ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 36, nº 6 (2000), p. 775-786
<http://www.numdam.org/item?id=AIHPB_2000__36_6_775_0>

Ann. Inst. Henri Poincaré, Probabilités et Statistiques 36, 6 (2000) 775–786

A class of generalized Ornstein transformations with the weak mixing property

by

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ABSTRACT. – It is shown that there is a class of Ornstein transformations, distinct from the Ornstein mixing class, which are almost surely weak mixing © 2000 Éditions scientifiques et médicales Elsevier SAS

AMS classification: Primary : 28D05, secondary : 47A35

RÉSUMÉ. – On montre qu'une classe de transformations d'Ornstein, distincte de la classe mélangeante d'Ornstein, est constituée de transformations presque sûrement faiblement mélangeante. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

In his 1967 paper [11], D.S. Ornstein constructed a random family of transformations, associating to every point ω in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ a map T_ω belonging to the class of rank one transformations, for which he could prove that almost surely T_ω was mixing.

In this paper we consider another construction of the Ornstein type, with less constraints (namely in Ornstein's case, some non explicit cutting

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parameters satisfying some growth condition ensuring mixing are shown to exist, whereas here we fix them in advance. We require however that some specific spreading of the spacers should occur infinitely often). In this context we are able to prove almost surely weak mixing. We do not know whether mixing can occur with positive probability, neither do we know whether the mixing property satisfies a 0–1 law. Our construction is sufficiently flexible to produce rigidity.

There are other natural examples of transformations indexed by a probability space, which are known to be almost surely rank one, for example the interval exchange maps. In this class, whether weak mixing is almost surely satisfied is an open question [10]. It is our hope that there is sufficient analogy between our construction and interval exchange maps (where the randomness comes from the Teichmüller flow which is very stochastic [12]) for our methods to extend to this latter case.

Let us note also that Ornstein had to impose some extra conditions for his transformations to be totally ergodic. We have proved in [4] that this extra condition is unnecessary. Let us remark also that Veech in [13] has proved that almost surely interval exchange maps are totally ergodic.

We will assume that the reader is familiar with the method of cutting and stacking for constructing rank one transformations.

2. RANK ONE TRANSFORMATION BY CONSTRUCTION

Using the cutting and stacking method described in [6,7], one can define inductively a family of measure preserving transformations, called rank one transformations, as follows

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the kth stage we divide the stack obtained at the $(k-1)$th stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the kth stage of our construction the number of spacers put above the jth column of the $(k-1)$th stack is $a_j^{(k-1)}$, $0 \leqslant a_j^{(k-1)} < \infty$, $1 \leqslant j \leqslant p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

Proceeding in this way we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or σ-finite depending on the number of spacers added.

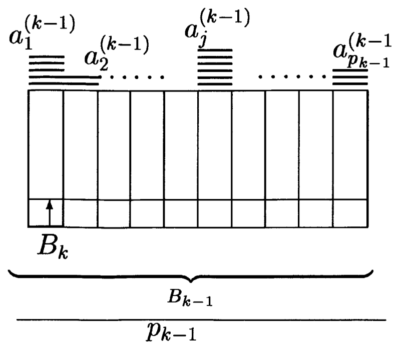

Fig. 1. kth-tower.

The construction of any rank one transformation thus needs two parameters $(p_k)_{k=0}^{\infty}$ (parameter of cutting and stacking) and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (parameter of spacers). We put

$$T \stackrel{\text{def}}{=} T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}.$$

3. ORNSTEIN'S CLASS OF TRANSFORMATIONS

In Ornstein's construction, the p_k's are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leqslant i \leqslant p_k - 1$, are chosen stochastically in a certain way (subject to certain bounds). This may be organized in various ways as noted by J. Bourgain in [1], in fact, let (t_k) be a sequence of positive integers such that $t_k \leqslant h_{k-1}$. We choose now independently, using the uniform distribution on the set $X_k = \{-\frac{t_k}{2}, \ldots, \frac{t_k}{2}\}$, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in $\mathbb{N}$. We put, for $1 \leqslant i \leqslant p_k$,

$$a_i^{(k)} = h_{k-1} + x_{k,i} - x_{k,i-1}, \quad \text{with } x_{k,0} = 0.$$

Then one sees that

$$h_{k+1} = p_k(h_k + h_{k-1}) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ completely determine the sequence of heights $(h_k)_{k=1}^{\infty}$. The total measure of the resulting measure space is finite if $\sum_{k=1}^{\infty} \frac{h_{k-1}}{h_k} + \sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty$. We will assume that this requirement is satisfied. Then we have $\sum_{k=0}^{\infty} \frac{1}{p_k} < \infty$. Indeed

