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A Berry–Esseen theorem on semisimple Lie groups

by

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ABSTRACT. – We give Berry–Esseen type of estimates for convolution powers of a probability density on a semisimple Lie group and we deduce gaussian estimates. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On donne des estimations de type Berry–Esseen pour des puissances de convolution d' une densité de probabilité sur un groupe de Lie semisimple et on en déduit des estimations gaussiennes. © 2000 Éditions scientifiques et médicales Elsevier SAS

Let μ be a probability measure on a connected semisimple non compact Lie group G with finite center. The asymptotic behavior of the convolution powers μ^{*n} has been studied, among others, by Bougerol [3] who has proved that under suitable conditions on μ there exists a sequence $a_n \approx \rho^{-n} n^{\nu}$ such that $a_n \mu^{*n}$ converges weakly to a certain measure. The aim of this paper is to improve such a result in the case the measure μ has a density f, giving an estimate of the rate of convergence of $f^{*n}(e)$ to $C_f \rho^n n^{-\nu}$. In the classical setting of $\mathbf{R}^n$ it is well known [4] that if $f \in L^1(\mathbf{R}^n)$ is the density of a probability measure with mean zero, variance Q, finite third moment and whose Fourier transform is integrable and if we denote by h_t^Q the Gaussian with variance Q then

$|f^{*n}(x) - h_n^Q(x)| \leqslant Cn^{-d-1/2}$. We prove an analogue of that result, namely that there exists a function F_n such that $\|f^{*n} - F_n\|_{L^\infty(G)} \leqslant C_\varepsilon \rho^n n^{-\nu-1/2+\varepsilon}$. F_n *does* depend on f, but if f is biinvariant, F_n is the classical binvariant heat kernel $\rho^n H_n^Q(e)$ for an appropriate Q while in the general case we can at least say that $F_n(e) = \rho^n H_n^Q(e)$. We remark that what is relevant for the applications it is the gain of the factor $n^{-1/2+\varepsilon}$ since both terms are known to be smaller than $\rho^n n^{-\nu}$.

Let us fix some notations and recall some well known facts about harmonic analysis on semisimple Lie groups. Let $\mathcal{G}$ be the Lie algebra of G and $\mathcal{G} = \mathcal{K} + \mathcal{P}$ a Cartan decomposition. We fix $\mathcal{A} \subset \mathcal{P}$ a maximal abelian subalgebra of $\mathcal{P}$ and we denote by Σ the root system of $(\mathcal{G}, \mathcal{A})$, by Σ^+ the set of positive roots and by Σ_0^+ the set of positive indivisible roots. Let $G = K(\exp \overline{\mathcal{A}^+})K$ be the Cartan decomposition associated to this choice of positive roots. A function f is called biinvariant if $f(k_1 g k_2) = f(g)\ \forall k_1, k_2 \in K$. We denote by $\mathcal{S}(K\backslash G/K)$ the set of smooth biinvariant rapidly decreasing functions and by $\{\phi_\lambda\}_{\lambda \in \mathcal{A}^*}$ the set of spherical functions. If $f \in \mathcal{S}(K\backslash G/K)$ its spherical transform is given by

$$\mathcal{F}f(\lambda) = \int_G \phi_\lambda(g) f(g)\,\mathrm{d}g.$$

The Fourier analysis of biinvariant functions is based on the following two formulas

$$\mathcal{F}(f * g)(\lambda) = \mathcal{F}f(\lambda)\mathcal{F}g(\lambda), \tag{1}$$

$$f(g) = \mathcal{F}^{-1}\big(\mathcal{F}f(\lambda)\big) = c_G \int_{\mathcal{A}^*} \mathcal{F}f(\lambda)\phi_{-\lambda}(g)\big|c(\lambda)\big|^{-2}\,\mathrm{d}\lambda, \tag{2}$$

where $c(\lambda)$ denotes the Harish–Chandra function. Finally given a positive definite matrix Q we denote by

$$H_n^Q = \mathcal{F}^{-1}\big(e^{-n\langle Q\lambda,\lambda\rangle}\big)$$

the heat kernel with covariance Q.

THEOREM. – *Let $f \in C_c^\infty(G)$ be the density of a symmetric probability measure. Then for every $\varepsilon > 0$ there exist a constant C_ε independent of n and a function F_n such that*

$$\big\|f^{*n} - F_n\big\|_{L^\infty(G)} \leqslant C_\varepsilon \rho^n n^{-\frac{d+2p+1}{2}+\varepsilon},$$

where ρ is the spectral gap of f (i.e., the L^2 norm of the convolution operator $h \to f * h$), d the dimension of A and p the cardinality of Σ_0^+. Moreover the function F_n satisfies the following properties:

1. *if f is biinvariant and $Q = -d^2\mathcal{F}f(0)/2\rho$ then*
$$F_n = \rho^n \mathcal{F}^{-1}\big(e^{-n\langle Q\lambda,\lambda\rangle}\big);$$
2. *$F_n(e) = \rho^n H_n^Q(e)$ for some Q whose dependence on f will be expressed in the proof.*

The proof follows closely Bougerol's method. If f is biinvariant its spherical transform is a Schwartz function on $\mathcal{A}^*$ whose first derivative vanishes at zero and with second derivative strictly negative definite at zero. We can then use the formulas (1) and (2) to repeat "ad verbatim"the classical proof for $\mathbf{R^n}$. To deal with the non biinvariant case we replace f^{*n} and F_n with their convolution with a biinvariant smooth function ϕ_ε. We observe that $f^{*n} * \phi_\varepsilon$ takes at the origin the same value as the biinvariant function $G_n = m_K * f^{*n} * \phi_\varepsilon * m_K$, m_K being the Haar measure on K. A difficulty arises in applying the Fourier method to estimate G_n since the spherical transform of G_n is not the n-power of the spherical transform of G_1. To recover partially this essential property we need to analyze certain representations of G that come into play in the definition of the spherical functions. This is done in Section 1, while in Section 2 we give the proof of the theorem. Section 3 contains some applications in particular a new proof of the gaussian estimates for semisimple groups.

