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Completely asymmetric Lévy processes confined in a finite interval

by

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ABSTRACT. – Consider a completely asymmetric Lévy process which has absolutely continuous transition probabilities. By harmonic transform, we establish the existence of the Lévy process conditioned to stay in a finite interval, called the confined process (the confined Brownian motion is F.B. Knight's Brownian taboo process). We show that the confined process is positive-recurrent and specify some useful identities concerning its excursion measure away from a point. We investigate the rate of convergence of the supremum process to the right-end point of the interval. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Lévy process, Completely asymmetric, Conditional law, h-transform, Excursion measure

RÉSUMÉ. – Considérons un processus de Lévy complètement asymétrique dont les probabilités de transition sont absolument continues. Nous établissons l'existence par transformée harmonique du processus conditionné à demeurer dans un intervalle fini, appelé le processus confiné (le mouvement brownien confiné est le processus tabou brownien de F.B. Knight). Nous montrons que le processus confiné est récurrent-positif et mettons en évidence quelques identités utiles concernant sa mesure d'excursion hors d'un point. Nous traitons la vitesse de convergence

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du processus des suprema vers la borne supérieure de l'intervalle.

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1. INTRODUCTION

A Lévy process is a stochastic process with càdlàg sample-paths and stationary independent increments. It is called completely asymmetric when it is real-valued and all its jumps have the same sign. Such processes have been studied by many authors, for their intrinsic properties (see Chapter VII in [1] and references quoted there) as well as their applications (e.g., [5,14]), for example in queues, dams, mathematical finance and insurance risks. The connection with branching processes [4, 10,12,13] is a further motivation for their study. Several papers [2,3,9, 16–18], have considered completely asymmetric Lévy processes before they leave a finite interval (viz. before extinction of a species and overcrowding, or before overflowing and draining of a dam) and the so-called two-sided exit problem.

For simplicity, fix $a > 0$ and suppose that X is a Lévy process with no positive jumps starting from $x \in (0, a)$; denote by T the first exit-time

$$T \doteq \inf\{t \geqslant 0\colon\ X_t \notin (0, a)\}.$$

The starting point of this work lies in the observation that the conditional laws $\mathbb{P}(\cdot|T > t)$ converge as $t \to \infty$. Our main purpose is to study the limiting distribution, denoted by $\mathbb{P}^{\updownarrow}$. In particular, we show the positive-recurrence under $\mathbb{P}^{\updownarrow}$, and determine the stationary probability, as well as Laplace transforms of first hitting times. We next study the excursion measure under $\mathbb{P}^{\updownarrow}$ away from a point. The last section is devoted to the rate of convergence of the supremum $S_t = \sup_{0 \leqslant s \leqslant t} X_s$ to a, both in law and pathwise.

2. PRELIMINARIES

This section reviews standard results on spectrally negative Lévy processes, and introduces the key tools developed in [3].

We use the canonical notation. Let $\Omega = \mathcal{D}([0, \infty), \mathbb{R})$ be the space of càdlàg functions $\omega\colon [0, \infty) \to \mathbb{R}$, endowed with Skorohod's topology and the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$, and $X=(X_t,\ t\geqslant 0)$ the coordinate process. Let $(\mathbb{P}_x)_{x\in\mathbb{R}}$ be a family of probability measures for which X is a Lévy process with no positive jumps starting from x. The trivial cases of deterministic processes and of negatives of subordinators are excluded. We further suppose the absolute continuity condition

$$\mathbb{P}_0(X_t\in dx)\ll dx \quad \text{for any } t>0. \tag{AC}$$

The Laplace transform of X_t is defined by

$$\mathbb{E}_0\left(e^{\lambda X_t}\right)=e^{t\psi(\lambda)}, \quad \lambda,t\geqslant 0,$$

where $\psi:[0,\infty)\to\mathbb{R}$ is known as the Laplace exponent. It is given by the Lévy–Khinchin formula

$$\psi(\lambda)=m\lambda+\frac{b}{2}\lambda^2+\int_{-\infty}^{0}\left(e^{\lambda r}-1-\lambda r\mathbf{1}_{\{r>-1\}}\right)\Lambda(dr), \tag{1}$$

where Λ denotes the Lévy measure of the process, and b its Gaussian coefficient. It is plain that ψ is a convex function with $\lim_{\lambda\to\infty}\psi(\lambda)=+\infty$. Denote by $\phi(0)$ its largest root. If $\phi(0)>0$, ψ has exactly two roots (0 and $\phi(0)$), otherwise ψ has a unique root $\phi(0)=0$. The right-inverse of ψ is denoted by $\phi:[0,\infty)\to[\phi(0),\infty)$.

We write

$$T_A=\inf\{t\geqslant 0:\ X_t\in A\},$$

for the first entrance time in a Borel set A. Recall that for $A=(-\infty,0]\cup[a,+\infty)$, we simply write $T=T_A$. It is known that

$$\mathbb{E}_0\left(e^{-qT_{\{x\}}}\right)=e^{-\phi(q)x}, \quad q,x\geqslant 0.$$

The paths of X have bounded variation a.s. if and only if $b=0$ and $\int_{-1}^{0}|r|\Lambda(dr)<\infty$. Otherwise X has unbounded variation a.s.

We now turn our attention to the two-sided exit problem which has its roots in Takács [18]. There exists a unique continuous function $W:\mathbb{R}_+\to\mathbb{R}_+$, with Laplace transform

$$\int_0^\infty e^{-\lambda x}W(x)\,dx=\frac{1}{\psi(\lambda)}, \quad \lambda>\phi(0),$$

such that for any $x \in (0, a)$,

$$\mathbb{P}_x(X_T = a) = W(x)/W(a). \tag{2}$$

The function W is strictly increasing and called the scale function. More precisely, the Laplace transform of T on $\{X_T = a\}$ is given by

$$\mathbb{E}_x\left(\mathrm{e}^{-qT}, X_T = a\right) = W^{(q)}(x)/W^{(q)}(a), \quad q > 0, \tag{3}$$

where $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}_+$ is the continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\psi(\lambda) - q} = \sum_{k \geqslant 0} q^k \psi^{-k-1}(\lambda), \quad \lambda > \phi(q).$$

For every fixed $x \geqslant 0$, the mapping $q \mapsto W^{(q)}(x)$ can be extended analytically to complex numbers by the identity

$$W^{(q)}(x) = \sum_{k \geqslant 0} q^k W^{\star k+1}(x), \tag{4}$$

where $W^{\star k}$ denotes the kth convolution power of W. The convergence of the series is plain from the inequality

$$W^{\star k+1}(x) \leqslant \frac{x^k W(x)^{k+1}}{k!}, \quad x \geqslant 0,\ k \in \mathbb{N}, \tag{5}$$

which follows from the monotonicity of W. Note that

$$W(0) = \lim_{\lambda \to \infty} \frac{\lambda}{\psi(\lambda)} = 0$$

if and only if X has unbounded variation (cf. Corollary VII.5 in [1]). It follows from (4) and (5) that for every q, $W^{(q)}(0) = 0$ if and only if X has unbounded variation.