$$\frac{\frac{h_{k-1}}{h_k}}{\frac{1}{p_{k-1}}} = \frac{p_{k-1}h_{k-1}}{p_{k-1}(h_{k-1}+h_{k-2}) + x_{k-1,p_{k-1}}}$$
$$= \frac{1}{(1+\frac{h_{k-2}}{h_{k-1}}) + \frac{x_{k-1,p_{k-1}}}{p_{k-1}h_{k-1}}} \to 1, \quad \text{as } k \to \infty.$$

We thus have a probability space of Ornstein transformations $\prod_{l=1}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \bigotimes_{l=1}^{\infty} P_l$, where $P_l \stackrel{\text{def}}{=} \bigotimes_{l=1}^{p_l-1} \mathcal{U}_l$; $\mathcal{U}_l$ is the uniform probability on X_l. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leqslant i \leqslant p_k - 1$, is the projection from Ω onto the ith coordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \leqslant i \leqslant p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=1}^{\infty}$ in Ω defines the spacers and therefore a rank one transformation which we denote by $T_{\omega,x}$ where $x = (x_{k,p_k})_{k=1}^{\infty}$ is *admissible* i.e.

$$\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty.$$

The definition above gives a more general definition of random construction due to Ornstein.

Now, let us recall that an automorphism is said to be totally ergodic if all its non-zero powers are ergodic. It is shown in [4] that the Ornstein transformations are almost surely totally ergodic using the fact that a measure preserving automorphism is totally ergodic if and only if no root of unity other than 1 is an eigenvalue. In fact it is proved in [4] that for a fixed $z \in \mathbb{T} \setminus \{1\} \equiv [0, 1) \setminus \{0\}$ the set $\{\omega\colon\ z$ is an eigenvalue of $T_\omega\}$ is a null set. By Van der Corput's inequality and Bernstein's inequality on the derivative of a trigonometric polynomial combined with the ingredients of [4] and [2] (all these are recalled below), we obtain a null set A such that for all $\omega \notin A$, T_ω has no eigenvalue other than 1 provided that for an infinity of n's we have $t_{k_n} = p_{k_n}$, and this implies the almost sure weak mixing property. For simplicity of notation we will denote a subsequence (k_n) by k and further subsequences of (k_n) will be denoted again by (k_n) if no confusion can arise. We mention also that in the particular case of the Ornstein transformations construted in [11] the construction of the Ornstein probabilistic space Ω is done step by step, defining simultanousuly the construction of the associated rank one transformations. Precisely, in order to construct the $(k+1)$th stage we apply the arithmetical lemma (see [9]) to get the cutting parameter $p_k \gg h_{k-1}$ and the set of good spacers parameter at the kth stage. Finally, in order to exhibit a mixing rank one transformation, Ornstein proved, using a combinatorial argument, that one can choose a deterministic sequence $x = (x_{k,p_k})_{k=1}^{\infty}$ in the space $\{1,2,3,4\}^{\mathbb{N}}$ so that a.e. $T_{\omega,x}$ is totally ergodic. But as mentioned above one can relax this condition by proving that almost surely Ornstein transformations are totally ergodic. We prove here that if we choose stochastically for infinitely many n's the numbers $(x_{n,i})_{i=1}^{p_n-1}$ in $X_n = \{-\frac{t_n}{2}, \ldots, \frac{t_n}{2}\}$ with $t_n = p_n$ then we have the following

THEOREM. – *With above assumption and for any fixed admissible sequence integers $x = (x_{k,p_k})_{k=1}^{\infty}$ the associated Ornstein's transformations are almost surely weak mixing.*

In the proof of the theorem we need the two following inequalities, and a combinatorial lemma due to Ornstein [11] (see also [9]) which we recall here.