1. THE REPRESENTATION T_G^λ

The results of this section are taken from [3]. For the reader's convenience we recall some of their proofs since under our more restrictive hypotheses they result to be simpler. We fix an Iwasawa decomposition of $G = K\exp\mathcal{A}N$ and if $g = k\exp an$, $k \in K$, $n \in N$, $a \in \mathcal{A}$ we denote by $K(g)$ the element k and by $H(g)$ the element a. Let M be the centralizer of A in K and $L^2(K/M)$ the space of functions in $L^2(K)$ that are M-right invariant and denote by $\tilde{\rho}$ half the sum of positive roots. For all $g \in G$ and $\lambda \in \mathcal{A}^*$ the map

$$\begin{aligned} T_g^\lambda : L^2(K/M) &\to L^2(K/M), \\ \phi(k) &\to e^{-(i\lambda+\tilde{\rho})H(g^{-1}k)}\phi\big(K(g^{-1}k)\big), \end{aligned}$$

is a well defined unitary operator and the correspondence $g \to T_g^\lambda$ is a unitary representation of G. If μ is a bounded measure on G we define the operator $\pi_\mu^\lambda : L^2(K/M) \to L^2(K/M)$ by

$$\pi_\mu^\lambda = \int_G T_g^\lambda \, \mathrm{d}\mu(g).$$

It is easy to see that $\pi_{\mu*\nu}^\lambda = \pi_\mu^\lambda \circ \pi_\nu^\lambda$ and $(\pi_\mu^\lambda)^* = \pi_{\mu^*}^\lambda$ where $\mathrm{d}\mu^*(x) = \overline{\mathrm{d}\mu(x^{-1})}$.

If $f \in C_c(G)$ we denote by π_f^λ the operator $\pi_{f\mathrm{d}g}^\lambda$. Using the standard decomposition of the Haar measure $\mathrm{d}g = e^{2\tilde{\rho}H(a)}\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$ we obtain

$$\begin{aligned}\pi_f^\lambda\phi(k_0) &= \int_K \int_{A\times N} e^{-(i\lambda-\tilde{\rho})H(a)} f\left(k_0 n^{-1}a^{-1}k^{-1}\right) \mathrm{d}a\,\mathrm{d}n\phi(k)\,\mathrm{d}k \\ &= \int_K F^\lambda(k,k_0)\phi(k)\,\mathrm{d}k,\end{aligned}$$

where $F^\lambda(kM, k_0M) = \int_A \mathcal{R}f(kM, a, k_0M)a^{i\lambda}\,\mathrm{d}a$ is the Euclidean Fourier transform of the Radon transform

$$\mathcal{R}f(kM, a, k_0M) = a^{\tilde{\rho}} \int_M \int_N f\left(kmank_0^{-1}\right) \mathrm{d}n\,\mathrm{d}m.$$

This shows that π_f^λ is a Hilbert–Schimtd operator whose norm, majorized by the L^2 norm of $F^\lambda(k,k_0)$, goes to zero when $|\lambda|$ goes to infinity, by the Riemann–Lebesgue lemma. Moreover $\int_{\mathcal{A}^*} \|\pi_f^\lambda\|_{HS}^2 |c(\lambda)|^{-2}\,\mathrm{d}\lambda < \infty$.

LEMMA 1. – *Let $f \in C_c(G)$ a density of a symmetric probability measure. Then*

1. *for each $\lambda \in \mathcal{A}^*$ there exists an orthonormal basis ϕ_i^λ in $L^2(K/M)$ and a sequence μ_i^λ, $|\mu_1^\lambda| \geqslant |\mu_2^\lambda| \geqslant \cdots$ such that*

$$\pi_f^\lambda = \sum_{i=1}^{\infty} \mu_i^\lambda \phi_i^\lambda \otimes \phi_i^\lambda;$$

2. *if $\lambda \neq 0$ the norm $\|\pi_f^\lambda\| = |\mu_1^\lambda|$ is strictly smaller than the norm $\|\pi_f^0\| = \mu_1^0$;*
3. *there is a neighborhood $V \subset A^*$ of the origin such that for all $\lambda \in V$ the first eigenvalue μ_1^λ is both positive and simple and the*

corresponding eigenfunction is strictly positive. Moreover $-\mu_1^\lambda$ *is not an eigenvalue (since* $(\mu_1^\lambda)^2$ *is a simple eigenvalue of* $(\pi_f^\lambda)^2 = \pi_{f*2}^\lambda$*).*

Proof. – 1) It follows from the spectral theorem for self adjoint compact operators, since the symmetry of f implies the symmetry of the operator π_f^λ.

