \mu(B_{u_1\ldots u_{i+1}})\colon i \leqslant k-1\}$ (which is independent of $\{N_v\colon v > u_1\ldots u_k\}$),

(b) $\{\mu(B_{v_1\ldots v_i}) - \mu(B_{v_1\ldots v_{i+1}})\colon i \leqslant k-1, v_1\ldots v_{i+1} \not< u_1\ldots u_k, |v_j| = \delta_j \ \forall j \leqslant i+1\}$ (which is also independent of $\{N_v\colon v > u_1\ldots u_k\}$), and

(c) $\{\mathbf{1}\{v_1\ldots v_k \in \mathbf{T}\}\colon |v_i| = \delta_i \ \forall i \leqslant k\}$ (which depends only on $\{N_v\colon |v| < n_k\}$).

Therefore $X_{u_1\ldots u_k}$ is independent of the family $\{\mathbf{1}\{v \in D_k\}\colon v \in \mathbf{U}\}$, so that it is independent of ${}^{\#}D_k$. It is then clear that $({}^{\#}D_k)$ $(k \geqslant 0)$ forms a branching process with varying environments; each individual $u_1\ldots u_k \in D_k$ $(k \geqslant 1)$ gives birth to $X_{u_1\ldots u_k}$ children whose distribution does not depend on the choice of the sequence $u_1\ldots u_k$ (but only on the generational number k). We shall claim that with positive probability, the genealogical tree D does not terminate at finite time. Put $m_0 = E\,{}^{\#}D_1$, $m_0^{(1+\varepsilon)} = E[({}^{\#}D_1)^{1+\varepsilon}]$ and, for $k \geqslant 1$,

$$m_k = EX_{u_1\ldots u_k}, \qquad m_k^{(1+\varepsilon)} = EX_{u_1\ldots u_k}^{1+\varepsilon}, \quad \varepsilon > 0.$$

By the argument of the proof of Theorem 3(ii) of [6] about the survival probability of a branching process in varying environments, it can be easily shown that for all $k \geqslant 1$ and all $0 < \varepsilon \leqslant 1$,

$$P({}^{\#}D_k > 0) \geqslant \left\{1 + \sum_{i=0}^{k-1} P_i^{-\varepsilon}\left[\frac{m_i^{(1+\varepsilon)}}{m_i^{1+\varepsilon}} - 1\right]\right\}^{-1/\varepsilon}, \tag{4.8}$$

where $P_0 = 1$, $P_i = \prod_{j=0}^{i-1} m_j$ if $i \geqslant 1$. Consequently $\lim_{k\to\infty} P({}^{\#}D_k > 0) > 0$ if for some $0 < \varepsilon \leqslant 1$,

$$\sum_{i=0}^{\infty} P_i^{-\varepsilon}\,\frac{m_i^{(1+\varepsilon)}}{m_i^{1+\varepsilon}} < \infty. \tag{4.9}$$

To prove (4.9), we need a lower bound of m_i and an upper bound of $m_i^{(1+\varepsilon)}$. Using

$$\mathbf{1}\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\}$$

$$\geqslant \mathbf{1}\{\mu(B_{u_1\dots u_k}) > \mathrm{e}^{-an_k}\} - \mathbf{1}\{\mu(B_{u_1\dots u_{k+1}}) > \mathrm{e}^{-an_{k+1}}\},$$

we obtain that

$$\begin{aligned} m_k &\geqslant E\mathbf{1}\{\mu(B_{u_1\dots u_k}) > \mathrm{e}^{-an_k}\}^{\#}\{u_{k+1} \in \mathbf{T}_{u_1\dots u_k}\colon |u_{k+1}| = \delta_{k+1}\} \\ &\quad - E \sum_{u_{k+1}\in \mathbf{T}_{u_1\dots u_k},\, |u_{k+1}|=\delta_{k+1}} \mathbf{1}\{\mu(B_{u_1\dots u_{k+1}}) > \mathrm{e}^{-an_{k+1}}\beta\} \\ &= E\mathbf{1}\{W > \mathrm{e}^{(\alpha-a)n_k}\}^{\#} z_{\delta_{k+1}} - P[W > \mathrm{e}^{(\alpha-a)n_{k+1}}] m^{\delta_{k+1}}, \end{aligned}$$

where the last equality holds because for each fixed $u_1 \dots u_{k+1} \in \mathbf{U}$ with $|u_i| = \delta_i$, the random variable $\mu(B_{u_1\dots u_{k+1}})$ is independent of the event $\{u_1 \dots u_{k+1} \in \mathbf{T}\}$, and $P\{\mu(B_{u_1\dots u_{k+1}}) > \mathrm{e}^{-an_{k+1}}\} = P[W > \mathrm{e}^{(\alpha-a)n_{k+1}}]$. Therefore for all $k \geqslant 1$,

$$m_k \geqslant \left(l_k - P[W > \mathrm{e}^{(\alpha-a)n_{k+1}}]\right) m^{\delta_{k+1}}, \tag{4.10}$$

where $l_k = E\mathbf{1}\{W > \mathrm{e}^{(\alpha-a)n_k}\} W_{(\delta_{k+1})}$, with $W_{(j)} = ({}^{\#}z_j) m^{-j}$ if $j \in \mathbb{N}^*$. Using $W_{(\delta_{k+1})} \geqslant W - |W_{(\delta_{k+1})} - W|$ and the lower bound of $P(W > x)$ (cf. (4.7)), we see that if $\mathrm{e}^{(\alpha-a)n_1} \geqslant x_0$, then for all $k \geqslant 1$,

