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Stochastic heat equation with white-noise drift

by

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ABSTRACT. – We study the existence and uniqueness of the solution for a one-dimensional anticipative stochastic evolution equation driven by a two-parameter Wiener process $W_{t,x}$ and based on a stochastic semigroup $p(s,t,y,x)$. The kernel $p(s,t,y,x)$ is supposed to be measurable with respect to the increments of the Wiener process on $[s,t] \times \mathbb{R}$. The results are based on L^p-estimates for the Skorohod integral. As a application we deduce the existence of a weak solution for the stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{v}(t,x)\,\frac{\partial u}{\partial x} + F(t,x,u)\,\frac{\partial^2 W}{\partial t \partial x},$$

where $\dot{v}(t,x)$ is a white-noise in time. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On étudie l'existence et l'unicité de la solution pour une équation d'évolution stochastique anticipative en une dimension, perturbée par un processus de Wiener à deux paramètres $W_{t,x}$ et conduite par un semigroupe stochastique $p(s,t,y,x)$. On suppose que le noyau

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$p(s,t,y,x)$ est mesurable par rapport aux accroissements du processus de Wiener dans l'intervalle $[s,t]\times\mathbb{R}$. Les résultats sont basés sur des estimations L^p pour l'intégrale de Skorohod. Comme application, on déduit l'existence d'une solution faible pour l'equation aux dérivées partielles stochastique

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\dot{v}(t,x)\,\frac{\partial u}{\partial x}+F(t,x,u)\,\frac{\partial^2 W}{\partial t\partial x},$$

où $\dot{v}(t,x)$ est un bruit blanc en temps.

1. INTRODUCTION

The purpose of this paper is to establish the existence and uniqueness of a solution for anticipative stochastic evolution equations of the form

$$u(t,x)=\int_{\mathbb{R}} p(0,t,y,x)\,u_0(y)\,dy + \int_{\mathbb{R}}\int_0^t p(s,t,y,x)F\big(s,y,u(s,y)\big)\,dW_{s,y}, \tag{1.1}$$

where $W=\{W(t,x), t\in[0,T], x\in\mathbb{R}\}$ is a zero mean Gaussian random field with covariance $\frac{1}{2}(s\wedge t)(|x|+|y|-|x-y|)$. We asume that $p(s,t,y,x)$ is a stochastic semigroup measurable with respect to the σ-field $\sigma\{W(r,x)-W(s,x), x\in\mathbb{R}, r\in[s,t]\}$. The stochastic integral in Eq. (1.1) is anticipative because the integrand is the product of the adapted factor $F(s,y,u(s,y))$, and of $p(s,t,y,x)$, which is adapted to the future increments of the random field W. We interpret this integral in the Skorohod sense (see [15]) which coincides in this case with a two-sided stochastic integral (see [14]). The choice of this notion of stochastic integral is motivated by the concrete example handled in Section 5, where $p(s,t,y,x)$ is the backward heat kernel of the random operator $\frac{d^2}{dx^2}+\dot{v}(t,x)\,\frac{d}{dx}$, $\dot{v}(t,x)$ being a white-noise in time. In this case, $u(t,x)$ turns out to be (see Section 6) a weak solution of the stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{v}(t,x)\,\frac{\partial u}{\partial x} + F(t,x,u)\,\frac{\partial^2 W}{\partial t\,\partial x}. \tag{1.2}$$

A stochastic evolution equation of the form (1.1) on $\mathbb{R}^d$ perturbed by a noise of the form $W(ds,y)\,dy$, where W is a random field with covariance $(s\wedge t)\,Q(x,y)$, Q being a bounded function, has been studied in [13]. Following the approach introduced in this paper we establish in Theorem 4.1 the existence and uniqueness of a solution to Eq. (1.1) with values in $L^p_M(\mathbb{R})$. Here $L^p_M(\mathbb{R})$ means the space of real-valued functions f such that $\int_{\mathbb{R}} e^{-M(x)}|f(x)|^p\,dx < \infty$ where $M > 0$ and $p \geqslant 2$. This theorem is a consequence of the estimates of the moments of Skorohod integrals of the form

$$\int_{\mathbb{R}}\int_0^t p(s,t,y,x)\phi(s,y)\,dW_{s,y},$$

obtained in Section 3 by means of the techniques of the Malliavin calculus.

2. PRELIMINARIES

For $s,t\in[0,T]$, $s\leqslant t$, we set $I^t=[0,t]\times\mathbb{R}$ and $I_s^t=[s,t]\times\mathbb{R}$. Consider a Gaussian family of random variables $W=\{W(A),\ A\in\mathcal{B}(I^T),\ \mu(A)<\infty\}$, defined on a complete probability space, with zero mean, and covariance function given by

$$E\big(W(A)W(B)\big)=\mu(A\cap B),$$

where μ denotes the Lebesgue measure on I^T. We will assume that $\mathcal{F}$ is generated by W and the P-null sets. For each $s,t\in[0,T]$, $s\leqslant t$, we will denote by $\mathcal{F}_{s,t}$ the σ-algebra generated by $\{W(A),\ A\subset[s,t]\times\mathbb{R},\ \mu(A)<\infty\}$ and the P-null sets. We say that a stochastic process $u=\{u(t,x),\ (t,x)\in I^T\}$ is *adapted* if $u(t,x)$ is $\mathcal{F}_{0,t}$-measurable for each (t,x). Set $H=L^2(I^T,\ \mathcal{B}(I^T),\mu)$ and denote by $W(h)=\int_{I^T}h\,dW$ the Wiener integral of a deterministic function $h\in H$.

In the sequel we introduce the basic notation and results of the stochastic calculus of variations with respect to W. For a complete exposition we refer to [2,11].

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F=f\big(W(h_1),\dots,W(h_n)\big), \tag{2.1}$$

where $n \geqslant 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $h_1, \ldots, h_n \in H$. Given a random variable F of the form (2.1), we define its derivative as the stochastic process $\{D_{t,x}F,\ (t,x) \in I^T\}$ given by

$$D_{t,x}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(W(h_1), \ldots, W(h_n)\big)h_i(t,x), \quad (t,x) \in I^T.$$

More generally, we can define the iterated derivative operator on a cylindrical random variable F by setting

$$D^n_{t_1,x_1,\ldots,t_n,x_n} F = D_{t_1,x_1} \cdots D_{t_n,x_n} F.$$

The iterated derivative operator D^n is a closable unbounded operator from $L^2(\Omega)$ into $L^2((I^T)^n \times \Omega)$ for each $n \geqslant 1$. We denote by $\mathbb{D}^{n,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|^2_{n,2} = \|F\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|D^i F\|^2_{L^2((I^T)^i \times \Omega)}.$$

If V is a real and separable Hilbert space we denote by $\mathbb{D}^{n,2}(V)$ the corresponding Sobolev space of V-valued random variables.

We denote by δ the adjoint of the derivative operator D. That is, the domain of δ (denoted by Dom δ) is the set of elements $u \in L^2(I^T \times \Omega)$ such that there exists a constant c satisfying

$$\left| E \int_{I^T} (D_{t,x}F)u(t,x)\,dt\,dx \right| \leqslant c\|F\|_{L^2(\Omega)},$$

for all $F \in \mathcal{S}$. If $u \in \operatorname{Dom}\delta$, $\delta(u)$ is the element in $L^2(\Omega)$ characterized by

$$E\big(\delta(u)F\big) = E\int_{I^T} (D_{t,x}\,F)u(t,x)\,dt\,dx, \quad F \in \mathcal{S}.$$

The operator δ is an extension of the Itô integral (see Skorohod [15]), in the sense that the set $L^2_a(I^T \times \Omega)$ of square integrable and adapted processes is included in Dom δ and the operator δ restricted to $L^2_a(I^T \times \Omega)$ coincides with the Itô stochastic integral defined in [16]. We will make use of the notation $\delta(u) = \int_{I^T} u(t,x)\,dW_{t,x}$ for any $u \in \operatorname{Dom}\delta$.

We recall that $\mathbb{L}^{1,2} := L^2(I^T, \mathbb{D}^{1,2})$ is included in the domain of δ, and for a process $u \in \mathbb{L}^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$E\delta(u)^2 = E\int_{I^T} u^2(t,x)\,dt\,dx + E\int_{I^T}\int_{I^T} D_{s,y}u(t,x)D_{t,x}u(s,y)\,dt\,dx\,ds\,dy.$$

We need the following results on the Skorohod integral:

PROPOSITION 2.1. – *Let $u \in \operatorname{Dom}\delta$ and consider a random variable $F \in \mathbb{D}^{1,2}$ such that $E(F^2 \int_{I^T} u(t,x)^2\,dt\,dx) < \infty$. Then*

$$\int_{I^T} Fu(t,x)\,dW_{t,x} = F\int_{I^T} u(t,x)\,dW_{t,x} - \int_{I^T} (D_{t,x}F)u(t,x)\,dt\,dx, \tag{2.2}$$

in the sense that $Fu \in \operatorname{Dom}\delta$ if and only if the right-hand side of (2.2) is square integrable.

PROPOSITION 2.2. – *Consider a process u in $\mathbb{L}^{1,2}$. Suppose that for almost all $(\theta,z) \in I^T$, the process $\{D_{\theta,z}u(s,y)\mathbb{1}_{[0,\theta]}(s),\ (s,y) \in I^T\}$ belongs to $\operatorname{Dom}\delta$ and, moreover,*

$$E\int_{I^T}\left|\int_{I^\theta} D_{\theta,z}u(s,y)\,dW_{s,y}\right|^2 d\theta\,dz < \infty.$$

Then u belongs to $\operatorname{Dom}\delta$ and we have the following expression for the variance of the Skorohod integral of u:

$$E\delta(u)^2 = E\int_{I^T} u^2(s,y)\,ds\,dy + 2E\int_{I^T} u(\theta,z)\left(\int_{I^\theta} D_{\theta,z}u(s,y)\,dW_{s,y}\right) d\theta\,dz. \tag{2.3}$$

We make use of the change-of-variables formula for the Skorohod integral:

THEOREM 2.3. – *Consider a process of the form*

$$X_t = \int_{I^t} u(s, y)\, dW_{s,y},$$

where

(i) $u \in \mathbb{L}^{2,2}$,

(ii) $u \in L^\beta(I^T \times \Omega)$, *for some* $\beta > 2$,

(iii) $\int_{I^T} u^2(s, y)\, ds\, dy < N$,

for some positive constant N. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a twice continuously differentiable function such that* F'' *is bounded. Then we have*

$$F(X_t) = F(0) + \int_{I^t} F'(X_s)u(s, y)\, dW_{s,y} + \frac{1}{2}\int_{I^t} F''(X_s)u^2(s, y)\, ds\, dy$$
$$+ \int_{I^t} F''(X_s)u(s, y)\left(\int_{I^s} D_{s,y}u(r, z)\, dW_{r,z}\right) ds\, dy. \tag{2.4}$$

Notice that under the assumptions of Theorem 2.3 the process X_t has a continuous version (see [2,5]) and, moreover, $\{F'(X_s)u(s, y), (s, y) \in I^T\}$ belongs to $\operatorname{Dom}\delta$.