Finally, we recall the expression of the q-resolvent density $u_q(x, y)$, of X killed at time T (cf. [17] or [3, p. 159])

$$u_q(x, y) = \frac{W^{(q)}(x) W^{(q)}(a - y)}{W^{(q)}(a)} - \mathbf{1}_{\{x \geqslant y\}} W^{(q)}(x - y). \tag{6}$$

We can now state the main result of [3].

THEOREM 2.1. – *Let*

$$P^t(x, A) = \mathbb{P}_x(X_t \in A, t < T),$$

be the transition kernel of X killed at time T and introduce

$$\rho \doteq \inf\{q \geqslant 0\colon\ W^{(-q)}(a) = 0\}.$$

Then ρ is finite and positive, and for any $q < \rho$ and $x \in (0, a)$, $W^{(-q)}(x) > 0$. Furthermore, the following assertions hold

(i) *ρ is a simple root of the entire function $q \mapsto W^{(-q)}(a)$;*
(ii) *P^t is ρ-recurrent and, more precisely, ρ-positive;*
(iii) *The function $W^{(-\rho)}$ is positive on $(0, a)$ and is ρ-invariant for P^t*

$$P^t W^{(-\rho)}(x) = \mathrm{e}^{-\rho t} W^{(-\rho)}(x), \quad \text{for any } x \in (0, a);$$

(iv) *The measure $\Pi(dx) = W^{(-\rho)}(a - x)\,dx$ on $(0, a)$ is ρ-invariant for P^t*

$$\Pi P^t(dx) = \mathrm{e}^{-\rho t}\Pi(dx);$$

(v) *There is a constant $c > 0$ such that, for any $x \in (0, a)$,*

$$\lim_{t\to\infty} \mathrm{e}^{\rho t} P^t(x, \cdot) = \frac{1}{c} W^{(-\rho)}(x)\Pi(\cdot),$$

in the sense of weak convergence.

For instance, in the case of a completely asymmetric stable process of index $\alpha \in (1, 2]$, i.e., $\psi(\lambda) = \lambda^\alpha$, we have

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha\big(qx^\alpha\big),$$

where E'_α stands for the derivative of the Mittag–Leffler function of parameter α

$$E_\alpha(y) = \sum_{n\geqslant 0} \frac{y^n}{\Gamma(1 + \alpha n)} \tag{7}$$

(cf. [2]). As a consequence $\rho = r(\alpha)a^{-\alpha}$, where $-r(\alpha)$ is the first negative root of E'_α.

More specifically, for $\alpha = 2$ and $\psi(\lambda) = \lambda^2/2$, the Brownian motion killed upon leaving $(0, a)$ has the following features, in agreement with [11]:

$$W^{(-q)}(x)=\sqrt{2}\,q^{-1/2}\sin\bigl(\sqrt{2q}\,x\bigr),\qquad c=2a^3/\pi^2,\qquad \rho=\pi^2/2a^2.$$

3. THE LAW $\mathbb{P}^\updownarrow$ OF THE CONFINED PROCESS

The main results of this section are collected in the following

THEOREM 3.1. – *Let $x\in(0,a)$.*

(i) *The conditional laws $\mathbb{P}_x(\cdot|T>t)$ converge as $t\to\infty$ to a limit denoted by $\mathbb{P}_x^\updownarrow$, in the sense that for any $s\geqslant 0$ and $\Lambda\in\mathcal{F}_s$,*

$$\lim_{t\to\infty}\mathbb{P}_x(\Lambda|T>t)=\mathbb{P}_x^\updownarrow(\Lambda).$$

(ii) *The probability measure $\mathbb{P}^\updownarrow$ can be expressed as a h-transform of $\mathbb{P}$ based on the $(\mathbb{P},(\mathcal{F}_t))$-martingale*

$$D_t=\mathrm{e}^{\rho t}\mathbf{1}_{\{t<T\}}\frac{W^{(-\rho)}(X_t)}{W^{(-\rho)}(x)},$$

that is

$$d\mathbb{P}_x^\updownarrow|_{\mathcal{F}_t}=D_t\,.\,d\mathbb{P}_x|_{\mathcal{F}_t}.$$

(iii) *Under $\mathbb{P}^\updownarrow$, X is a homogeneous strong Markov process. Its λ-resolvent $(\lambda>0)$ has density*

$$u_\lambda^\updownarrow(x,y)=\left(\frac{W^{(\lambda-\rho)}(x)W^{(\lambda-\rho)}(a-y)}{W^{(\lambda-\rho)}(a)}-\mathbf{1}_{\{x>y\}}W^{(\lambda-\rho)}(x-y)\right)\times\frac{W^{(-\rho)}(y)}{W^{(-\rho)}(x)}. \tag{8}$$

(iv) *Under $\mathbb{P}^\updownarrow$, X is positive-recurrent with stationary probability*

$$\mu(dx)=\frac{W^{(-\rho)}(x)\Pi(dx)}{c(a)}=p(x)\,dx,$$

where

$$p(x)=\frac{W^{(-\rho)}(x)W^{(-\rho)}(a-x)}{c(a)}$$

and

$$c(a)=\left.\frac{\partial W^{(\lambda)}(a)}{\partial\lambda}\right|_{\lambda=-\rho}$$

(*recall from Theorem* 2.1(i) *that* $c(a) \in (0, \infty)$). *Finally, the density* p *is continuous on* $[0, a]$, *vanishes both at* 0 *and* a, *is symmetric with respect to* $a/2$ *and is unimodal: it is monotone increasing on* $(0, a/2)$ *and monotone decreasing on* $(a/2, a)$.

Remarks. – (a) It can be easily checked that in the Brownian case, the law of the confined process coincides with that of Knight's Brownian taboo process [11], with taboo states $\{0, a\}$.

(b) A straightforward calculation shows that the constant c in Theorem 2.1(v) is equal to that in statement (iv) of the preceding theorem.

(c) The symmetry of the stationary density p can be explained as follows. Consider the excursions of the confined process away from $a/2$. The duality lemma (Lemma II.2 in [1]) entails that the negative of the time-reversed excursion is distributed as the initial excursion. The occupation measure of the excursion is therefore symmetric with respect to $a/2$. Now we conclude with Theorem XIX.46 in [8] which states that the stationary law is proportional to the occupation measure.

(d) We mention that the family of probability measures $(\mathbb{P}_\varepsilon^\updownarrow, \varepsilon \in (0, a))$ has a weak limit as $\varepsilon \to 0+$. This can be checked easily using the absolute continuity relation between $\mathbb{P}_\varepsilon^\updownarrow$ and the law $\mathbb{P}_\varepsilon^\uparrow$ of the Lévy process started at ε and conditioned to stay positive (see Section VII.3 in [1]). The interested reader is referred to the author's Ph.D. thesis (in preparation).

