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Chung-type functional laws of the iterated logarithm for tail empirical processes

by

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ABSTRACT. – We obtain functional laws of the iterated logarithm for the tail empirical process analogue to the Chung (1948) and Csáki (1980) limit laws for the Wiener process. The tail empirical process is defined by $h_n^{-1/2}\alpha_n(h_n u)$ for $0 \leq u \leq 1$, where α_n denotes the uniform empirical process based upon n independent uniform $(0, 1)$ random variables. Under appropriate assumptions on $h_n \rightarrow 0$, Mason (1988) showed that the sequence of functions $f_n = (2h_n \log \log n)^{-1/2}\alpha_n(h_n \cdot)$ is almost surely compact with respect to the topology defined by the sup-norm $\|\cdot\|$, and gave a characterization of the corresponding limit set \mathbb{K} . In this paper, we obtain an estimate of the rate of this limit law by evaluating $\liminf_{n \rightarrow \infty} (\log \log n) \|f_n - f\|$ for each $f \in \mathbb{K}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous établissons des lois fonctionnelles du logarithme itéré pour le processus empirique de queue analogues aux lois limites de Chung (1948) et Csáki (1980) pour le processus de Wiener. Le

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processus empirique de queue est défini par $h_n^{-1/2}\alpha_n(h_n u)$ pour $0 \leq u \leq 1$, où α_n désigne le processus empirique uniforme basé sur n variables aléatoires indépendantes de loi uniforme sur $[0, 1]$. Sous des hypothèses convenables portant sur $h_n \rightarrow 0$, Mason (1988) a montré que la suite de fonctions $f_n = (2h_n \log \log n)^{-1/2}\alpha_n(h_n \cdot)$ est presque sûrement relativement compacte pour la topologie définie par la norme uniforme $\|\cdot\|$, et caractérisé l'ensemble limite \mathbb{K} correspondant. Nous évaluons la vitesse de cette convergence en déterminant la valeur de $\liminf_{n \rightarrow \infty} (\log \log n) \|f_n - f\|$ pour chaque $f \in \mathbb{K}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let $\{U_n: n \geq 1\}$ be independent and identically distributed (i.i.d.) uniform $(0, 1)$ random variables. For each $n \geq 1$, define the uniform empirical process by

$$\alpha_n(t) = n^{1/2}(U_n(t) - t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where $U_n(t) = n^{-1}\#\{U_i \leq t: 1 \leq i \leq n\}$, and $\#A$ denotes the cardinality of A . Let $\{h_n: n \geq 1\}$ be positive constants fulfilling assumptions among the following (H.1)–(H.2)–(H.3) stated below. Here and elsewhere, we set $\log_1 u = \log_+ u = \log(u \vee e)$, and $\log_p u = \log_+(\log_{p-1} u)$ for $p \geq 2$.

(H.1) (i) $h_n \rightarrow 0$; (ii) $h_n \downarrow$; (iii) $nh_n \uparrow$;

(H.2) $nh_n / \log_2 n \rightarrow \infty$;

(H.3) $nh_n / (\log_2 n)^3 \rightarrow \infty$.

We are concerned with the limiting behavior as $n \rightarrow \infty$ of the *tail empirical process* defined by

$$h_n^{-1/2}\alpha_n(h_n u) \quad \text{for } 0 \leq u \leq 1. \tag{1.2}$$

It is now well known (see, e.g., Theorem 2.1 in Csörgő and Mason [13]) that, whenever $h_n \rightarrow 0$ together with $nh_n \rightarrow \infty$, we have the convergence in distribution

$$h_n^{-1/2}\alpha_n(h_n \cdot) \rightarrow_d W \quad \text{as } n \rightarrow \infty, \tag{1.3}$$

where W denotes the restriction on $[0, 1]$ of a standard Wiener process $\{W(t): t \geq 0\}$. As could be expected from (1.3), the description of

the strong limiting behavior of $h_n^{-1/2}\alpha_n(h_n\cdot)$ as $n \rightarrow \infty$, which we consider next, makes an instrumental use of Gaussian process theory. The following notation is needed for the statement of the corresponding results. Let $(B[0, 1], \mathcal{U})$ denote the set $B[0, 1]$ of all bounded functions f on $[0, 1]$, endowed with the uniform topology \mathcal{U} , induced by the sup-norm $\|f\| = \sup_{0 \leq u \leq 1} |f(u)|$. Let $AC_0[0, 1]$ denote the set of all absolutely continuous functions f on $[0, 1]$, with Lebesgue derivative $\dot{f}(u) = \frac{d}{du}f(u)$, and such that $f(0) = 0$. For each $f \in B[0, 1]$, set

$$|f|_{\mathbb{H}} = \begin{cases} \left\{ \int_0^1 \dot{f}(u)^2 \right\}^{1/2} & \text{if } f \in AC_0[0, 1], \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathbb{H} = \{f \in B[0, 1]: |f|_{\mathbb{H}} < \infty\}$ denote the Hilbert subspace of $AC_0[0, 1]$ with Hilbert norm $|\cdot|_{\mathbb{H}}$. The unit ball of \mathbb{H} constitutes the Strassen set (see, e.g., Strassen [37]), denoted by

$$\mathbb{K} = \{f \in AC_0[0, 1]: |f|_{\mathbb{H}} \leq 1\}. \tag{1.4}$$

We note for further use that the norm inequality

$$\|f\| \leq |f|_{\mathbb{H}}, \tag{1.5}$$

holds for each $f \in \mathbb{H}$ (see, e.g., Lemma 2.5, p. 2021 in Deheuvels [16]). Mason [29] proved the following functional law of the iterated logarithm (FLIL) for the tail empirical process.

THEOREM 1.1. – *Under (H.1) and (H.2), the sequence of random functions*

$$\left\{ \frac{h_n^{-1/2}\alpha_n(h_n\cdot)}{\sqrt{2 \log_2 n}} : n \geq 1 \right\}, \tag{1.6}$$

is almost surely relatively compact in $(B[0, 1], \mathcal{U})$, with limit set equal to \mathbb{K} .

The meaning of Theorem 1.1 is that the statements (L.1) and (L.2) below hold jointly with probability 1.

$$(L.1) \quad \lim_{n \rightarrow \infty} \left\{ \inf_{f \in \mathbb{K}} \left\| \frac{h_n^{-1/2}\alpha_n(h_n\cdot)}{\sqrt{2 \log_2 n}} - f \right\| \right\} = 0;$$

$$(L.2) \quad \liminf_{n \rightarrow \infty} \left\| \frac{h_n^{-1/2}\alpha_n(h_n\cdot)}{\sqrt{2 \log_2 n}} - f \right\| = 0 \quad \text{for each } f \in \mathbb{K}.$$

In this paper, we give an evaluation of the rate of convergence in (L.2) by proving the following main theorem. Below, we use the convention that $\lambda/0 = \infty$ for $\lambda > 0$.

THEOREM 1.2. – *Assume that (H.1) and (H.3) hold. Then, for each $f \in \mathbb{K}$, we have*

$$\liminf_{n \rightarrow \infty} (\log_2 n) \times \left\| \frac{h_n^{-1/2} \alpha_n(h_n \cdot)}{\sqrt{2 \log_2 n}} - f \right\| = \frac{\pi}{4 \sqrt{1 - |f|_{\mathbb{H}}^2}} \quad \text{a.s.} \quad (1.7)$$

The proof of Theorem 1.2 is postponed until Section 2. We anticipate the exposition of our arguments to note that it is not excessively difficult (even though technically involved) to establish (1.7) under (H.1), (H.3) and (H.4), where (H.4) denotes the condition

$$(H.4) \quad h_n (\log_2 n)^2 \rightarrow 0.$$

One of the major difficulties of the proof consists to extend the validity of (1.7) to sequences fulfilling (H.3) but not (H.4). We will treat this problem by establishing first some refinements to probability bounds given in de Acosta [1]. The latter may turn out to be of independent interest.

We briefly mention that the limiting constant in (1.7) is dependent of the use of the sup-norm $\|\cdot\|$ in the evaluation of *small ball* probabilities for the Wiener process (refer to Kuelbs, Li and Talagrand [26] and the references therein for a general approach to this problem). It would be of interest to obtain analogues of (1.7) for other norms. This, however, cannot be done without overcoming huge technical difficulties because of the lack of appropriate invariance principles (see, e.g., Berthet [5] for examples of the kind). We will therefore limit ourselves to the case of the sup-norm which has interest in and of itself, even though the arguments we will use later on are likely to be extended in a more general setting.

For further motivation of Theorem 1.2, we now discuss some of its consequences. Making use of the fact (following from the Arzela–Ascoli theorem) that \mathbb{K} is a compact subset of $(B[0, 1], \mathcal{U})$, a routine argument based upon Theorem 1.1 shows that for any \mathcal{U} -continuous functional Θ on $B[0, 1]$, under (H.1) and (H.2),

$$\limsup_{n \rightarrow \infty} \Theta \left(\frac{h_n^{-1/2} \alpha_n(h_n \cdot)}{\sqrt{2 \log_2 n}} \right) = \sup_{f \in \mathbb{K}} \Theta(f) \quad \text{a.s.} \quad (1.8)$$

In particular, when $\Theta(f) = \|f\|$ in (1.8), we get that, under (H.1) and (H.2),

$$\limsup_{n \rightarrow \infty} \frac{\|h_n^{-1/2} \alpha_n(h_n \cdot)\|}{\sqrt{2 \log_2 n}} = 1 \quad \text{a.s.} \tag{1.9}$$

The “lim inf” version of (1.9) is obtained by choosing $f = 0$ in (1.7), which yields, under (H.1) and (H.3),

$$\liminf_{n \rightarrow \infty} \sqrt{2 \log_2 n} \times \|h_n^{-1/2} \alpha_n(h_n \cdot)\| = \frac{\pi}{2} \quad \text{a.s.} \tag{1.10}$$

We now discuss the sharpness of the conditions (H.1) and (H.3) in Theorem 1.2. It is noteworthy (see, e.g., Deheuvels and Mason [18]) that Theorem 1.1 becomes invalid when $nh_n / \log_2 n = O(1)$. It is therefore hopeless to expect any extension of Theorem 1.2 to this case. This leaves us to consider sequences fulfilling (H.2) but not (H.3), i.e., such that

$$nh_n / \log_2 n \rightarrow \infty \quad \text{and} \quad nh_n / (\log_2 n)^3 = O(1).$$

Under these conditions, our arguments fail to give a general description of the form (1.7), and we conjecture that some additional regularity assumptions on $f \in \mathbb{K}$ should be needed to reach this goal. To treat the case of sequences fulfilling $nh_n = O((\log_2 n)^3)$ would necessitate the introduction of new technological arguments beyond the scope of the present paper. We leave therefore open the problem of finding the weakest possible conditions on $\{h_n: n \geq 1\}$ which imply the validity of (1.7) for an arbitrary $f \in \mathbb{K}$.

The analogues of the above results in the setting of Wiener processes are well known. Chung [10] proved the first law of the kind by showing that

$$\liminf_{T \rightarrow \infty} \sqrt{2 \log_2 T} \times \|T^{-1/2} W(T \cdot)\| = \frac{\pi}{2} \quad \text{a.s.} \tag{1.11}$$

The Wiener process analogue to Theorem 1.1 is the Strassen [37] FLIL. By extending seminal results of Csáki [11] (obtained under the restriction that \dot{f} is of bounded variation), de Acosta [1] established the full form of the Wiener process analogue of Theorem 1.2, by showing that, for each $f \in \mathbb{K}$,

$$\liminf_{T \rightarrow \infty} (\log_2 T) \times \left\| \frac{T^{-1/2} W(T \cdot)}{\sqrt{2 \log_2 T}} - f \right\| = \frac{\pi}{4 \sqrt{1 - |f|_{\mathbb{H}}^2}} \quad \text{a.s.} \tag{1.12}$$

A comparison of (1.7)–(1.10) with (1.11)–(1.12) illustrates well enough the connection of the latter results with our work. We refer to de Acosta [2], Csáki [12], Csörgő and Révész [14,15], Deheuvels and Mason [19], Grill [21], Mueller [33], Mogulskii [32], Révész [35], and the references therein for some related FLIL's and Chung-type limit laws.

