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OLIVIER GARET

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Gibbsian fields associated to exponentially decreasing quadratic potentials

by

Olivier GARET

Laboratoire de Statistique et Probabilités, EP CNRS 1765,
Université des Sciences et Technologies de Lille,
59655 Villeneuve d'Ascq Cedex - France
E-Mail: garet@jacta.univ-lille1.fr

ABSTRACT. – We study the set of Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ associated to interactions of Gaussian type which decrease exponentially. We obtain simple criteria for the existence and uniqueness of such measures in a subclass defined by support conditions. Moreover, we establish the connectedness of the set of admissible parameters for which there is uniqueness. More precise results on phase transition are given when the lattice is one-dimensional ($d = 1$). Finally, we verify the stability under small perturbations of the uniqueness property. © Elsevier, Paris

Key words: Gibbsian fields, Gaussian fields, phase transition.

RÉSUMÉ. – On étudie l'ensemble des mesures de Gibbs sur $\mathbb{R}^{\mathbb{Z}^d}$ associées à une interaction de type gaussien à décroissance exponentielle. On obtient des critères simples pour l'existence et l'unicité de telles mesures au sein d'une sous-classe définie par des conditions de support. On établit de plus la connexité de l'ensemble des paramètres admissibles pour lequels il y a unicité. Des résultats plus précis sur la transition de phase sont donnés quand le réseau est unidimensionnel ($d = 1$). Enfin, on vérifie la stabilité de la propriété d'unicité sous de petites perturbations. © Elsevier, Paris

Mots clés : champs gibbsiens, champs gaussiens, transition de phase.

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INTRODUCTION

The study of lattice random fields with prescribed Gaussian conditional distributions was initiated by Dobrushin in 1966. Gaussian fields on $\mathbb{R}^{\mathbb{Z}^d}$ are special Gibbsian fields of high relevance for the following reasons: At first, it is well known that Gaussian variables allow precise computations (which imply fine results), even when they are strongly dependent. So, lattice Gaussian fields are an example of completely solvable Gibbsian fields. An important remark is that Gaussian fields offer the only example of random fields (with non compact spin state) for which the structure of the set of Gibbs measures is precisely described. This remarkable work has been performed independently by Dobrushin and Künsch in 1980. Rozanov, Chay, Benfatto et al. had also contributed to this domain with various results.

Nevertheless, the structure result obtained by Dobrushin and Knsch is essentially abstract, since it links theoretically the set of Gibbs measures with the kernel –hard to compute– of a linear operator defined on a subset of $\mathbb{R}^{\mathbb{Z}^d}$. Dobrushin suggested in his paper that restrictions (like support conditions) on the set of considered Gibbs measures could allow to get more precise informations, and he obtained fine results for potential decreasing like the inverse of a polynomial.

In Section 1, under the assumptions that the interaction is exponentially decreasing, we give a necessary and sufficient condition for the uniqueness of a Gibbs measure in a large class in terms of the existence of a root of a function in an annulus. We show that the set of symmetric potentials for which we have existence and uniqueness of the Gibbs measure –with the same restrictions– is arcwise connected.

Then, in Section 2, we study the set of Gibbs measures on $\mathbb{R}^{\mathbb{Z}}$ associated to an exponentially decreasing interaction depending of some parameters: we find for which values of the parameters there is existence or uniqueness of a Gibbs measure and we completely describe the set of Gibbs measures.

Finally, in Section 3, we remark that under the restrictions we have made, we recover the stability of the uniqueness of Gibbs measures when the potential is subject to small perturbations. This stability, well known in the case of compact spin state space, is lost in the non compact case if we do not make any restriction on the set of Gibbs measures.

NOTATIONS

By a lattice random field, we denote a probability measure on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, *i.e.* an element of $\mathcal{P}(\Omega)$. As usually, for $i \in \mathbb{Z}^d$, the random variable X_i will denote the canonical projection on the i-th component.

Let us introduce the concept of Gibbs measure. Each $\omega \in \Omega$ can be considered as a map from $\mathbb{Z}^d$ to $\mathbb{R}$. We will denote ω_Λ its restriction to Λ. Then, when A and B are two disjoint subsets of $\mathbb{Z}^d$ and $(\omega, \eta) \in \mathbb{R}^A \times \mathbb{R}^B$, $\omega\eta$ denotes the concatenation of ω and η, that is the element $z \in \mathbb{R}^{A\cup B}$ such that

$$z_i = \begin{cases} \omega_i & \text{if } i \in A \\ \eta_i & \text{if } i \in B \end{cases}$$

For finite subset Λ of $\mathbb{Z}^d$, we define $\sigma(\Lambda)$ to be the σ-field generated by $\{X_i, i \in \Lambda\}$.