3. ESTIMATES FOR THE SKOROHOD INTEGRAL

We denote by C a generic constant that can change from one formula to another one. Let $p(s, t, y, x)$ be a random measurable function defined on $\{0 \leqslant s < t \leqslant T,\ x, y \in \mathbb{R}\} \times \Omega$. We will assume that the following conditions hold:

(H1) For all $0 \leqslant s < t \leqslant T,\ x, y \in \mathbb{R},\ p(s, t, y, x)$ is $\mathcal{F}_{s,t}$-measurable.

(H2) $p(s, t, y, x) \geqslant 0$, for each $0 \leqslant s < t \leqslant T,\ x, y \in \mathbb{R}$.

(H3) For all $0 \leqslant s < t \leqslant T,\ x \in \mathbb{R},\ \int_{\mathbb{R}} p(s, t, y, x)\, dy = 1$.

(H4) For each $s \in [0, T], y \in \mathbb{R},\ p(s, t, y, \cdot)$ is continuous in $t \in (s, T]$ with values in $L^2(\mathbb{R})$.

(H5) For all $0 \leqslant s < r < t \leqslant T$, and $x, y \in \mathbb{R}$

$$p(s, t, y, x) = \int_{\mathbb{R}} p(s, r, y, z)p(r, t, z, x)\, dz.$$

(H6) For all $0 \leqslant s < t \leqslant T,\ x, y \in \mathbb{R},\ p(s, t, y, x) \in \mathbb{D}^{1,2}$ and $p(s, t, \cdot, x)$ belongs to $\mathbb{D}^{1,2}(L^2(\mathbb{R}))$. Moreover, there exists a version of the derivative such that the following limit exists in $L^2(\Omega;\ L^2(\mathbb{R}))$ for each s, z, t, x

$$D^-_{s,z} p(s,t,\cdot,x) = \lim_{\varepsilon\downarrow 0} D_{s,z}\, p(s-\varepsilon,t,\cdot,x). \tag{3.1}$$

(H7) For all $0 \leqslant s < t \leqslant T,\ x, y \in \mathbb{R},\ p \geqslant 1$ there exist a nonnegative, $\mathcal{F}_{s,t}$-measurable random variable $V_p(s,t,x)$ and $\delta_p > 0$ such that

$$p(s,t,y,x) \leqslant V_p(s,t,x)\exp\left(-\frac{|x-y|^2}{\delta_p(t-s)}\right)$$

and such that for all $p \geqslant 1$, there exists a positive constant $C_{1,p}$ such that

$$\left\|V_p(s,t,x)\right\|_{L^p(\Omega)} \leqslant C_{1,p}|t-s|^{-1/2}.$$

(H8) For all $0 \leqslant s < t \leqslant T,\ \ x, y, z \in \mathbb{R}$, and $p \geqslant 1$ there exist a nonnegative, $\mathcal{F}_{s,t}$-measurable random variable $U_p(s,t,x)$, a constant $\gamma_p > 0$ and a nonnegative measurable deterministic function $f(y,z)$ such that

$$\text{(i)}\quad \left|D^-_{s,z} p(s,t,y,x)\right| \leqslant U_p(s,t,x)\exp\left(-\frac{|x-y|^2}{\gamma_p(t-s)}\right) f(y,z),$$

$$\text{(ii)}\quad \sup_y \int_{\mathbb{R}} f^2(y,z)\,dz := C_f < \infty,$$

$$\text{(iii)}\quad \left\|U_p(s,t,x)\right\|_{L^p(\Omega)} \leqslant C_{2,p}|t-s|^{-1},$$

for some positive constants $C_{2,p},\ C_f$.

The following lemma is a straightforward consequence of the above hypotheses:

LEMMA 3.1. – *Under the above hypotheses we have that for all* $0 \leqslant r < s < t \leqslant T,\ x, y, z \in \mathbb{R}$,

$$D_{s,y} p(r,t,z,x) = \int_{\mathbb{R}} \left(D^-_{s,y} p(s,t,u,x)\right) p(r,s,z,u)\,du. \tag{3.2}$$

Proof. – Taking into account the properties of the derivative operator and using hypotheses (H1), (H5) and (H6) we have that

$$D_{s,y}p(r,t,z,x) = D_{s,y}\int_{\mathbb{R}} p(r,s-\varepsilon,z,u)p(s-\varepsilon,t,u,x)\,du$$
$$= \int_{\mathbb{R}} p(r,s-\varepsilon,z,u)D_{s,y}p(s-\varepsilon,t,u,x)\,du.$$

Now, letting ε tend to zero and using hypotheses (H1), (H4), (H6) and (H8) we can easily complete the proof. □

We are now in a position to prove our estimates for the Skorohod integral. For all $M > 0$, we will denote by $L^p_M(I^T \times \Omega)$ the space of processes $\phi = \{\phi(s,y), (s,y) \in I^T\}$ such that

$$E\int_{I^T} \mathrm{e}^{-M|y|}|\phi(s,y)|^p\,ds\,dy < \infty.$$

THEOREM 3.2. – *Fix $p > 4$, $\alpha \in [0, \frac{p-4}{4p})$ and $M > 0$. Let $\phi = \{\phi(s,y), (s,y) \in I^T\}$ be an adapted process in $L^p_M(I^T \times \Omega)$. Assume that $p(s,t,y,x)$ is a stochastic kernel satisfying hypotheses* (H1)–(H8). *Then, for almost all $(t,x) \in I^T$, the process*

$$\{(t-s)^{-\alpha}p(s,t,y,x)\phi(s,y)\mathbb{1}_{[0,t]}(s), (s,y) \in I^T\}$$

belongs to $\mathrm{Dom}\,\delta$, *and*

$$\int_{\mathbb{R}} \mathrm{e}^{-M|x|}E\left|\int_{I^t}(t-s)^{-\alpha}p(s,t,y,x)\phi(s,y)\,dW_{s,y}\right|^p dx$$
$$\leqslant C\int_0^t (t-s)^{-\alpha-\frac{1}{4}-\frac{1}{p}}\left(\int_{\mathbb{R}} \mathrm{e}^{-M|y|}E|\phi(s,y)|^p\,dy\right)ds, \qquad (3.3)$$

for some positive constant C depending only on α, p, T, M, δ_p, γ_p, $C_{1,p}$, $C_{2,p}$ and C_f.

Proof. – The proof is based on the change-of-variables formula for the Skorohod integral stated in Theorem 3.2 for the function $|x|^p$ which leads to the inequality (3.5). From this inequality, the exponential estimates given in (H7) and (H8) and the equality provided by Lemma 3.1 allow us to deduce the inequality (3.8). Finally we conclude the proof integrating with respect to the measure $\mathrm{e}^{-M|x|}\,dx$ and using a Gronwall-like procedure.

First we show by means of an approximation argument that we can reduce the proof to the case of a simple process. Let us denote by $\mathcal{S}^a$ the space of simple and adapted processes of the form

$$\phi(s,y)=\sum_{i,j=0}^{m-1} F_{ij}\,\mathbb{1}_{(t_i,t_{i+1}]}(s)h_j(y),$$

where $0=t_0<t_1<\cdots<t_m=T$, $h_j\in C_K^\infty(\mathbb{R})$ and the F_{ij} are $\mathcal{F}_{0,t_i}$-measurable functions in $\mathcal{S}$. Let ϕ be an adapted process in $L_M^p(I^T\times\Omega)$. We can find a sequence ϕ^n of processes in $\mathcal{S}^a$ such that

$$\lim_{n\to\infty}\int_0^T\left(\int_{\mathbb{R}} \mathrm{e}^{-M|y|}E\left|\phi^n(s,y)-\phi(s,y)\right|^p dy\right)ds=0.$$

We can easily check that this implies the existence of a subsequence n_k such that for almost all $t\in[0,T]$

$$\lim_{k\to\infty}\int_0^t (t-s)^{-\alpha-\frac{1}{4}-\frac{1}{p}}\left(\int_{\mathbb{R}} \mathrm{e}^{-M|y|}E\left|\phi^{n_k}(s,y)-\phi(s,y)\right|^p dy\right)ds=0.$$

On the other hand, using the fact that $\alpha<\frac{1}{4}$ and hypothesis (H7) we have that

$$\begin{aligned}
A:=&\lim_{k\to\infty}E\int_0^T\left(\int_{\mathbb{R}}\mathrm{e}^{-M|x|}\left(\int_{I^t}(t-s)^{-2\alpha}p^2(s,t,y,x)\right.\right.\\
&\left.\left.\times\left|\phi^{n_k}(s,y)-\phi(s,y)\right|^2 ds\,dy\right)dx\right)dt\\
\leqslant{}& C_{1,2}^2\lim_{k\to\infty}\int_0^T\left(\int_{\mathbb{R}}\mathrm{e}^{-M|x|}\left(\int_{I^t}(t-s)^{-2\alpha-1}\exp\left(-\frac{2|x-y|^2}{\delta_2(t-s)}\right)\right.\right.\\
&\left.\left.\times E\left|\phi^{n_k}(s,y)-\phi(s,y)\right|^2 ds\,dy\right)dx\right)dt\\
={}& C_{1,2}^2\lim_{k\to\infty}\int_{I^T}E\left|\phi^{n_k}(s,y)-\phi(s,y)\right|^2\left(\int_{I_s^T}(t-s)^{-2\alpha-1}\right.\\
&\left.\times\exp\left(-M|x|-\frac{2|x-y|^2}{\delta_2(t-s)}\right)dt\,dx\right)ds\,dy.
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}} \exp\left(-M|x| - \frac{2|x-y|^2}{\delta_2(t-s)}\right) dx$$

$$= \int_{\mathbb{R}} \exp\left(-M|x+y| - \frac{2x^2}{\delta_2(t-s)}\right) dx$$

$$\leqslant \mathrm{e}^{-M|y|} \int_{\mathbb{R}} \exp\left(M|x| - \frac{2x^2}{\delta_2(t-s)}\right) dx \leqslant K_1\sqrt{t-s}\,\mathrm{e}^{-M|y|},$$

where $K_1 = \sqrt{2\pi\delta_2}\,\mathrm{e}^{M^2\delta_2 T/8}$. Then

$$A \leqslant C_{1,2}^2 K_1 \lim_{k\to\infty} \int_{I^T} \mathrm{e}^{-M|y|} E\left|\phi^{n_k}(s,y) - \phi(s,y)\right|^2 ds\, dy = 0.$$

Then, choosing a further subsequence (denoted again by n_k) we have that for almost all $(t,x) \in I^T$

$$\lim_{k\to\infty} E \int_{I^t} (t-s)^{-2\alpha} p^2(s,t,y,x)\left|\phi^{n_k}(s,y) - \phi(s,y)\right|^2 ds\, dy = 0.$$

This allows us to suppose that $\phi \in \mathcal{S}^a$. Fix $t_0 > t_1$ in $[0,T]$ and define

$$B_x(s,y) = (t_0 - s)^{-\alpha} p(s,t_1,y,x)\phi(s,y),$$
$$X(t,x) = \int_{I^t} B_x(s,y)\, dW_{s,y}, \quad t \in [0,t_1].$$