In the case where $|f|_{\mathbb{H}} = 1$, Theorem 1.2 yields a degenerate limit, and some other arguments are needed to provide the exact rates. This is a much more difficult problem since the methods of proof, as well as the limiting constants, depend heavily on regularity assumptions of f . We may cite, among others, the work of Goodman and Kuelbs [20] who showed, in the setting of the Wiener process analogue (1.12) to (1.7), that whenever $|f|_{\mathbb{H}} = 1$, there exists a constant $\gamma(f)$ such that

$$\gamma(f) = \liminf_{T \rightarrow \infty} (\log_2 T)^{2/3} \times \left\| \frac{T^{-1/2} W(T \cdot)}{\sqrt{2 \log_2 T}} - f \right\| < \infty \quad \text{a.s.} \quad (1.13)$$

The value of the constant $\gamma(f)$ has been explicitly evaluated in a series of important cases (see, for example, Grill [21]). In view of (1.7)–(1.11), we may expect to have (with the same limiting constant as in (1.13)) under (H.1) and (H.3), for each $f \in \mathbb{K}$ with $|f|_{\mathbb{H}} = 1$,

$$\liminf_{n \rightarrow \infty} (\log_2 n)^{2/3} \times \left\| \frac{h_n^{-1/2} \alpha_n(h_n \cdot)}{\sqrt{2 \log_2 n}} - f \right\| = \gamma(f) \quad \text{a.s.} \quad (1.14)$$

The proof of (1.14) would necessitate some different arguments than that used in the sequel for Theorem 1.2. We therefore leave it presently as an open conjecture, to be considered elsewhere.

The remainder of this paper is organized as follows. In the forthcoming Section 2.1, we derive the probability bounds which are used in Section 2.2 to prove Theorem 1.2.

2. PROOFS

2.1. Preliminary facts and notation

We will make use of the following notation and basic facts taken from the theory of Gaussian random functions. Some details may be found in the books of Ledoux and Talagrand [27] and Lifshits [28].

Let Z be a centered Gaussian vector with values in a real separable Banach space \mathbf{X} . Set $\mathbb{P}_Z(B) = \mathbb{P}(Z \in B)$ for each $B \in \mathcal{B}$, the σ -algebra

of Borel subsets of \mathbf{X} . It is well known that there exists a *kernel* (or *reproducing kernel Hilbert space* (RKHS)) \mathbb{H} , which is a linear subspace of \mathbf{X} endowed with a Hilbert norm $|h|_{\mathbb{H}} = \langle h, h \rangle_{\mathbb{H}}^{1/2}$, fulfilling the following properties. For each $h \in \mathbb{H}$, there exists a measurable linear form $\tilde{h} = \mathcal{I}h$ on \mathbf{X} , such that

$$\mathbb{P}_Z(B + h) = \int_B \exp\left(\tilde{h}(z) - \frac{1}{2}|h|_{\mathbb{H}}^2\right) \mathbb{P}_Z(dz) \quad \text{for each } B \in \mathcal{B}, \quad (2.1)$$

$$\int_{\mathbf{X}} \tilde{h}(z)^2 \mathbb{P}_Z(dz) = |h|_{\mathbb{H}}^2. \quad (2.2)$$

We refer to Kuelbs [24,25], and the references therein, for details concerning the construction of \mathbb{H} , and to Kuelbs, Li and Talagrand [26] for a description of the linear mapping $h \in \mathbb{H} \rightarrow \tilde{h} = \mathcal{I}h \in \mathbf{X}^*$. To be more explicit, if \mathbf{X}^* stands for the topological dual of \mathbf{X} (i.e. the space of continuous linear forms on \mathbf{X}), we consider the linear mapping $\mathcal{J} : \mathbf{X}^* \rightarrow \mathbf{X}$ defined by the Bochner integral

$$\mathcal{J}h^* = \mathbb{E}(Zh^*(Z)) = \int_{\mathbf{X}} zh^*(z) \mathbb{P}_Z(dz) \quad \text{for } h^* \in \mathbf{X}^*, \quad (2.3)$$

and the inner product on $\mathbb{H}^* := \mathcal{J}\mathbf{X}^*$ defined by

$$\langle \mathcal{J}g^*, \mathcal{J}h^* \rangle_{\mathbb{H}} = \mathbb{E}(g^*(Z)h^*(Z)) = \int_{\mathbf{X}} g^*(z)h^*(z) \mathbb{P}_Z(dz) \quad (2.4)$$

for $g^*, h^* \in \mathbf{X}^*$.

Given (2.3)–(2.4), the RKHS \mathbb{H} is the completion of \mathbb{H}^* in \mathbf{X} with respect to the norm $|h|_{\mathbb{H}} = \langle h, h \rangle_{\mathbb{H}}^{1/2}$, the latter being defined for each $h \in \mathbb{H}^*$ via (2.4). We stress the fact that an arbitrary $h \in \mathbb{H}$ does not necessarily belong to $\mathbb{H}^* = \mathcal{J}\mathbf{X}^*$. On the other hand, it is such that, for any $\varepsilon > 0$ there exists a $G_\varepsilon = \mathcal{J}G_\varepsilon^* \in \mathbb{H}^*$ with $G_\varepsilon^* \in \mathbf{X}^*$, such that

$$|h - G_\varepsilon|_{\mathbb{H}} < \varepsilon. \quad (2.5)$$

We will specialize here in the case where $\mathbf{X} = C_0[0, 1]$, the space of all continuous functions f on $[0, 1]$ with $f(0) = 0$, endowed with the uniform topology, denoted by \mathcal{U} , and generated by the sup-norm $\|f\| = \sup_{0 \leq t \leq 1} |f(t)|$. We will choose $Z = W$, where W denotes the restriction on $[0, 1]$ of a standard Wiener process $\{W(t) : t \geq 0\}$. In this case, $\mathbb{P}_Z =$

\mathbb{P}_W is the *Wiener measure* (see, e.g., Itô and McKean [22], Kuo [23]), and the RKHS \mathbb{H} consists of all absolutely continuous functions $h \in C_0[0, 1]$ with Lebesgue derivative $\dot{h} \in L_2[0, 1]$. The Hilbert norm (2.2), and the corresponding inner product (2.4) may then be defined on \mathbb{H} (see, e.g., Adler [3]) by

$$\|h\|_{\mathbb{H}} = \left\{ \int_0^1 \dot{h}(t)^2 dt \right\}^{1/2} \quad \text{and} \quad \langle g, h \rangle_{\mathbb{H}} = \int_0^1 \dot{g}(t)\dot{h}(t) dt, \quad (2.6)$$

which is in agreement with the notation introduced in Section 1. It is not easy in general to derive a simple characterization of the functions $g \in \mathbb{H}$ which belong to \mathbb{H}^* . In the present framework where $Z = W$ and $\mathbf{X} = C_0[0, 1]$, this may be achieved by a specialization of the arguments of Kuelbs, Li and Talagrand [26], as follows. We start by writing the almost surely uniformly convergent on $[0, 1]$ Karhunen–Loève expansion of W (see, e.g., Example 1.4.4, p. 42 in Ash and Gardner [4]), namely

$$W(t) = \sum_{n=1}^{\infty} \left\{ \left(n - \frac{1}{2} \right) \pi \right\}^{-1} \xi_n e_n(t) \quad \text{for } t \in [0, 1], \quad (2.7)$$

where $\{\xi_n: n \geq 1\}$ is an i.i.d. sequence of $N(0, 1)$ standard normal random variables, and $e_n(t) = 2^{1/2} \sin((n - \frac{1}{2})\pi t)$, for $n \geq 1$ and $t \in [0, 1]$. Below, we will work, without loss of generality, on the event of probability 1 on which the uniform convergence in (2.7) holds.

The functions $\{e_n: n \geq 1\}$ in (2.7) form an $L_2[0, 1]$ orthonormal sequence, since, for $m, n \geq 1$,

$$\int_0^1 e_m(t)e_n(t) dt = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Let, for $n \geq 1$, $a_n^* \in \mathbf{X}^*$ denote the \mathcal{U} -continuous linear form on $\mathbf{X} = C_0[0, 1]$, defined by

$$a_n^*(f) = \left\{ \left(n - \frac{1}{2} \right) \pi \right\} \int_0^1 f(t)e_n(t) dt \quad \text{for } f \in \mathbf{X} = C_0[0, 1]. \quad (2.9)$$

It follows obviously from (2.7)–(2.8)–(2.9) that, for all $n \geq 1$,

$$a_n^*(W) = \xi_n, \quad (2.10)$$

so that we readily infer from (2.3) and (2.9) that

$$\mathcal{J}a_n^* = \left\{ \left(n - \frac{1}{2} \right) \pi \right\}^{-1} e_n \quad \text{for } n \geq 1. \tag{2.11}$$

We may check that $\{\mathcal{J}a_n^*: n \geq 1\}$ constitutes an orthonormal sequence in \mathbb{H} from the following equalities, implied by (2.10). For each $m, n \geq 1$, we have

$$\langle \mathcal{J}a_m^*, \mathcal{J}a_n^* \rangle_{\mathbb{H}} = \mathbb{E}(a_m^*(W)a_n^*(W)) = \mathbb{E}(\xi_m \xi_n) = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Thus (see, e.g., pp. 1881–1882 in Kuelbs, Li and Talagrand [26]), for any $h \in \mathbb{H}$, the linear form $\tilde{h} = \mathcal{I}h$ in (2.1) is nothing else but

$$\tilde{h}(f) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k^*(f) \langle h, \mathcal{J}a_k^* \rangle_{\mathbb{H}} \quad \text{for } f \in \mathbf{X} = C_0[0, 1]. \tag{2.13}$$

We may render (2.13) even more explicit, by the observation, following from (2.6), (2.9), (2.11) and an integration by parts, that, for each $h \in \mathbb{H}$,

$$\begin{aligned} \langle h, \mathcal{J}a_k^* \rangle_{\mathbb{H}} &= \int_0^1 \dot{h}(t) 2^{1/2} \cos\left(\left(k - \frac{1}{2}\right)\pi t\right) dt \\ &= \int_0^1 h(t) \left\{ \left(k - \frac{1}{2}\right)\pi \right\} e_k(t) dt = a_k^*(h). \end{aligned} \tag{2.14}$$

By combining (2.13) with (2.14) we so obtain the basic formula, for each $h \in \mathbb{H}$,

$$\tilde{h}(f) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k^*(f) a_k^*(h) \quad \text{for } f \in \mathbf{X} = C_0[0, 1]. \tag{2.15}$$

Given (2.9) and (2.15), the following proposition is about the best we can do to characterize those $h \in \mathbb{H}$ which belong to \mathbb{H}^* .

PROPOSITION 2.1. – *The function $h \in \mathbb{H}$ belongs to $\mathbb{H}^* = \mathcal{J}\mathbb{H}$ if and only if the linear form $\tilde{h} = \mathcal{I}h$ defined by*

$$f \rightarrow \tilde{h}(f) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ \left(k - \frac{1}{2} \right) \pi \right\}^2 \left(\int_0^1 f(t) e_k(t) \right) \left(\int_0^1 h(t) e_k(t) \right), \tag{2.16}$$

constitutes a continuous linear form on $\mathbf{X} = C_0[0, 1]$ with respect to the uniform topology \mathcal{U} , in which case we have $h = \mathcal{J}\tilde{h}$.