$$\begin{aligned} l_k &\geqslant E\mathbf{1}\{W > \mathrm{e}^{(\alpha-a)n_k}\} W - E\mathbf{1}\{W > \mathrm{e}^{(\alpha-a)n_k}\} |W_{(\delta_{k+1})} - W| \\ &\geqslant \mathrm{e}^{(\alpha-a)n_k} P\{W > \mathrm{e}^{(\alpha-a)n_k}\} - r_k \\ &\geqslant \mathrm{e}^{-(\alpha-a)(b-1)n_k} - r_k, \end{aligned}$$

where $r_k = E\mathbf{1}\{W > \mathrm{e}^{(\alpha-a)n_k}\} |W_{(\delta_{k+1})} - W|$. By our condition, $EN^p < \infty$ for some $p \in (1, 2]$; therefore by Proposition 1.3, there is a constant $C > 0$ such that for all $j \in \mathbb{N}^*$, $E|W_{(j)} - W|^p \leqslant C m^{-(p-1)j}$. Using this together with the upper bound of $P(W > x)$ (cf. (4.7)), we obtain

$$\begin{aligned} r_k &\leqslant \left(P\{W > \mathrm{e}^{(\alpha-a)n_k}\}\right)^{1/q} \left(E[|W_{(\delta_{k+1})} - W|^p]\right)^{1/p} \quad \left(\text{where } \frac{1}{p} + \frac{1}{q} = 1\right) \\ &\leqslant \mathrm{e}^{-(\alpha-a)\underline{b}n_k/q} C^{1/p} m^{-(p-1)\delta_{k+1}/p} \\ &= C^{1/p} \exp\{-(\alpha-a)\underline{b}n_k/q - \alpha(p-1)\delta_{k+1}/p\}. \end{aligned}$$

It follows that if λ is large enough, say $\lambda \geqslant \lambda_0$, then for some constant $c_1 > 0$ and all $k \geqslant 1$,

$$l_k \geqslant c_1 \exp\{-(\alpha-a)(b-1)n_k\}.$$

Therefore by (4.10) together with the upper bound of $P(W > x)$ given in (4.7), if λ is large enough, say $\lambda \geqslant \lambda_1$ (it suffices to choose λ_1 such that $\lambda_1 \geqslant \lambda_0$ and that $(b-1) < \underline{b}\lambda_1$), then there is some constant $c_2 > 0$ such that for all $k \geqslant 1$,

$$\frac{m_k}{m^{\delta_{k+1}}} \geqslant c_2 \exp\{-(\alpha-a)(b-1)n_k\}. \tag{4.11}$$

Let $0 < \eta < 1$ be small enough such that $\varepsilon := (\underline{b} - 1)\eta \leqslant 1$. Using $X_{u_1\dots u_k} \leqslant^{\#} \{u_{k+1} \in T_{u_1\dots u_k}\colon |u_{k+1}| = \delta_{k+1}\}$ and Hölder's inequality (with $p' = 1/(1-\eta)$ and $q' = 1/\eta$), we have

$$
\begin{aligned}
m_k^{(1+\varepsilon)} &\leqslant E\big\{X_{u_1\dots u_k}^{1-\eta}\big({}^{\#}\{u_{k+1}\in \mathbf{T}_{u_1\dots u_k}\colon\ |u_{k+1}|=\delta_{k+1}\}\big)^{\varepsilon+\eta}\big\}\\
&\leqslant (EX_{u_1\dots u_k})^{1-\eta}\big\{E\big({}^{\#}\{u_{k+1}\in \mathbf{T}_{u_1\dots u_k}\colon\ |u_{k+1}|=\delta_{k+1}\}\big)^{(\varepsilon+\eta)/\eta}\big\}^{\eta}\\
&= m_k^{1-\eta}\big\{E\big[({}^{\#}z_{\delta_{k+1}})^{(\varepsilon+\eta)/\eta}\big]\big\}^{\eta}.
\end{aligned}
$$

Remarking that $\varepsilon+\eta=\underline{b}\eta$ and dividing the above display by $m_k^{1+\varepsilon}$, we obtain

$$
\frac{m_k^{(1+\varepsilon)}}{m_k^{1+\varepsilon}} \leqslant \big[EW_{(\delta_{k+1})}^{\underline{b}}\big]^{\eta}\left(\frac{m^{\delta_{k+1}}}{m_k}\right)^{\varepsilon+\eta}.
$$

Since $EN^{\underline{b}}<\infty$, the sequence $(W_{(j)})$ is bounded in $L^{\underline{b}}$. Therefore for some constant $C_1>0$ and all $k\geqslant 1$,

$$
\frac{m_k^{(1+\varepsilon)}}{m_k^{1+\varepsilon}} \leqslant C_1\left(\frac{m^{\delta_{k+1}}}{m_k}\right)^{\varepsilon+\eta}. \tag{4.12}
$$

Remark that $m_0=m^{n_1}$. It then follows from (4.11) and (4.12) that if $\lambda\geqslant\lambda_1$ is large enough and if $\varepsilon=(\underline{b}-1)\eta\leqslant 1$ is small enough, then for some constant $C_2=C_2(\lambda,\varepsilon)>0$ and all $k\geqslant 2$,

$$
\begin{aligned}
P_k^{-\varepsilon}\frac{m_k^{(1+\varepsilon)}}{m_k^{1+\varepsilon}} \leqslant C_2^k \exp\{&-\varepsilon\alpha n_k+\varepsilon(\alpha-a)(b-1)(n_1+\cdots+n_{k-1})\\
&+(\alpha-a)(b-1)(\varepsilon+\eta)n_k\}.
\end{aligned}\tag{4.13}
$$

Using this and the fact that $n_k=\lambda^k$ and $n_1+\cdots+n_{k-1}=(\lambda^k-\lambda)/(\lambda-1)$, it is easily seen that (4.9) holds whenever

$$
\varepsilon\alpha>\varepsilon(\alpha-a)(b-1)\frac{1}{\lambda-1}+(\alpha-a)(b-1)(\varepsilon+\eta). \tag{4.14}
$$