Denote $F(x) = |x|^p$. Let F_N be the increasing sequence of functions defined by

$$F_N(x) = \int_0^{|x|}\int_0^{y} \left(p(p-1)z^{p-2} \wedge N\right) dz\, dy.$$

Suppose first that $p(s,t_1,y,x)$ is an elementary backward-adapted process of the form

$$\sum_{i,j,k=1}^{n} H_{ijk}\beta_j(y)\gamma_k(z)\mathbb{1}_{(s_i,s_{i+1}]}(s),$$

where $H_{ijk} \in \mathcal{S}$, $\beta_j, \gamma_k \in C_K^\infty(\mathbb{R})$, $0 = s_1 < \cdots < s_{n+1} = t_1$, and H_{ijk} is $\mathcal{F}_{s_{i+1},t_1}$-measurable. Then we can apply Itô's formula (see Theorem 2.3) for the function F_N and the process $B_x(s,y)$, obtaining that for all $t < t_1$,

$$E\big(F_N(X(t,x))\big) = \frac{1}{2} E \int_{I^t} F_N''\big(X(s,x)\big) B_x^2(s,y)\, ds\, dy + E \int_{I^t} F_N''\big(X(s,x)\big) B_x(s,y) \bigg(\int_{I^s} D_{s,y} B_x(r,z)\, dW_{r,z} \bigg)\, ds\, dy. \tag{3.4}$$

Using hypotheses (H1), (H7) and (H8), Lemma 3.1 and the fact that ϕ is simple and adapted we can easily check that, for all $p(s,t,y,x)$ satisfying the hypotheses of the theorem, and for all $t < t_1$

$$\text{(i)} \quad E \int_{I^t} B_x^2(s,y)\, ds\, dy < \infty,$$

$$\text{(ii)} \quad E\bigg(\int_{I^t} B_x(s,y)\, dW_{s,y} \bigg)^2 < \infty,$$

$$\text{(iii)} \quad E \int_{I^t} \bigg| \int_{I^s} D_{s,y} B_x(r,z)\, dW_{r,z} \bigg|^2 ds\, dy < \infty.$$

This allows us to deduce that (3.4) still holds for every $p(s,t,y,x)$ satisfying hypotheses (H1) to (H8) and for all $t < t_1$. We can easily check that

$$F_N''(x) \leqslant \bigg(2^{1+\frac{1}{p}} + \frac{1}{p(p-1)} \bigg) \big(F_N(x)\big)^{(p-2)/p}.$$

Then we have that

$$E F_N\big(X(t,x)\big) \leqslant C_p \bigg\{ \frac{1}{2} E \int_{I^t} \big(F_N\big(X(s,x)\big)\big)^{(p-2)/p} B_x^2(s,y)\, ds\, dy + E \int_{I^t} \big(F_N\big(X(s,x)\big)\big)^{(p-2)/p} \bigg| \int_{\mathbb{R}} B_x(s,y) \times \bigg(\int_{I^s} D_{s,y} B_x(r,z)\, dW_{r,z} \bigg) dy \bigg| ds \bigg\}.$$

Hölder's inequality gives us that

$$
\begin{aligned}
&EF_N\big(X(t,x)\big)\\
&\quad\leqslant C_p\Bigg\{\int_0^t \frac{1}{2}\big(EF_N(X(s,x))\big)^{(p-2)/p}\bigg(E\bigg|\int_{\mathbb{R}} B_x^2(s,y)\,dy\bigg|^{p/2}\bigg)^{2/p} ds\\
&\qquad+\int_0^t \big(EF_N(X(s,x))\big)^{(p-2)/p}\\
&\qquad\times\bigg(E\bigg|\int_{\mathbb{R}} B_x(s,y)\bigg(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\bigg)dy\bigg|^{p/2}\bigg)^{2/p} ds\Bigg\}.
\end{aligned}
$$

Applying the lemma of [17, p. 171] we obtain that

$$
\begin{aligned}
&EF_N\big(X(t,x)\big)\\
&\quad\leqslant C_p\Bigg\{\int_0^t \frac{1}{2}\bigg(E\bigg|\int_{\mathbb{R}} B_x^2(s,y)\,dy\bigg|^{p/2}\bigg)^{2/p} ds\\
&\qquad+\int_0^t \bigg(E\bigg|\int_{\mathbb{R}} B_x(s,y)\bigg(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\bigg)dy\bigg|^{p/2}\bigg)^{2/p} ds\Bigg\}^{p/2}.
\end{aligned}
$$

Fatou's lemma gives us that, letting N tend to infinity

$$
\begin{aligned}
&E\big|X(t,x)\big|^p\\
&\quad\leqslant C_p\Bigg\{\int_0^t \frac{1}{2}\bigg(E\bigg|\int_{\mathbb{R}} B_x^2(s,y)\,dy\bigg|^{p/2}\bigg)^{2/p} ds\\
&\qquad+\int_0^t \bigg(E\bigg|\int_{\mathbb{R}} B_x(s,y)\bigg(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\bigg)dy\bigg|^{p/2}\bigg)^{2/p} ds\Bigg\}^{p/2}\\
&\quad=: C_p\bigg(\frac{1}{2}I_1+I_2\bigg)^{p/2}.
\end{aligned}
\tag{3.5}
$$

The term I_1 can be estimated using (H7) and Hölder's inequality:

$$
\begin{aligned}
I_1&\leqslant \int_0^t (t-s)^{-2\alpha}\bigg(E\bigg|\int_{\mathbb{R}} p^2(s,t_1,y,x)\phi^2(s,y)\,dy\bigg|^{p/2}\bigg)^{2/p} ds\\
&\leqslant \int_0^t (t-s)^{-2\alpha}\bigg(E\bigg|\int_{\mathbb{R}} \exp\bigg(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\bigg)\\
&\qquad\times V_p^2(s,t_1,x)\phi^2(s,y)\,dy\bigg|^{p/2}\bigg)^{2/p} ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C_{1,2}^2 \int_0^t (t-s)^{-2\alpha-1}\left(E\left|\int_{\mathbb{R}} \exp\left(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\right)\phi^2(s,y)\,dy\right|^{p/2}\right)^{2/p} ds \\
&\leqslant C \int_0^t (t-s)^{-2\alpha-\frac{1}{2}-\frac{1}{p}} \\
&\quad \times \left(\int_{\mathbb{R}} \exp\left(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\right) E\left|\phi(s,y)\right|^p dy\right)^{2/p} ds. \qquad (3.6)
\end{aligned}
$$

On the other hand, using Lemma 3.1 and Proposition 2.1 yields

$$
\begin{aligned}
&\int_{\mathbb{R}} B_x(s,y)\left(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\right) dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\phi(s,y) \\
&\qquad \times \left(\int_{I^s} (t_0-r)^{-\alpha}\left[D_{s,y}p(r,t_1,z,x)\right]\phi(r,z)\,dW_{r,z}\right) dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\phi(s,y) \\
&\qquad \times \left(\int_{I^s} (t_0-r)^{-\alpha}\left[\int_{\mathbb{R}} p(r,s,z,u)D_{s,y}^{-}p(s,t_1,u,x)\,du\right]\right. \\
&\qquad \left.\times\, \phi(r,z)\,dW_{r,z}\right) dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\phi(s,y)\left[\int_{\mathbb{R}} D_{s,y}^{-}p(s,t_1,u,x)\right. \\
&\qquad \left.\times \left(\int_{I^s} (t_0-r)^{-\alpha} p(r,s,z,u)\phi(r,z)\,dW_{r,z}\right) du\right] dy.
\end{aligned}
$$

Let us denote $Y(s,u) := \int_{I^s} (t_0-r)^{-\alpha} p(r,s,z,u)\phi(r,z)\,dW_{r,z}$. Notice that $X(t_1,x) = Y(t_1,x)$. We have proved that

$$
\begin{aligned}
&\int_{\mathbb{R}} B_x(s,y)\left(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\right) dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\phi(s,y)\left[\int_{\mathbb{R}} D_{s,y}^{-}p(s,t_1,u,x)Y(s,u)\,du\right] dy,
\end{aligned}
$$

and then

$$I_2 \leqslant \int_0^t (t_0 - s)^{-\alpha} \left(E \left| \int_{\mathbb{R}} p(s, t_1, y, x)\phi(s, y) \times \left[\int_{\mathbb{R}} D^-_{s,y} p(s, t_1, u, x) Y(s, u)\, du \right] dy \right|^{p/2} \right)^{2/p} ds.$$

Applying the estimates given in (H7) and (H8) we obtain

$$\begin{aligned}
&E\left| \int_{\mathbb{R}} p(s, t_1, y, x)\phi(s, y) \left[\int_{\mathbb{R}} D^-_{s,y} p(s, t_1, u, x) Y(s, u)\, du \right] dy \right|^{p/2} \\
&= E\left| \int_{\mathbb{R}^2} p(s, t_1, y, x) D^-_{s,y} p(s, t_1, u, x)\phi(s, y) Y(s, u)\, du\, dy \right|^{p/2} \\
&\leqslant E\left| \int_{\mathbb{R}^2} V_p(s, t_1, x) U_p(s, t_1, x) f(u, y) \right. \\
&\quad \left. \times \exp\left(-\frac{|y - x|^2}{\delta_p(t_1 - s)} - \frac{|x - u|^2}{\gamma_p(t_1 - s)} \right) \left| \phi(s, y) Y(s, u) \right| du\, dy \right|^{p/2} \\
&\leqslant C|t_1 - s|^{-3p/4} E\left| \int_{\mathbb{R}^2} f(u, y) \exp\left(-\frac{|y - x|^2}{\delta_p(t_1 - s)} - \frac{|x - u|^2}{\gamma_p(t_1 - s)} \right) \right. \\
&\quad \left. \times \left| \phi(s, y) Y(s, u) \right| du\, dy \right|^{p/2}.
\end{aligned}$$

Then, Schwartz and Hölder's inequalities yield

$$\begin{aligned}
&E\left| \int_{\mathbb{R}} p(s, t_1, y, x)\phi(s, y) \left[\int_{\mathbb{R}} D^-_{s,y} p(s, t_1, u, x) Y(s, u)\, du \right] dy \right|^{p/2} \\
&\leqslant C(t_1 - s)^{-3p/4} E\left(\left| \int_{\mathbb{R}^2} Y^2(s, u) f^2(u, y)\, \mathrm{e}^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)}}\, du\, dy \right|^{p/4} \right. \\
&\quad \left. \times \left| \int_{\mathbb{R}^2} \phi^2(s, y)\, \mathrm{e}^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)} - \frac{2|y-x|^2}{\delta_p(t_1 - s)}}\, du\, dy \right|^{p/4} \right) \\
&\leqslant C(t_1 - s)^{-5p/8} E\left(\left| \int_{\mathbb{R}} Y^2(s, u)\, \mathrm{e}^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)}}\, du \right|^{p/4} \right. \\
&\quad \left. \times \left| \int_{\mathbb{R}} \phi^2(s, y)\, \mathrm{e}^{-\frac{2|y-x|^2}{\delta_p(t_1 - s)}}\, dy \right|^{p/4} \right)
\end{aligned}$$

$$\leqslant C(t_1-s)^{-5p/8}\biggl(E\biggl|\int_{\mathbb{R}} Y^2(s,u)\,\mathrm{e}^{-\frac{|x-u|^2}{\gamma_p(t_1-s)}}\,du\biggr|^{p/2} + E\biggl|\int_{\mathbb{R}} \phi^2(s,y)\,\mathrm{e}^{-\frac{2|y-x|^2}{\delta_p(t_1-s)}}\,dy\biggr|^{p/2}\biggr)$$

$$\leqslant C(t_1-s)^{-\frac{3p}{8}-\frac{1}{2}}E\biggl(\int_{\mathbb{R}} |Y(s,y)|^p\,\mathrm{e}^{-\frac{|x-y|^2}{\gamma_p(t_1-s)}}\,dy + \int_{\mathbb{R}} |\phi(s,y)|^p\,\mathrm{e}^{-\frac{2|x-y|^2}{\delta_p(t_1-s)}}\,dy\biggr).$$

As a consequence,

$$I_2 \leqslant C\int_0^t (t_0-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}}\biggl(\int_{\mathbb{R}} \exp\biggl(-\frac{|x-y|^2}{(\frac{\delta_p}{2}\vee\gamma_p)(t_1-s)}\biggr) \times E\bigl(|\phi(s,y)|^p+|Y(s,y)|^p\bigr)\,dy\biggr)^{2/p} ds. \tag{3.7}$$

Putting (3.6) and (3.7) into (3.5) and using the fact that $\alpha < (p-4)/(4p)$ we obtain that

$$E|X(t,x)|^p \leqslant C\int_0^t (t_0-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}}\biggl(\int_{\mathbb{R}} \exp\biggl(-\frac{|x-y|^2}{c(t_1-s)}\biggr) \times E\bigl(|\phi(s,y)|^p+|Y(s,y)|^p\bigr)\,dy\biggr)\,ds \tag{3.8}$$

where $c=\max(\delta_p,\gamma_p)$.