Proof. – Let $h \in \mathbb{H}$ be arbitrary. We first observe that, because of (2.9), (2.16) is equivalent to (2.15). Next, by setting $f = W$ in (2.15), we infer from (2.10), that

$$\begin{aligned} \tilde{h}(W) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k^*(h) \xi_k =_d N(0, \sigma_h^2) \\ \text{where } \sigma_h^2 &= \sum_{k=1}^{\infty} a_k^*(h)^2 = |h|_{\mathbb{H}}^2, \end{aligned} \tag{2.17}$$

which is (2.2) in the present framework. Finally, we make use of the fact (see, e.g., (2.5) in Kuelbs, Li and Talagrand [26]) that $h = \mathcal{J}h^*$ for some $h^* \in \mathbf{X}^*$, if and only if there exists a $\Theta = \Theta(h) \subseteq C_0[0, 1]$ with $\mathbb{P}_W(\Theta) = 0$, such that, whenever $W \notin \Theta$,

$$\tilde{h}(W) = h^*(W). \tag{2.18}$$

Since (2.18) entails that $\tilde{h} = h^*$ on $C_0[0, 1] - \Theta$, the conclusion is straightforward. \square

The most simple non-trivial example of functions fulfilling the conditions of Proposition 2.1 is given (see, e.g., (3.1) in Kuelbs, Li and Talagrand [26]) by the functions $h \in \mathbb{H}$ with Lebesgue derivative $\frac{d}{dt}h = \dot{h}$ of bounded variation on $[0, 1]$, in which case we may write $h = \mathcal{J}h^*$ and $\tilde{h} = h^* = \mathcal{I}h$, with

$$\tilde{h}(\ell) = h^*(\ell) = \ell(1)\dot{h}(1) - \int_0^1 \ell(t)\dot{h}(t) dt \quad \text{for } \ell \in C_0[0, 1]. \tag{2.19}$$

We note further, that for $Z = W$ and $\mathbf{X} = C_0[0, 1]$, (2.1) is the celebrated Cameron–Martin formula (see, e.g., Cameron and Martin [8] and Proposition 2.1 in de Acosta [1]).

In the following sequence of lemmas, we state some useful facts, taken from the literature. The first two of these lemmas hold for arbitrary Z and \mathbf{X} as above. In general, for $A \subseteq \mathbf{X}$, $f \in \mathbf{X}$ and $\lambda \in \mathbb{R}$, we set $f + \lambda A = \{f + \lambda g : g \in A\}$.

LEMMA 2.1. – For any closed convex symmetric subset C of \mathbf{X} , and for any $f \in \mathbf{X}$, we have

$$\mathbb{P}_Z(f + C) \leq \mathbb{P}_Z(C). \tag{2.20}$$

Proof. – This is Anderson’s inequality (see, e.g., Theorem 9, p. 135 in Lifshits [28]). \square

LEMMA 2.2. – For any symmetric Borel subset B of \mathbf{X} , and for any $h \in \mathbb{H}$, we have

$$\mathbb{P}_Z(h + B) \geq \mathbb{P}_Z(B) \exp\left(-\frac{1}{2}|h|_{\mathbb{H}}^2\right). \tag{2.21}$$

Proof. – This inequality, due to Borell [6] (see, e.g., Lemma 2.1 in Deheuvels and Lifshits [17]), follows readily from (2.1). \square

The bounds given in (2.20)–(2.21) are not quite sufficient for our needs. We will make use of the following sharper inequalities due to de Acosta [1]. Below, we specialize to $Z = W$, $\mathbf{X} = C_0[0, 1]$, and set $\mathbf{U} = \{f \in C_0[0, 1] : \|f\| \leq 1\}$.

LEMMA 2.3. – For any $h \in \mathbb{H}$, $\rho > 0$ and $G = \mathcal{J}G^* \in \mathbb{H}^*$, with $G^* \in \mathbf{X}^* = C_0[0, 1]^*$, we have

$$\begin{aligned} & \mathbb{P}_W(h + \rho\mathbf{U}) \\ & \leq \mathbb{P}_W(\rho\mathbf{U}) \exp\left(-\frac{1}{2}\{|h|_{\mathbb{H}}^2 - |h - G|_{\mathbb{H}}^2\} + 2\rho \sup_{\ell \in \mathbf{U}} |G^*(\ell)|\right). \end{aligned} \tag{2.22}$$

Moreover, for any Borel subset A of \mathbf{X} ,

$$\begin{aligned} & \mathbb{P}_W(\rho\mathbf{U})\mathbb{P}_W(h + A \cap \{\rho\mathbf{U}\}) \geq \mathbb{P}_W(A \cap \{\rho\mathbf{U}\})^2 \\ & \times \exp\left(-\frac{1}{2}\{|h|_{\mathbb{H}}^2 + |h - G|_{\mathbb{H}}^2\} - 2\rho \sup_{\ell \in \mathbf{U}} |G^*(\ell)|\right). \end{aligned} \tag{2.23}$$

Proof. – This variant of Proposition 2.2 of de Acosta [1] is a consequence of (2.1). \square

Remark 2.1. – (i) We note that the condition that $G = \mathcal{J}G^* \in \mathbb{H}^*$, with $G^* \in \mathbf{X}^*$, is essential to ensure the finiteness of $\sup_{\ell \in \mathbf{U}} |G^*(\ell)|$ in (2.22)–(2.23).

(ii) Under the additional assumption that $h \in \mathbb{H}$ belongs to \mathbb{H}^* , we may choose $G = h$ and $G^* = h^* = \mathcal{I}h$ in (2.22)–(2.23), to obtain, via (2.21), that

$$\begin{aligned} & \mathbb{P}_W(\rho\mathbf{U}) \exp\left(-\frac{1}{2}|h|_{\mathbb{H}}^2\right) \\ & \leq \mathbb{P}_W(h + \rho\mathbf{U}) \leq \mathbb{P}_W(\rho\mathbf{U}) \exp\left(-\frac{1}{2}|h|_{\mathbb{H}}^2 + 2\rho \sup_{\ell \in \mathbf{U}} |h^*(\ell)|\right), \end{aligned} \quad (2.24)$$

which, in turn, implies readily that, as $\rho \rightarrow 0$

$$\mathbb{P}_W(h + \rho\mathbf{U}) = (1 + o(1))\mathbb{P}_W(\rho\mathbf{U}) \exp\left(-\frac{1}{2}|h|_{\mathbb{H}}^2\right). \quad (2.25)$$

We observe that (2.25) is essentially identical to a uniform bound version of Theorem 2 of Borovkov and Mogulskii [7] (see also Mogulskii [31] and Nagaev [34]), the latter being given under the additional restrictions that \dot{h} is bounded, and with a derivative in $L_1[0, 1]$.

(iii) We will not use (2.23) in our forthcoming proofs of Theorems 1.2 and 1.3. This result is stated here because of the fact that it is related to (2.22), and of interest by itself.

(iv) The bound (2.20) yields readily the rough inequality

$$\mathbb{P}_W(h + \rho\mathbf{U}) \leq \mathbb{P}_W(\rho\mathbf{U}),$$

which, in view of (2.22) and the arguments given later on, suffices for deriving the rate in (1.7).

LEMMA 2.4. – *For any specified $r > 0$, we have*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-2} \log \mathbb{P}_W(\lambda^{-1}r\mathbf{U}) = -\frac{\pi^2}{8r^2}. \quad (2.26)$$

Proof. – It is well known (see, e.g., Chung [10]) that, as $x \rightarrow 0$ with $x > 0$,

$$\mathbb{P}_W(x\mathbf{U}) = \mathbb{P}(\|W\| \leq x) = \frac{4(1 + o(1))}{\pi} \exp\left(-\frac{\pi^2}{8x^2}\right), \quad (2.27)$$

from where (2.26) is straightforward, by setting $x = \lambda^{-1}r$ in (2.27). \square

The next proposition, directly inspired by Theorem 3.3 in de Acosta [1], will be instrumental in the proof of our results. Denote by $I(t) = t$ the identity function.

PROPOSITION 2.2. – For any $\varepsilon > 0$, $r > 0$ and $f \in \mathbb{H}$, there exists a $\lambda_0 = \lambda_0(\varepsilon, r, f) > 0$ such that, for all $\lambda \geq \lambda_0$ and $|\gamma| \leq \varepsilon\lambda^2/(32r)$,

$$\begin{aligned} & \mathbb{P}\left(\|W - \lambda\{f + \gamma I\}\| \leq \frac{r}{\lambda}\right) \\ & \leq \exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} + |f + \gamma I|_{\mathbb{H}}^2 - \varepsilon\right\}\right), \end{aligned} \tag{2.28}$$

and, for all $\lambda \geq \lambda_0$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{P}\left(\|W - \lambda\{f + \gamma I\}\| \leq \frac{r}{\lambda}\right) \\ & \geq \exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} + |f + \gamma I|_{\mathbb{H}}^2 + \varepsilon\right\}\right). \end{aligned} \tag{2.29}$$

Proof. – Fix any $\varepsilon > 0$, $r > 0$ and $f \in \mathbb{H}$. Making use of (2.5) taken with $h = f$, choose a $g = G_{\frac{1}{2}\sqrt{\varepsilon}} = \mathcal{J}g^* \in \mathbb{H}^*$ with $g^* \in C_0[0, 1]^*$ in such a way that $|f - g|_{\mathbb{H}}^2 < \varepsilon/4$. Next, observe that the function $I_\gamma(t) = \gamma I(t) = \gamma t$ has constant Lebesgue derivative γ , the latter being trivially of bounded variation on $[0, 1]$. Thus, by (2.19), we may write $I_\gamma = \mathcal{J}I_\gamma^*$, where $I_\gamma^* = \tilde{I}_\gamma = \mathcal{I}I_\gamma$ is a \mathcal{U} -continuous linear form on $C_0[0, 1]$, defined by

$$I_\gamma^*(\ell) = \gamma\ell(1) - \gamma \int_0^1 \ell(t) dt \quad \text{for } \ell \in C_0[0, 1]. \tag{2.30}$$

It follows readily from (2.30) and the definition of $\mathbf{U} = \{f \in C_0[0, 1]; \|f\| \leq 1\}$ that

$$\sup_{\ell \in \mathbf{U}} |I_\gamma^*(\ell)| = 2|\gamma|. \tag{2.31}$$

Next, set $G = \lambda\{g + I_\gamma\}$. By linearity of \mathcal{J} , we have $G = \mathcal{J}G^*$, where $G^* = \lambda\{g^* + I_\gamma^*\}$. Moreover, it follows from (2.31) that

$$\sup_{\ell \in \mathbf{U}} |G^*(\ell)| \leq |\lambda| \sup_{\ell \in \mathbf{U}} |g^*(\ell)| + |\lambda| \sup_{\ell \in \mathbf{U}} |I_\gamma^*(\ell)| =: |\lambda|(K + 2|\gamma|). \tag{2.32}$$

Note that we may assume, without loss of generality, that $K = K(\varepsilon, f) = \sup_{\ell \in \mathbf{U}} |g^*(\ell)|$ depends on ε and f only, through a proper choice of $g = \mathcal{J}g^* \in \mathbb{H}^*$ fulfilling $|f - g|_{\mathbb{H}}^2 \leq \varepsilon/4$. Making use of (2.26), we see that there exists a $\lambda_1 = \lambda_1(r, \varepsilon)$ such that, for all $\lambda \geq \lambda_1$, we have the

inequalities

$$\exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} + \frac{\varepsilon}{4}\right\}\right) \leq \mathbb{P}_W(\lambda^{-1}r\mathbf{U}) \leq \exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} - \frac{\varepsilon}{4}\right\}\right). \tag{2.33}$$