VAN DER CORPUT'S INEQUALITY ([8]). – *Let $u_0, \ldots, u_{N-1}$ be complex numbers, and let H be an integer with $0 \leqslant H \leqslant N-1$. Then*

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} u_n\right|^2 \leqslant \frac{N+H}{N^2(H+1)}\sum_{n=0}^{N-1}|u_n|^2 + 2\frac{N+H}{N^2(H+1)^2}\sum_{h=1}^{H}(H+1-h)\,\mathrm{Re}\left(\sum_{n=0}^{N-h-1} u_{n+h}\overline{u_n}\right)$$

where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

BERNSTEIN'S INEQUALITY. – *If $P_n(\theta)$ is a trigonometric polynomial of order not higher than n then*

$$\sup_{\theta\in[0,1)} \left|P_n'(\theta)\right| \leqslant n \sup_{\theta\in[0,1)} \left|P_n(\theta)\right|.$$

A COMBINATORIAL LEMMA. – *Let m, n be positive integers, $n < m$. Consider the set E of $m-n$ pairs*

$$(1, 1+n), (2, 2+n), (3, 3+n), \ldots, (m-n, m).$$

We can divide E into two disjoint sets E_1 and E_2, each containing at least $[\frac{m-n}{4}]$ pairs, such that no integer occurs in more than one pair in each E_i, $i = 1, 2$.

We need also the following characterization of the eigenvalues of a rank one transformation due to Choksi and Nadkarni [3, Theorem 4]:

Let T be a rank one transformation with parameters $(p_k, (a_i^{(k)})_{i=1}^{p_k})_{k=0}^{\infty}$. Put

$$P_k(\theta) = \frac{1}{p_k}\sum_{j=0}^{p_k-1} \mathrm{e}^{2\pi i (jh_k + \sum_{i=0}^{j} a_i^{(k)})\theta} \quad \textit{with } a_0^{(k)} = 0.$$

Then $\mathrm{e}^{2\pi i\theta}$ is an eigenvalue of T if and only if

$$\sum_{k=1}^{\infty}\left(1 - |P_k(\theta)|^2\right) < \infty.$$

We apply this criterion to the setting on hand. Put

$$P_k(\theta) = P_k(\theta, \omega) = \frac{1}{p_k}\sum_{j=0}^{p_k-1} \mathrm{e}^{2\pi i (j(h_k + h_{k-1}) + x_{k,j})\theta},$$

and fix a positive integer H. Assume from now that $t_k = p_k$ and let k such that: $p_k > H$, so by Van der Corput's inequality we have

$$\begin{aligned}\left|P_k(\theta, \omega)\right|^2 \leqslant{}& \frac{p_k + H}{p_k(H+1)} \\ &+ 2\frac{p_k + H}{{p_k}^2(H+1)^2}\sum_{h=1}^{H}(H+1-h) \\ &\times \mathrm{Re}\left(\sum_{n=0}^{p_k-h-1} \mathrm{e}^{2\pi i h(h_k + h_{k-1})\theta}\mathrm{e}^{2\pi i (x_{k,n+h} - x_{k,n})\theta}\right). \qquad (1)\end{aligned}$$

Let

$${}_hQ_k(\theta, \omega) \stackrel{\text{def}}{=} {}_hQ_k(\theta) \stackrel{\text{def}}{=} \frac{1}{p_k - h}\sum_{n=0}^{p_k-h-1} \mathrm{e}^{2\pi i (x_{k,n+h} - x_{k,n})\theta}.$$

Using the Combinatorial lemma (with $n = h$ and $m = p_k$) we write

$${}_hQ_k(\theta) = {}_hQ_k^{(1)}(\theta) + {}_hQ_k^{(2)}(\theta),$$

where

$$ {}_hQ_k^{(j)}(\theta)=\frac{1}{p_k-h}\sum_{(n,n+h)\in E_j}\mathrm{e}^{2\pi i(x_{k,n+h}-x_{k,n})\theta},\quad j\in\{1,2\}, $$

and

$$ \left[\frac{p_k-h}{4}\right]\leqslant \operatorname{card}(E_j)\leqslant p_k-h\leqslant p_k,\quad j=1,2. $$

For fixed $\delta>0$, we get in the following lemma the uniform estimate with respect to $\theta\in[\delta,1-\delta]$ for the fourth moment of $({}_hQ_k^{(j)})$ (see [2]).