Now we make t tend to t_1 and use Fatou's lemma to obtain

$$E|Y(t_1,x)|^p \leqslant C\int_0^{t_1} (t_1-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}}\biggl(\int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{c(t_1-s)}}\,E\bigl(|\phi(s,y)|^p+|Y(s,y)|^p\bigr)\,dy\biggr)\,ds.$$

Using a Gronwall-like iterative procedure we have that

$$E|Y(t,x)|^p \leqslant C\int_0^t (t-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}}\biggl(\int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{c(t-s)}}\,E|\phi(s,y)|^p\,dy\biggr)\,ds,$$

for all $0 \leqslant t \leqslant t_0$. Finally for any fixed $t \in [0, T)$ letting the parameter t_0 in the definition of $Y(t, x)$ to converge to t and integrating with respect to the measure $e^{-M|x|}\,dx$ leads to the desired result. □

Let us now consider the following additional condition on the stochastic kernel $p(s, t, y, x)$:

$(\text{H9})_{\text{M}}$ There exists a constant $C_M > 0$ such that

$$\sup_{0 \leqslant r \leqslant T} E\left(\sup_{T \geqslant s \geqslant r} \int_{\mathbb{R}} e^{-M|x|} p(r, s, y, x)\,dx \right) \leqslant C_M\, e^{-M|y|}.$$

We will denote by $L_M^p(\mathbb{R})$ the space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} e^{-M|x|} |f(x)|^p\,dx < \infty$.

THEOREM 3.3. – *Fix $p > 8$ and $M > 0$. Let $\phi = \{\phi(s, y), (s, y) \in I^T\}$ be an adapted process in $L_M^p(I^T \times \Omega)$. Assume that $p(s, t, y, x)$ is a stochastic kernel satisfying conditions* (H1) *to* (H8) *and* $(\text{H9})_{\text{M}}$. *Then for all $t \in [0, T]$, the process $\{p(s, t, y, x)\phi(s, y)\mathbb{1}_{[0,t]}(s),\ (s, y) \in I^T\}$ belongs to* Dom δ *for almost all $x \in \mathbb{R}$, and the stochastic process*

$$Z = \left\{ Z_t = \int_{I^t} p(s, t, y, \cdot)\phi(s, y)\,dW_{s,y},\, t \in [0, T] \right\}$$

posseses a continuous version with values in $L_M^p(\mathbb{R})$. Moreover,

$$E\left(\sup_{0 \leqslant t \leqslant T} \int_{\mathbb{R}} e^{-M|x|} \left| \int_{I^t} p(s, t, y, x)\phi(s, y)\,dW_{s,y} \right|^p dx \right) \leqslant C \int_{I^T} e^{-M|y|} E\big|\phi(s, y)\big|^p\,ds\,dy, \tag{3.9}$$

for some positive constant C depending only on T, p, M, $C_{1,p}$, $C_{2,p}$, C_f, γ_p, δ_p and C_M.

Proof. – Using the same arguments as in the proof of Theorem 3.2 we can assume that the process ϕ is simple. Fix $0 < \alpha < (p - 4)/(4p)$ and define

$$Y(r, u) = \int_{I^r} (r - s)^{-\alpha} p(s, r, y, u)\phi(s, y)\,dW_{s,y}.$$

As

$$p(s,t,y,x) = C_\alpha \int\limits_{I_s^t} (t-r)^{\alpha-1}(r-s)^{-\alpha} p(s,r,y,u)p(r,t,u,x)\,dr\,du,$$

with $C_\alpha = \sin\pi\alpha/\pi$ it is easy to show that

$$Z_t(x) = C_\alpha \int\limits_{I^t} (t-r)^{\alpha-1} p(r,t,u,x) Y(r,u)\,dr\,du.$$

Then we have that for any $t < t'$ and $\alpha \in (\frac{1}{p}, \frac{p-4}{4p})$

$$\begin{aligned}
&\int\limits_{\mathbb{R}} e^{-M|x|} \big|Z_{t'}(x) - Z_t(x)\big|^p\,dx \\
&\quad\leqslant \int\limits_{\mathbb{R}} e^{-M|x|} \Bigg| \int\limits_{I_t^{t'}} (t-r)^{\alpha-1} p(r,t',u,x) Y(r,u)\,dr\,du \Bigg|^p dx \\
&\qquad + \int\limits_{\mathbb{R}} e^{-M|x|} \Bigg| \int\limits_{I_t} (t-r)^{\alpha-1} \big[p(r,t',u,x) - p(r,t,u,x)\big] \\
&\qquad \times Y(r,u)\,dr\,du \Bigg|^p dx \\
&\quad\leqslant C_{\alpha,p}(t'-t)^{\alpha-\frac{1}{p}} \int\limits_{I^{t'}} e^{-M|x|} \Bigg| \int\limits_{\mathbb{R}} p(r,t',u,x) Y(r,u)\,du \Bigg|^p dr\,dx \\
&\qquad + C_{\alpha,p} t^{\alpha-\frac{1}{p}} \int\limits_{I^t} e^{-M|x|} \Bigg| \int\limits_{\mathbb{R}} \big[p(r,t',u,x) - p(r,t,u,x)\big] \\
&\qquad \times Y(r,u)\,du \Bigg|^p dr\,dx \\
&\quad\leqslant C_{\alpha,p}(t'-t)^{\alpha-\frac{1}{p}} \int\limits_{I^T} \big|Y(r,u)\big|^p \Bigg(\sup_{T \geqslant s \geqslant r} \int\limits_{\mathbb{R}} e^{-M|x|} p(r,s,u,x)\,dx \Bigg) dr\,du \\
&\qquad + C \int\limits_{I^t} \Bigg(\int\limits_{\mathbb{R}} e^{-M|x|} \big|p(r,t',u,x) - p(r,t,u,x)\big|\,dx \Bigg) \big|Y(r,u)\big|^p\,du\,dr.
\end{aligned}$$

Both summands in the above expression converge to zero almost-surely as $|t'-t| \to 0$. In fact, by hypothesis $(\mathrm{H9})_{\mathrm{M}}$ and Theorem 3.2 the first summand can be written as $C_{\alpha,p}(t'-t)^{\alpha-(1/p)} Z$ where $E(Z) < \infty$.

For the second summand we apply the dominated convergence theorem and hypothesis (H4). On the other hand, taking $t=0$ we obtain (3.8). □

We will also need the following L^2-estimate for the Skorohod integral.

THEOREM 3.4. – *Fix $M>0$. Let $\phi=\{\phi(s,y),\ (s,y)\in I^T\}$ be an adapted random field in $L^2_M(I^T\times\Omega)$. Assume that $p(s,t,y,x)$ is a stochastic kernel satisfying conditions* (H1)–(H8). *Then, for almost all $(t,x)\in I^T$, the process*

$$\{p(s,t,y,x)\phi(s,y)\mathbb{1}_{[0,t]}(s),\ (s,y)\in I^T\}$$

belongs to Dom δ *and we have that*

$$\int_{\mathbb{R}} \mathrm{e}^{-M|x|} E\left|\int_{I^t} p(s,t,y,x)\phi(s,y)\,dW_{s,y}\right|^2 dx$$
$$\leqslant C\int_0^t (t-s)^{-3/4}\left(\int_{\mathbb{R}} \mathrm{e}^{-M|x|}E|\phi(s,y)|^2\,dy\right)ds, \tag{3.10}$$

for some positive constant C depending only on T, M, $C_{1,2}$, $C_{2,2}$, C_f, δ_2 and γ_2.

Proof. – Using the same arguments as in the proof of Theorem 3.2 we can assume that $\phi\in\mathcal{S}^a$. Fix $(t,x)\in I^T$ and define

$$B_{t,x}(s,y)=p(s,t,y,x)\phi(s,y)\mathbb{1}_{[0,t]}(s),$$
$$X(t,x)=\int_{I^t} B_{t,x}(s,y)\,dW_{s,y}.$$

By the isometry properties of the Skorohod integral (Proposition 2.2) we have that

$$\int_{\mathbb{R}} \mathrm{e}^{-M|x|}E|X(t,x)|^2\,dx$$
$$=\int_{\mathbb{R}} \mathrm{e}^{-M|x|}\left(\int_{I^t} E|B_{t,x}(s,y)|^2\,ds\,dy\right)dx$$
$$+2\int_{\mathbb{R}} \mathrm{e}^{-M|x|}E\left[\int_{I^t} B_{t,x}(s,y)\left(\int_{I^s} D_{s,y}B_{t,x}(r,z)\,dW_{r,z}\right)ds\,dy\right]dx$$
$$=I_1+2I_2. \tag{3.11}$$

By hypothesis (H7) we have that

$$I_1 \leqslant \int_{\mathbb{R}} \mathrm{e}^{-M|x|} \Big(\int_{I^t} E\big|V_2(s,t,x)\big|^2 \times \exp\Big(-\frac{2|x-y|^2}{\delta_2(t-s)}\Big) E\big|\phi(s,y)\big|^2 \, ds\, dy \Big) dx$$

$$\leqslant C_{1,2}^2 \int_{I^t} (t-s)^{-1} E\big|\phi(s,y)\big|^2 \times \Big(\int_{\mathbb{R}} \exp\Big(-M|x| - \frac{2|x-y|^2}{\delta_2(t-s)}\Big) dx \Big) ds\, dy$$

$$\leqslant C_{1,2}^2 K_1 \int_{I^t} (t-s)^{-1/2} \mathrm{e}^{-M|y|} E\big|\phi(s,y)\big|^2 \, ds\, dy. \tag{3.12}$$