Set $\lambda_0 = \lambda_0(\varepsilon, r, f) = \max\{\lambda_1, \sqrt{16rK/\varepsilon}\}$, so that each choice of $\lambda \geq \lambda_0$ fulfills

$$\frac{4rK}{\lambda^2} \leq \frac{\varepsilon}{4}. \tag{2.34}$$

Let us now select an arbitrary $\lambda \geq \lambda_0$, and choose any γ fulfilling

$$|\gamma| \leq \varepsilon\lambda^2/(32r). \tag{2.35}$$

We note for further use that (2.34)–(2.35) jointly imply that

$$\frac{4r}{\lambda^2}(K + 2|\gamma|) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{2.36}$$

Given these choices of λ and γ , we set $h = \lambda\{f + I_\gamma\}$, $\rho = \lambda^{-1}r$, $G = \lambda\{g + I_\gamma\}$, and $G^* = \lambda\{g^* + I_\gamma^*\}$ in (2.22). Making use of the observation that $h - G = \lambda\{f - g\}$, we infer readily from (2.22), (2.32), (2.33), (2.36) and $|f - g|_{\mathbb{H}}^2 \leq \varepsilon/4$, the following chain of inequalities.

$$\begin{aligned} & \mathbb{P}(\|W - \lambda\{f + \gamma I\}\| \leq \lambda^{-1}r) \\ &= \mathbb{P}_W(h + \rho\mathbf{U}) \\ &\leq \mathbb{P}_W(\rho\mathbf{U}) \exp\left(-\frac{\lambda^2}{2}\left\{|f + \gamma I|_{\mathbb{H}}^2 - |f - g|_{\mathbb{H}}^2 - \frac{4r}{\lambda^2}(K + 2|\gamma|)\right\}\right) \\ &\leq \mathbb{P}_W(\lambda^{-1}r\mathbf{U}) \exp\left(-\frac{\lambda^2}{2}\left\{|f + \gamma I|_{\mathbb{H}}^2 - \frac{3\varepsilon}{4}\right\}\right) \\ &\leq \exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} + |f + \gamma I|_{\mathbb{H}}^2 - \varepsilon\right\}\right), \end{aligned} \tag{2.37}$$

which yields (2.28).

To prove (2.29), we apply (2.21), taken with $B = \lambda^{-1}r\mathbf{U}$ and $h = \lambda\{f + \gamma I\}$. We so obtain, via (2.33), that, uniformly over all $\lambda \geq \lambda_1$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{P}(\|W - \lambda\{f + \gamma I\}\| \leq \lambda^{-1}r) \\ &= \mathbb{P}_W(h + B) \geq \mathbb{P}_W(B) \exp\left(-\frac{1}{2}|h|_{\mathbb{H}}^2\right) \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{P}_W(\lambda^{-1}r\mathbf{U}) \exp\left(-\frac{\lambda^2}{2}\left\{|f + \gamma I|_{\mathbb{H}}^2\right\}\right) \\
 &\geq \exp\left(-\frac{\lambda^2}{2}\left\{\frac{\pi^2}{4r^2} + |f + \gamma I|_{\mathbb{H}}^2 + \varepsilon\right\}\right), \tag{2.38}
 \end{aligned}$$

which is (2.29). \square

2.2. Proof of Theorem 1.2

In this section, we consider, at times, sequences $\{h_n: n \geq 1\}$ fulfilling the assumption (H.5) below.

$$(H.5) \quad nh_n(\log_3 n)^2/(\log_2 n)^3 \rightarrow \infty.$$

It will become obvious that the main difficulty in this part of our proof corresponds to when

$$h_n \rightarrow 0 \quad \text{and} \quad h_n(\log_2 n)^2 \not\rightarrow 0. \tag{2.39}$$

This question will be addressed to later on, in Remark 2.3.

The invariance principle stated in the next lemma will be instrumental. Let $I(t) = t$ denote identity.

LEMMA 2.5. – *On a suitable probability space, we may define $\{\alpha_n: n \geq 1\}$ jointly with a sequence $\{W_n(t): t \geq 0\}$, $n = 1, 2, \dots$, of standard Wiener processes, such that the following property holds. There exist universal positive constants C_1, C_2, C_3 such that, for all $n^{-1} \leq h \leq 1$ and $x \geq 0$,*

$$\begin{aligned}
 &\mathbb{P}(n^{1/2}\|\alpha_n(hI) - \{W_n(hI) - hIW_n(1)\}\| \geq C_1 \log(nh) + x) \\
 &\leq C_2 \exp(-C_3x). \tag{2.40}
 \end{aligned}$$

Proof. – This is Theorem 1 of Mason and van Zwet [30]. \square

PROPOSITION 2.3. – *Assume (H.1) and (H.3) and let $\{W_n: n \geq 1\}$ be as in Lemma 2.5. Then, for each $\varepsilon > 0$, there exists an $N_0 = N_0(\varepsilon) < \infty$ such that, for all $n \geq N_0$,*

$$\begin{aligned}
 &\mathbb{P}\left(\left\|\frac{\alpha_n(h_n I)}{\sqrt{2h_n \log_2 n}} - \frac{W_n(h_n I) - h_n I W_n(1)}{\sqrt{2h_n \log_2 n}}\right\| \geq \frac{\varepsilon}{\log_2 n}\right) \\
 &\leq \exp(-2 \log_2 n). \tag{2.41}
 \end{aligned}$$

Proof. – The probability in the LHS of (2.41) is

$$P_n := \mathbb{P}\left(n^{1/2}\|\alpha_n(hI) - \{W_n(hI) - hIW_n(1)\}\| \geq \varepsilon \sqrt{\frac{2nh_n}{\log_2 n}}\right). \tag{2.42}$$

Set now $h'_n = \min\{h_n, n^{-1}(\log_2 n)^6\}$ and $h''_n = \max\{h_n, n^{-1}(\log_2 n)^6\}$. We first observe that there exists an $N'_0 < \infty$ such that, for all $n \geq N'_0$,

$$(2/C_3) \log_2 n \geq 6C_1 \log_3 n + (\log C_2)/C_3 \geq C_1 \log(nh'_n) + (\log C_2)/C_3,$$

and, because of (H.3),

$$2nh'_n \geq \{4/(C_3\varepsilon)\}^2 (\log_2 n)^3.$$

Thus, for all $n \geq N'_0$, we have

$$\varepsilon \sqrt{\frac{2nh'_n}{\log_2 n}} \geq \frac{4}{C_3} \log_2 n \geq C_1 \log(nh'_n) + \frac{1}{C_3} (2 \log_2 n + \log C_2). \quad (2.43)$$

On the other hand, since $nh''_n \geq (nh''_n)^{1/3} (\log_2 n)^4$, there exists an $N''_0 < \infty$ such that, for all $n \geq N''_0$,

$$\begin{aligned} \varepsilon \sqrt{\frac{2nh''_n}{\log_2 n}} &\geq 2^{1/2} \varepsilon (nh''_n)^{1/6} (\log_2 n)^{3/2} \\ &\geq C_1 \log(nh''_n) + \frac{1}{C_3} (2 \log_2 n + \log C_2). \end{aligned} \quad (2.44)$$

By combining (2.43) and (2.44) with (2.42), we obtain readily that, for all $n \geq N_0 := N'_0 \vee N''_0$,

$$\begin{aligned} P_n &\leq \mathbb{P} \left(n^{1/2} \|\alpha_n(h_n I) - \{W_n(h_n I) - h_n I W_n(1)\}\| \right. \\ &\quad \left. \geq C_1 \log(nh_n) + \frac{1}{C_3} (2 \log_2 n + \log C_2) \right), \end{aligned}$$

which, by (2.40), is less than or equal to $\exp(-2 \log_2 n)$, yielding (2.41). \square

Remark 2.2. – It is noteworthy that the condition (H.3) that $nh_n/(\log_2 n)^3 \rightarrow \infty$ is essential for the validity of (2.41). As will become obvious later on, this evaluation turns out to be a crucial step in the proof of Theorem 1.2. Therefore, we cannot expect to treat the case where $nh_n = O((\log_2 n)^3)$ by the present methodology.

Letting from now on $\{W_n: n \geq 1\}$ be as in Lemma 2.5 and Proposition 2.3, we set for $n \geq 1$

$$\Psi_n^*(t) = \frac{\alpha_n(h_n t)}{\sqrt{2h_n \log_2 n}}, \quad (2.45)$$

$$\begin{aligned} \Psi_n(t) &= \frac{W_n(h_n t) - h_n t W_n(1)}{\sqrt{2h_n \log_2 n}} \\ &= \frac{W_n(h_n t) - h_n t W_n(h_n)}{\sqrt{2h_n \log_2 n}} + t h_n^{1/2} \left\{ \frac{W_n(h_n) - W_n(1)}{\sqrt{2 \log_2 n}} \right\} \\ &= \Psi_n^{(1)}(t) + t \rho_n \quad \text{for } 0 \leq t \leq 1, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} \Psi_n^{(1)}(t) &= \frac{W_n(h_n t) - h_n t W_n(h_n)}{\sqrt{2h_n \log_2 n}} \quad \text{and} \\ \rho_n &= h_n^{1/2} \left\{ \frac{W_n(h_n) - W_n(1)}{\sqrt{2 \log_2 n}} \right\}. \end{aligned} \tag{2.47}$$

Note for further use that, whenever $0 < h_n < 1$, $\Psi_n^{(1)}$ and ρ_n are independent.

In the sequel, $\theta \in \mathbb{R}$ and $\delta \in \mathbb{R}$ will denote fixed constants, whose values will be specified in each application. We will make an instrumental use of the integer sequences

$$\begin{aligned} n_k &= \lfloor \exp\{k(\log_+ k)^\theta\} \rfloor \quad \text{and} \quad m_k = \lfloor \exp\{2k(\delta + \log_2 k)\} \rfloor \\ &\text{for } k \geq 0, \end{aligned} \tag{2.48}$$

where $\lfloor u \rfloor \leq u < \lfloor u \rfloor + 1$ denotes the integer part of u .