LEMMA 3.1. – *There exist a constant $C_\delta>0$ such that*

$$ \sup_{\{\theta\in[\delta,1-\delta]\}}\int_\Omega \left|{}_hQ_k^{(j)}(\theta)\right|^4 d\mathbb{P}\leqslant\frac{C_\delta}{(p_k-h)^2}. $$

Proof. – First set

$$ Y_p^j=x_{k,p+h}-x_{k,p},\quad \text{if } (p,p+h)\in E_j. $$

By the property of E_1 and E_2, for fixed j, the random variables (Y_p^j) are independent and identically distributed, and since $x_{k,p+h}$ and $x_{k,p}$ are independent and uniformly distributed we have

$$ \begin{aligned} \left|\int \mathrm{e}^{2\pi i\theta Y_p^j}\right| &= \left|\int \mathrm{e}^{2\pi i\theta x_{k,p+h}}\int \mathrm{e}^{-2\pi i\theta x_{k,p}}\right| = \left|\frac{1}{p_k}\sum_{s=0}^{p_k}\mathrm{e}^{2\pi i s\theta}\right|^2 \\ &= \frac{1}{p_k^2}\left|\frac{1-\mathrm{e}^{2\pi i(p_k+1)\theta}}{1-\mathrm{e}^{2\pi i\theta}}\right|^2\leqslant\frac{1}{p_k^2}\frac{1}{\sin^2(\pi\delta)}. \end{aligned} \tag{2} $$

Now observe that

$$ \begin{aligned} \mathbb{E}\left(\left|{}_hQ_k^{(j)}\right|^4\right) &= \int_\Omega \left|{}_hQ_k^{(j)}\right|^4 d\mathbb{P} = \mathbb{E}\left({}_hQ_k^{(j)2}\,\overline{{}_hQ_k^{(j)2}}\right) \\ &= \frac{1}{(p_k-h)^4}\sum_{q,r,s,t}\mathbb{E}\left(\mathrm{e}^{2\pi i\theta(Y_q^j+Y_r^j-Y_s^j-Y_t^j)}\right) \\ &= \frac{1}{(p_k-h)^4}\left(\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4\right), \end{aligned} $$

where

$$\sum_{\nu} = \sum_{q,r,s,t;\ \mathrm{card}(\{q,r,s,t\})=\nu} \mathbb{E}\big(\mathrm{e}^{2\pi i\theta(Y_q^j+Y_r^j-Y_s^j-Y_t^j)}\big), \quad \nu = 1,2,3,4.$$

The modulus of each term in the sum for $\sum_\nu$ is at most one, so $\frac{1}{(p_k-h)^4}|\sum_1|$ is less than $\frac{1}{(p_k-h)^3}$ and $\frac{1}{(p_k-h)^4}|\sum_2|$ is less than $\frac{1}{(p_k-h)^2}$.

It is best to break $\sum_3$ into two sums, $\sum_3{}'$ and $\sum_3{}''$, where $\sum_3{}'$ consists of those terms of $\sum_3$ for which $q=r$ or $s=t$ and $\sum_3{}''$ consists of those terms of $\sum_3$ for which $|\{q,r\}\cap\{s,t\}|=1$. Then by (2)

$$\begin{aligned}
\Big|\sum\nolimits_3{}'\Big| &= 2\Big|\mathrm{Re}\Big(\sum_{q\neq s\neq t} \mathbb{E}\big(\mathrm{e}^{2\pi i\theta(2Y_q^j-Y_s^j-Y_t^j)}\big)\Big|\\
&\leqslant 2\Big|\sum_{q\neq s\neq t} \mathbb{E}\big(\mathrm{e}^{2\pi i\theta(2Y_q^j)}\big)\mathbb{E}\big(\mathrm{e}^{-2\pi i\theta Y_s^j}\big)\mathbb{E}\big(\mathrm{e}^{-2\pi i\theta Y_t^j}\big)\Big|\\
&\leqslant 2(p_k-h)^3\big|\mathbb{E}\big(\mathrm{e}^{2\pi i\theta Y_1^j}\big)\big|^2\\
&\leqslant 2\frac{1}{(p_k-h)}\,\frac{1}{\sin^4(\pi\delta)}.
\end{aligned}$$

$|\mathbb{E}(\mathrm{e}^{2\pi i\theta(2Y_q^j)})|$ is at most one. We have proved that

$$\frac{1}{(p_k-h)^4}\Big|\sum\nolimits_3{}'\Big| \leqslant 2\frac{1}{(p_k-h)^5}\,\frac{1}{\sin^4(\pi\delta)}.$$