3) We will give the proof in the real setting, since $\ker(\pi_f^0 - \mu_1^0 I)$ is stable by $\phi \to \bar{\phi}, |\phi|, \Re\phi$. Consider $L^2(K/M)$ as a Banach lattice with the usual order relation $\phi \leqslant \psi$ iff $\phi(x) \leqslant \psi(x)$ a.e. and denote by $\phi^+ = \max\{\phi, 0\}$ and $\phi^- = \min\{\phi, 0\}$. We say that an operator P is positive (respectively, strictly positive) if $P\phi \geqslant 0$ if $\phi \geqslant 0$ (respectively, $P\phi > 0$ if $\phi > 0$). A vector subspace A is called an ideal if the condition $\phi \in L^2(K/M)$, $g \in A$ and $|\phi| \leqslant g$ implies that $\phi \in A$. A positive operator S is called irreducible if the ideal generated by the orbit $\{S^n x\}_{n\in N}$ is dense for every $x > 0$. We claim that the operator π_f^0 is irreducible. This is equivalent to prove that for every ϕ and $\psi \in L^2(K/M)$, $\phi, \psi > 0$ there exists an integer n such that $\langle (\pi_f^0)^n \phi, \psi\rangle > 0$. If not $\langle T_g^0 \phi, \psi\rangle$ would be zero on the union of the supports of f^{*n}, i.e., on all G by hypothesis. In particular for $g \in K$ this amounts to saying that the convolution $\phi *_K \psi$ would be zero which is clearly absurd since $\psi, \phi > 0$. Let us consider

$$T = \frac{\pi_f^0}{\mu_1^0}.$$

We have that

$$T = P + \sum_{i=k+1}^{\infty} \frac{\mu_i^0}{\mu_1^0} \phi_i^0 \otimes \phi_i^0,$$

where P denotes the projection onto the eigenspace associated to μ_0^1. We want to show that P is one dimensional. Observe that $P = \lim_{n\to\infty} T^n$ is a positive operator. Moreover it is immediate to see that $PT = TP = P$ and the image of P is given by $\{\phi\colon\ T\phi = \phi\}$. It follows that $A = \{\phi\colon\ P|\phi| = 0\}$ is a closed ideal of $L^2(K/M)$ which is invariant by T. By the irreducibility and the positivity of T we deduce that $A = 0$. P is thus a strictly positive operator and this implies that the image of P is a sublattice of $L^2(K/M)$ i.e. it contains the positive and negative part of its elements. The principal ideals generated by $(P\phi)^+$ and $(P\phi)^-$ are thus invariant by T and since they cannot be both dense either $(P\phi)^+$ or $(P\phi)^-$ should be zero. We deduce that $P(L^2(K/M)$ is a totally ordered

Banach space and thus is isomorphic to **R** (by Lemma 3.4 II in [8]). Since for small λ, π_f^λ is a small perturbation of π_f^0 we obtain 3).

2) Suppose that $\|\pi_f^\lambda\| = \|\pi_f^0\| = \mu_1^0$. Then there exists ϕ such that $\pi_f^\lambda\phi = \mu_1^0\phi$ and thus

$$|\pi_f^\lambda\phi| = \mu_1^0|\phi|. \tag{3}$$

It is clear that $|\pi_f^\lambda\phi| \leqslant \pi_f^0|\phi|$. It follows that $\pi_f^0|\phi| \geqslant \mu_1^0|\phi|$, but since $\|\pi_f^0\| = \mu_1^0$ we have

$$\pi_f^0|\phi| = \mu_1^0|\phi|. \tag{4}$$

Comparing (3) and (4) we deduce that $\forall k \in K$

$$\int_G e^{-\tilde{\rho}H(g^{-1}k)}\left|\phi\big(K(g^{-1}k)\big)\right| f(g)\,\mathrm{d}g = \left|\int_G e^{-(i\lambda+\tilde{\rho})H(g^{-1}k)}\phi\big(K(g^{-1}k)\big) f(g)\,\mathrm{d}g\right|.$$

Since both sides are continuous nonzero functions of k it follows that

$$e^{-i\lambda H(g)}\phi\big(K(g)\big) = \left|\phi\big(K(g)\big)\right| \neq 0$$

for some $k \in K$ and for every g in a subset of G of positive measure, which is possible only if $\lambda = 0$.

Let V a neighborhood like in the previous lemma. Then for $\lambda \in V$ we can write

$$\pi_f^\lambda = \mu_1^\lambda P_f^0 + \sum_{i=2}^{\infty} \mu_i^\lambda \phi_i^\lambda \otimes \phi_i^\lambda$$

with $|\mu_i^\lambda| < \mu_1^\lambda$, $\forall i \geqslant 2$. Observe that

$$\left(\pi_f^\lambda\right)^n = \left(\mu_1^\lambda\right)^n P_f^0 + \sum_{i=2}^{\infty} \left(\mu_i^\lambda\right)^n \phi_i^\lambda \otimes \phi_i^\lambda = \left(\mu_1^\lambda\right)^n P_f^\lambda + Q_{n,f}^\lambda,$$

where

$$\left\|Q_{n,f}^\lambda\right\| = \left|\mu_2^\lambda\right|^n < \left(\mu_1^\lambda\right)^n < \left(\mu_1^0\right)^n. \qquad \square$$

LEMMA 2. – *The function*

$$s : A^* \to R^+,$$
$$\lambda \to \mu_1^\lambda,$$

is smooth in a neighborhood V' of the origin and satisfies:

$$\frac{ds}{d\lambda}(0)=0, \qquad \frac{d^2 s}{d\lambda^2}(0)=-2Q,$$

where Q is a positive definite bilinear form on A^*.