LEMMA 2.6. – Assume that $h_n \rightarrow 0$, and let $\{\rho_n: n \geq 1\}$ be as in (2.47). Then, there exists an $N_1 < \infty$ such that, for all $n \geq N_1$, $0 < h_n < 1$ and

$$\mathbb{P}(|\rho_n| \geq 2h_n^{1/2}) \leq \exp(-2 \log_2 n). \tag{2.49}$$

Proof. – Define N_1 as the smallest value of $m \geq 1$ such that $0 < h_n < 1$ for all $n \geq m$. Note that $2\sqrt{2 \log_2 n} \geq 1$ for all $n \geq 1$. Let Y denote a standard normal $N(0, 1)$ random variable. Making use of the bound $\mathbb{P}(|Y| \geq y) \leq \exp(-y^2/2)$ for $y \geq 1$, we see that, for all $n \geq N_1$,

$$\mathbb{P}(|\rho_n| \geq 2h_n^{1/2}) = P\left(|Y| \geq \frac{2\sqrt{2 \log_2 n}}{\sqrt{1 - h_n}}\right) \leq \exp(-4 \log_2 n),$$

which is more than needed for (2.49). \square

Remark 2.3. – Some heuristical comments will be helpful to render more explicit the technical problems which appear in this part of our proof. Grossly speaking, everything would be greatly simplified if

we could ignore the term $t\rho_n$ in (2.46) and work directly on $\Psi_n^{(1)}$. Unfortunately, this is not possible in general, and for the following reasons. An application of the Borel–Cantelli lemma in combination with (2.48) and (2.49) shows that (independently of the choice of $\theta \in \mathbb{R}$) $\rho_{n_k} = O(h_{n_k}^{1/2})$ a.s. as $k \rightarrow \infty$. This is about the best we can do without an explicit knowledge of the dependence structure of $\{W_n: n \geq 1\}$ with respect to $n \geq 1$. Such an information, however, would not help if it were available. Indeed, the replacement of the Mason and van Zwet [30] invariance principle, in Lemma 2.5, by a Kiefer process approximation such as given in Castelle and Laurent-Bonvalot [9], does not allow any improvement to the rate $\rho_{n_k} = O(h_{n_k}^{1/2})$. The fact that $h_n^{1/2}$ is not negligible with respect to $1/\log_2 n$ for sequences $\{h_n: n \geq 1\}$ fulfilling $h_n(\log_2 n)^2 \rightarrow \infty$, renders therefore necessary the use of a special argument to take the term $t\rho_n$ in (2.46) into account. This will be achieved in the following Lemmas 2.7–2.10, where are established the appropriate probability bounds.

In view of (2.45) and (2.46), introduce the events, for $r > 0$ and $f \in \mathbb{H}$,

$$A_n(f, r) = \left\{ \left\| \frac{W_n(h_n I)}{\sqrt{2h_n \log_2 n}} - f \right\| \leq \frac{r}{2 \log_2 n} \right\}, \tag{2.50}$$

$$\begin{aligned} B_n(f, r) &= \left\{ \left\| \frac{W_n(h_n I) - h_n I W_n(h_n)}{\sqrt{2h_n \log_2 n}} - f \right\| \leq \frac{r}{2 \log_2 n} \right\} \\ &= \left\{ \left\| \Psi_n^{(1)} - f \right\| \leq \frac{r}{2 \log_2 n} \right\}, \end{aligned} \tag{2.51}$$

$$\begin{aligned} C_n(f, r) &= \left\{ \left\| \frac{W_n(h_n I) - h_n I W_n(1)}{\sqrt{2h_n \log_2 n}} - f \right\| \leq \frac{r}{2 \log_2 n} \right\} \\ &= \left\{ \left\| \Psi_n - f \right\| \leq \frac{r}{2 \log_2 n} \right\} \\ &= \left\{ \left\| \Psi_n^{(1)} + \rho_n I - f \right\| \leq \frac{r}{2 \log_2 n} \right\}, \end{aligned} \tag{2.52}$$

$$\begin{aligned} D_n(f, r) &= \left\{ \left\| \frac{\alpha_n(h_n I)}{\sqrt{2h_n \log_2 n}} - f \right\| \leq \frac{r}{2 \log_2 n} \right\} \\ &= \left\{ \left\| \Psi_n^* - f \right\| \leq \frac{r}{2 \log_2 n} \right\}. \end{aligned} \tag{2.53}$$

LEMMA 2.7. – Assume that $h_n \rightarrow 0$. Then, for any $0 < \varepsilon < 1$, there exists an $N_2 = N_2(\varepsilon) < \infty$ such that, for all $n \geq N_2$, $r > 0$ and $f \in B[0, 1]$,

$$A_n(f, r) \subseteq B_n(f - h_n f(1)I, r(1 + \varepsilon)), \tag{2.54}$$

$$B_n(f - h_n f(1)I, r(1 - \varepsilon)) \subseteq A_n(f, r). \tag{2.55}$$

Proof. – To establish (2.54), choose $N_2 = N_2(\varepsilon)$ as the minimal value of $m \geq 1$ such that, for all $n \geq m$, $h_n < \varepsilon$. If $A_n(f, r)$ holds, then

$$\begin{aligned} \left\| \frac{W_n(h_n I)}{\sqrt{2h_n \log_2 n}} - f \right\| &\leq \frac{r}{2 \log_2 n} \\ \Rightarrow \left\| \frac{h_n I W_n(h_n)}{\sqrt{2h_n \log_2 n}} - h_n f(1)I \right\| &\leq \frac{r h_n}{2 \log_2 n}, \end{aligned}$$

and hence, by the triangle inequality, for all $n \geq N_2$,

$$\begin{aligned} \left\| \frac{W_n(h_n I) - h_n I W_n(h_n)}{\sqrt{2h_n \log_2 n}} - \{f - h_n f(1)I\} \right\| \\ \leq \frac{r(1 + h_n)}{2 \log_2 n} \leq \frac{r(1 + \varepsilon)}{2 \log_2 n}, \end{aligned}$$

so that $B_n(f - h_n f(1)I, r(1 + \varepsilon))$ is satisfied.

For (2.55), we observe that, if f, g are functions such that, for some $\delta > 0$ and $0 < h_n < 1$,

$$\begin{aligned} \|\{g - f\} - h_n I\{g(1) - f(1)\}\| \\ = \|\{g - h_n g(1)I\} - \{f - h_n f(1)I\}\| \leq \delta, \end{aligned}$$

then we must have $|g(1) - f(1)|(1 - h_n) \leq \delta$. The triangle inequality implies therefore that

$$\|g - f\| \leq \delta + \frac{\delta h_n}{1 - h_n} = \frac{\delta}{1 - h_n}.$$

Set now $n \geq N_2$, so that $h_n < \varepsilon < 1$ and $(1 - \varepsilon)/(1 - h_n) < 1$. An application of the above inequalities, taken with $g = W_n(h_n I)/\sqrt{2h_n \log_2 n}$ and $\delta = r(1 - \varepsilon)/(2 \log_2 n)$ leads to (2.55). \square

LEMMA 2.8. – Assume that $h_n \rightarrow 0$. Then for each $r_1 > 0$, $f \in \mathbb{K}$, $0 < \varepsilon_0 < 1$ and $0 < \varepsilon_1 < 1$, there exists an $N_3 = N_3(\varepsilon_0, \varepsilon_1, r_1, f)$ such that, for all $n \geq N_3$ and $|\rho| \leq 2h_n^{1/2}$, we have

$$\begin{aligned} \mathbb{P}(B_n(f + \rho I, r_1(1 + \varepsilon_0))) \\ \geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 + 2\varepsilon_1 \right\}\right), \end{aligned} \tag{2.56}$$

$$\begin{aligned} & \mathbb{P}(B_n(f + \rho I, r_1(1 - \varepsilon_0))) \\ & \leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 - 2\varepsilon_1 \right\}\right). \end{aligned} \quad (2.57)$$

Proof. – Recall the definition (2.50) of $A_n(f, r)$. Fix any $f \in \mathbb{K}$, $0 < \varepsilon_0 < 1$, $0 < \varepsilon_1 < 1$ and $r_1 > 0$, and set $\lambda = \sqrt{2 \log_2 n}$ in (2.28)–(2.29). We infer from Proposition 2.2 and the fact that $h_n^{-1/2} W_n(h_n I) =_d W$ the existence of an $N'_3 = N'_3(\varepsilon_1, r_1, f)$ such that, for all $n \geq N'_3$ and $|\gamma| \leq \varepsilon_1 \lambda^2 / (32r_1) = \{\varepsilon_1 / (16r_1)\} \log_2 n$,

$$\begin{aligned} & \mathbb{P}(A_n(f + \gamma I, r_1)) \\ & \geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f + \gamma I|_{\mathbb{H}}^2 + \varepsilon_1 \right\}\right), \end{aligned} \quad (2.58)$$

$$\begin{aligned} & \mathbb{P}(A_n(f + \gamma I, r_1)) \\ & \leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f + \gamma I|_{\mathbb{H}}^2 - \varepsilon_1 \right\}\right). \end{aligned} \quad (2.59)$$

Set now $\gamma = \rho + h_n f(1)$, where $\rho \in J_n := [-2h_n^{1/2}, 2h_n^{1/2}]$ is arbitrary (but non-random). Let $N''_3 = N''_3(\varepsilon_1, r_1, f)$ denote the minimal value of $m \geq 1$ such that, for all $n \geq m$,

$$0 < h_n < 1 \wedge (\varepsilon_1/9)^2 \quad \text{and} \quad \{\varepsilon_1 / (16r_1)\} \log_2 n \geq 1.$$

By (1.5), the assumption that $f \in \mathbb{K}$ implies that $|f(1)| \leq 1$. Thus, for all $n \geq N''_3$ and $\rho \in J_n$,

$$|\gamma| = |\rho + h_n f(1)| \leq 2h_n^{1/2} + h_n \leq 3h_n^{1/2} < \varepsilon_1/3 < 1. \quad (2.60)$$

We obtain readily from (1.5), (2.6) and (2.60), that, for all $n \geq N''_3$,

$$\begin{aligned} ||f + \gamma I|_{\mathbb{H}}^2 - |f|_{\mathbb{H}}^2| &= |2\gamma \langle I, f \rangle_{\mathbb{H}} + \gamma^2 |I|_{\mathbb{H}}^2| = |2\gamma f(1) + \gamma^2| \\ &\leq 2|\gamma| + \gamma^2 \leq 3|\gamma| \leq 9h_n^{1/2} \leq \varepsilon_1. \end{aligned} \quad (2.61)$$

In view of (2.61), (2.58) and (2.59), we have, for all $n \geq N''_3$, $f \in \mathbb{K}$ and $\rho \in J_n$,

$$\begin{aligned} & \mathbb{P}(A_n(f + h_n f(1)I + \rho I, r_1)) \\ & \geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 + 2\varepsilon_1 \right\}\right), \end{aligned} \quad (2.62)$$

$$\begin{aligned} & \mathbb{P}(A_n(f + h_n f(1)I + \rho I, r_1)) \\ & \leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 - 2\varepsilon_1 \right\}\right). \end{aligned} \quad (2.63)$$

By combining (2.50)–(2.51) with (2.62)–(2.63) and (2.54)–(2.55), taken with $\varepsilon = \varepsilon_0$ and the formal change of f into $f + h_n f(1)$, we obtain readily that, for all

$$n \geq N_3(\varepsilon_0, \varepsilon_1, r_1, f) := N_2(\varepsilon_0) \vee N'_3(\varepsilon_1, r_1, f) \vee N''_3(\varepsilon_1, r_1, f),$$

and $\rho \in J_n$, we have the inequalities

$$\mathbb{P}(B_n(f + \rho I, r_1(1 + \varepsilon_0))) \geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 + 2\varepsilon_1 \right\}\right),$$

$$\mathbb{P}(B_n(f + \rho I, r_1(1 - \varepsilon_0))) \leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 - 2\varepsilon_1 \right\}\right).$$

This being (2.56)–(2.57), the proof of the lemma is completed. \square

LEMMA 2.9. – Assume that $h_n \rightarrow 0$. Then, for each $r_2 > 0$, $f \in \mathbb{K}$ and $0 < \varepsilon_1 < 1$, there exists an $N_4 = N_4(\varepsilon_1, r_2, f)$ such that, for all $n \geq N_4$ and $|\rho| < 2h_n^{1/2}$, we have

$$\begin{aligned} & \mathbb{P}(C_n(f, r_2) | \rho_n = -\rho) \\ & \geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 + 3\varepsilon_1 \right\}\right), \end{aligned} \tag{2.64}$$

$$\begin{aligned} & \mathbb{P}(C_n(f, r_2) | \rho_n = -\rho) \\ & \leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 - 3\varepsilon_1 \right\}\right). \end{aligned} \tag{2.65}$$