Similarly, by (2) we have

$$\begin{aligned}
\frac{1}{(p_k-h)^4}\Big|\sum\nolimits_3{}''\Big| &= \frac{4}{(p_k-h)^4}\Big|\sum_{q\neq r,s\neq t,r=s} \mathbb{E}\big(\mathrm{e}^{2\pi i\theta(Y_q^j-Y_s^j)}\big)\Big|\\
&\leqslant \frac{4}{(p_k-h)^5}\frac{1}{\sin^4(\pi\delta)}.
\end{aligned}$$

To estimate $\sum_4$ we will first write it in terms of ordered summation indices, $q>r>s>t$. There are then six types of terms according to the position of the two positive signs among $\pm Y_q^j \pm Y_r^j \pm Y_s^j \pm Y_t^j$. These can be coalesced into three types of terms by adding conjugates, to give

$$\begin{aligned}
\Big|\sum\nolimits_4\Big| = 8\Big|\mathrm{Re}\Big(\sum_{q>r>s>t} &\mathbb{E}\big(\mathrm{e}^{2\pi i\theta(Y_q^j+Y_r^j-Y_s^j-Y_t^j)}\big)\\
&+\mathbb{E}\big(\mathrm{e}^{2\pi i\theta(Y_q^j-Y_r^j+Y_s^j-Y_t^j)}\big)+\mathbb{E}\big(\mathrm{e}^{2\pi i\theta(Y_q^j-Y_r^j-Y_s^j+Y_t^j)}\big)\Big)\Big|
\end{aligned}$$

$$\leqslant 24(p_k - h)^4 \left|\mathbb{E}\left(e^{2\pi i\theta(Y_q^j + Y_r^j - Y_s^j - Y_t^j)}\right)\right|$$
$$\leqslant 24 \frac{4}{(p_k - h)^4} \frac{1}{\sin^8(\pi\delta)}.$$

Combining these estimates, we obtaint

$$\sup_{\{\theta\in[\delta,1-\delta]\}} \int_{\Omega} \left|{}_hQ_k^{(j)}(\theta)\right|^4 d\mathbb{P} \leqslant \frac{C_\delta}{(p_k - h)^2}.$$

for some constant C_δ, and the lemma is proved. □

Applying Bernstein inequality and the previous lemma we obtain the following crucial lemma.

LEMMA 3.2. – $\sup_{\{\delta\leqslant\theta\leqslant1-\delta\}} |{}_hQ_k^{(j)}(\theta,\omega)| \to 0$ *almost surely as* $k \to \infty$.

Proof. – Recall that $t_k \leqslant p_k$ ($k = k_n$), so the degree of ${}_hQ_k^{(j)}(\theta,\omega)$ is at most p_k. Therefore, by Bernstein inequality we have

$$\left|\frac{d}{d\theta}\left({}_hQ_k^{(j)}\right)(\theta,\omega)\right| \leqslant p_k \sup_{t\in[0,1)} \left|{}_hQ_k^{(j)}(\theta,\omega)\right| \leqslant p_k$$

for any θ in $[0, 1)$. Now put

$$M_k(\omega) = \sup_{\delta\leqslant\theta\leqslant1-\delta} \left|{}_hQ_k^{(j)}(\theta,\omega)\right|,$$

and $S_{k,\delta} = \{n/p_k^{9/7}\colon n/p_k^{9/7}$ in $[\delta, 1-\delta]$, $n \in \mathbb{Z}\}$. Let $\theta_0 \in [\delta, 1-\delta]$ be a point where the supremum of ${}_hQ_k^{(j)}(.,\omega)$ is taken (θ_0 depends on δ, ω and k), and let $\theta \in S_{k,\delta}$ such that $|\theta - \theta_0| < 1/p_k^{9/7}$. Therefore by Bernstein inequality

$$\left|{}_hQ_k^{(j)}(\theta_0,\omega)\right| \leqslant \left|{}_hQ_k^{(j)}(\theta,\omega)\right| + \left|{}_hQ_k^{(j)}(\theta,\omega) -{}_h Q_k^{(j)}(\theta_0,\omega)\right|$$
$$\leqslant \left|{}_hQ_k^{(j)}(\theta,\omega)\right| + \frac{1}{{p_k}^{2/7}}.$$

So, we have

$$\overline{\lim}\sup_{\{\delta\leqslant\theta\leqslant1-\delta\}} \left|{}_hQ_k^{(j)}(\theta,\omega)\right| = \overline{\lim}\sup_{\{n:\delta\leqslant n/p_k^{9/7}\leqslant1-\delta\}} \left|{}_hQ_k^{(j)}(n/p_k^{9/7},\omega)\right|.$$