Proof. – See [3] Proposition 2.2.7. Without loss of generality we can suppose that $V=V'$. □

Remark. – The fact that μ_1^0 is equal to the spectral gap ρ follows from the *principes de majoration* of C. Herz, as popularized by N. Lohoué [7].

2. PROOF OF THE THEOREM

One interest of the representations T_g^λ is that we can express the spherical functions as their coefficients, i.e.,

$$\phi_\lambda(g)=\langle T_g^\lambda 1, 1\rangle,$$

where 1 denotes the function on K that has constant value 1. Thus the spherical transform of a function $f\in\mathcal{S}(K\backslash G/K)$can be written

$$\mathcal{F}f(\lambda)=\int_G f(g)\phi_\lambda(g)\,dg=\int_G f(g)\langle T_g^\lambda 1,1\rangle\,dg=\langle\pi_f^\lambda 1,1\rangle. \tag{5}$$

Let $\|\cdot\|_K$ denote the scalar product given by the killing form and ϕ_ε the symmetric biinvariant function given by

$$\phi_\varepsilon(g)=c_G\int_{A^*} e^{-(\|\lambda\|_K)^\varepsilon}\phi_\lambda(g)\left|c(\lambda)\right|^{-2}d\lambda,$$

ϕ_ε is an approximation of the identity, i.e., $\lim_{\varepsilon\to 0}\phi_\varepsilon * f(g)=f(g)$ for all f K-invariant continuous and integrable. Using the binvariance of ϕ_ε we easily deduce that the associated operator $\pi_{\phi_\varepsilon}^\lambda$ satisfies

$$\begin{aligned}\left\|\pi_{\phi_\varepsilon}^\lambda\right\|&=\mathcal{F}\phi_\varepsilon(\lambda)\leqslant 1,\\ \pi_{\phi_\varepsilon}^\lambda&=\pi_{\phi_\varepsilon * m_K}^\lambda=\pi_{\phi_\varepsilon}^\lambda\pi_{m_K}^\lambda,\\ \pi_{\phi_\varepsilon}^\lambda 1&=e^{-\varepsilon\|\lambda\|}1,\end{aligned}$$

where m_K is the Haar measure on K and $\pi_{m_K}^\lambda\phi=\langle\phi,1\rangle 1$. Note that

$$f^{*n}*\phi_\varepsilon(g)=f^{*n}*\phi_\varepsilon*\delta_{g^{-1}}(e), \tag{6}$$

where δ_g is the delta function at g. Using the unimodularity of G and the biinvariance of ϕ_ε we obtain that (6) is equal to

$$m_K * \phi_\varepsilon * \delta_{g^{-1}} * f^{*n} * m_K(e) = G_n(e).$$

The Fourier inversion formula (1) and (5) give

$$G_n(e) = c_G \int_{A^*} \mathcal{F}(G_n)(\lambda)\left|c(\lambda)\right|^{-2} \mathrm{d}\lambda = c_G \int_{A^*} \left\langle \left(\pi_f^\lambda\right)^n 1, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda 1\right\rangle \mathrm{d}\lambda.$$

Let us define

$$F_n(g) = c_G \int_{A^*} \left(\mu_1^0\right)^n e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle T_{g^{-1}}^\lambda \phi_1^0, \phi_1^0\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda,$$

$$\begin{aligned}
F_n * \phi_\varepsilon(g) &= F_n * \phi_\varepsilon * \delta_{g^{-1}}(e) = F_n * (\delta_g * \phi_\varepsilon)\check{}\,(e) \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} \int_G e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle \phi_1^0, T_h^\lambda \phi_1^0\right\rangle \delta_g * \phi_\varepsilon(h)\, \mathrm{d}h \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle \phi_1^0, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda \phi_1^0\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle \phi_1^0, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda \pi_{m_K}^\lambda \phi_1^0\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle \phi_1^0, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda \langle \phi_1^0\, 1\rangle 1\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_1^0}} \left\langle \langle 1, \phi_1^0\rangle \phi_1^0, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda 1\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&= c_G \left(\mu_1^0\right)^n \int_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0} n - \varepsilon \|\lambda\|_K^2} \left\langle P_f^0, T_g^\lambda 1\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda.
\end{aligned}$$

Let V be a neighborhood of the origin with the properties of Lemma 1. Then

$$\begin{aligned}
&\int_{V^c} \left\langle \left(\pi_f^\lambda\right)^n 1, T_g^\lambda \pi_{\phi_\varepsilon}^\lambda 1\right\rangle \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \\
&\qquad \leqslant \sup_{\|\lambda\| \geqslant \varepsilon_0} \left\|\pi_f^\lambda\right\|^{n-2} \left\|\pi_{\phi_\varepsilon}^\lambda\right\| \int_{A^*} \left\|\pi_f^\lambda\right\|^2 \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda
\end{aligned}$$

$$\leqslant C\left(\frac{\sup_{\|\lambda\|\geqslant\varepsilon_0}\|\pi_f^\lambda\|}{\mu_1^0}\right)^{n-2}(\mu_1^0)^n.$$

By Lemma 1 the factor $(\sup_{\|\lambda\|\geqslant\varepsilon_0}\|\pi_f^\lambda\|/\mu_1^0)^n$ goes to zero faster than any power of n. In a similar way we see that

$$(\mu_1^0)^n\int_{V^c} e^{-\frac{\langle Q\lambda,\lambda\rangle n}{\mu_0^1}}\langle P_f^0 1, T_g^\lambda\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda\leqslant C(\mu_1^0)^n n^{-\frac{d+2p+1}{2}}.$$