Proof. – Recall the definitions (2.51) and (2.52) of $B_n(f, r)$ and $C_n(f, r)$. Since (2.46) entails that $\Psi_n^{(1)}$ and ρ_n are independent when $0 < h_n < 1$, we obtain in this case that

$$\mathbb{P}(C_n(f, r_2) | \rho_n = -\rho) = \mathbb{P}(B_n(f + \rho I, r_2)). \tag{2.66}$$

Fix now $f \in \mathbb{K}$, $r_2 > 0$ and $0 < \varepsilon_1 < 1$. Select an $0 < \varepsilon_0 < 1$ such that $r_1 = r_2/(1 + \varepsilon_0)$ fulfills the inequality

$$0 < \frac{\pi^2}{4r_1^2} - \frac{\pi^2}{4r_2^2} = \frac{\pi^2}{4r_2^2} \{(1 + \varepsilon_0)^2 - 1\} < \varepsilon_1. \tag{2.67}$$

Given these choices of r_1 and ε_0 , we infer from (2.67) and Lemma 2.8 that, for all $n \geq N_3(\varepsilon_0, \varepsilon_1, r_1, f)$ and $|\rho| \leq 2h_n^{1/2}$,

$$\begin{aligned}
\mathbb{P}(B_n(f + \rho I, r_2)) &= \mathbb{P}(B_n(f + \rho I, r_1(1 + \varepsilon_0))) \\
&\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_1^2} + |f|_{\mathbb{H}}^2 + 2\varepsilon_1 \right\}\right) \\
&\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 + 3\varepsilon_1 \right\}\right),
\end{aligned}$$

which, when combined with (2.66), readily yields (2.64). The proof of (2.65) is very similar, and therefore omitted. \square

LEMMA 2.10. – Assume (H.1) and (H.3). Then, for each $r > 0$, $f \in \mathbb{K}$ and $0 < \varepsilon_1 < 1$, there exists an $N_5 = N_5(\varepsilon_1, r, f)$ such that, for all $n \geq N_5$, we have

$$\begin{aligned}
\mathbb{P}(D_n(f, r)) &\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r^2} + |f|_{\mathbb{H}}^2 + 4\varepsilon_1 \right\}\right) \\
&\quad - 2 \exp(-2 \log_2 n),
\end{aligned} \tag{2.68}$$

$$\begin{aligned}
\mathbb{P}(D_n(f, r)) &\leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r^2} + |f|_{\mathbb{H}}^2 - 4\varepsilon_1 \right\}\right) \\
&\quad + 2 \exp(-2 \log_2 n).
\end{aligned} \tag{2.69}$$

Proof. – Fix any $f \in \mathbb{K}$, $r > 0$, $0 < \varepsilon_1 < 1$, and recall the definitions (2.52) and (2.53) of $C_n(f, r)$ and $D_n(f, r)$. Let \mathbb{I}_E stand for the indicator function of the event E . We first write, via (2.49) and (2.64), that, for each choice of $r_2 > 0$ and all $n \geq N_4(\varepsilon_1, r_2, f) \vee N_1$,

$$\begin{aligned}
\mathbb{P}(C_n(f, r_2)) &\geq \mathbb{P}(C_n(f, r_2) \cap \{|\rho_n| \leq 2h_n^{1/2}\}) \\
&= \mathbb{E}(P(C_n(f, r_2)|\rho_n) \mathbb{I}_{\{|\rho_n| \leq 2h_n^{1/2}\}}) \\
&\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 + 3\varepsilon_1 \right\}\right) \times \mathbb{P}(|\rho_n| \leq 2h_n^{1/2}) \\
&\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 + 3\varepsilon_1 \right\}\right) - \mathbb{P}(|\rho_n| > 2h_n^{1/2}) \\
&\geq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 + 3\varepsilon_1 \right\}\right) - \exp(-2 \log_2 n).
\end{aligned} \tag{2.70}$$

Likewise, we infer from (2.49) and (2.65) that, for all $n \geq N_4(\varepsilon_1, r_2, f) \vee N_1$,

$$\begin{aligned}
\mathbb{P}(C_n(f, r_2)) &\leq \mathbb{P}(C_n(f, r_2) \cap \{|\rho_n| \leq 2h_n^{1/2}\}) + \mathbb{P}(|\rho_n| > 2h_n^{1/2}) \\
&= \mathbb{E}(P(C_n(f, r_2)|\rho_n) \mathbb{I}_{\{|\rho_n| \leq 2h_n^{1/2}\}}) + \mathbb{P}(|\rho_n| > 2h_n^{1/2})
\end{aligned}$$

$$\begin{aligned} &\leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 - 3\varepsilon_1 \right\}\right) + \mathbb{P}(|\rho_n| > 2h_n^{1/2}) \\ &\leq \exp\left(-(\log_2 n) \left\{ \frac{\pi^2}{4r_2^2} + |f|_{\mathbb{H}}^2 - 3\varepsilon_1 \right\}\right) + \exp(-2\log_2 n). \end{aligned} \tag{2.71}$$

We next choose ε in such a way that $0 < \varepsilon < 1 \wedge r$, and

$$\frac{\pi^2}{4(r - \varepsilon)^2} - \varepsilon_1 \leq \frac{\pi^2}{4r^2} \leq \frac{\pi^2}{4(r + \varepsilon)^2} + \varepsilon_1. \tag{2.72}$$

Recalling the definitions (2.45)–(2.46) of Ψ_n, Ψ_n^* and (2.52)–(2.53) of $C_n(f, r), D_n(f, r)$, we make use of Proposition 2.3 to show, via (2.41), that the following inequalities hold for all $n \geq N_0(\varepsilon)$.

$$\begin{aligned} &\mathbb{P}(C_n(f, r - \varepsilon)) - \exp(-2\log_2 n) \\ &\leq \mathbb{P}\left(\|\Psi_n - f\| \leq \frac{r - \varepsilon}{\log_2 n}\right) - \mathbb{P}\left(\|\Psi_n - \Psi_n^*\| \geq \frac{\varepsilon}{\log_2 n}\right) \\ &\leq \mathbb{P}(D_n(f, r)) = \mathbb{P}\left(\|\Psi_n^* - f\| \leq \frac{r}{\log_2 n}\right) \\ &\leq \mathbb{P}\left(\|\Psi_n - f\| \leq \frac{r + \varepsilon}{\log_2 n}\right) + \mathbb{P}\left(\|\Psi_n - \Psi_n^*\| \geq \frac{\varepsilon}{\log_2 n}\right) \\ &\leq \mathbb{P}(C_n(f, r + \varepsilon)) + \exp(-2\log_2 n). \end{aligned} \tag{2.73}$$

To obtain (2.68), we set $r_2 = r - \varepsilon$ in (2.70) and combine the inequality so obtained with (2.72) and (2.73). The proof of (2.69) is achieved likewise by setting $r_2 = r + \varepsilon$ in (2.71), and making use of (2.72) and (2.73). \square

We have now in hand most of the necessary ingredients to achieve the proof of the lower bound part of Theorem 1.2. We start with the following proposition. Recall from (2.48) that $n_k = \lfloor \exp\{k(\log_+ k)^\theta\} \rfloor$ for $k \geq 0$.

PROPOSITION 2.4. – *Under (H.1) and (H.3), for each $\theta \in \mathbb{R}$ and $f \in \mathbb{K}$, we have*

$$\liminf_{k \rightarrow \infty} (2\log_2 n_k) \left\| \frac{\alpha_{n_k}(h_{n_k} I)}{\sqrt{2h_{n_k} \log_2 n_k}} - f \right\| \geq \frac{\pi}{2\sqrt{1 - |f|_{\mathbb{H}}^2}} \quad \text{a.s.} \tag{2.74}$$

Proof. – Fix any $f \in \mathbb{K}$, $0 < \varepsilon_1 < 1$, and set

$$r = r(\varepsilon_1) = \frac{\pi}{2\sqrt{1 + 5\varepsilon_1 - |f|_{\mathbb{H}}^2}} \Leftrightarrow \frac{\pi^2}{4r^2} + |f|_{\mathbb{H}}^2 = 1 + 5\varepsilon_1.$$

This, in combination with (2.69) shows that

$$\mathbb{P}(D_{n_k}(f, r(\varepsilon_1))) \leq \exp(-(1 + \varepsilon_1) \log_2 n_k) + 2 \exp(-2 \log_2 n_k),$$

which, by (2.48), is summable in k . An application of the Borel–Cantelli lemma in combination with (2.53) shows therefore that

$$\liminf_{k \rightarrow \infty} (2 \log_2 n_k) \left\| \frac{\alpha_{n_k}(h_{n_k} I)}{\sqrt{2h_{n_k} \log_2 n_k}} - f \right\| \geq r(\varepsilon_1) \quad \text{a.s.}$$

Since a choice of $\varepsilon_1 > 0$ arbitrarily small renders $r(\varepsilon_1)$ arbitrarily close to the RHS of (2.74), the conclusion follows. \square

Of course, we would like to show that (2.74) holds ultimately in $n \rightarrow \infty$ instead of being true only along the sequence n_k . To do so, we will need to specify the choice of θ in the definition of n_k , in order to “bridge the gaps” between n_k and n_{k+1} . Towards this aim, we will borrow the following facts from Mason [29]. Note that Fact 2.2 below is (1.9) which we state here for convenience (see, e.g., (2.9) p. 500 and (1.4.i) p. 493, in Mason [29]).

FACT 2.1. – For each integer $v \geq 1$, $0 < a < 1$ and $r > 2\sqrt{2a}$,

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq m \leq v} \sup_{0 \leq s \leq a} |m^{1/2} \alpha_m(s)| \geq r v^{1/2}\right) \\ & \leq 2\mathbb{P}\left(\sup_{0 \leq s \leq a} |v^{1/2} \alpha_v(s)| \geq \frac{1}{2} r v^{1/2}\right). \end{aligned} \tag{2.75}$$

FACT 2.2. – Under (H.1–2), we have

$$\limsup_{n \rightarrow \infty} \left\| \frac{\alpha_n(h_n I)}{\sqrt{2h_n \log_2 n}} \right\| = 1 \quad \text{a.s.} \tag{2.76}$$

The next fact is a simple consequence of Inequality 2, p. 444 in Shorack and Wellner [36], in combination with the following observations. Let $\mathbf{h}(x) = x \log x - x + 1$ for $x > 0$, $\mathbf{h}(x) = 1$ for $x = 1$ and $\mathbf{h}(x) = \infty$ for $x < 0$. Set $\psi(\lambda) = 2\mathbf{h}(1 + \lambda)/\lambda^2$ for $\lambda \neq 0$, and $\psi(\lambda) = 1$

for $\lambda = 0$. The following properties of $\psi(\lambda)$ follow from Proposition 1, p. 441 in Shorack and Wellner [36] when $\lambda > -1$. Since $\psi(\lambda) = 2$ for $\lambda = -1$ and $\psi(\lambda) = \infty$ for $\lambda < -1$, they are straightforward for $\lambda \leq -1$.