Notice that $\varepsilon+\eta=\varepsilon\underline{b}/(\underline{b}-1)$. We can always choose $\lambda\geqslant\lambda_1$ large enough for (4.14) to be true if

$$
\alpha>(\alpha-a)(b-1)\frac{\underline{b}}{\underline{b}-1}, \tag{4.15}
$$

which is equivalent to

$$
a>a_0:=\alpha\left[1-\left(\frac{\underline{b}-1}{\underline{b}}\right)\frac{1}{b-1}\right]. \tag{4.16}
$$

Notice that $a_0\to\alpha_0$ if $\underline{b}\to p_+$ and $b\to\bar{p}_+$. So if $\alpha>a>\alpha_0$, then we can choose $1<\underline{b}<p_+$ and $b>\bar{p}_+$ for which $\alpha>a>a_0$. We have therefore proved that if $\alpha>a>\alpha_0$, then we can choose $\lambda\in\mathbb{N}^*$ large enough such that (4.9) holds, so that

$$
P\left(\bigcap_{k=1}^{\infty}\{D_k\neq\emptyset\}\right)=\lim_{k\to\infty}P({}^{\#}D_k>0)>0.
$$

Let $\partial D = \{u \in \partial\mathbf{T}\colon \forall k \geqslant 1,\ u|n_k \in D_k\}$ be the set of infinite descendants of the non-homogeneous branching process $(D_k)_k$. What we have proved above implies that

$$P(\partial D \neq \emptyset) > 0 \tag{4.17}$$

if $\alpha > a > \alpha_0$ and if $\lambda \in \mathbb{N}^*$ is large enough. If $u \in \partial D$, then by the definition of ∂D and D_k, for all $k \geqslant 1$,

$$\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}.$$

Adding up the consecutive inequalities, we obtain

$$\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_l}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_l} \quad \text{if } l > k.$$

Letting $l \to \infty$ gives

$$\mu(B_{u_1\ldots u_k}) \geqslant \mathrm{e}^{-an_k}.$$

Clearly this implies $\underline{d}(\mu, u) \leqslant a$. Therefore writing

$$A_a = \{u \in \partial\mathbf{T}\colon \underline{d}(\mu, u) \leqslant a\},$$

we have $\partial D \subset A_a$, so $\{\partial D \neq \emptyset\} \subset \{A_a \neq \emptyset\}$. Hence by (4.17), if $\alpha > a > \alpha_0$, then

$$P(\{A_a \neq \emptyset\}) > 0. \tag{4.18}$$

By the monotonicity of the event $\{A_a \neq \emptyset\}$ in a, if (4.18) holds for some $a = a_1$ then it also holds for all $a > a_1$. Therefore (4.18) holds for all $a > \alpha_0$.

Now by considering the sub-trees of $\mathbf{T}$ beginning with the nodes $i \in \{1, \ldots, N\}$, it can be easily checked that the probability $q_a := P(A_a = \emptyset)$ is a fixed point of $f(x) = \sum_{i=1}^{\infty} p_i x^i$. Since f has only two fixed points 0 and 1 on $[0, 1]$ (recall that $p_0 = 0$), the assertion $q_a < 1$ (cf. (4.18)) implies $q_a = 0$. Therefore we have proved that for all $a > a_0$, a.s. $A_a \neq \emptyset$. Hence a.s. $A_a \neq \emptyset$ for all rational $a > \alpha_0$. By the monotonicity of A_a (in a), this implies that a.s. $A_a \neq \emptyset$ for all $a > \alpha_0$.

(ii) The proof of part (ii) is similar: by (0.7) the conclusion is evident if $p_- = \infty$; so we assume $p_- < \infty$, fix $a > \alpha$, and consider the the events $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) \leqslant \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) \leqslant \mathrm{e}^{-an_k}\}$ and $\{W \leqslant \mathrm{e}^{-(a-\alpha)n_k}\}$ instead of $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) > \mathrm{e}^{-an_k}\}$ and $\{W > \mathrm{e}^{(\alpha-a)n_k}\}$ $(k \geqslant 1)$ respectively, using

$$c_3 x^{p_-} \leqslant P(W \leqslant x) \leqslant c_4 x^{p_-} \quad \text{and} \quad c_5 x^{p_-+1} \leqslant EW\mathbf{1}\{W \leqslant x\} \leqslant c_6 x^{p_-+1},$$

where c_i $(3 \leqslant i \leqslant 6)$ are some positive constants independent of x, $0 < x \leqslant 1$. Here to see that $c_5 x^{p_-+1} \leqslant EW\,\mathbf{1}\{W \leqslant x\}$, it suffices to take $\eta \in (0, 1)$ small enough such that $c_5 := \eta(c_3 - \eta^{p_-} c_4) > 0$, remarking that for all $x \in (0, 1]$,

$$\begin{aligned} EW\mathbf{1}\{Z \leqslant x\} &\geqslant EW\,\mathbf{1}\{\eta x < W \leqslant x\} \\ &\geqslant \eta x[P(W \leqslant x) - P(W \leqslant \eta x)] \geqslant \eta x\bigl(c_3 - \eta^{p_-} c_4\bigr) x^{p_-}. \end{aligned} \tag{4.19}$$

The displays corresponding to (4.11), (4.14) and (4.16) are, respectively,

$$m_k m^{-\delta_{k+1}} \geqslant c_7 \exp\{-(a-\alpha)(p_- + 1)n_k\}, \tag{4.20}$$

$$\varepsilon\alpha > \varepsilon(a-\alpha)(p_- + 1)\frac{1}{\lambda - 1} + (a-\alpha)(p_- + 1)(\varepsilon + \eta) \tag{4.21}$$

and

$$a < \bar{a}_0 := \alpha\left[1 + \left(\frac{\underline{b} - 1}{\underline{b}}\right)\frac{1}{p_- + 1}\right]. \qquad \square \tag{4.22}$$

Proof of Theorem 4.1. – The assertions of part (a) follow easily from those of part (b). In part (b), the assertion (i) is a direct consequence of Lemma 4.1, and holds without the conditions of the theorem; in the assertion (ii), the conclusion for $\underline{d}(\mu, u)$ is a combination of Lemmas 4.1 and 4.2, and also holds without the conditions of the theorem, while the conclusion for $\overline{d}(\mu, u)$ follows from Lemma 4.3(ii), remarking that the number $\bar{\alpha}_0$ defined in that lemma is strictly greater than α; the assertion (iii) comes immediately from Lemma 4.3(i).