Moreover

$$\begin{aligned}
&\int_V\langle(\pi_f^\lambda)^n 1, T_g^\lambda\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda\\
&\quad=\int_V\langle s(\lambda)^n P_f^\lambda 1, T_g^\lambda\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda\\
&\qquad+\int_V\langle Q_{n,f}^\lambda 1, T_g^\lambda\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda.
\end{aligned}$$

The second term of this sum can be estimated by

$$\begin{aligned}
&\sup_{\lambda\in V}\|Q_{n,f}^\lambda\|\int_V\|T_g^\lambda\|\|\pi_{\phi_\varepsilon}^\lambda\||c(\lambda)|^{-2}\,\mathrm{d}\lambda\\
&\quad\leqslant C\left(\frac{\sup_{\lambda\in V}\|Q_{n,f}^\lambda\|}{\mu_1^0}\right)^n(\mu_1^0)^n\leqslant Cn^{-\frac{d+2p+1}{2}}(\mu_1^0)^n.
\end{aligned}$$

Thus modulo an error term smaller than $Cn^{-\frac{d+2p+1}{2}}(\mu_1^0)^n$

$$\begin{aligned}
&\|f^{*n}*\phi_\varepsilon-F_n*\phi_\varepsilon\|_{L^\infty(G)}\\
&\quad\leqslant c_G\left|\int_V s(\lambda)^n\langle P_f^\lambda 1, T_g^\lambda\,\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda\right.\\
&\qquad\left.-(\mu_1^0)^n\int_V e^{-\frac{n\langle Q\lambda,\lambda\rangle}{\mu_1^0}}\langle P_f^0 1, T_g^\lambda\pi_{\phi_\varepsilon}^\lambda 1\rangle|c(\lambda)|^{-2}\,\mathrm{d}\lambda\right|\\
&\quad\leqslant c_G\int_V\left\|s(\lambda)^n P_f^\lambda 1-(\mu_1^0)^n e^{-\frac{n\langle Q\lambda,\lambda\rangle}{\mu_1^0}}P_f^0 1\right\||c(\lambda)|^{-2}\,\mathrm{d}\lambda
\end{aligned}$$

the change of variable $\lambda\to\frac{\lambda}{\sqrt{n}}$ transforms the above integral in

$$(\mu_1^0)^n n^{-\frac{d+2p}{2}} \int\limits_{V\cdot\sqrt{n}} \left\| \left(\frac{s(\lambda/\sqrt{n})}{\mu_1^0}\right)^n P_f^{\frac{\lambda}{\sqrt{n}}} 1 - e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0}} P_f^0 1 \right\| n^p \left| c\left(\frac{\lambda}{\sqrt{n}}\right)\right|^{-2} \mathrm{d}\lambda$$

$$\leqslant (\mu_1^0)^n n^{-\frac{d+2p}{2}} \int\limits_{V\cdot\sqrt{n}} \left| \left(\frac{s(\lambda/\sqrt{n})}{\mu_1^0}\right)^n - e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0}} \right| n^p \left| c\left(\frac{\lambda}{\sqrt{n}}\right)\right|^{-2} \mathrm{d}\lambda$$

$$+ (\mu_1^0)^n n^{-\frac{d+2p}{2}} \int\limits_{V\cdot\sqrt{n}} e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0}} \left\| P_f^{\frac{\lambda}{\sqrt{n}}} 1 - P_f^0 1 \right\| n^p \left| c\left(\frac{\lambda}{\sqrt{n}}\right)\right|^{-2} \mathrm{d}\lambda$$

$$= I + II.$$

Using the Taylor expansion of $s(\lambda)$ we obtain

$$\frac{s(\lambda/\sqrt{n})}{\mu_1^0} = 1 - \frac{\langle Q\lambda,\lambda\rangle}{n\mu_1^0} + \mathcal{O}(\|\lambda/\sqrt{n}\|_K^3) \leqslant e^{-c\frac{\|\lambda\|^2}{n}}, \quad \forall\lambda \in V\cdot\sqrt{n}. \tag{7}$$

In order to estimate the first integral we will use the inequality

$$(a^n - b^n) \leqslant n\left|(a-b)\right| r^{n-1}, \quad \text{where } r = \max(a,b)$$

with

$$a = \frac{s(\lambda/\sqrt{n})}{\mu_1^0}, \qquad b = e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0 n}}.$$

It is clear from (7) that r satisfies $r^n \leqslant Ce^{-c\|\lambda\|^2}$ while

$$n\left|\frac{s(\lambda/\sqrt{n})}{\mu_1^0} - e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0 n}}\right|$$
$$\leqslant n\left|\frac{s(\lambda/\sqrt{n})}{\mu_1^0} - 1 + \frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0 n}\right| + n\left|1 - \frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0 n} - e^{-\frac{\langle Q\lambda,\lambda\rangle}{\mu_1^0 n}}\right|$$
$$\leqslant C\left(\frac{\|\lambda\|^3}{\sqrt{n}} + \frac{\|\lambda\|^4}{n}\right).$$