On the other hand, using the same arguments as in the proof of Theorem 3.2 it is easy to show that

$$I_2 = E \int_0^t \int_{\mathbb{R}^2} \mathrm{e}^{-M|x|} p(s,t,y,x) \phi(s,y) \times \Big(\int_{\mathbb{R}} D_{s,y}^{-} p(s,t,u,x) X(s,u)\, du \Big) dx\, dy\, ds$$

$$= E \int_0^t \int_{\mathbb{R}^3} \mathrm{e}^{-M|x|} p(s,t,y,x) D_{s,y}^{-} p(s,t,u,x) \phi(s,y) X(s,u)\, dx\, dy\, du\, ds$$

$$\leqslant \int_{I^t} \mathrm{e}^{-M|x|} \big(E\big|V_2(s,t,x)\big|^2\big)^{1/2} \big(E\big|U_2(s,t,x)\big|^2\big)^{1/2} \times \Big[\int_{\mathbb{R}^2} \exp\Big(-\frac{|x-y|^2}{\delta_2(t-s)} - \frac{|x-u|^2}{\gamma_2(t-s)}\Big) f(u,y) \times E\big|\phi(s,y) X(s,u)\big| \, dy\, du \Big] ds\, dx$$

$$\leqslant C_{1,2} C_{2,2} \int_{I^t} \mathrm{e}^{-M|x|} (t-s)^{-3/2} \times \Big(\int_{\mathbb{R}^2} E\big|X(s,u)\big|^2 f(u,y)^2 \mathrm{e}^{-\frac{|x-u|^2}{\gamma_2(t-s)}} \, du\, dy \Big)^{1/2}$$

$$
\times \left(\int_{\mathbb{R}^2} E\left|\phi(s,y)\right|^2 \mathrm{e}^{-\frac{|x-u|^2}{\gamma_2(t-s)}-\frac{2|x-y|^2}{\delta_2(t-s)}}\,du\,dy\right)^{1/2} ds\,dx
$$

$$
\leqslant C \int_{I^t} \mathrm{e}^{-M|x|}(t-s)^{-5/4}\left(\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{|x-y|^2}{(\frac{\delta}{2}\vee\gamma_2)(t-s)}}
$$

$$
\times \left(E\left|X(s,y)\right|^2 + E\left|\phi(s,y)\right|^2\right) dy\right) ds\,dx
$$

$$
\leqslant C \int_0^t (t-s)^{-3/4}\left(\int_{\mathbb{R}} \mathrm{e}^{-M|y|}\left(E\left|X(s,y)\right|^2\right.\right.
$$

$$
\left.\left. + E\left|\phi(s,y)\right|^2\right) dy\right) ds. \tag{3.13}
$$

Now, substituting (3.13) and (3.12) into (3.11) and using an iteration argument the result follows. □

Using the same arguments it is easy to show the following result.

COROLLARY 3.5. – *Let $\phi = \{\phi(s,y),\ (s,y) \in I^T\}$ be an adapted process in $L^2(I^T \times \Omega)$. Assume that $p(s,t,y,x)$ is a random function satisfying hypotheses* (H1)–(H8). *Then, for almost all $(t,x) \in I^T$, the process*

$$
\{p(s,t,y,x)\phi(s,y)\mathbb{1}_{[0,t]}(s),\ (s,y) \in I^T\}
$$

belongs to Dom δ *and*

$$
\int_{\mathbb{R}} E\left| \int_{I^t} p(s,t,y,x)\phi(s,y)\,dW_{s,y}\right|^2 dx
$$

$$
\leqslant C \int_0^t (t-s)^{-3/4}\left(\int_{\mathbb{R}} E\left|\phi(s,y)\right|^2 dy\right) ds, \tag{3.14}
$$

for some positive constant C depending only on T, $C_{1,2}$, $C_{2,2}$, C_f, δ_2 and γ_2.

4. EXISTENCE AND UNIQUENESS OF SOLUTION FOR STOCHASTIC EVOLUTION EQUATIONS WITH A RANDOM KERNEL

Our purpose in this section is to prove the existence and uniqueness of solution for the following anticipating stochastic evolution equation

$$u(t,x)=\int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy + \int_{\mathbb{R}} p(s,t,y,x)F\big(s,y,u(s,y)\big)\,dW_{s,y}, \tag{4.1}$$

where $p(s,t,y,x)$ is a stochastic kernel satisfying conditions (H1) to (H8) and $(\mathrm{H9})_{\mathrm{M}}$, $u_0:\mathbb{R}\to\mathbb{R}$ is the initial condition and $F:[0,T]\times\mathbb{R}^2\times\Omega\to\mathbb{R}$ is a random field. Let us consider the following hypotheses.

(F1) F is measurable with respect to the σ-field $\mathcal{B}([0,t]\times\mathbb{R}^2)\otimes\mathcal{F}_{0,t}$, when restricted to $[0,t]\times\mathbb{R}^2\times\Omega$, for each $t\in[0,T]$.

(F2) For all $t\in[0,T]$, $x,y,z\in\mathbb{R}$

$$\big|F(t,y,x)-F(t,y,z)\big|\leqslant C|x-z|,$$

for some positive nonrandom constant C.

$(\mathrm{F3})^{\mathrm{p}}_{\mathrm{M}}$ For all $t\in[0,T]$, $x\in\mathbb{R}$,

$$\big|F(t,x,0)\big|\leqslant h(x),$$

for some $h\in L^p_M(\mathbb{R})$.

We are now in a position to prove the main result of this paper.

THEOREM 4.1. – *Fix $M>0$ and $p>8$. Let u_0 be a function in $L^p_M(\mathbb{R})$. Consider an adapted random field $F(s,y,x)$ satisfying conditions* (F1) *to* $(\mathrm{F3})^{\mathrm{p}}_{\mathrm{M}}$ *and a stochastic kernel $p(s,t,y,x)$ satisfying hypotheses* (H1) *to* (H8) *and* $(\mathrm{H9})_{\mathrm{M}}$. *Then, there exists a unique adapted random field $u=\{u(t,x),\ (t,x)\in I^T\}$ in $L^2_M(I^T\times\Omega)$ that is solution of* (4.1). *Moreover,*

(i) *$\{u(t,\cdot),\ t\in[0,T]\}$ is continuous a.s. as a process with values in $L^p_M(\mathbb{R})$ and*

$$E\bigg(\sup_{0\leqslant t\leqslant T}\int_R \mathrm{e}^{-M|x|}\big|u(t,x)\big|^p\,dx\bigg)\leqslant C, \tag{4.2}$$

for some positive constant C depending only on T, p, M, $C_{1,p}$, $C_{2,p}$, C_f, δ_p, γ_p and C_M.

(ii) *If, moreover, u_0 and h belong to $L^2(\mathbb{R})$, then $u\in L^2(I^T\times\Omega)$.*

Proof of existence and uniqueness. – Suppose that u and v are two adapted solutions of (4.1) in $L^2_M(I^T\times\Omega)$, for some $M>0$. Then, for every $t\in[0,T]$ we can write

$$\int_{\mathbb{R}} e^{-M|x|} E\left|u(t,x)-v(t,x)\right|^2 dx$$
$$= \int_{\mathbb{R}} e^{-M|x|} E\left|\int_{I^t} p(s,t,y,x)\big(F(s,y,u(s,y)) - F(s,y,v(s,y))\big)\,dW_{s,y}\right|^2 dx.$$

By Theorem 3.4 and the Lipschitz condition on F we have that

$$\int_{\mathbb{R}} e^{-M|x|} E\left|u(t,x)-v(t,x)\right|^2 dx \leqslant \int_0^t (t-s)^{-3/4}\left(\int_{\mathbb{R}} e^{-M|y|} E\left|u(s,y)-v(s,y)\right|^2 dy\right) ds.$$

Appliying an iteration argument we obtain that

$$\int_{\mathbb{R}} e^{-M|x|} E\left|u(t,x)-v(t,x)\right|^2 dx \leqslant C\int_0^t \left(\int_{\mathbb{R}} e^{-M|y|} E\left|u(s,y)-v(s,y)\right|^2 dy\right) ds,$$

from which we deduce that $\int_{\mathbb{R}} e^{-M|x|} E|u(t,x)-v(t,x)|^2\,dx = 0$. Consider now the Picard approximations

$$\begin{cases} u^0(t,x) = \displaystyle\int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy, \\ u^n(t,x) = \displaystyle\int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy \\ \qquad\qquad + \displaystyle\int_{I^t} p(s,t,y,x)F(s,y,u^{n-1}(s,y))\,dW_{s,y}. \end{cases}$$

By hypothesis (H1), $u^0(t,x)$ is adapted. On the other hand, using hypotheses (H3) and $(\mathrm{H9})_{\mathrm{M}}$ we have that

$$E\left(\int_{\mathbb{R}} e^{-M|x|}\left|\int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy\right|^2 dx\right)$$
$$\leqslant E\left(\int_{\mathbb{R}} |u_0(y)|^2\left(\int_{\mathbb{R}} e^{-M|x|} p(0,t,y,x)\,dx\right) dy\right)$$

$$\leqslant \int_{\mathbb{R}} e^{-M|y|} |u_0(y)|^2 \, dy.$$

Now, using induction on n and Theorem 3.4 it is easy to show that u^n is adapted and belongs to $L^2_M(I^T \times \Omega)$. Using an inductive argument we can easily show that

$$\sum_{n=0}^{\infty} E\left(\int_{\mathbb{R}} e^{-M|x|} |u^{n+1}(t,x) - u^n(t,x)|^2 \, dx \right) < \infty,$$

and the limit u of the sequence u^n provides the solution.

Proof of (i). – Using the same arguments as in the proof of the existence we can see that the solution u belongs to $L^p_M(I^T \times \Omega)$. Now we have to show that the following two terms are a.s. continuous in $L^p_M(\mathbb{R})$:

$$A_1(t) = \int_{\mathbb{R}} p(0,t,y,x) u_0(y) \, dy,$$

$$A_2(t) = \int_{I^t} p(s,t,y,x) F\big(s,y,u(s,y)\big) \, dW_{s,y}.$$

In order to prove the continuity of A_1, note that hypothesis $(\mathrm{H9})_{\mathrm{M}}$ implies that, for all φ and ϕ in $L^p_M(\mathbb{R})$

$$E\left(\sup_{0 \leqslant t \leqslant T} \int_{\mathbb{R}} e^{-M|x|} \left| \int_{\mathbb{R}} p(0,t,y,x) \big(\varphi(y) - \phi(y)\big) \, dy \right|^p dx \right)$$

$$\leqslant E\left(\sup_{0 \leqslant t \leqslant T} \int_{\mathbb{R}} |\varphi(y) - \phi(y)|^p \left(\int_{\mathbb{R}} e^{-M|x|} p(0,t,y,x) \, dx \right) dy \right)$$

$$\leqslant \int_{\mathbb{R}} e^{-M|y|} |\varphi(y) - \phi(y)|^p \, dy.$$

Hence, we can assume that u_0 is a smooth function with compact support. In this case

$$\int_{\mathbb{R}} e^{-M|x|} \left| \int_{\mathbb{R}} \big(p(0,t+\varepsilon,y,x) - p(0,t,y,x)\big) u_0(y) \, dy \right|^p dx$$

$$\leqslant 2^{p-1} \|u_0\|_\infty \int_{\mathbb{R} \times K} e^{-M|x|} \big|p(0,t+\varepsilon,y,x) - p(0,t,y,x)\big| \, dx \, dy,$$

which tends to zero almost-surely for all t by hypotheses (H4) and $(\mathrm{H9})_{\mathrm{M}}$, as in the proof of Theorem 3.3. The continuity of A_2 is an immediate consequence of Theorem 3.3. Finally, using a recurrence argument it is easy to prove that the Picard aproximations u^n satisfy that

$$\sum_{n=0}^{\infty} E\left(\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}} e^{-M|x|} \left|u^{n+1}(t,x) - u^n(t,x)\right|^p dx \right) < \infty,$$

from where (4.2) follows.