It remains to prove part (c). Since a.s. $\underline{d}(\mu, u) = \alpha$ for μ-a.e. $u \in \partial\mathbf{T}$, we have a.s. $\sup_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \geqslant \alpha$; by Lemma 4.2, $\sup_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \leqslant \alpha$. So we have proved the first assertion without the conditions of the theorem. By Lemma 4.1, a.s. $\inf_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \geqslant (1 - 1/p_+)\alpha$; by Lemma 4.3(i), a.s. $\inf_{u\in\partial\mathbf{T}} \underline{d}(\mu, u) \leqslant (1 - 1/p_+)\alpha$ if $EN^{1+\delta} < \infty$ for some $\delta > 0$ and if (3.3) holds. This gives the second assertion. $\square$

5. An equivalent of m_n

We shall see that Theorem 2.1 can be improved when W has exponential left tail. Assume $p_1 = 0$ and write

$$\underline{m} = \operatorname{ess\,inf} N \quad \text{and} \quad \beta_- = 1 - \log m / \log \underline{m}. \tag{5.1}$$

Then $-\infty < \beta_- < 0$. Define

$$r_- = \sup\{t \geqslant 0\colon \ E\exp(tW^{1/\beta_-}) < \infty\}; \tag{5.2}$$

just as in (3.2), an equivalent definition of r_- is

$$r_- = \liminf_{x\to 0} \frac{-\log P\{W < x\}}{x^{1/\beta_-}}. \tag{5.3}$$

It is known that $0 < r_- < \infty$ (whenever $p_1 = 0$). We shall sometimes need the condition that

$$r_- = \lim_{x\to 0} \frac{-\log P\{W < x\}}{x^{1/\beta_-}}. \tag{5.4}$$

THEOREM 5.1. – *Assume $p_1 = 0$, let β_- and r_- be defined in* (5.1) *and* (5.3), *and put $C_- := (\alpha/r_-)^{\beta_-}$. Then a.s.*

$$\liminf_{n\to\infty} \frac{m^n m_n}{n^{\beta_-}} = C_-. \tag{5.5}$$

If furthermore (5.4) *holds, then the* lim inf *above is in fact a* lim: *we have a.s.*

$$\lim_{n\to\infty} \frac{m^n m_n}{n^{\beta_-}} = C_-. \tag{5.6}$$

Remark. – The result (5.6) can be re-written as $\lim_{n\to\infty} \inf_{u\in\partial\mathbf{T}} \mu(B_{u|n})/\psi_-(|B_{u|n}|) = C_-$, where $\psi_-(t) = t^\alpha(\log(1/t))^{\beta_-}$. This result is similar to a property of the occupation mesure of a stable subordinator with index $\alpha \in (0,1)$, cf. Theorem 1 of Hawkes [8] and the display (3.1) of Hu and Taylor [10] .

Proof of Theorem 5.1. – (a) We first prove that a.s. $\liminf_{n\to\infty} \frac{m^n m_n}{n^{\beta_-}} \geqslant C_-$. Let $0 < C < C_-$ be arbitrarily fixed, and let $\varepsilon > 0$ be small enough such that $(r_- - \varepsilon)C^{1/\beta_-} > \alpha$. This is possible since $r_- C^{1/\beta_-} > r_- C_-^{1/\beta_-} = \alpha$. (Recall that $\beta_- < 0$.) By (5.3), for all n large enough,

$$P\{W < n^{\beta_-} C\} \leqslant \exp\{-(r_- - \varepsilon)C^{1/\beta_-} n\}.$$

Therefore by Proposition 1.2,

$$P\left[\frac{m^n m_n}{n^{\beta_-}} < C\right] \leqslant e^{n\alpha} P\{W < n^{\beta_-} C\} \leqslant \exp\{-[(r_- - \varepsilon)C^{1/\beta_-} - \alpha]n\}.$$

Since $(r_- - \varepsilon)C^{1/\beta_-} - \alpha > 0$, the series $\sum_{n=1}^\infty P[\frac{m^n m_n}{n^{\beta_-}} < C]$ converges, so that the conclusion follows by Borel–Cantelli's lemma and by letting $C \to C_-$.

(b) We next prove that a.s. $\liminf_{n\to\infty} \frac{m^n m_n}{n^{\beta_-}} \leqslant C_-$. Let $\infty > C > C_-$ be arbitrarily fixed, and let $\varepsilon > 0$ be small enough such that

$$\rho := e^{-(r_- + \varepsilon)C^{1/\beta_-}} > e^{-\alpha} = 1/m \tag{5.7}$$

(this is possible because $r_- C^{1/\beta_-} < r_- C_-^{1/\beta_-} = \alpha$). Since (5.3) also holds with x replaced by $n^{\beta_-} C$ $(n \to \infty)$, there are infinitely many $n \in \mathbb{N}$ such that

$$P(W \leqslant n^{\beta_-} C) \geqslant e^{-(r_- + \varepsilon)C^{1/\beta_-} n} = \rho^n, \tag{5.8}$$

so that by Proposition 1.2, for all these n,

$$P\left[\frac{m^n m_n}{n^{\beta_+}} > C\right] = f_n(1 - P\{W \leqslant n^{\beta_-} C\}) \leqslant f_n(1 - \rho^n). \tag{5.9}$$

Therefore by Proposition (1.1),

$$\liminf_{n\to\infty} P\left[\frac{m^n m_n}{n^{\beta_-}} < C\right] \leqslant \lim_{n\to\infty} f_n(1 - \rho^n) = 0.$$

Using $P(\liminf_{n\to\infty}[\frac{m^n m_n}{n^{\beta_-}} < C] \leqslant \liminf_{n\to\infty} P[\frac{m^n m_n}{n^{\beta_-}} < C]$ and then letting $C \to C_-$, we obtain the desired conclusion.