Proof. – (ii) We start with proving that D is a $\mathbb{P}$-martingale using the Markov property under the probability $\mathbb{P}$:

$$\begin{aligned}\mathbb{E}_x(D_{t+s}|\mathcal{F}_t) &= \frac{\mathrm{e}^{\rho(t+s)}}{W^{(-\rho)}(x)}\,\mathbb{E}_x\big(\mathbf{1}_{\{t+s<T\}}W^{(-\rho)}(X_{t+s})|\mathcal{F}_t\big)\\ &= \frac{\mathrm{e}^{\rho(t+s)}}{W^{(-\rho)}(x)}\,\mathbf{1}_{\{t<T\}}\mathbb{E}_{X_t}\big(\mathbf{1}_{\{s<T\}}W^{(-\rho)}(X_s)\big).\end{aligned}$$

The martingale property of D thus follows from Theorem 2.1(iii).

(i) To prove the convergence of the conditional laws, pick $s, t > 0$. One easily deduces from Theorem 2.1 that for every $x, y \in (0, a)$ and $t > 0$,

$$\lim_{s\to\infty}\frac{\mathbb{P}_y(T > s)}{\mathbb{P}_x(T > t + s)} = \mathrm{e}^{\rho t}\,\frac{W^{(-\rho)}(y)}{W^{(-\rho)}(x)} \quad \text{a.s.}$$

Therefore the variables

$$D_{t,s} = \mathbf{1}_{\{T>t\}}\,\frac{\mathbb{P}_{X_t}(T > s)}{\mathbb{P}_x(T > t + s)}$$

converge a.s. to D_t as $s \to \infty$. Since

$$\mathbb{E}_x(D_{t,s}) = \mathbb{E}_x(1|T > t+s) = 1 = \mathbb{E}_x(D_t),$$

it follows from Scheffe's lemma that the preceding convergence holds in L^1. We deduce that for every $Y \in L^\infty(\mathcal{F}_t)$, $\mathbb{E}_x(YD_{t,s})$ converges to $\mathbb{E}_x(YD_t)$. By the Markov property, this means:

$$\lim_{s\to\infty} \mathbb{E}_x(Y|T > t+s) = \mathbb{E}_x(YD_t) = \mathbb{E}_x^{\updownarrow}(Y).$$

(iii) The homogeneous Markov property under $\mathbb{P}^{\updownarrow}$ is straightforward and the fact that h-transforms preserve the strong Markov property is well known ([7, Theorem XVI.28 p. 329]).

The semigroup of the confined process is given by

$$\begin{aligned} P_t^{\updownarrow}(x,dy) &= P^t(x,dy)\,\mathrm{e}^{\rho t}\,\frac{W^{(-\rho)}(y)}{W^{(-\rho)}(x)} \\ &= \mathbb{P}_x(X_t \in dy, t < T)\,\mathrm{e}^{\rho t}\,\frac{W^{(-\rho)}(y)}{W^{(-\rho)}(x)}. \end{aligned}$$

This yields the following expression for the λ-resolvent density $u_\lambda^{\updownarrow}(x,y)$

$$u_\lambda^{\updownarrow}(x,y) = u_{\lambda-\rho}(x,y)W^{(-\rho)}(y)/W^{(-\rho)}(x).$$

The desired formula now follows from (6) for every $\lambda > \rho$, and then for $0 \leqslant \lambda \leqslant \rho$ thanks to the resolvent equation.

(iv) Clearly the measure μ has a finite mass. That it is invariant for $P_t^{\updownarrow}$ for any t is immediate from Theorem 2.1(iv).

The computation of the normalizing constant $c(a)$ relies on the observation that

$$c(a) = W^{(-\rho)} \star W^{(-\rho)}(a),$$

and on the expansion (4).

It is obvious that the density p is symmetric, continuous, and vanishes at the endpoints of $(0,a)$, for $W^{(-\rho)}$ is continuous and vanishes at a.

It remains to show the unimodality: an appeal to the forthcoming Proposition 5.1 allows us to differentiate $W^{(-\rho)}$ at any point in $(0,\infty)$. Now thanks to the forthcoming Proposition 4.2, we know that for any $y \in (0,a)$, the function

$$\eta \mapsto \frac{W^{(-\rho)}(y+\eta)}{W^{(-\rho)}(\eta)}$$

is decreasing on $(0, a-y)$. Using logarithmic derivatives, we get

$$\frac{W^{(-\rho)\prime}(\eta+y)}{W^{(-\rho)}(\eta+y)} - \frac{W^{(-\rho)\prime}(\eta)}{W^{(-\rho)}(\eta)} \leqslant 0,$$

therefore the function

$$\eta \mapsto \frac{W^{(-\rho)\prime}(\eta)}{W^{(-\rho)}(\eta)}$$

is decreasing on $(0, a)$, and

$$\frac{p'(x)}{p(x)} = \frac{W^{(-\rho)\prime}(x)}{W^{(-\rho)}(x)} - \frac{W^{(-\rho)\prime}(a-x)}{W^{(-\rho)}(a-x)}$$

is hence nonnegative on $(0, a/2)$ and nonpositive on $(a/2, a)$. $\square$

In the sequel, we will frequently use the fact that for every finite stopping time S and $Y \in L_+(\mathcal{F}_S)$,

$$\mathbb{P}_x^{\updownarrow}(Y) = \mathbb{E}_x(Y D_S).$$

We now give some expressions for the Laplace transforms of first hitting times under $\mathbb{P}^{\updownarrow}$ that will be useful in the sequel.

PROPOSITION 3.2. – *For any $0 < b < x < c < a$, the following hold:*
(i) *Two-sided exit problem under $\mathbb{P}_x^{\updownarrow}$. If $T' \doteq \inf\{t \geqslant 0\colon\ X_t \notin (b, c)\}$,*

$$\mathbb{E}_x^{\updownarrow}\big(\mathrm{e}^{-qT'}\mathbf{1}_{\{X_{T'}=c\}}\big) = \frac{W^{(-\rho)}(c)}{W^{(-\rho)}(x)}\,\frac{W^{(q-\rho)}(x-b)}{W^{(q-\rho)}(c-b)}.$$

(ii) *Passage time at an upper level*

$$\mathbb{E}_x^{\updownarrow}\big(\exp(-qT_{\{c\}})\big) = \frac{W^{(-\rho)}(c)}{W^{(-\rho)}(x)}\,\frac{W^{(q-\rho)}(x)}{W^{(q-\rho)}(c)}.$$

(iii) *Passage time below a lower level*

$$\begin{aligned}
&\mathbb{E}_x^{\updownarrow}\big(\mathbf{1}_{\{X_{T_{(0,b)}-}\in dy\}}\mathbf{1}_{\{\Delta X_{T_{(0,b)}}\in dz\}}\exp(-qT_{(0,b)})\big)\\
&\quad= \frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\bigg(\frac{W^{(q-\rho)}(x-b)W^{(q-\rho)}(a-y)}{W^{(q-\rho)}(a-b)}\\
&\qquad\qquad - \mathbf{1}_{\{x\geqslant y\}}W^{(q-\rho)}(x-y)\bigg)\,dy\,\Lambda(dz).
\end{aligned}$$

Proof. – (i) and (ii) follow readily from (3) and the remark preceding the proposition.