- (i) $\psi(\lambda) \uparrow 1$ as $\lambda \downarrow 0$;
- (ii) $\psi(\lambda) \geq 1/(1 + \lambda/3)$ for all $\lambda > -1$;
- (iii) $\psi(\lambda) = (1 + o(1))\lambda^{-1} \log \lambda$ as $\lambda \rightarrow \infty$;
- (iv) $\psi(\lambda) \leq \psi(-\lambda)$ for all $\lambda \geq 0$. (2.77)

FACT 2.3. – Let $0 < a \leq 1/2$, $n \geq 1$ and $\lambda \geq 0$. Then

$$\mathbb{P}\left(\sup_{0 \leq s \leq a} |\alpha_n(t)| \geq \lambda/(1 - a)\right) \leq 2 \exp\left(-\frac{\lambda^2}{2a(1 - a)} \psi\left(\frac{\lambda}{a\sqrt{n}}\right)\right). \tag{2.78}$$

The next proposition will be proved in a slightly more general setting as before, in view of possible extensions of our results. Recall the assumption

$$(H.5) \quad nh_n(\log_3 n)^2/(\log_2 n)^3 \rightarrow \infty.$$

PROPOSITION 2.5. – Let (H.1) and (H.5) hold. Then, if $\theta = -3$ in the definition (2.48) of n_k , we have

$$\lim_{k \rightarrow \infty} (2 \log_2 n_k) \times \left\{ \max_{n_{k-1} < n \leq n_k} \left\| \frac{\alpha_n(h_n I)}{\sqrt{2h_n \log_2 n}} - \frac{\alpha_{n_k}(h_{n_k} I)}{\sqrt{2h_{n_k} \log_2 n_k}} \right\| \right\} = 0 \quad a.s. \tag{2.79}$$

Proof. – The proof is decomposed into three parts, where we consider, at times, different values of θ .

Part I. Let $\theta \in \mathbb{R}$ be arbitrary. Since $n_k = \exp\{k(\log k)^\theta\} + O(1)$ as $k \rightarrow \infty$, straightforward Taylor expansions show that, for $\theta \neq 0$, as $k \rightarrow \infty$,

$$\frac{n_{k-1}}{n_k} = \exp(-(\log k)^\theta - (1 + o(1))\theta (\log k)^{\theta-1}), \tag{2.80}$$

$$\log_2 n_k = (1 + o(1)) \log k \quad \text{and} \quad \frac{\log_2 n_{k-1}}{\log_2 n_k} = 1 - \frac{1 + o(1)}{k \log k} \rightarrow 1. \tag{2.81}$$

We make use of (ii)–(iii) in (H.1) to write that, for all $k \geq 3$ and $n_{k-1} \leq n \leq n_k$,

$$\begin{aligned} h_{n_k} &\leq h_n \leq h_{n_{k-1}} \leq (n_k/n_{k-1})h_{n_k} \quad \text{and} \\ n_{k-1}h_{n_k} &\leq n_{k-1}h_{n_{k-1}} \leq nh_n \leq n_k h_{n_k}. \end{aligned} \tag{2.82}$$

When $\theta < 0$, (2.80)–(2.81) and (2.82) jointly imply that, ultimately as $k \rightarrow \infty$, for $n_{k-1} \leq n \leq n_k$,

$$\begin{aligned} 0 \leq h_n - h_{n_k} &\leq h_{n_{k-1}} - h_{n_k} \leq h_{n_k} \left(\frac{n_k}{n_{k-1}} - 1 \right) \\ &= (1 + o(1)) h_{n_k} (\log_2 n_k)^\theta \\ &\leq 2h_{n_k} (\log_2 n_k)^\theta. \end{aligned} \tag{2.83}$$

Since $\log_2 n_k = (1 + o(1)) \log k$ as $k \rightarrow \infty$, (2.80) and (2.82) imply, when $\theta < 0$, that for all large k ,

$$\begin{aligned} \max_{n_{k-1} < n \leq n_k} \left| 1 - \sqrt{\frac{nh_n \log_2 n}{n_k h_{n_k} \log_2 n_k}} \right| &\leq \left| 1 - \sqrt{\frac{n_{k-1} \log_2 n_{k-1}}{n_k \log_2 n_k}} \right| \\ &\leq (\log_2 n_k)^\theta. \end{aligned} \tag{2.84}$$

By combining (2.76) and (2.81) with (2.84), we obtain, in turn, that, if $\theta < -1$,

$$\begin{aligned} &\limsup_{k \rightarrow \infty} (2 \log_2 n_k) \\ &\times \left\{ \max_{n_{k-1} < n \leq n_k} \left\| \frac{n^{1/2} \alpha_n(h_n I)}{\sqrt{2nh_n \log_2 n}} - \frac{n^{1/2} \alpha_n(h_n I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} \right\| \right\} \\ &\leq 2 \times \left\{ \limsup_{n \rightarrow \infty} \left\| \frac{\alpha_n(h_n I)}{\sqrt{2h_n \log_2 n}} \right\| \right\} \\ &\times \left\{ \lim_{k \rightarrow \infty} (\log_2 n_k)^{1+\theta} \right\} = 0 \quad \text{a.s.} \end{aligned} \tag{2.85}$$

Part II. Below, we will set $v_k = n_k - n_{k-1}$ for notational convenience, and select an arbitrary $\varepsilon > 0$. We observe from (2.82) that $h_n \leq h_{n_{k-1}}$ for all $n_{k-1} \leq n \leq n_k$. Thus, making use of the equality in distribution, for $1 \leq n < n_k$,

$$n_k^{1/2} \alpha_{n_k} - n^{1/2} \alpha_n =_d (n_k - n)^{1/2} \alpha_{n_k - n},$$

we infer from (2.75), taken with $v = v_k$, $a = h_{n_{k-1}}$ and $r = r_k$ defined below, that, when $\theta < -1$,

$$\begin{aligned} P_k(\varepsilon) &:= \mathbb{P} \left(\max_{n_{k-1} \leq n \leq n_k} \left\| \frac{n_k^{1/2} \alpha_{n_k}(h_n I) - n^{1/2} \alpha_n(h_n I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} \right\| \geq \frac{\varepsilon}{2 \log_2 n_k} \right) \\ &\leq \mathbb{P} \left(\max_{n_{k-1} \leq n \leq n_k} \left\| \frac{n_k^{1/2} \alpha_{n_k}(h_{n_{k-1}} I) - n^{1/2} \alpha_n(h_{n_{k-1}} I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} \right\| \geq \frac{\varepsilon}{2 \log_2 n_k} \right) \end{aligned}$$

$$\begin{aligned} &\leq 2\mathbb{P}\left(\|v_k^{1/2}\alpha_{v_k}(h_{n_{k-1}}I)\| \geq \frac{\varepsilon}{2}\left\{\frac{n_k h_{n_k}}{2\log_2 n_k}\right\}^{1/2}\right) \\ &=: 2\mathbb{P}\left(\|v_k^{1/2}\alpha_{v_k}(h_{n_{k-1}}I)\| \geq \frac{1}{2}r_k v_k^{1/2}\right). \end{aligned} \tag{2.86}$$

We note that $r = r_k$ in (2.86) fulfills the condition $r \geq 2\sqrt{2a} = 2\sqrt{2h_{n_{k-1}}}$ required in Fact 2.1 for (2.75). This follows from the observation, via (2.82)–(2.83), that $h_{n_k} = (1 + o(1))h_{n_{k-1}}$. This implies, in turn, that ultimately as $k \rightarrow \infty$,

$$\begin{aligned} r_k &= \varepsilon \left\{ \frac{n_k h_{n_k}}{2(n_k - n_{k-1}) \log_2 n_k} \right\}^{1/2} \\ &= 2\sqrt{2h_{n_{k-1}}} \times \frac{(1 + o(1))\varepsilon}{4} \times (\log_2 n_k)^{-(1+\theta)/2} \\ &> 2\sqrt{2h_{n_{k-1}}}, \end{aligned}$$

where we have used the fact that $(\log_2 n_k)^{-(1+\theta)/2} \rightarrow \infty$ for $\theta < -1$.

From now on we set $\theta = -3$ and make use of (2.78), taken with $n = v_k$, $a = h_{n_{k-1}}$ and $\lambda = \frac{1}{2}r_k(1 - h_{n_{k-1}})$. Observe that, for these choices of λ and a , for all large k

$$\begin{aligned} \frac{\lambda^2}{2a(1-a)} &= \frac{(1 + o(1))\varepsilon^2}{16} \times (\log_2 n_k)^{-(1+\theta)}, \\ &= \frac{(1 + o(1))\varepsilon^2}{16} \times (\log_2 n_k)^2 > \frac{2\log_2 n_k}{\psi(1)}, \end{aligned} \tag{2.87}$$

and, likewise, assuming only (H.2), for all large k ,

$$\begin{aligned} \frac{\lambda}{a\sqrt{n}} &= \frac{(1 + o(1))\varepsilon}{4} \times \left\{ \frac{2\log_2 n_k}{n_k h_{n_k}} \right\}^{1/2} \times (\log_2 n_k)^{-(3+\theta)/2} \\ &= \frac{(1 + o(1))\varepsilon}{4} \times \left\{ \frac{2\log_2 n_k}{n_k h_{n_k}} \right\}^{1/2} < 1. \end{aligned} \tag{2.88}$$

By combining (2.87) with (2.88), we have, for all large k ,

$$\frac{\lambda^2}{2a(1-a)} \psi\left(\frac{\lambda}{a\sqrt{n}}\right) \geq \frac{2\log_2 n_k}{\psi(1)} \times \psi(1) = 2\log_2 n_k.$$

By (2.78) and (2.86), this implies in turn, via (2.77), that, for the above choices of λ and a , and all large k ,

$$P_k(\varepsilon) \leq 2\mathbb{P}(\|\alpha_{v_k}(h_{n_k}I)\| \geq \lambda/(1-a)) \leq 4\exp(-2\log_2 n_k),$$

which is summable in k . This, when combined with the Borel–Cantelli lemma and the fact that $\varepsilon > 0$ may be chosen arbitrarily small, entails that

$$\lim_{k \rightarrow \infty} (2 \log_2 n_k) \times \left\{ \max_{n_{k-1} \leq n \leq n_k} \left\| \frac{n_k^{1/2} \alpha_{n_k}(h_n I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} - \frac{n^{1/2} \alpha_n(h_n I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} \right\| \right\} = 0 \quad \text{a.s.} \quad (2.89)$$

Part III. In this last part of our proof, we fix $\theta = -3$. Our aim is to show that

$$\lim_{k \rightarrow \infty} (2 \log_2 n_k) \times \left\{ \max_{n_{k-1} \leq n \leq n_k} \left\| \frac{n_k^{1/2} \alpha_{n_k}(h_{n_k} I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} - \frac{n_k^{1/2} \alpha_{n_k}(h_n I)}{\sqrt{2n_k h_{n_k} \log_2 n_k}} \right\| \right\} = 0 \quad \text{a.s.} \quad (2.90)$$

Given (2.85), (2.89) and (2.90), the conclusion (2.79) will follow from an application of the triangle inequality. To establish (2.90) it suffices to show that

$$\lim_{k \rightarrow \infty} \sqrt{\frac{\log_2 n_k}{h_{n_k}}} \times \sup_{0 \leq t \leq h_{n_k}} \|\alpha_{n_k}(t + \ell_{n_k} I) - \alpha_{n_k}(t)\| = 0 \quad \text{a.s.}, \quad (2.91)$$

where we set

$$\ell_n = 2h_n(\log_2 n)^\theta = 2h_n/(\log_2 n)^3, \quad (2.92)$$

and make use of (2.83), to check that, for all large k and $n_{k-1} < n \leq n_k$,

$$0 \leq h_n - h_{n_k} \leq \ell_{n_k}.$$

Set $M_n = \lceil 3(\log_2 n)^3 \rceil$. It is readily verified that, for all large n , $(M_n - 1)\ell_n \geq h_n + \ell_n$, whence, by (2.92),

$$\begin{aligned} & \sqrt{\frac{\log_2 n}{h_n}} \times \sup_{0 \leq t \leq h_n} \|\alpha_n(t + \ell_n I) - \alpha_n(t)\| \\ &= \frac{\sqrt{2}}{(\log_2 n)\sqrt{\ell_n}} \times \sup_{0 \leq t \leq h_n} \|\alpha_n(t + \ell_n I) - \alpha_n(t)\| \\ &\leq \frac{4\sqrt{2}}{(\log_2 n)\sqrt{\ell_n}} \max_{1 \leq j \leq M_n} \|\alpha_n((j-1)\ell_n + \ell_n I) - \alpha_n((j-1)\ell_n)\|. \end{aligned} \quad (2.93)$$

In view of (2.93) and the distributional equality $\alpha_n((j-1)\ell_n + \ell_n I) - \alpha_n((j-1)\ell_n) =_d \alpha_n(\ell_n I)$, an application of the Borel–Cantelli lemma

shows that the proof of (2.91) may be reduced to show that, for each $\varepsilon > 0$

$$\sum_k M_{n_k} Q_{n_k}(\varepsilon) < \infty \Leftrightarrow \sum_k (\log_2 n_k)^3 Q_{n_k}(\varepsilon) < \infty, \tag{2.94}$$

where

$$Q_n(\varepsilon) = \mathbb{P}(\|\alpha_n(\ell_n I)\| \geq \varepsilon \sqrt{\ell_n} (\log_2 n)). \tag{2.95}$$

Now, we make use of (2.78), taken with $a = \ell_n$ and $\lambda = \lambda_n := (1 - \ell_n)\varepsilon\sqrt{\ell_n} (\log_2 n)$, to evaluate $Q_n(\varepsilon)$. We distinguish the following two cases.