(c) We finally prove that if (5.4) holds, then a.s. $\limsup_{n\to\infty} \frac{m^n m_n}{n^{\beta_-}} \leqslant C_-$. Let C and ε be as in the proof (b) above. By (5.4), we know that (5.8) and so (5.9) holds for all

$n \in \mathbb{N}$ large enough; by Proposition (1.1), this implies that the series $\sum_{n=1}^{\infty} P[\frac{m^n m_n}{n^{\beta_-}} > C]$ converges, so that the conclusion follows by Borel–Cantelli's lemma and by letting $C \to C_-$. □

6. An equivalent of M_n

Just as in the case for m_n, Theorem 3.1 can also be improved when W has exponential right tail. Write

$$\overline{m} = \text{ ess sup} N \quad \text{and} \quad \beta_+ = 1 - \log m / \log \overline{m}. \tag{6.1}$$

Then $0 < \beta_+ \leqslant 1$. (By convention, $\beta_+ = 1$ if $\overline{m} = \infty$.) Define

$$r_+ = \sup\{t \geqslant 0\colon\ E \exp(t W^{1/\beta_+}) < \infty\}, \tag{6.2}$$

or, equivalently,

$$r_+ = \liminf_{x \to \infty} \frac{-\log P\{W > x\}}{x^{1/\beta}}. \tag{6.3}$$

Of course $r_+ \in [0, \infty]$. We shall sometimes need the condition that

$$r_+ = \lim_{x \to \infty} \frac{-\log P\{W > x\}}{x^{1/\beta_+}}. \tag{6.4}$$

The first part of the following theorem was proved in Liu and Shieh [17]. But for convenience of readers, we shall give a complete proof of the theorem. The result is the counter part of Theorem 5.1.

THEOREM 6.1. – *Let $\beta_+ \in (0, 1]$ and $r_+ \in [0, \infty]$ be defined in* (6.1) *and* (6.3), *and put $C_+ = (\alpha / r_+)^{\beta_+}$. Then a.s.*

$$\limsup_{n \to \infty} \frac{m^n M_n}{n^{\beta_+}} = C_+. \tag{6.5}$$

If furthermore (6.4) *holds, then the* lim sup *above is in fact a* lim: *we have a.s.*

$$\lim_{n \to \infty} \frac{m^n M_n}{n^{\beta_+}} = C_+. \tag{6.6}$$

Remarks. – (i) If either $\overline{m} < \infty$ or $E \exp(tN) < \infty$ for some but not all $t > 0$, then $0 < r_+ < \infty$ (cf. [13]), so that $0 < C_+ < \infty$, and hence Theorem 6.1 improves Theorem 3.1.

(ii) If N is of geometric distribution: $P(N = k) = p(1-p)^{k-1}$ for some $p \in (0, 1)$ and all $k \geqslant 1$, we have $C_+ = 1$; in this case (6.6) was proved by Hawkes [9, Theorem 3].

(iii) As in the case for m_n (cf. the remark following Theorem 5.1), we may rewrite the result (6.6) as $\lim_{n \to \infty} \sup_{u \in \partial \mathbf{T}} \mu(B_{u|n}) / \psi_+(|B_{u|n}|) = C_+$, where $\psi_+(t) = t^{\alpha}(\log(1/t))^{\beta_+}$; in this form the result is consistent with some well-known uniform asymptotic laws associated with Brownian motions or stable processes, see for example [12, Théorème 52.2, p. 172], [8, Theorem 2] and [22, Lemma 2.3 and Corollary 5.2].

Proof of Theorem 6.1. – The proof is similar to that of Theorem 5.1.

We first prove that a.s. $\limsup_{n\to\infty} \frac{m^n M_n}{n^{\beta_+}} \leqslant C_+$. If $C_+ = \infty$ (i.e. $r_+ = 0$), there is nothing to prove. Assume $C_+ < \infty$ (i.e. $r_+ > 0$), and let $\infty > C > C_+$ be arbitrarily fixed. Let $\varepsilon > 0$ be small enough such that $(r_+ - \varepsilon)C^{1/\beta_+} > \alpha$. This is possible since $r_+ C^{1/\beta_+} > r_+ C_+^{1/\beta_+} = \alpha$. By Proposition 1.2, we have

$$P\left[\frac{m^n M_n}{n^{\beta_+}} > C\right] \leqslant \mathrm{e}^{n\alpha} P\{W > n^{\beta_+} C\};$$

by (6.3), we have, for all n large enough,

$$P\{W > n^{\beta_+} C\} \leqslant \exp\{-(r - \varepsilon) C^{1/\beta_+} n\}.$$

Therefore $\sum_{n=1}^{\infty} P[\frac{m^n M_n}{n^{\beta_+}} > C] < \infty$ and the conclusion follows by Borel–Cantelli's lemma and by letting $C \to C_+$.