Since it is well known that

$$\left| n^p \left| c\left(\frac{\lambda}{\sqrt{n}}\right)\right|^{-2} - \prod_{\alpha\in\Sigma_0} \langle\alpha,\lambda\rangle^2 \right| \leqslant \frac{C\lambda}{\sqrt{n}}$$

we obtain that the integrand is smaller than

$$\frac{C}{\sqrt{n}}\|\lambda\|^q e^{-c\|\lambda\|^2},$$

and thus

$$|I| \leqslant C\left(\mu_1^0\right)^n n^{-\frac{d+2p+1}{2}}.$$

To prove that $|II|$ satisfies the same bound we just need to observe that the application

$$\lambda \to P_f^\lambda$$

is C^1 with respect to the operator norm and thus

$$\left\|P_f^{\frac{\lambda}{\sqrt{n}}}1 - P_f^0 1\right\| \leqslant \frac{C\lambda}{\sqrt{n}}.$$

The proof of the theorem follows for f K-invariant. To deal with a general f we need to use the full strength of Bougerol's method. Let $\tilde{g}$ be a continuous symmetric biinvariant function with compact support. Since $f \in C_c(G)$ its support generates the group and thus there exists r such that $f^{*r} \geqslant \alpha\tilde{g} = g$. Define $\beta = f^{*r} - g$, write L_h for π_h^0 and notice that

$$\|L_g\| = a < \left(\mu_1^0\right)^r, \qquad \|L_\beta\| = b < \left(\mu_1^0\right)^r.$$

The proof is easy. Suppose that $a = (\mu_1^0)^r$. By Lemma 1 we deduce that there exists $\phi > 0$ such that $L_g\phi = (\mu_1^0)^r\phi$. Since $L_{f^{*r}}\phi$ majorizes $L_g\phi$ and there exists ψ such that $L_{f^{*r}}\psi = (\mu_1^0)^r\psi$ we deduce that $L_{f^{*r}}\phi = (\mu_1^0)^r\phi$. This implies that $L_\beta\phi = 0$ which is absurd since $\beta > 0$ and $\phi > 0$. Let $q \in \mathbf{N}$ and write

$$f^{*r} = \frac{\ln q^{1/4}}{q}g + \left(\left(1 - \frac{\ln q^{1/4}}{q}\right)g + \beta\right) = g_q + \beta_q.$$

Obviously $\|L_{g_q}\| \leqslant (a\ln q^{1/4})/q$ and

$$\begin{aligned}\beta_q &= \left(1 - \frac{\ln q^{1/4}}{q}\right)g + \left(1 - \frac{\ln q^{1/4}}{q}\right)\beta + \frac{\ln q^{1/4}}{q}\beta\\ &= \left(1 - \frac{\ln q^{1/4}}{q}\right)f^{*r} + \frac{\ln q^{1/4}}{q}\beta\end{aligned}$$

thus $\|L_{\beta_q}\| \leqslant (1 - \frac{c \ln q^{1/4}}{q})(\mu_1^0)^r$. Observe that our proof implies that if g is a symmetric biinvariant function with compact support and ν_1 and ν_2 are symmetric bounded measures with compact support then

$$\begin{aligned} &\left\| f^{*n} * \nu_1 * g * \nu_2 - F_n * \nu_1 * g * \nu_2 \right\|_{L^\infty} \\ &\quad \leqslant C n^{-\frac{d+2p+1}{2}} (\mu_1^0)^n \|L_{\nu_1}\| \|L_{\nu_2}\| \|L_g\|. \end{aligned} \tag{8}$$

Let q be such that $qr \approx n$ and ν a symmetric positive bounded measure with compact support

$$\begin{aligned} &\left\| f^{*n+qr} * \nu - F_{n+qr} * \nu \right\|_{L^\infty} \\ &\quad \leqslant \left\| f^{*n} * (f^{*qr} - \beta_q^{*q}) * \nu - F_n * (f^{*qr} - \beta_q^{*q}) * \nu \right\|_{L^\infty} \\ &\qquad + \left\| f^{*n} * \beta_q^{*q} * \nu - F_n * \beta_q^{*q} * \nu \right\|_{L^\infty} \\ &\qquad + \left\| F_n * f^{*qr} * \nu - F_{n+qr} * \nu \right\|_{L^\infty}. \end{aligned}$$

Observe that it follows from the expression of the q power of a non commutative binomial that

$$f^{*qr} - \beta_q^{*q} = \sum_k \nu_1^k * g_q * \nu_2^k.$$

Thus (8) gives for the first term the bound

$$\begin{aligned} &\left\| f^{*n} * (f^{*qr} - \beta_q^{*q}) * \nu - F_n * (f^{*qr} - \beta_q^{*q}) * \nu \right\|_{L^\infty} \\ &\quad \leqslant n^{-\frac{2p+d+1}{2}} (\mu_1^0)^n \|L_\nu\| \sum_{k=0}^{q} \binom{q}{k} \|L_{\beta_q}\|^k \|L_{g_q}\|^{q-k} \\ &\quad \leqslant n^{-\frac{2p+d+1}{2}} (\mu_1^0)^n \|L_\nu\| \left(\|L_{\beta_q}\| + \|L_{g_q}\|\right)^q \\ &\quad \leqslant C n^{-\frac{2p+d+1}{2}} q^{\frac{1}{4}} (\mu_1^0)^{n+qr} \|L_\nu\|. \end{aligned}$$