Proof of existence in $L^2(I^T \times \Omega)$. – Using hypothesis (H7) we have that

$$\begin{aligned} E \int_{I^T} \left| \int_{\mathbb{R}} p(0,t,y,x) u_0(y)\, dy \right|^2 dt\, dx \\ &\leqslant E \int_{I^T} \left|u_0(y)\right|^2 \left(\int_{\mathbb{R}} p(0,t,y,x)\, dx \right) dt\, dy \\ &\leqslant T \int_{\mathbb{R}} \left|u_0(y)\right|^2 dy. \end{aligned}$$

Using now induction on n and Corollary 3.5 it is easy to show that $\int_{I^T} E|u^n(t,x)|^2\, dt\, dx < \infty$ and that u^n is a Cauchy sequence in $L^2(I^T \times \Omega)$. This implies that u belongs to $L^2(I^T \times \Omega)$. □

For every $p \geqslant 1$, $0 < \varepsilon \leqslant p$ and $K > 0$ we denote by $W^{p,\varepsilon}(K)$ the set of continuous functions $f : [-K, K] \to \mathbb{R}$ such that

$$\|f\|^p_{p,\varepsilon,K} := \int_{[-K,K]^2} \frac{|f(x) - f(z)|^p}{|x - z|^{2+\varepsilon}}\, dx\, dz < \infty.$$

Notice that if $f \in W^{p,\varepsilon}(K)$, then f is Hölder continuous in $[-K, K]$ with order ε/p.

Now our purpose is to prove that, under some suitable hypotheses, the solution $u(t,\cdot)$ belongs to $W^{p,\varepsilon}(K)$, for some $p \geqslant 1$, $0 < \varepsilon \leqslant p$ and all $K > 0$.

THEOREM 4.2. – *Fix $p > 4$ and $M > 0$. Let u_0 be a function in $L^p_M(\mathbb{R})$. Consider an adapted random field $F(s,y,x)$ satisfying hypotheses* (F1) *to* $(\mathrm{F3})^{\mathrm{p}}_{\mathrm{M}}$ *and a stochastic kernel $p(s,t,y,x)$ satisfying* (H1) *to* (H8) *and* $(\mathrm{H9})_{\mathrm{M}}$. *Then, the solution $u(t,x)$ constructed in Theorem* 4.1 *belongs a.s., as a function in x, to $W^{p,\varepsilon}(K)$, for all $p > 8$, $\varepsilon < \frac{p}{2} - 3$ and $K > 0$.*

Proof. – We have to show that the following two terms belong to $W^{p,\varepsilon}(K)$:

$$B_1(x) = \int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy,$$

$$B_2(x) = \int_{I^t} p(s,t,y,x)F\big(s,y,u(s,y)\big)\,dW_{s,y}.$$

Using Minkowski's inequality we have that

$$\begin{aligned}
&E\int_{[-K,K]^2} \frac{|B_1(x)-B_1(z)|^p}{|x-z|^{2+\varepsilon}}\,dx\,dz \\
&= \int_{[-K,K]^2} |x-z|^{-2-\varepsilon} \\
&\quad \times E\left|\int_{\mathbb{R}} \big[p(0,t,y,x)-p(0,t,y,z)\big]u_0(y)\,dy\right|^p dx\,dz \\
&\leqslant \int_{[-K,K]^2} |x-z|^{-2-\varepsilon} \\
&\quad \times \left(\int_{\mathbb{R}} \big\|p(0,t,y,x)-p(0,t,y,z)\big\|_p |u_0(y)|\,dy\right)^p dx\,dz.
\end{aligned}$$

Taking into account estimate (5.10) and the same arguments as in [4, p. 17] it is easy to show that $\forall\, 0\leqslant s<t\leqslant T,\ x,y,z\in\mathbb{R},\ \beta\in[0,1]$ and $p\geqslant 1$,

$$\begin{aligned}
&\big\|p(s,t,y,x)-p(s,t,y,z)\big\|_p \\
&\quad \leqslant K|x-z|^{\beta}(t-s)^{-\frac{1}{2}(\beta+1)} \\
&\qquad \times \left[\exp\left(-\frac{|y-x|^2}{c(t-s)}\right)+\exp\left(-\frac{|y-z|^2}{c(t-s)}\right)\right],
\end{aligned} \tag{4.3}$$

for some $K, c>0$. This gives us that, taking $\beta=1$

$$\begin{aligned}
&E\int_{[-K,K]^2} \frac{|B_1(x)-B_1(z)|^p}{|x-z|^{2+\varepsilon}}\,dx\,dz \\
&\leqslant Ct^{-p} \int_{[-K,K]^2} |x-z|^{p-2-\varepsilon}
\end{aligned}$$

$$\times \left(\int_{\mathbb{R}} \exp\left(-\frac{|y-x|^2}{ct} \right) |u_0(y)|\, dy \right)^p dx\, dz$$

$$\leqslant Ct^{-p} \int_{[-K,K]} \left(\int_{\mathbb{R}} \exp\left(-\frac{|y-x|^2}{ct} \right) |u_0(y)|\, dy \right)^p dx$$

$$\leqslant Ct^{-\frac{p}{2}-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-M|y|} |u_0(y)|^p\, dy < \infty,$$

which gives us that $B_1(x)$ belongs a.s. to $W^{p,\varepsilon}(K)$. On the other hand, as in the proof of Theorem 3.3 we can write for $\alpha \in (0, \frac{p-4}{4p})$

$$B_2(x) = C_\alpha \int_{I^t} (t-r)^{\alpha-1} p(r,t,u,x) Y(r,u)\, dr\, du,$$

where

$$Y(r,u) := \int_{I^r} (r-s)^{-\alpha} p(s,r,y,u) F\big(s,y,u(s,y)\big)\, dW_{s,y}.$$

This gives us that

$$E \int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\, dx\, dz = CE \int_{[-K,K]^2} |x-z|^{-2-\varepsilon}$$

$$\times \left| \int_{I^t} (t-r)^{\alpha-1} \big[p(r,t,u,x) - p(r,t,u,z) \big] Y(r,u)\, dr\, du \right|^p dx\, dz.$$

Using Minkowski's inequality and the estimate (4.3) we obtain that

$$E \int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\, dx\, dz$$

$$\leqslant C \int_{[-K,K]^2} |x-z|^{\beta p-2-\varepsilon} \left(\int_{I^t} (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}} \exp\left(-\frac{|u-x|^2}{c(t-r)} \right) \right.$$

$$\left. \times \big\| Y(r,u) \big\|_p\, dr\, du \right)^p dx\, dz$$

$$\leqslant C \int_{[-K,K]} \left[\int_0^t (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}} \left(\int_{\mathbb{R}} \exp\left(-\frac{|u-x|^2}{c(t-r)} \right) \right. \right.$$

$$\times \left\|Y(r,u)\right\|_p du\Big) dr\Bigg]^p dx$$

$$= C\left[\int_0^t (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}}\left(\int_{[-K,K]}\left(\int_{\mathbb{R}} \exp\left(-\frac{|u-x|^2}{c(t-r)}\right)\right.\right.\right.$$

$$\left.\left.\left.\times \left\|Y(r,u)\right\|_p du\right)^p dx\right)^{1/p} dr\right]^p.$$

Using Hölder's inequality we obtain

$$E\int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\,dx\,dz$$

$$\leqslant C\left(\int_0^t (t-r)^{\alpha-1-\frac{\beta}{2}}\left(\int_{\mathbb{R}} \mathrm{e}^{-M|u|}E\left|Y(r,u)\right|^p du\right)^{1/p} dr\right)^p$$

$$\leqslant Ct^{p\alpha-\beta\frac{p}{2}-1}\int_{I^t} \mathrm{e}^{-M|u|}E\left|Y(r,u)\right|^p dr\,du,$$

provided $\alpha > \frac{1}{p} + \frac{\beta}{2}$. Finally, from the proof of Theorem 3.2 and the facts that $u_0 \in L^p(\mathbb{R})$ and $\alpha < (p-4)/(4p)$ it is easy to show that $\int_{I^t} \mathrm{e}^{-M|u|}E|Y(r,u)|^p\,dr\,du < \infty$, which allows us to complete the proof. We have made use of the following conditions

$$p\beta > \varepsilon + 1, \qquad \alpha > \frac{1}{p} + \frac{\beta}{2}, \qquad \alpha < \frac{p-4}{4p}.$$

We can easily check that thanks to the fact that $p > 8$ we can take α and β such that these inequalities hold. □

5. ESTIMATES FOR THE HEAT KERNEL WITH WHITE-NOISE DRIFT

In this section, following the approach of [13] we construct and estimate the heat kernel of the random operator $\frac{d^2}{dx^2} + \dot{v}(t,x)\,\frac{d}{dx}$, where $v = \{v(t,x),\ t \in [0,T],\ x \in \mathbb{R})\}$ is a zero mean Gaussian field which is Brownian in time. The differential $\dot{v}(t,x)\,dt := v(dt,x)$ is interpreted in the backward Itô sense. More precisely, we assume that v can be represented as

$$v(t,x)=\int_{I^t} g(x,y)\,dW_{s,y}, \tag{5.1}$$

where $g:\mathbb{R}^2\to\mathbb{R}$ is a measurable function, differentiable with respect to x, satisfying the following condition

$$\sup_x \int_{\mathbb{R}} \left(g(x,y)^2+\frac{\partial g}{\partial x}(x,y)^2 \right) dy<\infty. \tag{5.2}$$

Set $G(x,y)=\int_{\mathbb{R}} g(x,z)g(y,z)\,dz$ and let us introduce the following *coercivity* condition:

(C1) $\Sigma(x):=1-\frac{1}{2}G(x,x)\geqslant\varepsilon>0$, for all $x\in\mathbb{R}$ and for some $\varepsilon>0$.

Let $b=\{b(t),\ t\in[0,T]\}$ be a Brownian motion with variance $2t$ defined on another probability space $(\mathcal{W},\mathcal{G},Q)$. Consider the following *backward stochastic differential equation* on the product probability space $(\Omega\times\mathcal{W},\ \mathcal{F}\otimes\mathcal{G},\ P\times Q)$:

$$\varphi_{t,s}(x)=x-\int_s^t\int_{\mathbb{R}} g\big(\varphi_{t,r}(x),y\big)\,dW_{r,y}+\int_s^t\sqrt{\Sigma\big(\varphi_{t,r}(x)\big)}\,db_r. \tag{5.3}$$

Thanks to condition (5.2), applying Theorems 3.4.1 and 4.5.1 in [7], one can prove that (5.3) has a solution $\varphi=\{\varphi_{t,s}(x),\ 0\leqslant s\leqslant t\leqslant T,\ x\in\mathbb{R}\}$ continuous in the three variables and verifying

$$\varphi_{r,s}\big(\varphi_{t,r}(x)\big)=\varphi_{t,s}(x), \tag{5.4}$$

for all $s<r<t,\ x\in\mathbb{R}$.