We next prove that $\limsup_{n\to\infty} \frac{m^n M_n}{n^{\beta_+}} \geqslant C_+$ a.s., and that $\liminf_{n\to\infty} \frac{m^n M_n}{n^{\beta_+}} \geqslant C_+$ a.s. if (6.4) holds. If $C_+ = 0$ (i.e. $r_+ = \infty$), there is nothing to prove. So we assume $C_+ > 0$ (i.e. $r_+ < \infty$). Let $0 < C < C_+$ be arbitrarily fixed. By Proposition 1.2, we have

$$P\left[\frac{m^n M_n}{n^{\beta_+}} < C\right] = f_n\left(1 - P\{W \geqslant n^{\beta_+} C\}\right). \tag{6.7}$$

Let $\varepsilon > 0$ be small enough such that $\rho := \mathrm{e}^{-(r_+ + \varepsilon) C^{1/\beta_+}} > \mathrm{e}^{-\alpha} = 1/m$. This is possible because $r_+ C^{1/\beta_+} < r_+ C_+^{1/\beta_+} = \alpha$. Then

$$P\left(W \geqslant n^{\beta_+} C\right) \geqslant \mathrm{e}^{-(r+\varepsilon) C^{1/\beta_+} n} = \rho^n$$

for infinitely many $n \in \mathbb{N}$ by (6.3), and for all $n \in \mathbb{N}$ large enough if (6.4) holds. Therefore by Proposition 1.1, we see that $\liminf_{n\to\infty} P[\frac{m^n M_n}{n^{\beta_+}} < C] = 0$, and that $\sum_{n=1}^{\infty} P[\frac{m^n M_n}{n^{\beta_+}} < C] < \infty$ if (6.4) holds. This implies that $\limsup_{n\to\infty} \frac{m^n M_n}{n^{\beta_+}} \geqslant C$ a.s., and that $\limsup_{n\to\infty} \frac{m^n M_n}{n^{\beta_+}} \geqslant C$ a.s. if (6.4) holds. Letting $C \to C^+$ gives the desired conclusion. □

Let us give an example where Theorem 6.1 applies easily. If the probability generating function of N has the form

$$f(s) = s/\left[m - (m-1)s^k\right]^{1/k},$$

where $m > 1$, and $k \in \mathbb{N}^*$ is a positive integer, then W has a $\Gamma(1/k, 1/k)$ distribution with density

$$d(u) = \frac{k^{1/k}}{\Gamma(1/k)} u^{1/k - 1} \mathrm{e}^{-u/k}, \quad u > 0$$

(see [7, p. 17]), $m = f'(1)$, $\alpha = \log m$, $\beta_+ = 1$, $r_+ = 1/k$, $C_+ = k \log m$, and the condition (6.4) holds. So by Theorem 6.1,

$$\lim_{n\to\infty} \frac{m^n M_n}{n} = k \log m \quad \text{a.s.}$$

If $k = 1$ (i.e. the geometric case), this was proved by Hawkes [9].

7. More on uniform bounds of local dimensions

The results in Sections 5 and 6 can be used to obtain uniform bounds for the local dimension of μ. Since $m_n \leqslant \mu(B_{u|n}) \leqslant M_n$ for all $u \in \partial\mathbf{T}$, by Theorem 6.1, we have

$$\sup_{u\in\partial\mathbf{T}} \limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_+}} \leqslant C_+ \quad \text{a.s.}; \tag{7.1}$$

and by Theorem 5.1, we have

$$\inf_{u\in\partial\mathbf{T}} \liminf_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_+}} \geqslant C_- \quad \text{a.s. if } p_1 = 0. \tag{7.2}$$

The following result shows that C_+ and C_- are the exact uniform bounds:

THEOREM 7.1. –
(i) *If* (6.4) *holds, then*

$$\sup_{u\in\partial\mathbf{T}} \limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_+}} = C_+ \quad a.s.$$

(ii) *Assume that* $p_1 = 0$ *and* $EN^p < \infty$ *for all* $p > 1$. *If* (5.4) *holds, then*

$$\inf_{u\in\partial\mathbf{T}} \liminf_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_-}} = C_- \quad a.s.$$

The proof relies on the following result, together with (7.1) and (7.2).

PROPOSITION 7.1. –
(i) *A.s.* $\{u \in \partial\mathbf{T}\colon \limsup_{n\to\infty} \frac{m^n\mu(B_{u|n})}{n^{\beta_+}} \geqslant a\} \neq \emptyset$ *for all* $0 < a < \underline{C}_+ := (\frac{\alpha}{\bar{r}_+})^{\beta_+}$, *if* $\bar{r}_+ := \limsup_{x\to\infty} \frac{-\log P(Z>x)}{x^{1/\beta_+}} < \infty$;
(ii) *a.s.* $\{u \in \partial\mathbf{T} : \liminf_{n\to\infty} \frac{m^n\mu(B_{u|n})}{n^{\beta_-}} \leqslant a\} \neq \emptyset$ *for all* $a > \overline{C}_- := (\frac{\alpha}{\bar{r}_-})^{\beta_-}(\frac{p_+-1}{p_+})^{\beta_-}$, *if* $\bar{r}_- := \limsup_{x\to\infty} \frac{-\log P(W<x)}{x^{1/\beta_-}} < \infty$ *and if* $EN^{1+\varepsilon} < \infty$ *for some* $\varepsilon > 0$. (*Where* $\frac{p_+-1}{p_+}$ *is interpreted to be* 1 *if* $p_+ = \infty$.)