The estimate of the last term is easier since (modulo $\mathcal{O}(n^{-\frac{2p+d+1}{2}} \times (\mu_1^0)^{n+qr} \|L_\nu\|)$) we have

$$\begin{aligned} &\left\| F_n * f^{*qr} * \nu - F_{n+qr} * \nu \right\| \\ &\quad \leqslant C (\mu_1^0)^{n+qr} \|L_\nu\| \int_{\mathcal{A}^*} \left| \left(\frac{s(\lambda)}{\mu_1^0} \right)^{qr} - e^{-qr \frac{\langle Q\lambda, \lambda \rangle}{\mu_1^0}} \right| e^{-n \frac{\langle Q\lambda, \lambda \rangle}{\mu_1^0}} |c(\lambda)|^{-2} \, \mathrm{d}\lambda \\ &\quad \leqslant C \|L_\nu\| (\mu_1^0)^{n+qr} n^{-\frac{d+2p+1}{2}}. \end{aligned}$$

To control the second term we consider a symmetric biinvariant function h with compact support that majorizes f^{*r}. Then

$$
\begin{aligned}
f^{*n} * \beta_q^{*q} * \nu &\leqslant f^{*n} * h * \beta_q^{*(q-1)} * \nu \\
&\leqslant C\|L_{\beta_q}\|^q \|L_\nu\| \int\limits_{A^*} \|\pi_f^\lambda\|^n |c(\lambda)|^{-2}\,\mathrm{d}\lambda \\
&\leqslant Cq^{-\frac{1}{4}} n^{-\frac{2p+d}{2}} (\mu_1^0)^{n+qr} \|L_\nu\|.
\end{aligned}
$$

The same bound is valid for $F_n * \beta_q^{*q} * \nu$ and thus we obtain that

$$
\|f^{*n} * \nu - F_n * \nu\|_{L^\infty} \leqslant \|L_\nu\| (\mu_1^0)^n n^{-\frac{d+2p+1}{2}+\frac{1}{4}}. \tag{9}
$$

Now we can use recursively this information to get the desired bound.

In fact if we choose $g_q = (\ln q^{1/8}/q)g$ and we use (9) to estimate the second term we gain another factor $n^{-1/8}$ in the final Berry–Esseen estimate. Repeating the same procedure for $g_q = (\ln q^{1/2^k}/q)g$ we are done.

3. APPLICATIONS

COROLLARY 1. – *Let f be as in the theorem. Then for every $\varepsilon > 0$ there exists a constant C_ε such that*

$$
\left|f^{*n}(e) - C_f \rho^n n^{-\frac{2p+d}{2}}\right| \leqslant C_\varepsilon \rho^n n^{-\frac{2p+d+1}{2}+\varepsilon},
$$

where

$$
C_f = c_G \int\limits_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle}{\rho}} \prod_{\alpha\in\Sigma_0} \langle\alpha,\lambda\rangle^2\,\mathrm{d}\lambda.
$$

Proof. – It is enough to show that

$$
\left|F_n(e) - c_G \rho^n n^{-\frac{2p+d}{2}} \int\limits_{A^*} e^{-\frac{\langle Q\lambda,\lambda\rangle}{\rho}} \prod_{\alpha\in\Sigma_0} \langle\alpha,\lambda\rangle^2\,\mathrm{d}\lambda\right| \leqslant C\rho^n n^{-\frac{2p+d+1}{2}},
$$

whose easy proof is essentially a simplified version of the proof of the theorem. □

The function F_n plays the role of the heat kernel and it would be interesting to prove sharp gaussian estimates similar to those proved by Anker for the biinvariant heat kernel. Although we cannot use the same techniques we show how deduce from the Berry–Esseen estimates an easy proof of the gaussian estimates for f^{*n} proved by N. Varopoulos in a more general setting.

COROLLARY 2. – *Let f as in the theorem and denote by $d_G(\cdot,\cdot)$ the distance induced by some left invariant riemannian structure on G. Then*

$$f^{*n}(g) \leqslant C\rho^n n^{-\frac{d+2p}{2}} e^{-\frac{c d_G(e,g)^2}{n}}.$$

Proof. – Notice that for $d_G(g,e) = \|g\| \leqslant Cn$

$$\begin{aligned} \left|\langle T_g^\lambda \phi_1^0, \phi_1^0\rangle\right| &= \left|\int_K e^{-(i\lambda+\tilde\rho)H(g^{-1}k)} \phi_1^0\big(K(g^{-1}k)\big)\phi_1^0(k)\,\mathrm{d}k\right| \\ &\leqslant M\int_K e^{-\tilde\rho H(g^{-1}k)}\,\mathrm{d}k \leqslant Ce^{-c d_G(g,e)} \leqslant Ce^{-\frac{c d_G(g,e)^2}{n}}. \end{aligned}$$

Thus

$$F_n(g) = \rho^n \int_{\mathcal{A}^*} \langle T_g^\lambda \phi_1^0, \phi_1^0\rangle e^{-\frac{\langle Q\lambda,\lambda\rangle}{\rho}} \left|c(\lambda)\right|^{-2} \mathrm{d}\lambda \leqslant C\rho^n n^{-\frac{2p+d}{2}} e^{-\frac{d_G(g,e)^2}{n}}.$$