(iii) We now set $0 < b < x < a$, $0 < b < y < a$, $0 < y + z < b$, and compute, by the compensation formula applied to the Poisson point process of jumps, the following quantity, where we temporarily replaced $T_{(0,b)}$ by T'_b

$$\begin{aligned}
&\mathbb{E}_x^{\updownarrow}\left(\mathbf{1}_{\{X_{T'_b-}\in dy\}}\mathbf{1}_{\{\Delta X_{T'_b}\in dz\}}\exp\left(-qT'_b\right)\right)\\
&= \mathbb{E}_x\left(\mathbf{1}_{\{X_{T'_b-}\in dy\}}\mathbf{1}_{\{\Delta X_{T'_b}\in dz\}}\,\mathrm{e}^{(\rho-q)T'_b}\mathbf{1}_{\{T'_b<T\}}\,\frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\right)\\
&= \mathbb{E}_x\left(\sum_{t\geqslant 0}\mathbf{1}_{\{X_{t-}\in dy\}}\mathbf{1}_{\{\Delta X_t\in dz\}}\,\mathrm{e}^{(\rho-q)t}\mathbf{1}_{\{X_s\in(b,a)\forall s<t\}}\,\frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\right)\\
&= \mathbb{E}_x\left(\int_0^\infty dt\,\Lambda(dz)\mathbf{1}_{\{X_t\in dy\}}\,\mathrm{e}^{(\rho-q)t}\mathbf{1}_{\{X_s\in(b,a)\forall s\leqslant t\}}\,\frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\right)\\
&= \Lambda(dz)\left(\int_0^\infty dt\,\mathrm{e}^{(\rho-q)t}\,P^t_{(a-b)}(x-b,dy-b)\frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\right)\\
&= \Lambda(dz)\,\frac{W^{(-\rho)}(y+z)}{W^{(-\rho)}(x)}\,u^{(q-\rho)}_{(a-b)}(x-b,dy-b),
\end{aligned}$$

where $u^q_{(a-b)}$ still denotes the q-resolvent density of the process, but this time killed as it exits from $(-\infty,0]\cup[a-b,+\infty)$. The result follows thanks to (6). □

4. EXCURSION MEASURE AWAY FROM A POINT

Recall that a point $x\in(0,a)$ is said regular (for itself) under $\mathbb{P}^{\updownarrow}$ if

$$\mathbb{P}_x^{\updownarrow}\left(\inf\{s>0\colon\ X_s=x\}=0\right)=1.$$

It is obvious that x is regular under $\mathbb{P}^{\updownarrow}$ if and only if it is regular under $\mathbb{P}$, hence if and only if X has unbounded variation under $\mathbb{P}$ (cf. [3, Corollary VII.5]). We assume this holds throughout this section.

4.1. The excursion measure $n_x^{\ddagger}$

The local time at level x, say L^x, is defined as the occupation density at x

$$L_t^x = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{\{|X_s-x|<\varepsilon\}}\,ds. \tag{9}$$

Let $(\tau_s, s\geqslant 0)$ be its right-continuous inverse

$$\tau_s = \inf\{t>0\colon\ L_t^x > s\},\qquad s\geqslant 0.$$

We consider the excursion process $e=(e_s, s\geqslant 0)$ of X away from $\{x\}$

$$e_s = \begin{cases} (X_{\tau_{s-}+u},\ 0\leqslant u<\tau_s-\tau_{s-}) & \text{if } \tau_{s-}<\tau_s,\\ \Upsilon & \text{otherwise,}\end{cases}$$

where Υ is an additional isolated point. According to a famous theorem by Itô, $(e_s, s\geqslant 0)$ is a Poisson point process valued in the space $\mathcal{E}$ of excursions away from $\{x\}$, that is the set of càdlàg paths ε with generic lifetime $V=V(\varepsilon)$. Its characteristic measure is denoted by n_x under $\mathbb{P}$ (and $n_x^{\ddagger}$ under $\mathbb{P}^{\ddagger}$), and is called the excursion measure away from $\{x\}$.

PROPOSITION 4.1. – *For every nonnegative measurable functional F we have*

$$n_x^{\ddagger}(F(\varepsilon)) = n_x\big(F(\varepsilon)\,\mathrm{e}^{\rho V}\mathbf{1}_{\{V<T\}}\big).$$

Proof. – As $X_{\tau_1}=x$ a.s., the probability measure $\mathbb{P}_x^{\ddagger}$ is absolutely continuous on $\mathcal{F}_{\tau_1}$ with respect to $\mathbb{P}_x$, with density $\mathbf{1}_{\{\tau_1<T\}}\mathrm{e}^{\rho\tau_1}$. It follows from the compensation formula of excursion theory that for every nonnegative measurable functional F,

$$n_x^{\ddagger}(F(\varepsilon)) = \mathbb{E}_x^{\ddagger}\Big(\sum_{0\leqslant s\leqslant 1} F(e_s)\Big) = \mathbb{E}_x\Big(\Big(\sum_{0\leqslant s\leqslant 1} F(e_s)\Big)\mathbf{1}_{\{\tau_1<T\}}\mathrm{e}^{\rho\tau_1}\Big).$$

Now from Theorem 3.1(ii) and the fact that $X_{\tau_t}=x$ a.s., the process $(\mathbf{1}_{\{\tau_t<T\}}\mathrm{e}^{\rho\tau_t}, t\geqslant 0)$ is a $(\mathbb{P},\mathcal{F}_{\tau_t})$-martingale. By optional projection, we can express the preceding quantity as

$$\begin{aligned}&\mathbb{E}_x\Big(\sum_{0\leqslant s\leqslant 1} F(e_s)\mathbf{1}_{\{\tau_s<T\}}\,\mathrm{e}^{\rho\tau_s}\Big)\\ &\quad= \mathbb{E}_x\Big(\sum_{0\leqslant s\leqslant 1} F(e_s)\mathbf{1}_{\{\tau_{s-}<T\}}\,\mathrm{e}^{\rho\tau_{s-}}\,\mathrm{e}^{\rho V(e_s)}\mathbf{1}_{\{V(e_s)<T\circ\theta_{\tau_{s-}}\}}\Big),\end{aligned}$$

where θ stands for the shift operator. The compensation formula of excursion theory now yields

$$n_x^{\ddagger}\big(F(\varepsilon)\big) = \mathbb{E}_x\left(\int_0^1 \mathbf{1}_{\{\tau_s<T\}}\,\mathrm{e}^{\rho\tau_s}\,ds\right) n_x\big(F(\varepsilon)\,\mathrm{e}^{\rho V}\mathbf{1}_{\{V<T\}}\big).$$

Our claim follows as $\mathbb{E}_x(\mathbf{1}_{\{\tau_s<T\}}\,\mathrm{e}^{\rho\tau_s}) = 1$. $\square$