For every finite Λ in $\mathbb{Z}^d$, let Φ_Λ be a real-valued $\sigma(\Lambda)$-measurable function. The family $(\Phi_\Lambda)_\Lambda$, when Λ describes the finite subsets of $\mathbb{Z}^d$, is called an interaction potential, or simply a potential. For a finite subset Λ of $\mathbb{Z}^d$, the quantity

$$H_\Lambda = \sum_{B:\ B\cap\Lambda\neq\emptyset} \Phi_B$$

is called the Hamiltonian on the volume Λ. Usually, H_Λ can be defined only on a subset of $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that there exists a subset $\tilde{\Omega}$ of Ω such that

$$\forall \text{ finite } \Lambda\ \forall \omega \in \tilde{\Omega} \qquad \sum_{B:\ B\cap\Lambda\neq\emptyset} |\Phi_B(\omega)| < +\infty$$

$(H_\Lambda)_\Lambda$ is called the Hamiltonian.

We define the range of the interaction to be the supremum of the diameters of the subsets Λ for which Φ_Λ does not identically vanish.

We now define the so called partition function Z_Λ: denoting by λ the Lebesgue's measure on the real line, we let

$$Z_\Lambda(\omega) = \int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) d\lambda^{\otimes\Lambda}(\eta_\Lambda)$$

By convention, we set $\exp(-H_\Lambda(\eta_\Lambda\omega_{\Lambda^c})) = 0$ when the Hamiltonian is not defined.

We suppose that for each ω in $\tilde{\Omega}$, we have $0 < Z_\Lambda(\omega) < +\infty$. Then, we can define for each bounded measurable function f and for each $\omega \in \tilde{\Omega}$,

$$T_\Lambda f(\omega) = \frac{\int_{\mathbb{R}^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) f(\eta_\Lambda \omega_{\Lambda^c}) d\lambda^{\otimes \Lambda}(\eta_\Lambda)}{Z_\Lambda(\omega)}.$$

The operator T_Λ is a kernel, generalizing the Markovian ones, $\mathbb{Z}^d$ playing the role of the time. If a measure μ on Ω is such that $\mu(\tilde{\Omega}) = 1$, we say that μ is a Gibbs measure or a Gibbsian field when for each bounded measurable function f and each finite subset Λ of $\mathbb{Z}^d$, we have

$$E_\mu(f | (X_i)_{i \in \Lambda^c}) = T_\Lambda f \quad \mu \text{ a.s.}$$

1. EXISTENCE AND UNIQUENESS RESULTS FOR QUADRATIC INTERACTIONS

1.1 The quadratic Hamiltonian

We now introduce the three parameters of a quadratic potential.

Let $J : \mathbb{Z}^d \to \mathbb{R}$ be an even function such that $\sum_{i \in \mathbb{Z}^d} |J(i)| < +\infty$, $h \in \mathbb{R}$ and β be a positive real number, physically considered as the inverse of the absolute temperature.

Given these parameters, we deal with Gibbsian random fields μ associated to the potential $\Phi^{J,h,\beta}$ defined on Ω by

$$\Phi_\Lambda^{J,h,\beta}(\omega) = \begin{cases} \beta(\frac{1}{2} J(0)\omega_i^2 + h\omega_i) & \text{if } \Lambda = \{i\} \\ \beta J(i-j)\omega_i \omega_j & \text{if } \Lambda = \{i,j\}, i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Then, the corresponding Hamiltonian function is equal to

$$H_\Lambda(\omega) = \frac{\beta}{2} \sum_{i \in \Lambda} \omega_i [h + \sum_{j \in \Lambda} J(i-j)\omega_j] + \beta \sum_{i \in \Lambda, j \in \Lambda^c} J(i-j)\omega_i \omega_j. \quad (1)$$

We can define

$$\tilde{\Omega} = \{\omega \in \mathbb{R}^{\mathbb{Z}^d} \quad \forall i \in \mathbb{Z}^d \quad \sum_{j \in \mathbb{Z}^d} |J(i-j)\omega_j| < +\infty\}.$$

On $\tilde{\Omega}$, H_Λ is well defined. It is clear that it could not be possible to take a larger $\tilde{\Omega}$, so this is a canonical choice.