Proof. – (i) The argument is similar to that of the proof of Lemma 4.3. Instead of (4.7), we have

$$c_8 \exp\{-r x^{1/\beta_+}\} \leqslant P(W > x) \leqslant c_9 \exp\{-\underline{r} x^{1/\beta_+}\}, \tag{7.3}$$

where $0 < \underline{r} < r_+$ and $\infty > r > \bar{r}_+$ are arbitrarily fixed, $c_8, c_9 > 0$ are some constants independent of $x \in (0, \infty)$; this implies clearly that for each $\delta > 0$, there is some constant $c_{10} > 0$ such that for all $x \in (0, \infty)$,

$$E[\mathbf{1}\{W > x\}W] \geqslant c_{10} \exp\{-(r+\delta)x^{1/\beta_+}\}. \tag{7.4}$$

Instead of the events $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > \mathrm{e}^{-an_k} - \mathrm{e}^{-an_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) > \mathrm{e}^{-an_k}\}$ and $\{W > \mathrm{e}^{(\alpha-a)n_k}\}$ considered in the proof of Lemma 4.3, we now consider the events $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > an_k^{\beta_+} m^{-n_k} - an_{k+1}^{\beta_+} m^{-n_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) > an_k^{\beta_+} m^{-n_k}\}$ and $\{W > an_k^{\beta_+}\}$, where the value of $a > 0$ is to be determined. The displays corresponding to (4.11), (4.15) and (4.16) are

$$\frac{m_k}{m^{\delta_{k+1}}} \geqslant c_{11} \exp\{-(r+\delta)a^{1/\beta_+} n_k\}, \tag{7.5}$$

$$\alpha > (r+\delta)a^{1/\beta_+} \frac{\underline{b}}{\underline{b}-1} \tag{7.6}$$

and

$$a < \left(\frac{\alpha}{r+\delta}\right)^{\beta} \left(\frac{\underline{b}-1}{\underline{b}}\right)^{\beta}, \tag{7.7}$$

respectively. The conclusion then follows, remarking that the right hand side of the last display tends to $(\frac{\alpha}{\bar{r}_+})^{\beta}(\frac{p_+-1}{p_+})^{\beta} = \underline{C}_+$ (notice that $r_+ < \infty$ implies $p_+ = \infty$) when $\delta \to 0$, $r \to \bar{r}_+$ and $\underline{b} \to p_+$.

(ii) The argument is very similar to the above one, by considering the events $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) \leqslant an_k^{\beta_-} m^{-n_k} - an_{k+1}^{\beta_-} m^{-n_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) \leqslant an_k^{\beta_-} m^{-n_k}\}$ and $\{W \leqslant an_k^{\beta_-}\}$ instead of $\{\mu(B_{u_1\ldots u_k}) - \mu(B_{u_1\ldots u_{k+1}}) > an_k^{\beta_+} m^{-n_k} - an_{k+1}^{\beta_+} m^{-n_{k+1}}\}$, $\{\mu(B_{u_1\ldots u_k}) > an_k^{\beta_+} m^{-n_k}\}$ and $\{W > an_k^{\beta_+}\}$ respectively. (For a lower bound of $E[\mathbf{1}\{W \leqslant x\}W]$ we use an argument similar to (4.19).) □

Proof of Theorem 7.1. – For part (i), the upper bound is given in (7.1). For the lower bound, by Proposition 7.2 (i), if $r_+ < \infty$ ($\Leftrightarrow C_+ > 0$) and if (6.4) holds, then

$$\sup_{u \in \partial \mathbf{T}} \limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_+}} \geqslant C_+ \quad \text{a.s.};$$

if $r_+ = \infty$ ($\Leftrightarrow C_+ = 0$), the inequality is evident.

The proof of part (ii) is similar: the lower bound is given by (7.2), while the upper bound comes from Proposition 7.2(ii). □

The bounds $\inf_{u\in\partial\mathbf{T}} \limsup_{n\to\infty} \frac{m^n\mu(B_{u|n})}{n^{\beta_+}}$ and $\sup_{u\in\partial\mathbf{T}} \liminf_{n\to\infty} \frac{m^n\mu(B_{u|n})}{n^{\beta_-}}$ are easier to get, but less interesting because they are respectively 0 and ∞ under some mild conditions, as is explained in the following. By Proposition 3.1(ii) of [13] and its proof, we know that:

(a) if $\theta > 0$ and $EW^{1+\theta} < \infty$, then $\lim_{n\to\infty} \frac{m^n\mu(B_{u|n})}{n^{1/\theta}} = 0$ for P-a.e. $\omega \in \Omega$ and μ-a.e. $u \in \partial\mathbf{T}$;

(b) if $\theta < 0$ and $EW^{1+\theta} < \infty$, then $\lim_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{1/\theta}} = \infty$ for P-a.e. $\omega \in \Omega$ and μ-a.e. $u \in \partial\mathbf{T}$.

This implies clearly that:

(c) if $p_+ = \infty$, then $\inf_{u\in\partial\mathbf{T}} \limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_+}} = 0$ a.s.;

(d) if $p_1 = 0$ ($\Leftrightarrow p_- = \infty$), then $\sup_{u\in\partial\mathbf{T}} \liminf_{n\to\infty} \frac{m^n \mu(B_{u|n})}{n^{\beta_-}} = \infty$ a.s.

Of course, (a) and (b) are more precise than (c) and (d), and the conditions in (c) and (d) can be relaxed.

8. Exact local dimension at typical $u \in \partial\mathbf{T}$

Recall that (cf. (0.7)) for P-almost all $\omega \in \Omega$ and μ_ω-almost all $u \in \partial\mathbf{T}$, μ_ω has lower local dimension α: $\underline{d}(\mu_\omega, u) = \alpha$. But this gives only a rough idea about large values of $\mu(B_{u|n})$ at a typical $u \in \partial\mathbf{T}$: it says that for P-almost all $\omega \in \Omega$ and μ_ω-almost all $u \in \partial\mathbf{T}$,

$$\limsup_{n\to\infty} m^{n\delta} \mu_\omega(B_{u|n}) = \begin{cases} 0 & \text{if } \delta < 1, \\ \infty & \text{if } \delta > 1. \end{cases}$$

A deeper question is to find the exact dimension of large values of $\mu_\omega(B_{u|n})$: that is, find a function ϕ such that for P-almost all $\omega \in \Omega$ and μ_ω-almost all $u \in \partial\mathbf{T}$,

$$\limsup_n m^n \mu(B_{u|n})/\phi(n) = c \quad \text{for some constant } 0 < c < \infty.$$

In [9], Hawkes solved this question in the case where N has a geometric distribution on $\mathbb{N}^*$ [9, Theorem 4], and conjectured that there would be a similar result in a general case under some conditions [9, p. 382]. The following result shows that this is indeed the case whenever the number r_+ defined by (6.2) is strictly positive and finite.