4.2. Some calculations under $n_x^{\ddagger}$

The purpose of this section is to present some useful formulas involving the local time L^x and the excursion measure $n_x^{\ddagger}$. For every excursion ε away from $\{x\}$, we denote its height by $m(\varepsilon)$

$$m(\varepsilon) = \sup_{u\leqslant V(\varepsilon)} (\varepsilon_u - \varepsilon_0) = \sup_{u\leqslant V(\varepsilon)} \varepsilon_u - x.$$

Recall that τ stands for the inverse of the local time L^x. As well known, τ is a subordinator whose Laplace exponent $\phi_x^{\ddagger}$ is defined by

$$\mathbb{E}_x^{\ddagger}(\mathrm{e}^{-\lambda\tau_t}) = \exp\big(-t\phi_x^{\ddagger}(\lambda)\big), \quad \lambda\geqslant 0.$$

PROPOSITION 4.2. – *For any nonnegative λ and any $\eta\in[0,a-x]$,*

$$n_x^{\ddagger}(1-\mathbf{1}_{\{m<\eta\}}\,\mathrm{e}^{-\lambda V}) = \frac{W^{(\lambda-\rho)}(x+\eta)}{W^{(\lambda-\rho)}(x)\,W^{(\lambda-\rho)}(\eta)}.$$

In particular, for any nonnegative λ,

$$\phi_x^{\ddagger}(\lambda) = \frac{W^{(\lambda-\rho)}(a)}{W^{(\lambda-\rho)}(x)\,W^{(\lambda-\rho)}(a-x)},$$

and for any $\eta\in[0,a-x]$,

$$n_x^{\ddagger}(m>\eta) = \frac{W^{(-\rho)}(x+\eta)}{W^{(-\rho)}(\eta)\,W^{(-\rho)}(x)}.$$

Proof. – The last two assertions are straightforward consequences of the first (taking $\eta = a-x$ and $\lambda = 0$ respectively).

To prove the first assertion we start with observing the following identity

$$n_x^{\ddagger}(1-\mathbf{1}_{\{m<\eta\}}\,\mathrm{e}^{-\lambda V}) = \left[\mathbb{E}_x^{\ddagger}\left(\int_0^{T_{x+\eta}} \mathrm{e}^{-\lambda t}\,dL_t^x\right)\right]^{-1}. \tag{10}$$

Indeed we have

$$\mathbb{E}_x^{\ddagger}\left(\int_0^{T_{x+\eta}} \mathrm{e}^{-\lambda t}\,dL_t^x\right) = \mathbb{E}_x^{\ddagger}\left(\int_0^{\infty} ds\,\mathrm{e}^{-\lambda\tau_s}\mathbf{1}_{\{\tau_s<T_{x+\eta}\}}\right)$$
$$= \int_0^{\infty} ds\,\mathbb{E}_x^{\ddagger}\left(\exp\left(-\sum_{0\leqslant u\leqslant s}\lambda(\tau_u-\tau_{u-})\chi_{\{m(e_u)<\eta\}}\right)\right),$$

where we wrote $\chi_A(\omega)=1$ if $\omega\in A$ and $\chi_A(\omega)=\infty$ otherwise. By the exponential formula, the foregoing quantity is thus equal to

$$\int_0^{\infty} ds\,\exp(-sn_x^{\ddagger}(1-\exp(-\lambda V\chi_{\{m<\eta\}}))) = [n_x^{\ddagger}(1-\mathbf{1}_{\{m<\eta\}}\,\mathrm{e}^{-\lambda V})]^{-1},$$

which establishes (10).

The next step consists in proving the following identity

$$\mathbb{E}_y^{\ddagger}\left(\int_0^{\infty} \mathrm{e}^{-\lambda t}\,dL_t^x\right) = u_\lambda^{\ddagger}(y,x). \tag{11}$$

Indeed recall ([1, Proposition V.2]) that if θ is an independent exponential r.v. with parameter $\lambda>0$, then

$$(2\varepsilon)^{-1}\int_0^{\theta} \mathbf{1}_{\{|X_s-x|<\varepsilon\}}\,ds$$

converges to L_θ^x in $L^2(\mathbb{P})$. As a consequence, provided that $\lambda>\rho$, the convergence also holds in $L^2(\mathbb{P}^{\ddagger})$. In particular,

$$\mathbb{E}_y^{\ddagger}\left(\int_0^{\infty} \mathrm{e}^{-\lambda t}\,dL_t^x\right) = \lim_{\varepsilon\to0+}\frac{1}{2\varepsilon}\,\mathbb{E}_y^{\ddagger}\left(\int_0^{\infty} dt\,\mathrm{e}^{-\lambda t}\mathbf{1}_{\{|X_t-x|<\varepsilon\}}\right)$$
$$= \lim_{\varepsilon\to0+}\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} u_\lambda^{\ddagger}(y,u)\,du = u_\lambda^{\ddagger}(y,x)$$

(recall we assumed that the Lévy process has unbounded variation, which ensures that $W^{(\lambda-\rho)}(0)=0$. It follows from Theorem 3.1(iii) that $u_\lambda^\ddagger(y,\cdot)$ is continuous). This proves (11) for $\lambda>\rho$ which can be extended to $0<\lambda\leqslant\rho$ thanks to the resolvent equation.

We are now able to complete the proof of our statement. The identity (11) and the Markov property enable us to write

$$\mathbb{E}_x^\ddagger\left(\int_0^\infty \mathrm{e}^{-\lambda s}\,dL_s^x \,|\, \mathcal{F}_t\right)=\int_0^t \mathrm{e}^{-\lambda s}\,dL_s^x+\mathrm{e}^{-\lambda t}u_\lambda^\ddagger(X_t,x),$$

which entails, by an application of the optional sampling theorem at $T_{x+\eta}$, that

$$\mathbb{E}_x^\ddagger\left(\int_0^{T_{x+\eta}} \mathrm{e}^{-\lambda s}\,dL_s^x\right)=u_\lambda^\ddagger(x,x)-u_\lambda^\ddagger(x+\eta,x)\mathbb{E}_x^\ddagger\left(\mathrm{e}^{-\lambda T_{x+\eta}}\right).$$

Theorem 3.1 and Proposition 3.2(ii), jointly with the expression (8) for $u_\lambda^\ddagger(x,y)$ yield the result. □

The previous proposition enables us to specify the asymptotic behaviour of the local time. Recall from Theorem 3.1(iv) that the stationary probability μ of the confined process is absolutely continuous with a continuous density p.

COROLLARY 4.3. – *For any $x\in(0,a)$, we have a.s.*

$$\lim_{t\to\infty} L_t^x/t=p(x)=\frac{\mu(dx)}{dx}.$$

Proof. – This is essentially an application of the ergodic theorem (see also Section XIX.46 in [8]). Let us give a quick argument. We deduce from Proposition 4.2 that $\phi_x^\ddagger$ has a finite right-derivative at 0, equal to

$$\frac{c(a)}{W^{(-\rho)}(x)W^{(-\rho)}(a-x)}=p(x)^{-1}.$$

Hence, using

$$\mathbb{E}_x^\ddagger(\tau_t)=tp(x)^{-1},$$

and $(\tau_t, t \geqslant 0)$ being a Lévy process, the law of large numbers entails that a.s.,

$$\lim_{t\to\infty} \frac{L_t^x}{t} = \lim_{t\to\infty} \frac{t}{\tau_t} = p(x). \qquad \square$$

5. CONVERGENCE RATE OF THE SUPREMUM

5.1. Introduction

We still assume throughout this section that X has paths of unbounded variation. We write $S_t = \sup\{X_s; s \in [0,t]\}$ for the supremum. The fact that the confined process is recurrent implies that S_t converges to a and our purpose is to investigate the rate of convergence.