For fixed (J, h, β), we denote by $\mathfrak{G}^{\beta}_{J,h}$ the set of Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ associated to the Hamiltonian given in (1). If $\mathfrak{G}^{\beta}_{J,h}$ contains more than one point, we say that phase transition occurs. $\mathfrak{G}^{\beta}_{J,h}$ is a convex set whose extreme points are called pure phases. (For general results on Gibbs measures, see [7].)

Sometimes, we also will call J a potential.

More notations are necessary to recall the following propositions 1 and 2 simultaneously obtained by Dobrushin [5] and Künsch [10] which are the basis of our finer analysis.

For $z = (z_1, ..., z_d) \in \mathbb{C}^d$ and $n = (n_1, ..., n_d) \in \mathbb{Z}^d$, we set

$$z^n = \prod_{i=1}^{n} z_i^{n_i} \text{ and } |n| = \sum_{i=1}^{d} |n_i|$$

$$\mathbb{U} = \{z \in \mathbb{C}^d, \ \forall i \in \{1, \dots, d\} \ |z_i| = 1\}$$

We introduce $\hat{J}$, the dual function of J, defined on a subset of $\mathbb{C}^d$ by

$$\hat{J}(z) = \sum_{n \in \mathbb{Z}^d} J(n) z^n \tag{2}$$

whenever the considered series is absolutely convergent. Since J is summable, it is clear that $\hat{J}$ always defines a continuous map on $\mathbb{U}$.

PROPOSITION 1. – *Given (J, h, β), the set of Gibbs measures $\mathfrak{G}^{\beta}_{J,h}$ contains at least one element if and only if the following three conditions are fulfilled*

1. $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$.
2. $\int_{\mathbb{U}} \frac{1}{\hat{J}(z)} dz < \infty$, *where dz is the normalized Haar measure on the torus $\mathbb{U}$.*
3. $M_h^J = \{(u_n)_{n \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} : \forall k \in \mathbb{Z}^d \sum_{n \in \mathbb{Z}^d}, \ J(n) u_{k+n} = h\} \neq \emptyset$.

Remark. – In the proof of Proposition 1 (see [10] or [5]), it appears that the first condition implies that $0 < Z_\Lambda(\omega) < +\infty$ for $\omega \in \tilde{\Omega}$.

PROPOSITION 2. – *The assumptions of Proposition 1 being verified, the pure phases are the Gaussian measures μ on $\mathbb{R}^{\mathbb{Z}^d}$ with covariance $(i, j) \mapsto \int_{\mathbb{U}} \frac{z^{i-j}}{\beta \hat{J}(z)} dz$ and whose mean value vector $(E_\mu(X_n))_{n \in \mathbb{Z}^d}$ belongs to M_h^J.*

Remark. – The general theory of Gibbsian fields states that $\mathfrak{G}^{\beta}_{J,h}$ is a Choquet's simplex, that is every $\mu \in \mathfrak{G}^{\beta}_{J,h}$ can be represented as a mixture of pure phases. Proposition 2 thus implies that, in case of existence, the

Gibbs measures associated to this interaction are exactly those which can be written as the convolution of the centered gaussian measure whose covariance is given in Proposition 2 with any measure whose support is include in M_h^J. Then, the lack of phase transition is equivalent M_h^J to be a singleton, or by linearity, M_0^J to be equal to $\{0\}$ – since M_0^J is a linear space, we denote the constant sequence equal to zero by 0. ■

1.2 Uniqueness under certain growth assumptions

Except in the case of finite range interaction and for $d = 1$ which has been investigated by Künsch (see [10]), it is very difficult to give a complete description of M_0^J.In the second section of this article, we will completely describe M_0^J for a particular exponentially decreasing potential, but we should remember that such description is not possible in the general case.

Nevertheless, it has a physical sense to consider in M_0^J only the elements which do not increase too fast. For example, Dobrushin considers in [5] the class of slowly increasing sequences. Under the assumption that the potential is exponentially decreasing, we consider here a very large subset of M_0^J: the sequences which increase not faster than a reference exponential sequence.

Since our method is general, we will first state a result for a general class of "sub-exponential" potential called potential of type $\mathcal{E}$. It includes the result of Dobrushin mentioned above (see Corollary 3) and the case of exponentially decreasing potentials (Corollaries 4 and 5).

Let us introduce some definitions.

We say that a sequence $a =: (a_n)_{n\in\mathbb{N}}$ of positive numbers is *of type* $\mathcal{E}$ if it verifies

- $a_0 = 1$.
- $(a_n)_n$ is non decreasing.
- (a_n) is sub-multiplicative, *i.e.* $\forall n, p \in \mathbb{N}\ a_{n+p} \leq a_n a_p$.