THEOREM 8.1. – *Let $\beta_+ \in (0, 1]$ and $r_+ \in [0, \infty]$ be defined by* (6.1) *and* (6.2). *Then for P-almost all $\omega \in \Omega$ and for μ_ω-almost all $u \in \partial\mathbf{T}$, we have*

$$\limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{(\log n)^{\beta_+}} = \frac{1}{r_+^{\beta_+}}. \tag{8.1}$$

Proof. – The upper bound is easy, and is a consequence of (3.4a) of [13]. We therefore need only to prove that with probability 1,

$$\limsup_{n\to\infty} \frac{m^n \mu(B_{u|n})}{(\log n)^{\beta_+}} \geqslant \frac{1}{r_+^{\beta_+}} \quad \text{for } \mu_\omega\text{-a.e. } u \in \partial\mathbf{T}(\omega). \tag{8.2}$$

If $r_+ = \infty$, there is nothing to prove. Suppose that $r_+ < \infty$. It was proved in [13, Theorem 1] that a.s.

$$\phi_+ - H(\partial\mathbf{T}) = r_+^{\beta_+} W = r_+^{\beta_+} \mu_\omega(\partial\mathbf{T}), \tag{8.3}$$

where $\phi_+(t) = t^\alpha (\log\log \frac{1}{t})^{\beta_+}$, and $\phi_+ - H(.)$ denotes the ϕ_+-Hausdorff measure. Similarly, we can prove that a.s. for all $u \in \mathbf{T}(\omega)$,

$$\phi_+ - H(B_u) = r_+^{\beta_+} \mu_\omega(B_u). \tag{8.4}$$

Therefore for almost all ω,

$$\phi_+ - H(A) = r_+^{\beta_+}\mu_\omega(A) \quad \text{for all Borel set } A \subset \partial\mathbf{T}(\omega). \tag{8.5}$$

Fix ω for which (8.5) holds. By an argument similar to that used in the proof of Theorem 5.3 of Dai and Taylor [4], we can easily prove that for all Borel $A \subset \partial\mathbf{T}(\omega)$,

$$\mu_\omega(A) \inf_{u\in A} \liminf_{n\to\infty} \frac{\phi_+(|B_{u|n}|)}{\mu_\omega(B_{u|n})} \leqslant \phi_+\text{-}H(A). \tag{8.6}$$

Using (8.4), this implies that

$$\inf_{u\in A} \liminf_{n\to\infty} \frac{\phi_+(|B_{u|n}|)}{\mu_\omega(B_{u|n})} \leqslant r_+^{\beta_+} \quad \text{if } \mu_\omega(A) > 0. \tag{8.7}$$

Let us deduce from (8.7) that

$$\liminf_{n\to\infty} \frac{\phi_+(|B_{u|n}|)}{\mu_\omega(B_{u|n})} \leqslant r_+^{\beta_+} \quad \text{for } \mu_\omega\text{-a.e. } u \in \partial\mathbf{T}(\omega). \tag{8.8}$$

Of course, it suffices to prove that for each $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{\phi_+(|B_{u|n}|)}{\mu_\omega(B_{u|n})} \leqslant r_+^{\beta_+} + \varepsilon \quad \text{for } \mu_\omega\text{-a.e. } u \in \partial\mathbf{T}(\omega). \tag{8.9}$$

In fact, if this were not true, there would exist a number $\varepsilon_0 > 0$ and a Borel set A with $\mu_\omega(A) > 0$, such that for all $u \in A$,

$$\liminf_{n\to\infty} \frac{\phi_+(|B_{u|n}|)}{\mu_\omega(B_{u|n})} \geqslant r_+^{\beta_+} + \varepsilon_0,$$

which is a contradiction with (8.7). Therefore (8.9), so that (8.8) holds. Notice that (8.8) is just (8.2), so the proof is finished. □

The exact lower local dimension of μ is of course closely related to the exact Hausdorff dimension of its support. It is well-known that we can deduce the exact dimension of the support by the exact local dimension of the measure. Our argument in the proof above shows that we can also do the contrary.

Similarly, the exact upper local dimension is also closely related to the exact packing dimension. Liu [15] proved that if $p_1 = 0$, then the correct function for packing measure is

$$\phi_-(t) = t^\alpha \left(\log\log \frac{1}{t}\right)^{\beta_-}.$$

One might expect to prove the following: if $p_1 = 0$, then for P-almost all $\omega \in \Omega$ and for μ_ω almost all $u \in \partial\mathbf{T}$,

$$\liminf_{n\to\infty} \frac{m^n \mu_\omega(B_{u|n})}{(\log n)^{\beta_-}} = \frac{1}{r_-^{\beta_-}}.$$

The lower bound is easy: in the same way as in the proof of (3.4(a)) of Liu [13], we can prove that if $p_1 = 0$, then for P-almost all $\omega \in \Omega$ and for μ_ω almost all $u \in \partial\mathbf{T}$,

$$\liminf_{n\to\infty} \frac{m^n \mu_\omega(B_{u|n})}{(\log n)^{\beta_-}} \geqslant \frac{1}{r_-^{\beta_-}}.$$

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