Analogously, we might consider the rate of convergence of the infimum to 0. However, recalling Remark (c) following Theorem 3.1, we know that if s and i stand for the supremum and infimum of the generic excursion away from $\{a/2\}$ respectively, then i and $a - s$ are equally distributed. The very same arguments used in the sequel for the supremum process then apply to the infimum process, and we shall therefore focus on S.

Our study relies heavily on the following statement which will be proved at the end of this section.

PROPOSITION 5.1. – *The following assertions hold*

(i) *The mapping* $(x,q) \mapsto W^{(q)}(x)$ *is of class* C^1 *on* $(0,\infty) \times (-\infty,\infty)$.

(ii) *The mapping*

$$x \mapsto \rho(x) = \inf\{q > 0\colon\ W^{(-q)}(x) = 0\}$$

is strictly decreasing and of class C^1 *on* $(0,\infty)$. *As a consequence, the set*

$$\mathcal{D} = \{x > 0\colon\ \rho'(x) \neq 0\}$$

is an open and everywhere dense set of $(0,\infty)$.

(iii) *For every* $a > 0$,

$$\rho'(a)c(a) = \left.\frac{\partial W^{(-\rho)}(x)}{\partial x}\right|_{x=a},$$

where $c(a)$ *was defined in Proposition* 3.1.

(iv) *In the stable case of index* α, $\psi(\lambda)=\lambda^\alpha$,

$$\rho'(a)=-\frac{\alpha r(\alpha)}{a^{\alpha+1}}$$

(*recall the notations following Theorem* 2.1), *and therefore* $\mathcal{D}=(0,\infty)$.

In fact, this proposition still holds in the bounded variation case, provided the Lévy measure Λ of X under $\mathbb{P}$ has no atoms. However, to avoid technicalities, we stick to the unbounded variation case.

Let $f:[0,\infty)\to(0,\infty)$ be a decreasing function and write

$$l_f=\liminf_{t\to\infty}\frac{a-S_t}{f(t)},\qquad L_f=\limsup_{t\to\infty}\frac{a-S_t}{f(t)}.$$

Finally recall that a real function g is said to be slowly varying at infinity if for any $\lambda>0$,

$$\lim_{t\to\infty}\frac{g(\lambda t)}{g(t)}=1.$$

THEOREM 5.2. – *The following three assertions hold*

(i) *If* $a\in\mathcal{D}$, *then* $t(a-S_t)$ *converges in distribution as* $t\to\infty$ *to an exponential r.v. with parameter* $|\rho'(a)|$.

(ii)

$$\mathbb{P}_x^\updownarrow\big(a-S_t<f(t)\ \textit{i.o. as}\ t\to\infty\big)=0\quad\textit{or}\quad 1,$$

according whether $\int^\infty W^{(-\rho)}(a-f(s))\,ds$ *converges or diverges. As a consequence,*

$$\int^\infty f(s)\,ds<\infty\Rightarrow l_f=+\infty\quad a.s.,$$

and provided that $a\in\mathcal{D}$,

$$\int^\infty f(s)\,ds=\infty\Rightarrow l_f=0\quad a.s.$$

(iii) *Assume further that* $t\mapsto tf(t)$ *is increasing and slowly varying at infinity, and let*

$$\Gamma_f\doteq\inf\left\{\gamma\geqslant 0:\ \int^\infty dt\,f(t)\,\mathrm{e}^{-\gamma tf(t)}<\infty\right\},$$

with the convention $\inf\emptyset = +\infty$. *If* $a \in \mathcal{D}$, *then*

$$L_f = \frac{\Gamma_f}{|\rho'(a)|} \quad a.s.$$

Remarks. – (a) Let

$$I_f(\gamma) \doteq \int^{\infty} dt\, f(t)\,\mathrm{e}^{-\gamma t f(t)}.$$

One easily sees that if $\Gamma_f < \infty$, I_f is finite (and decreasing) on (Γ_f, ∞), and that if $\Gamma_f > 0$, $I_f = \infty$ on $[0, \Gamma_f)$. This remark is implicitly used in the proof of (iii).

(b) We give an idea for the value of Γ_f. If $\ln_k$ denotes the kth iterated logarithm, then for

$$f(t) = \begin{cases} t^{-1}\ln_3 t, & \Gamma_f = \infty, \\ t^{-1}\ln_2 t, & \Gamma_f = 1, \\ t^{-1}\ln t, & \Gamma_f = 0. \end{cases}$$

In particular, one observes that the slowly varying condition in (iii) is not stringent: $tf(t) = \ln t$ is already varying too fast since $\Gamma_f = 0$.

We see that if $f(t) = t^{-1}\ln_2 t$, then

$$l_f = 0 \quad \text{and} \quad L_f = \frac{1}{|\rho'(a)|} \quad \text{a.s.},$$

whereas if $f(t) = t^{-1}$, or $f(t) = t^{-1}\ln_k t$, with $k \geqslant 3$,

$$l_f = 0 \quad \text{and} \quad L_f = \infty \quad \text{a.s.}$$

(c) Recall that for $\alpha \in (1,2]$, $-r(\alpha)$ still denotes the first negative root of E'_α, where E_α stands for the Mittag–Leffler function of parameter α given by (7). In the stable case of index $\alpha \in (1,2]$, for any $a > 0$,

$$\limsup_{t\to\infty} \frac{t(a - S_t)}{\ln_2 t} = \frac{a^{\alpha+1}}{\alpha r(\alpha)} \quad \text{a.s.},$$

which yields for the confined Brownian motion

$$\limsup_{t\to\infty} \frac{t(a - S_t)}{\ln_2 t} = \frac{a^3}{\pi^2} \quad \text{a.s.}$$

Each of the next three subsections is devoted to the proof of some part of Theorem 5.2. Proposition 5.1 is established in the ultimate one.

5.2. Convergence in distribution of $t(a - S_t)$

The argument uses the next elementary lemma.