Then we have the following result

PROPOSITION 5.1. – *Let v be a Gaussian random field of the form* (5.1) *where the function g satisfies the coercivity condition* (C1) *and assume that g is three times continuously differentiable in x and satisfies*

$$\sup_x \sum_{k=0}^{3}\int_{\mathbb{R}} \left|g^{(k)}(x,y)\right|^2 dy<\infty.$$

Then there is a version of the density

$$p(s,t,y,x)=\frac{Q(\varphi_{t,s}(x)\in dy)}{dy}$$

which satisfies conditions (H1) *to* (H8) *and* $(\mathrm{H9})_{\mathrm{M}}$ *for each $M>0$.*

Proof. – Let us denote by δ^b and D^b the divergence and derivative operators with respect to the Brownian motion b. Applying the integration-by-parts formula of Malliavin calculus with respect to the Brownian motion b we obtain

$$p(s,t,y,x) = E_Q\big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}} H_{t,s}(x)\big), \tag{5.5}$$

where

$$H_{t,s}(x) = \delta^b\left(\frac{D^b\varphi_{t,s}(x)}{\|D^b\varphi_{t,s}(x)\|^2}\right).$$

Hypothesis (H1) follows easily from the expression (5.5) because $\varphi_{t,s}(x)$ is $\mathcal{F}_{s,t}\otimes\mathcal{G}$-measurable. The fact that $y \mapsto p(s,t,y,x)$ is the probability density of $\varphi_{t,s}(x)$, which has a continuous version in all the variables $x, y \in \mathbb{R}$, $0 \leqslant s < t \leqslant T$, imply (H2), (H3) and (H4). Hypothesis (H5) is a consequence of the flow property (5.4).

Applying the derivative operator to (5.5) yields

$$\begin{aligned} &D_{r,z}\, p(s,t,y,x) \\ &\quad = E_Q\big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}} D_{r,z}H_{t,s}(x)\big) + E_Q\big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}} \Psi_{t,s}(x)\big), \end{aligned} \tag{5.6}$$

where

$$\Psi_{t,s}(x) = \delta^b\left(\frac{D^b\varphi_{t,s}(x)}{\|D^b\varphi_{t,s}(x)\|^2}\, D_{r,z}\,\varphi_{t,s}(x) H_{t,s}(x)\right).$$

Then hypothesis (H6) follows easily from Eq. (5.6). Conditions (H7), (H8) and $(\mathrm{H9})_{\mathrm{M}}$ will be proved in the following lemmas. □

LEMMA 5.2. – *The stochastic kernel $p(s,t,y,x)$ satisfies condition* (H7) *with the constant $\delta_p = p/K$, for any $K < 1/8$.*

Proof. – By (5.5) the kernel p can be expressed as

$$\begin{aligned} p(s,t,y,x) &= E_Q\big(\mathbb{1}_{\{B_{t,s}(x)>y-x\}} H_{t,s}(x)\big) &(5.7)\\ &= -E_Q\big(\mathbb{1}_{\{-B_{t,s}>x-y\}} H_{t,s}(x)\big), &(5.8) \end{aligned}$$

where $B_{t,s}(x) = \varphi_{t,s}(x) - x$. Since B and $-B$ have the same distribution, it is sufficient to consider the expression in (5.7) and assume that $x \leqslant y$. Using the trivial bound

$$\mathbb{1}_{\{B>a\}} \leqslant \exp\left(\frac{(KB^2)}{p(t-s)}\right)\exp\left(-\frac{(Ka^2)}{p(t-s)}\right)$$

for any $a \geqslant 0$, $K > 0$ we obtain

$$p(s,t,y,x) \leqslant \mathrm{e}^{-\frac{K|x-y|^2}{p(t-s)}} V_p(s,t,x),$$

where

$$V_p(s,t,x) = E_Q\left(\exp\left(\frac{(KB_{t,s}(x)^2)}{p(t-s)}\right)\left|H_{t,s}(x)\right|\right).$$

We only need to calculate $E|V_p(s,t,x)|^p$. By Schwartz's inequality

$$E\left|V_p(s,t,x)\right|^p \leqslant \left(E\exp\left(\frac{(2KB_{t,s}(x)^2)}{(t-s)}\right)E\left|H_{t,s}(x)\right|^{2p}\right)^{1/2}.$$

Note that, if we fix t and let s vary, $B_{t,s}(x)$ becomes a backward martingale with quadratic variation

$$\begin{aligned}\left\langle B_{t,\cdot}(x)\right\rangle_s &= \int_s^t\int_{\mathbb{R}} g^2(\varphi_{t,r}(x),y)\,dy\,dr + 2\int_s^t \Sigma(\varphi_{t,r}(x))\,dr \\ &= 2(t-s).\end{aligned}$$

This gives us that $B_{t,\cdot}(x)$ is a Brownian motion with variance 2, and then, for any $K < 1/8$

$$E\exp\left(\frac{2K}{(t-s)}B_{t,s}(x)^2\right) = \frac{1}{\sqrt{1-8K}}. \tag{5.9}$$

On the other hand, it is known (see [13], proof of Proposition 10, (5.5)) that

$$\left(E\left|H_{t,s}(x)\right|^{2p}\right)^{1/2} \leqslant C_p(t-s)^{-p/2},$$

and now the proof is complete. □

LEMMA 5.3. – *The stochastic kernel* $p(s,t,y,x)$ *satisfies condition* (H8) *with the constant* $\gamma_p = p/K$, *for any* $K < 1/8$.

Proof. – We express $D^-_{s,z}\,p(s,t,y,x)$ as in [13] as

$$\begin{aligned}D^-_{s,z}p(s,t,y,x) &= -\frac{\partial}{\partial y}\left[p(s,t,y,x)g(y,z)\right] \\ &= -\frac{\partial p}{\partial y}(s,t,y,x)g(y,z) - p(s,t,y,x)\frac{\partial g}{\partial y}(y,z).\end{aligned}$$

Since g and $\partial g/\partial y$ satisfy condition (H8) (ii) and p satisfies the bound (H7), we only need to show

$$\left|\frac{\partial p}{\partial y}(s,t,y,x)\right| \leqslant U_p(s,t,x)\exp\left(-\frac{|x-y|^2}{\gamma_p(t-s)}\right), \tag{5.10}$$

where $\|U_p(s,t,x)\|_{L^p(\Omega)} \leqslant C_p(t-s)^{-1}$. Now taking the derivative $\partial/\partial y$ inside the formula (5.5) for p and integrating by parts we obtain

$$\frac{\partial p}{\partial y}(s,t,y,x) = E_Q\left(\mathbb{1}_{\{B_{t,s}(x)>y-x\}}H'_{t,s}(x)\right),$$

where

$$H'_{t,s}(x) = \delta^b\left(\frac{D^b\varphi_{b,s}(x)}{\|D^b\varphi_{t,s}(x)\|^2}H_{t,s}(x)\right).$$

The proof of Proposition 11 in [13] indicates that $\|H'_{b,s}(x)\|_q \leqslant C_q(t-s)^{-1}$ for all $q \geqslant 1$. Therefore, the estimates on $E\exp(2KB^2_{t,s}(x)/(t-s))$ from the proof of Lemma 5.2 yield the lemma. □

LEMMA 5.4. – *The stochastic kernel $p(s,t,y,x)$ satisfies condition* (H9)$_{\mathrm{M}}$ *for all $M>0$.*

Proof. – By Eq. (4.16) in [13] we know that

$$p(s,t,y,x) = q(s,t,y,x) + \int_{I_s^t}\left[\int_{\mathbb{R}} g(z,y)\frac{\partial p}{\partial z}(s,r,y,z)q(r,t,z,x)\,dz\right]dW_{r,y},$$

where

$$q(s,t,y,x) := \frac{1}{2\sqrt{\pi(t-s)}}\exp\left(-\frac{|y-x|^2}{4\sqrt{t-s}}\right).$$

This gives us that

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{e}^{-M|x|}p(s,t,y,x)\,dx\\ &\quad= \int_{\mathbb{R}} \mathrm{e}^{-M|x|}q(s,t,y,x)\,dx + \int_{I_s^t}\left[\int_{\mathbb{R}} g(z,y)\frac{\partial p}{\partial z}(s,r,y,z)\right.\\ &\quad\quad\left.\times\left(\int_{\mathbb{R}} \mathrm{e}^{-M|x|}q(r,t,z,x)\,dx\right)dz\right]dW_{r,y}\\ &\quad=: T_1 + T_2.\end{aligned}$$

Since $q(r,t,z,x)$ satisfies the forward heat equation in the variables (t,x), we obtain the following by integrating by parts

$$\int_{\mathbb{R}} e^{-M|x|} q(r,t,z,x)\,dx = e^{-M|z|} + \int_r^t \left(\int_{\mathbb{R}} e^{-M|x|} \frac{\partial q}{\partial \theta}(r,\theta,z,x)\,dx \right) d\theta$$

$$= e^{-M|z|} + M^2 \int_r^t \left(\int_{\mathbb{R}} e^{-M|x|} q(r,\theta,z,x)\,dx \right) d\theta$$

$$- 2M \int_r^t q(r,\theta,z,0)\,d\theta.$$

Fubini's stochastic theorem allows us then to write

$$T_2 = \int_{I_s^t} \left[\int_{\mathbb{R}} g(z,y) \frac{\partial p}{\partial z}(s,r,y,z)\, e^{-M|z|}\,dz \right] dW_{r,y}$$

$$+ M^2 \int_s^t \left[\int_{I_s^\theta} \left(\int_{\mathbb{R}} g(z,y) \frac{\partial p}{\partial z}(s,r,y,z) \right.\right.$$

$$\left.\left. \times \left(\int_{\mathbb{R}} e^{-M|x|} q(r,\theta,z,x)\,dz \right) dW_{r,y} \right) \right] d\theta$$

$$- 2M \int_s^t \left[\int_{I_s^\theta} \left(\int_{\mathbb{R}} g(z,y) \frac{\partial p}{\partial z}(s,r,y,z) q(r,\theta,z,0)\,dz \right) dW_{r,y} \right] d\theta.$$

From (4.16) in [13] it follows that

$$T_2 = \int_{I_s^t} \left[\int_{\mathbb{R}} g(z,y) \frac{\partial p}{\partial z}(s,r,y,z)\, e^{-M|z|}\,dz \right] dW_{r,y}$$

$$+ M^2 \int_s^t \left(\int_{\mathbb{R}} e^{-M|x|} \left[p(s,\theta,y,x) - q(s,\theta,y,x) \right] dx \right) d\theta$$

$$- 2M \int_s^t \left[p(s,\theta,y,0) - q(s,\theta,y,0) \right] d\theta.$$

Using integration-by-parts formula it follows that

$$T_2 = M \int_{I_s^t} \left[\int_{\mathbb{R}} g(z,y) p(s,r,y,z)\, e^{-M|z|}\,dz \right] dW_{r,y}$$

$$- \int_{I_s^t} \left[\int_{\mathbb{R}} \frac{\partial g}{\partial z}(z,y) p(s,r,y,z)\, e^{-M|z|}\,dz \right] dW_{r,y}$$

$$+ M^2 \int_s^t \left(\int_{\mathbb{R}} e^{-M|x|} [p(s,\theta,y,x) - q(s,\theta,y,x)]\, dx \right) d\theta$$

$$- 2M \int_s^t [p(s,\theta,y,0) - q(s,\theta,y,0)]\, d\theta.$$

It is easy to show that for all $M > 0$,

$$\int_s^T E|p(s,\theta,y,0)|\, d\theta + \int_s^T q(s,\theta,y,0)\, d\theta + \left(E\left| \int_{\mathbb{R}} e^{-M|x|} p(s,\theta,y,x)\, dx \right|^2 \right)^{1/2} \leqslant C_{M,T} e^{-M|y|}.$$