Case 1. When $\sqrt{n\ell_n} \geq 18/\varepsilon$, we make use of (2.77)(ii) to obtain, via the inequality $\psi(\lambda) \geq 1/(1 + \lambda/3)$, that, ultimately as $n \rightarrow \infty$,

$$\begin{aligned} Q_n(\varepsilon) &\leq 2 \exp\left(-\frac{\varepsilon^2(1 - \ell_n)}{2} (\log_2 n)^2 / \left\{1 + \frac{3\varepsilon(1 - \ell_n)(\log_2 n)}{\sqrt{n\ell_n}}\right\}\right) \\ &= 2 \exp\left(-(1 + o(1)) \frac{\varepsilon}{6} (\log_2 n) \sqrt{n\ell_n}\right) \leq 2 \exp(-2 \log_2 n). \end{aligned} \tag{2.96}$$

Case 2. When $\sqrt{n\ell_n} \leq 18/\varepsilon$, it holds that $\lambda_n/(\ell_n\sqrt{n}) = (1 + o(1)) (\log_2 n)/\sqrt{n\ell_n} \rightarrow \infty$. Since (H.5) entails that $n\ell_n(\log_3 n)^2 \rightarrow \infty$, we infer from (2.77)(iii),

$$\psi\left(\frac{\lambda_n}{\ell_n\sqrt{n}}\right) = (1 + o(1)) \frac{\sqrt{n\ell_n}}{\log_2 n} \log\left(\frac{\log_2 n}{\sqrt{n\ell_n}}\right) = (1 + o(1)) \frac{\sqrt{n\ell_n} \log_3 n}{\log_2 n}.$$

It follows therefore from (2.78) that, ultimately as $n \rightarrow \infty$,

$$\begin{aligned} Q_n(\varepsilon) &\leq 2 \exp\left(-\frac{\varepsilon^2(1 - \ell_n)}{2} (\log_2 n)^2 / \psi\left(\frac{\lambda_n}{\ell_n\sqrt{n}}\right)\right) \\ &= 2 \exp\left(-(1 + o(1)) \frac{\varepsilon^2}{2} (\log_2 n) \{n\ell_n(\log_3 n)^2\}^{1/2}\right) \\ &\leq 2 \exp(-2 \log_2 n). \end{aligned} \tag{2.97}$$

By combining (2.96)–(2.97) in Cases 1–2, we see that, for all large n ,

$$Q_n(\varepsilon) \leq 2 \exp(-2 \log_2 n).$$

This, in turn, readily implies (2.94) and completes the proof of the proposition. \square

The following arguments are directed towards proving the upper bound part of Theorem 1.2. From now on, we make use of the sequence m_k defined in (2.48).

LEMMA 2.11. – *Let (H.1) and (H.3) hold. Then, for each $0 < \varepsilon < 1$, if we choose $\delta \geq \log(4/\varepsilon)$ in the definition (2.48) of m_k , we have*

$$\limsup_{k \rightarrow \infty} (2 \log_2 m_k) \left\{ \frac{m_{k-1}}{m_k} \right\}^{1/2} \times \left\| \frac{\alpha_{m_{k-1}}(h_{m_k} I)}{\sqrt{2h_{m_k} \log_2 m_k}} \right\| \leq \varepsilon \quad \text{a.s.} \quad (2.98)$$

Proof. – Fix any $0 < \varepsilon < 1$. Recalling from (2.48) that $m_k = \lfloor \exp\{2k \times (\delta + \log_2 k)\} \rfloor$, we see that, as $k \rightarrow \infty$,

$$\frac{m_{k-1}}{m_k} = (1 + o(1)) \times \frac{e^{-2\delta}}{(\log k)^2} = (1 + o(1)) \times \frac{e^{-2\delta}}{(\log_2 m_k)^2} \rightarrow 0. \quad (2.99)$$

Thus, we have

$$\lim_{k \rightarrow \infty} (2 \log_2 m_k) \left\{ \frac{m_{k-1}}{m_k} \right\}^{1/2} = 2e^{-\delta}, \quad (2.100)$$

and we are done if we can show that

$$\limsup_{k \rightarrow \infty} \frac{\|\alpha_{m_{k-1}}(h_{m_k} I)\|}{\sqrt{2h_{m_k} \log_2 m_k}} \leq \varepsilon e^\delta / 2 \quad \text{a.s.} \quad (2.101)$$

Towards this end, we fix an arbitrary $\eta > 0$ and consider

$$R_k(\eta) = \mathbb{P} \left(\|\alpha_{m_{k-1}}(h_{m_k} I)\| \geq \eta \sqrt{2h_{m_k} \log_2 m_k} \right). \quad (2.102)$$

Then, we make use of (2.78), taken with $n = m_{k-1}$, $a = h_{m_k}$ and $\lambda = \eta \sqrt{2h_{m_k} \log_2 m_k}$. We infer from (2.99) that, for these choices of n , a and λ , we have, as $k \rightarrow \infty$,

$$\begin{aligned} \frac{\lambda^2}{2a(1-a)} &= (1 + o(1)) \eta^2 (\log_2 m_k), \\ \frac{\lambda}{a\sqrt{n}} &= (1 + o(1)) \eta \sqrt{2} \times \left\{ \frac{m_k}{m_{k-1}} \right\}^{1/2} \times \left\{ \frac{\log_2 m_k}{m_k h_{m_k}} \right\}^{1/2} \\ &= (1 + o(1)) \eta e^\delta \sqrt{2} \times \left\{ \frac{(\log_2 m_k)^3}{m_k h_{m_k}} \right\}^{1/2} \rightarrow 0. \end{aligned}$$

Since this entails, via (2.77), that $\psi(\lambda/(a\sqrt{n})) \rightarrow 1$, it follows from (2.78) that, for all k sufficiently large,

$$R_k(\eta) \leq 2 \exp\left(-\frac{\eta^2}{2} \log_2 m_k\right), \tag{2.103}$$

which is summable in k when $\eta \geq 2$. Thus, by the Borel–Cantelli lemma, we have (2.101) when δ is chosen in such a way that $\varepsilon e^\delta/2 \geq 2$, which is equivalent to $\delta \geq \log(4/\varepsilon)$. \square

PROPOSITION 2.6. – *Let (H.1) and (H.3) hold. Then, for each $0 < \varepsilon < 1$, if we choose $\delta > \log(4/\varepsilon)$ in the definition (2.48) of m_k , we have, for each $f \in \mathbb{K}$,*

$$\liminf_{k \rightarrow \infty} (2 \log_2 m_k) \left\| \frac{\alpha_{m_k}(h_{m_k} I)}{\sqrt{2h_{m_k} \log_2 m_k}} - f \right\| \leq \frac{\pi}{2\sqrt{1 - |f|_{\mathbb{H}}^2}} + \varepsilon \quad \text{a.s.} \tag{2.104}$$

Proof. – Since (2.104) is trivial when $|f|_{\mathbb{H}} = 1$, we assume without loss of generality that $|f|_{\mathbb{H}} < 1$. By (2.98) and the triangle inequality, to prove (2.104), we need only show that

$$\begin{aligned} \liminf_{k \rightarrow \infty} (2 \log_2 m_k) \left\| \frac{m_k^{1/2} \alpha_{m_k}(h_{m_k} I) - m_{k-1}^{1/2} \alpha_{m_{k-1}}(h_{m_k} I)}{\sqrt{2m_k h_{m_k} \log_2 m_k}} - f \right\| \\ \leq \frac{\pi}{2\sqrt{1 - |f|_{\mathbb{H}}^2}} \quad \text{a.s.} \end{aligned} \tag{2.105}$$

Towards this aim, we choose an arbitrary ε_1 such that $0 < \varepsilon_1 < (1 - |f|_{\mathbb{H}}^2)/5$. For such an ε_1 , we set

$$r = r(\varepsilon_1) = \frac{\pi}{2\sqrt{1 - 5\varepsilon_1 - |f|_{\mathbb{H}}^2}} \Leftrightarrow \frac{\pi^2}{4r^2} + |f|_{\mathbb{H}}^2 = 1 - 5\varepsilon_1.$$

Set for convenience $\mu_k = m_k - m_{k-1}$. We have the distributional equality

$$m_k^{1/2} \alpha_{m_k}(h_{m_k} I) - m_{k-1}^{1/2} \alpha_{m_{k-1}}(h_{m_k} I) =_d \mu_k^{1/2} \alpha_{\mu_k}(h_{m_k} I).$$

Moreover, the random processes $m_k^{1/2} \alpha_{m_k}(h_{m_k} I) - m_{k-1}^{1/2} \alpha_{m_{k-1}}(h_{m_k} I)$ are independent. Since (2.99) entails that $\mu_k = (1 + o(1))m_k$ as $k \rightarrow \infty$, the Borel–Cantelli lemma reduces the proof of (2.105) to show that, for each

choice of ε_1 such that $0 < \varepsilon_1 < (1 - |f|_{\mathbb{H}}^2)/5$, we have

$$\sum_k S_k(\varepsilon_1) = \infty, \quad (2.106)$$

where we set

$$S_k(\varepsilon_1) = \mathbb{P} \left(\left\| \frac{\alpha_{\mu_k}(h_{m_k} I)}{\sqrt{2h_{m_k} \log_2 \mu_k}} - f \right\| \leq r(\varepsilon_1) \right). \quad (2.107)$$

Since $\mu_k = (1 + o(1))m_k$ as $k \rightarrow \infty$, it is readily checked from (2.68) that for all k sufficiently large

$$\begin{aligned} S_k(\varepsilon_1) &\geq \exp \left(-(\log_2 m_k) \left\{ \frac{\pi^2}{4r^2} + |f|_{\mathbb{H}}^2 + 4\varepsilon_1 \right\} \right) \\ &\quad - 2 \exp(-2 \log_2 m_k) \\ &= \exp \left(-(\log_2 m_k) \{1 - \varepsilon_1\} \right) - 2 \exp(-2 \log_2 m_k), \end{aligned}$$

which readily implies (2.106), as sought. \square

Proof of Theorem 1.2. – Combine Propositions 2.4, 2.5 and 2.6, and observe that, in (2.104), we may choose $\varepsilon > 0$ arbitrarily small. \square

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