LEMMA 5.3. – *Fix $y \in (0, \infty]$ and let $R : [0, y) \to [0, \infty)$ be an increasing function. Next consider a Poisson point process $(Z_s, s \geqslant 0)$ on $[0, y)$ with characteristic measure dR. For every $t > 0$, set $i_t = \inf_{0 \leqslant s \leqslant t} Z_s$, where we agree that $\inf \emptyset = y$. Then*

$$\mathbb{E}(\mathrm{e}^{-\lambda i_t}) = 1 - \lambda \int_0^y du\, \mathrm{e}^{-\lambda u}\, \mathrm{e}^{-tR(u)}, \quad \lambda > 0.$$

Proof. – This is immediate from the identity

$$\mathbb{P}\big(Z_s > x \text{ for all } s \in [0, t]\big) = \exp\big(-tR(x)\big), \quad x \in (0, y). \qquad \square$$

Fix $x \in (0, a)$ and recall the notation in Section 4 involving the excursions away from $\{x\}$. Our argument relies on the elementary observation that $S_{\tau_t} - x$ is the maximum of the excursion heights $(m(e_s))$ for $s \leqslant t$.

Recall from Proposition 4.2 that

$$R_x(u) = \frac{W^{(-\rho)}(a - u)}{W^{(-\rho)}(x) W^{(-\rho)}(a - u - x)}, \quad u \in [0, a - x),$$

is the distribution function of the measure $n_x^{\ddagger}(a - m - x \in \cdot)$. From Proposition 5.1, the function R_x is of class C^1 and its derivative at 0 is positive (because $a \in \mathcal{D}$).

The point process $(Z_s, s \geqslant 0)$ defined by

$$Z_s = \begin{cases} a - m(e_s) - x & \text{if } \tau_{s-} < \tau_s, \\ \infty & \text{otherwise}, \end{cases}$$

is a Poisson point process with characteristic measure dR_x. We deduce from the previous lemma that

$$\mathbb{E}_x^{\ddagger}(\mathrm{e}^{-\lambda t(a - S_{\tau_t})}) = 1 - \lambda \int_0^{ty} dv\, \mathrm{e}^{-\lambda v}\, \mathrm{e}^{-tR_x(v/t)},$$

hence by dominated convergence,

$$\lim_{t\to\infty} \mathbb{E}_x^{\updownarrow}\left(\mathrm{e}^{-\lambda t(a-S_{\tau_t})}\right) = 1 - \lambda \int_0^{\infty} dv\, \mathrm{e}^{-\lambda v}\, \mathrm{e}^{-R_x'(0)v}$$
$$= \frac{R_x'(0)}{\lambda + R_x'(0)}.$$

Now let $\delta < p(x)$. According to Corollary 4.3, one has $\tau_{\delta t} < t$ for sufficiently large t, so that

$$\liminf_{t\to\infty} \mathbb{E}_x^{\updownarrow}\left(\mathrm{e}^{-\lambda t(a-S_t)}\right) \geqslant \frac{R_x'(0)}{\lambda\delta^{-1} + R_x'(0)}.$$

So letting $\delta \to p(x)$, we get that

$$\liminf_{t\to\infty} \mathbb{E}_x^{\updownarrow}\left(\mathrm{e}^{-\lambda t(a-S_t)}\right) \geqslant \frac{p(x)R_x'(0)}{\lambda + p(x)R_x'(0)}.$$

On the other hand, it follows from Proposition 5.1(iii) that

$$p(x)R_x'(0) = -p(x)\frac{\frac{\partial}{\partial x} W^{(-\rho)}(a)}{W^{(-\rho)}(x)W^{(-\rho)}(a-x)}$$
$$= -p(x)\frac{\rho'(a)c(a)}{W^{(-\rho)}(x)W^{(-\rho)}(a-x)} = \left|\rho'(a)\right|.$$

Proceeding similarly with $\delta > p(x)$, we get

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}\left(\mathrm{e}^{-\lambda t(a-S_t)}\right) = \frac{p(x)R_x'(0)}{\lambda + p(x)R_x'(0)} = \frac{|\rho'(a)|}{\lambda + |\rho'(a)|}.$$

5.3. Lower rate of convergence

We prove the second assertion. As $W^{(-\rho)}$ is of class C^1 (cf. Proposition 5.1), $\int^{\infty} f(s)\,ds < \infty$ implies $\int^{\infty} W^{(-\rho)}(a - f(s))\,ds < \infty$. Replacing f by λf, and then letting $\lambda \to \infty$, we get the result. The same method then applies to the converse assertion now letting $\lambda \to 0$: indeed if $a \in \mathcal{D}$, the derivative at a of $W^{(-\rho)}$ is negative, hence $\int^{\infty} f(s)\,ds = \infty$ implies that $\int^{\infty} W^{(-\rho)}(a - f(s))\,ds = \infty$.

We now prove the first assertion. Let

$$N_t = \sum_{s\leqslant t} \mathbf{1}_{\{m(e_s) > a - f(s) - x\}}$$

be the number of points of the excursion process before time t whose absolute maximum exceeds $a - f(L^x)$ ($m =$ absolute maximum $-x$).

We know N_t is a Poisson variable of parameter

$$\int_0^t n_x^{\ddagger}(m > a - f(s) - x)\, ds = \int_0^t \frac{W^{(-\rho)}(a - f(s))}{W^{(-\rho)}(a - f(s) - x)W^{(-\rho)}(x)}\, ds.$$

Therefore, N_∞ is a.s. infinite, that is $S_{\tau_t} > a - f(t)$ i.o. as $t \to \infty$, if $\int^\infty W^{(-\rho)}(a - f(s))\, ds = \infty$. Otherwise, the last event is evanescent.

To get the result with S_t, recall from Corollary 4.3 that as $t \to \infty$, $\tau_t \sim t/p(x)$ a.s. for some positive and finite constant $p(x)$. Observing that the integral criterion of the theorem remains unchanged when replacing f by $t \mapsto f(\lambda t)$, one deduces easily that the events $\{S_t > a - f(t)$ i.o. as $t \to \infty\}$ and $\{S_{\tau_t} > a - f(t)$ i.o. as $t \to \infty\}$ have the same probability.

5.4. Upper rate of convergence

Consider the Poisson point process Z with characteristic measure dR_x defined in Section 5.2. Recall that the function R_x is C^1 and that its derivative is positive on some interval $[0, \delta)$. As $a - S_{\tau_t} = \min\{Z_s; s \leqslant t\}$ for sufficiently large t, the path of $(R_x(a - S_{\tau_t}), t \geqslant 0)$ coincides with that of $(v_t, t \geqslant 0)$, where v_t is the minimum on $[0, t]$ of a Poisson point process with characteristic measure the uniform distribution on $(0, 1)$.