Now set

$$A_k^{(j,h)} = \{\omega\colon \sup_{\{n:\ \delta \leqslant n/p_k^{9/7} \leqslant 1-\delta\}} \left| {}_h Q_k^{(j)}\left(n/p_k^{9/7}, \omega\right)\right| \leqslant 1/{p_k}^{1/7}\}.$$

Then by Chebychev's inequality and Lemma 3.1

$$\mathbb{P}\left(\left(A_k^{(j,h)}\right)^c\right) \leqslant \sum_{\{n:\ \delta \leqslant n/p_k^{9/7} \leqslant 1-\delta\}} \mathbb{P}\left\{\left| {}_h Q_k^{(j)}\left(\frac{n}{p_k^{9/7}}, \omega\right)\right|^2 \geqslant \frac{1}{{p_k}^{2/7}}\right\}$$

$$\leqslant C_\delta \frac{p_k^{13/7}}{(p_k - h)^2} \sim C_\delta \frac{1}{p_k^{1/7}}.$$

The proof is completed by using the Borel–Cantelli lemma: for a subsequence (p_{k_n}) such that: $p_{k_n} \geqslant n^8$ for all n,

$$\sum_n \mathbb{P}\left(\left(A_{k_n}^{(j,h)}\right)^c\right) \leqslant C_\delta \sum_n n^{-8/7} < \infty$$

and so $\sup_{\{\delta \leqslant \theta \leqslant 1-\delta\}} |{}_h Q_{k_n}^{(j)}(\theta, \omega)| \to 0$ almost surely as $n \to \infty$. The null set depends on the countable set of h and j so we have

$$\sup_{\{\delta \leqslant \theta \leqslant 1-\delta\}} \left|{}_h Q_{k_n}^{(j)}(\theta, \omega)\right| \to 0 \quad \text{almost surely as } n \to \infty,$$

and the lemma follows. □

We present now the proof of the theorem.

Proof of the theorem. – By Lemma 3.2 we have

$$\sup_{\{\delta \leqslant \theta \leqslant 1-\delta\}} \left|{}_h Q_{k_n}^{(j)}(\theta, \omega)\right| \to 0 \quad \text{as } n \to \infty\ \mathbb{P}\text{-a.s.}$$

So by (1), we obtain

$$\overline{\lim} \sup_{\{\delta \leqslant \theta \leqslant 1-\delta\}} \left|P_{k_n}(\theta, \omega)\right|^2 \leqslant \frac{1}{H+1} \quad \mathbb{P}\text{-a.s.} \tag{3}$$

and since (3) holds for every positive integer H, for any $\delta > 0$ we have

$$\overline{\lim} \sup_{\{\delta \leqslant \theta \leqslant 1-\delta\}} \left|P_{k_n}(\theta, \omega)\right|^2 = 0 \quad \mathbb{P}\text{-a.s.}$$

Thus

$$\sum_{k=1}^{\infty}\left(1-|P_k(\theta)|^2\right)=\infty \quad \mathbb{P}\text{-a.s., for any } \delta \leqslant \theta \leqslant 1-\delta.$$

So we have proved that there exists a set of full measure Ω_δ such that if ω is in Ω_δ then eigenvalue of T_ω cannot belong to $[\delta, 1-\delta]$ i.e.

$$\mathbb{P}\{\omega\colon\ e(T_\omega)\cap[\delta,1-\delta]\neq\emptyset\}=0,$$

where $e(T_\omega)$ is the group of eigenvalues of T_ω. Now, let

$$\Omega_m=\left\{\omega\colon\ e(T_\omega)\cap\left[\frac{1}{m},1-\frac{1}{m}\right]=\emptyset\right\}$$

and $\Omega'=\bigcap_{m>1}\Omega_m$, then $\Omega'=\{\omega\colon\ e(T_\omega)=\{0\}\}$ and $\mathbb{P}(\Omega')=1$. This completes the proof of the theorem. □

ACKNOWLEDGEMENTS

This work benefits from the suggestion of the referee of [4] and many conversations I have had with J.-P. Thouvenot, J. De Sam Lazaro, M. Lemańczyk and F. Parreau suggested a number of corrections and improvements in both content and language. To all of them go my heartfelt thanks.

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