To finish the proof we need the following estimate valid in any unimodular group whose proof will be given at the end of the corollary

$$f^{*n}(g) \leqslant C\rho^n e^{-c\frac{d_G(g,e)^2}{n}}. \tag{10}$$

Thus for $\|g\| \leqslant Cn$ we have that

$$\left|f^{*n}(g) - F_n(g)\right| \leqslant C\rho^n e^{-c\frac{\|g\|^2}{n}},$$

by interpolation with the Berry–Esseen estimate we obtain

$$\left|f^{*n}(g) - F_n(g)\right| \leqslant C\rho^n n^{-\frac{2p+d}{2}} e^{-c\frac{\|g\|^2}{n}},$$

thus

$$f^{*n}(g) \leqslant C\rho^n n^{-\frac{2p+d}{2}} e^{-c\frac{\|g\|^2}{n}}.$$

This ends the proof since for $\|g\| > Cn$ we have $f^{*n}(g) = 0$. The proof of (10) is an immediate consequence of the following

LEMMA 3. – *Let $f \in C(G)$ be the density of a probability measure which is symmetric and compactly supported. Let K be the associated convolution operator on $L^2(G)$ and ρ its norm. Let K_s denote the* *operator whose kernel is given by*

$$k_s(x,y)=e^{-sd(x,x_0)}f\left(xy^{-1}\right)e^{sd(y,x_0)},$$

where x_0 is some fixed point in G. Then there exists a positive constant C such that $\forall s\in\mathbf{R}$ we have

$$\left\|K_s^n\right\|_{2\to 2}\leqslant C\rho^n\exp\left(Cs^2n+1\right). \tag{11}$$

Proof. – Using the fact that the support of f is compact it easy to see that

$$\left\|K_s^n\right\|_{L^p\to L^p}\leqslant \|K\|^n_{L^p\to L^p}e^{|s|r_0n},\qquad s\in\mathbf{R}, \tag{12}$$

that for $p=2$ gives (11) for every $|s|>c$. What remains to be done is to give the proof of (11) for $|s|\leqslant c$ Let $\langle\cdot,\cdot\rangle$ denote the ordinary scalar product in $L^2(G)$ and $\|\cdot\|$ the corresponding norm. Using the fact that f is compactly supported by Taylor's theorem we have

$$\begin{aligned}\bigl\langle(I-K_s)h,h\bigr\rangle=\bigl\langle(I-K)h,h\bigr\rangle+s\int&\bigl(d(x,x_0)-d(y,x_0)\bigr)\\ &\times f\left(xy^{-1}\right)h(x)h(y)\,\mathrm{d}x\,\mathrm{d}y+\mathcal{O}\left(s^2\right)\|h\|^2,\\ &|s|<c,\ h\in L^2(G).\end{aligned}$$

Since f is symmetric the first integral vanishes and thus we have

$$\bigl\langle(I-K_s)h,h\bigr\rangle\geqslant(1-\rho)\|h\|^2-Cs^2\|h\|^2. \tag{13}$$

Let us consider the perturbated semigroup

$$T_{s,t}h(x)=e^{-sd(x,x_0)}\exp(I-K)t\left(e^{sd(x_0,\cdot)}h(\cdot)\right).$$

(13) implies that

$$\|T_{s,t}\|_{L^2\to L^2}\leqslant\exp(\rho-1)t\exp\left(cs^2t\right).$$

Our goal is to get the discrete analogue of the above inequality. Toward that we shall follow closely [5]. Let us consider $\mathcal{E}(n)=\{i\in 2\mathbf{N}$: $n-\sqrt{n}\leqslant i\leqslant n\}$ and $f\in L^2(G)$ *nonnegative*. We have that

$$\left\|e^{-t}\sum_{i\in\mathcal{E}(n)}K_s^i f\right\|^2\leqslant\|T_{s,t}f\|^2\leqslant\exp 2(\rho-1)t\exp\left(cs^2t\right)\|f\|^2. \tag{14}$$

Using the trivial estimate

$$K_{-s}^{j} f(x) \leqslant e^{Cj|s|} K_s^j f(x)$$

and (12) we deduce that

$$\begin{aligned}\left\| e^{-t} \sum_{i\in\mathcal{E}(n)} \frac{t^i}{i!} K_s^i f \right\|^2 &= e^{-2t} \sum_{i\in\mathcal{E}(n)} \sum_{j\in\mathcal{E}(n)} \frac{t^{i+j}}{i!j!} \langle K_s^i f, K_s^j f\rangle \\ &\geqslant e^{-(2t+C|s|\sqrt{n})} \sum_{i\in\mathcal{E}(n)} \sum_{j\in\mathcal{E}(n)} \frac{t^{i+j}}{i!j!} \|K_s^{(i+j)/2} f\|^2 \\ &\geqslant e^{-(2t+C|s|\sqrt{n})} \rho^{-2n} \sum_{i\in\mathcal{E}(n)} \sum_{j\in\mathcal{E}(n)} \frac{(\rho t)^{i+j}}{i!j!} \|K_s^n f\|^2 \\ &\geqslant e^{-(2t+C|s|\sqrt{n})} \rho^{-2n} \|K_s^n f\|^2 \left(\sum_{i\in\mathcal{E}(n)} \frac{(\rho t)^i}{i!} \right)^2 .\end{aligned}$$

If we put $t = n$ in the last inequality and in (14) we have

$$\|K_s^n\|_{L^2\to L^2} \leqslant \rho^n \exp\bigl(C(s^2 n + 1)\bigr) \left(e^{\rho n} \left(\sum_{i\in\mathcal{E}(n)} \frac{(n\rho)^i}{i!} \right)^{-1} \right), \tag{15}$$

which gives (11) for $|s| \leqslant c$ since, by Stirling's formula, the last factor in (15) is majorized by a constant independent of n. □

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