An appeal to the extremal process $(u_t, t \geqslant 0)$ defined in [15] is the key step of this proof. One observes easily that such a process starting from $u_0 > 0$ has the same law as $(u_0 \wedge v_t, t \geqslant 0)$. The function f being decreasing at infinity and $t \mapsto tf(t)$ being increasing at infinity, Theorem 3 of [15] ensures that $u_t > f(t)$ i.o. as $t \to \infty$ with probability 0 or 1 according whether $I_f(1)$ converges or diverges. Suppose for instance that $I_f(1) = \infty$. Then

$$\limsup_{t\to\infty} \frac{R_x(a - S_{\tau_t})}{f(t)} \geqslant 1 \quad \text{a.s.}$$

Now let $\lambda > 1$. We have

$$\begin{aligned}\limsup_{t\to\infty} \frac{R_x(a - S_{\tau_t})}{f(t)} &= \limsup_{t\to\infty} \frac{R_x'(0)(a - S_{\tau(\lambda p(x)t)})}{f(\lambda p(x)t)} \\ &\leqslant \limsup_{t\to\infty} \frac{R_x'(0)(a - S_t)}{f(\lambda p(x)t)} \\ &= \limsup_{t\to\infty} \lambda p(x) R_x'(0) \frac{a - S_t}{f(t)},\end{aligned}$$

where the inequality is due to Corollary 4.3 and the last equality to the slow variation of $t \mapsto tf(t)$. Recalling that $p(x)R'_x(0) = |\rho'(a)|$, and letting $\lambda \to 1$, we obtain that $L_f \geqslant |\rho'(a)|^{-1}$ a.s.

In the case when $I_f(1) < \infty$, one can prove the opposite inequality in the same way.

To derive the required result, pick $\lambda > \Gamma_f$ ($\Gamma_f < \infty$). We can then apply the foregoing result to λf: $I_{\lambda f}(1)$ converges, so $L_f = \lambda L_{\lambda f}$ is bounded above a.s. by $\lambda|\rho'(a)|^{-1}$. Letting $\lambda \to \Gamma_f$, we get

$$L_f \leqslant \frac{\Gamma_f}{|\rho'(a)|} \quad \text{a.s.},$$

which obviously holds if $\Gamma_f = \infty$.

With $\lambda < \Gamma_f$ ($\Gamma_f > 0$), one can prove the opposite inequality. Hence

$$L_f = \frac{\Gamma_f}{|\rho'(a)|} \quad \text{a.s.}$$

5.5. Proof of Proposition 5.1

(i) We first prove that $W = W^{(0)}$ is of class C^1 on $(0, \infty)$. Recall the following expression for W, p. 195 in [1]:

$$W(x) = K \exp\left(-\int_x^\infty \nu(t, \infty)\, dt\right),$$

where K is a positive constant, and ν the characteristic measure of the point Poisson process of excursion heights of $S - X$ away from $\{0\}$ under $\mathbb{P}_0$. In particular, W is differentiable with derivative

$$W'(x) = W(x)\nu(x, \infty).$$

To show that W is C^1 on $(0, \infty)$, it suffices to prove that ν has no atoms. Suppose then that $\nu(\{x\}) > 0$ for some $x > 0$, that is $\tilde{n}(m = x) > 0$, where $\tilde{n}$ denotes the characteristic measure of the point process of excursions of $S - X$ away from $\{0\}$ under $\mathbb{P}_0$. Recall that in the unbounded variation case, x is regular for $(-\infty, x)$ under $\mathbb{P}$, and apply the strong Markov property at T_x under the conditional law $\tilde{n}(\cdot|m = x)$ ($0 \leqslant T_x < V$ with full $\tilde{n}(\cdot|m = x)$-measure since $S - X$ has only positive jumps). As the excursion visits (x, ∞) a.s. immediately after T_x, we get the contradiction.

We now prove that $(x,q)\mapsto W^{(q)}(x)$ is of class C^1 on $(0,\infty)\times(-\infty,\infty)$. It suffices to show that $(x,q)\mapsto W^{(q)}(x)$ has partial derivatives jointly continuous in q and x. To this aim, we shall use the expansion of $W^{(q)}$ as a power series and prove the uniform convergence of the partial derivatives of partial sums. For the derivative with respect to q, its existence and continuity follow readily from the expansion of $q\mapsto W^{(q)}(x)$ in the form (4), majoration (5), and the continuity of the scale function.

We turn to $x\mapsto W^{(q)}(x)$. We now know that $W^{\star k+1}$ is of class C^1 on $(0,\infty)$. Its derivative is nonnegative and

$$\begin{aligned}\left(W^{\star k+1}\right)'(x) &= W'\star W^{\star k}(x)\\ &\leqslant \int_0^x dy\, W'(x-y)\,\frac{y^{k-1}}{(k-1)!}\,W(y)^k\\ &\leqslant \frac{x^{k-1}}{(k-1)!}\,W(x)^{k+1},\end{aligned}$$

for the scale function is increasing. It follows that the series

$$W^{(q)\prime}(x)=\sum_{k\geqslant 0}q^k\left(W^{\star k+1}\right)'(x)$$

is uniformly convergent on compact sets of $(0,\infty)\times(-\infty,\infty)$ and our claim is proven.

(ii) and (iii) Consider the mapping $x\mapsto\rho(x)$ when x varies.

One easily sees that ρ is strictly decreasing for one knows that for $h>0$, and $q\leqslant\rho(a+h)$, $W^{(-q)}$ is positive on $(0,a+h)$, hence by definition of $\rho(a)$, $\rho(a)>\rho(a+h)$.

Furthermore, the continuity of $(x,q)\mapsto W^{(q)}(x)$ and the definition of $\rho(a)$ ensure the right-continuity of ρ. Specifically $W^{(-\rho(a+))}(a)=0$ from $W^{(-\rho(a+\varepsilon))}(a+\varepsilon)=0$ $(\varepsilon>0)$, and we get the result since $\rho(a+)\leqslant\rho(a)$.

In order to obtain the left-continuity, assume for a moment that $\rho(a)\neq\rho(a-)$. Then for any $q\in[\rho(a),\rho(a-)]$, we know that $W^{(-q)}(a-\varepsilon)>0$ $(\varepsilon>0)$, hence $W^{(-q)}(a)\geqslant 0$. Moreover, we know that for any $q'<\rho(a)$, the definition of $\rho(a)$ entails that $W^{(-q')}(a)>0$. Hence in a neighbourhood of $\rho(a)$, $q\mapsto W^{(-q)}(a)$ is nonnegative. This contradicts Theorem 2.1(i) according to which $\rho(a)$ is a simple root of $q\mapsto W^{(-q)}(a)$.

Eventually, we know that $x \mapsto \rho(x)$ is continuous, strictly decreasing, and satisfies

$$W^{(-\rho(x))}(x) = 0.$$

Since $(x, q) \mapsto W^{(q)}(x)$ is C^1 on $(0, \infty) \times (-\infty, \infty)$, and has positive partial derivative with respect to q at $(a, -\rho(a))$, we may apply the implicit function theorem at this point. Since ρ is continuous, it coincides locally with the implicit function, hence ρ is C^1 on $(0, \infty)$ and

$$\rho'(a) = \frac{\partial}{\partial x} W^{(-\rho(a))}(a) / \frac{\partial}{\partial q} W^{(-\rho(a))}(a) = \frac{\partial}{\partial x} W^{(-\rho(a))}(a)/c(a).$$

(iv) In the stable case of index $\alpha \in (1, 2]$, recall that $\rho(a) = r(\alpha)a^{-\alpha}$, so that the derivative of $a \mapsto \rho(a)$ is equal to $-\alpha r(\alpha)a^{-\alpha-1}$ and therefore never vanishes, that is

$$\mathcal{D} = (0, \infty).$$

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