By Burkholder's inequality it follows that

$$\begin{aligned} &E\left(\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}} e^{-M|x|} p(s,t,y,x)\, dx \right) \\ &\leqslant C_{M,T} \Bigg\{ e^{-M|y|} + \left(E \int_{I_s^T} \left(\int_{\mathbb{R}} \frac{\partial g}{\partial z}(z,y) p(s,r,y,z)\, e^{-M|z|}\, dz \right)^2 dr\, dy \right)^{1/2} \\ &\quad + \left(E \int_{I_s^T} \left(\int_{\mathbb{R}} g(z,y) p(s,r,y,z)\, e^{-M|z|}\, dz \right)^2 dr\, dy \right)^{1/2} \Bigg\} \\ &\leqslant C_{M,T} \left\{ e^{-M|y|} + \left(E \int_s^T \left| \int_{\mathbb{R}} p(s,r,y,z)\, e^{-M|z|}\, dz \right|^2 dr \right)^{1/2} \right\} \\ &\leqslant C_{M,T}\, e^{-M|y|}, \end{aligned}$$

which gives us $(\mathrm{H9})_{\mathrm{M}}$, thanks to (H7). Now the proof is complete. □

6. EQUIVALENCE OF EVOLUTION AND WEAK SOLUTIONS

Assume the notations of Section 5. By (4.15) in [13] we know that $p(s,t,y,x)$ is the fundamental solution (in the variables t and x) of the equation

$$du_t = \frac{\partial^2 u}{\partial x^2}(t,x)\, dt + v(dt,x) \frac{\partial u}{\partial x}(t,x). \tag{6.1}$$

Our purpose in this section is to study the following stochastic partial differential equation

$$du_t = \frac{\partial^2 u}{\partial x^2}(t,x)\,dt + v(dt,x)\,\frac{\partial u}{\partial x}(t,x) + F\big(t,x,u(t,x)\big)\frac{\partial W}{\partial x}(dt), \tag{6.2}$$

with initial condition $u_0 : \mathbb{R} \to \mathbb{R}$. Let us introduce the following definition.

DEFINITION 6.1. – *Let $u = \{u(t,x),\ (t,x) \in I^t\}$ be an adapted process. We say that u is a weak solution of* (6.2) *if for every $\psi \in \mathcal{C}_K^\infty(\mathbb{R})$ and $t \in [0,T]$ we have*

$$\begin{aligned}\int_{\mathbb{R}} \psi(x)u(t,x)\,dx &= \int_{\mathbb{R}} \psi(x)u_0(x)\,dx + \int_{I^t} \psi''(x)u(s,x)\,ds\,dx \\ &\quad - \int_{\mathbb{R}} \psi(x)\Big(\int_{I^t} u(s,x)\,\frac{\partial g}{\partial x}(x,y)\,dW_{s,y}\Big)\,dx \\ &\quad - \int_{\mathbb{R}} \psi'(x)\Big(\int_{I^t} u(s,x)g(x,y)\,dW_{s,y}\Big)\,dx \\ &\quad + \int_{I^t} \psi(x)u(s,x)\,dW_{s,x}. \end{aligned} \tag{6.3}$$

Now we have the following result.

THEOREM 6.2. – *Under the hypotheses of Theorem* 4.1(ii)*, the solution $u = \{u(t,x),\ (t,x) \in I^T\}$ of* (1.1) *is a weak solution of* (6.2).

Proof. – Suppose that u is the solution of (1.1). Let $\{e_k,\ k \geqslant 1\}$ be a complete orthonormal system in $L^2(\mathbb{R})$. For all $m \geqslant 1$ and $(t,x) \in I^T$ we define

$$u^m(t,x) = \int_{\mathbb{R}} p(0,t,y,x)u_0(y)\,dy + \sum_{k=1}^m \int_{I^t} \Big(\int_{\mathbb{R}} p(s,t,z,x)F_s(z)e_k(z)\,dz\Big)e_k(y)\,dW_{s,y}, \tag{6.4}$$

where $F_s(z) := F(s,z,u(s,z))$. The stochastic process $u^m(t,x)$ is well-defined because $\{(\int_{\mathbb{R}} p(s,t,z,x)F_s(z)e_k(z)\,dz)e_k(y)\mathbb{1}_{I^t}(s,y)\}$ belongs to the domain of δ for each $k \geqslant 1$. This property can be proved by the arguments used in the proofs of Theorems 3.2 and 3.4. By (4.15) in [13] we know that for all $0 \leqslant s < t \leqslant T,\ x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} p(s,t,y,x) f(y)\,dy = f(x) + \int_s^t \left(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}(s,r,y,x) f(y)\,dy \right) dr + \int_s^t \left(\int_{\mathbb{R}} \frac{\partial p}{\partial x}(s,r,y,x) f(y)\,dy \right) v(dr,x).$$

This gives us that

$$\begin{aligned} u^m(t,x) &= u_0(x) + \int_0^t \left(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}(0,r,y,x) u_0(y)\,dy \right) dr \\ &\quad + \int_0^t \left(\int_{\mathbb{R}} \frac{\partial p}{\partial x}(0,r,y,x) u_0(y)\,dy \right) v(dr,x) \\ &\quad + \sum_{k=1}^m \int_{I^t} F_s(x) e_k(x) e_k(y)\,dW_{s,y} \\ &\quad + \sum_{k=1}^m \int_{I^t} \left[\int_s^t \left(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}(s,r,z,x) F_s(z) e_k(z)\,dz \right) dr \right] \\ &\quad \times e_k(y)\,dW_{s,y} \\ &\quad + \sum_{k=1}^m \int_{I^t} \left[\int_s^t \left(\int_{\mathbb{R}} \frac{\partial p}{\partial x}(s,r,z,x) F_s(z) e_k(z)\,dz \right) v(dr,x) \right] \\ &\quad \times e_k(y)\,dW_{s,y}. \end{aligned}$$

Let ψ be a test function in $C_K^\infty(\mathbb{R})$. Using integration by parts formula and Fubini's theorem it is easy to obtain that

$$\begin{aligned} &\int_{\mathbb{R}} u^m(t,x)\psi(x)\,dx \\ &\quad = \int_{\mathbb{R}} \psi(x) u_0(x)\,dx + \int_{I^t} \psi''(x) \left(\int_{\mathbb{R}} p(0,r,y,x) u_0(y)\,dy \right) dr\,dx \\ &\qquad - \int_{I^t} \psi'(x) \left(\int_{\mathbb{R}} p(0,r,y,x) u_0(y)\,dy \right) v(dr,x)\,dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{I^t}\psi(x)\Big(\int_{\mathbb{R}}p(0,r,y,x)u_0(y)\,dy\Big)\operatorname{div}v(dr,x)\,dx\\
&+\sum_{k=1}^{m}\int_{I^t}\Big(\int_{\mathbb{R}}\psi(x)F_s(x)\,\mathrm{e}_k(x)\,dx\Big)e_k(y)\,dW_{s,y}\\
&+\sum_{k=1}^{m}\int_{\mathbb{R}}\psi''(x)\Bigg[\int_0^t\Big(\int_{I^r}\Big(\int_{\mathbb{R}}p(s,r,z,x)F_s(z)e_k(z)\,dz\Big)\\
&\times e_k(y)\,dW_{s,y}\Big)\,dr\Bigg]\,dx\\
&-\sum_{k=1}^{m}\int_{\mathbb{R}}\psi'(x)\Bigg[\int_0^t\Big(\int_{I^r}\Big(\int_{\mathbb{R}}p(s,r,z,x)F_s(z)e_k(z)\,dz\Big)\\
&\times \mathrm{e}_k(y)\,dW_{s,y}\Big)v(dr,x)\Bigg]\,dx\\
&-\sum_{k=1}^{m}\int_{\mathbb{R}}\psi(x)\Bigg[\int_0^t\Big(\int_{I^r}\Big(\int_{\mathbb{R}}p(s,r,z,x)F_s(z)e_k(z)\,dz\Big)\\
&\times \mathrm{e}_k(y)\,dW_{s,y}\Big)\operatorname{div}v(dr,x)\Bigg]\,dx.
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
&\int_{\mathbb{R}}u^m(t,x)\psi(x)\,dx\\
&\quad=\int_{\mathbb{R}}\psi(x)u_0(x)\,dx+\sum_{k=1}^{m}\int_{I^t}\Big(\int_{\mathbb{R}}\psi(x)F_s(x)e_k(x)\,dx\Big)e_k(y)\,dW_{s,y}\\
&\qquad+\int_{I^t}\psi''(x)u^m(r,x)\,dr\,dx\\
&\qquad-\int_{\mathbb{R}}\psi'(x)\Big(\int_{I^t}u^m(r,x)g(x,y)\,dW_{r,y}\Big)\,dx\\
&\qquad-\int_{\mathbb{R}}\psi(x)\Big(\int_{I^t}u^m(r,x)\,\frac{\partial g}{\partial x}(x,y)\,dW_{r,y}\Big)\,dx.
\end{aligned}
\tag{6.5}
$$

Notice that

$$\lim_m E\left|\sum_{k=1}^m \int_{I^t}\left(\int_{\mathbb{R}} \psi(x)F_s(x)e_k(x)\,dx\right)e_k(y)\,dW_{s,y} - \int_{I^t} \psi(y)F_{s,y}\,dW_{s,y}\right|^2$$

$$= \lim_m E\int_0^t \sum_{k=m+1}^{\infty}\left(\int_{\mathbb{R}} \psi(x)F_s(x)e_k(x)\,dx\right)^2 ds = 0.$$

In order to complete the proof it suffices to show that for any smooth and cylindrical random variable $G \in \mathcal{S}$ we have

$$\lim_m E\left(G\int_{\mathbb{R}} u^m(t,x)\psi(x)\,dx\right) = E\left(G\int_{\mathbb{R}} u(t,x)\psi(x)\,dx\right),$$

$$\lim_m E\left(G\int_{I^t} \psi''(x)u^m(r,x)\,dr\,dx\right) = E\left(G\int_{I^t} \psi''(x)u(r,x)\,dr\,dx\right),$$

$$\lim_m E\left(G\int_{\mathbb{R}} \psi'(x)\left(\int_{I^t} u^m(r,x)g(x,y)\,dW_{r,y}\right)dx\right)$$

$$= E\left(G\int_{\mathbb{R}} \psi'(x)\left(\int_{I^t} u(r,x)g(x,y)\,dW_{r,y}\right)dx\right),$$

and

$$\lim_m E\left(G\int_{\mathbb{R}} \psi(x)\left(\int_{I^t} u^m(r,x)\,\frac{\partial g}{\partial x}(x,y)\,dW_{r,y}\right)dx\right)$$

$$= E\left(G\int_{\mathbb{R}} \psi(x)\left(\int_{I^t} u(r,x)\,\frac{\partial g}{\partial x}(x,y)\,dW_{r,y}\right)dx\right).$$

These convergences are easily checked using the duality relationship between the Skorohod integral and the derivative operator. □

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