Then, we define the characteristic exponent r_a of the sequence a by

$$r_a = \inf_{n\geq 1} a_n^{1/n}.$$

It is clear that $0 \leq r_a < +\infty$. We will see that we have in fact $1 \leq r_a < +\infty$.

Basic example: Given a real number $\alpha \geq 1$, the sequence

$$a = (a_n)_{n\geq 0} = (\alpha^n)_{n\geq 0} \tag{3}$$

is of type $\mathcal{E}$ and its characteristic exponent is α. We call it the reference sequence of exponent α.

We set

1.
$$A_a = \{(u_n)_{n\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}\ \|u\|_{A_a} = \sum_{n\in\mathbb{Z}^d} |u_n|a_{|n|} < +\infty\}$$

A_a will be the space where the potential lives.

2.
$$B_a = \left\{(u_n)_{n\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}\ \|u\|_{B_a} = \sup_{n\in\mathbb{Z}^d} \frac{|u_n|}{a_{|n|}} < +\infty\right\}$$

We will consider the intersection of M_0^J and B_a.

3. We need the annulus

$$\mathbb{U}_r = \left\{z \in \mathbb{C}^d\ \forall i \in [1..d]\ \frac{1}{r} \leq |z_i| \leq r\right\} \text{ with } r \geq 1.$$

(Remark that $\mathbb{U}_1 = \mathbb{U}$.)

Remark. – For $J \in A_a$, $\hat{J}$, defined by (2), is well defined on $\mathbb{U}_{r_a}$.

Proof. – For all $z \in \mathbb{U}_{r_a}$, we have

$$\forall k \in [1..d],\ \forall n \in \mathbb{Z}^d,\ |z_i^{n_i}| \leq r_a^{|n_i|},\ \text{hence } |z^n| \leq r_a^{|n|}$$

so

$$\forall n \neq 0\ |J(n)z^n| \leq |J(n)r_a^{|n|}| \leq |J(n)a_{|n|}|$$

which implies the convergence of the series.

Let us recall what is the convolution of two sequences.

Given two sequences $u = (u_n)_{n\in\mathbb{Z}^d}, v = (v_n)_{n\in\mathbb{Z}^d}$ such that

$$\forall n \in \mathbb{Z}^d \sum_{k\in\mathbb{Z}^d} |u_k v_{n-k}| < +\infty,$$

the convolution $u * v$ of u and v is defined by

$$\forall n \in \mathbb{Z}^d,\ (u * v)_n = \sum_{k\in\mathbb{Z}^d} u_k v_{n-k}.$$

We now formulate two easy lemmas, whose simple proofs are omitted. Just remark that the proof of the first one uses the assumption of sub-multiplicativity of the sequence a, which will never more appear.

LEMMA 1. – $(A_a, ||.||_{A_a}, *)$ *is an unital commutative Banach algebra.*

LEMMA 2.

$$\forall u, v \in A_a \ \forall z \in \mathbb{U}_{r_a} \ \widehat{u * v}(z) = \hat{u}(z)\hat{v}(z)$$

Let us recall some useful results of the theory of Banach algebras (for a concise exposition, one can see for example [11], chapter 4.)

Let B be a Banach algebra with e as unit: we denote by $G(B)$ the subset of invertible elements in B. It is easy to see that $G(B)$ is open. When x belongs to B, the spectrum of x is the following subset of $\mathbb{C}$:

$$\text{Spec}(x) = \{\lambda \in \mathbb{C}, \ x \dot{-} \lambda e \notin G(B)\}.$$

It is a compact subset of $\mathbb{C}$. We denote by $\rho(x)$ the spectral radius of x, *i.e* the maximum modulus of the elements in the spectrum of x. Recall the spectral radius formula:

$$\rho(x) = \inf_{n \geq 1} ||x^n||^{\frac{1}{n}}.$$

Let Δ be the set of the continuous homomorphisms from B to $\mathbb{C}$ different from the constant 0 and which respect the algebra structure: such homomorphisms are called characters. A powerful result of Gelfand's theory is the following:

$$\text{Spec}(x) = \{\chi(x); \ \chi \in \Delta\}.$$

Lemma (2) shows that, for all z in $\mathbb{U}_{r_a}$, we can define a character χ_z by

$$c \mapsto \chi_z(c) = \hat{c}(z). \tag{4}$$

Let us now state the main theorem of this subsection.

THEOREM 1. – *Let $J \in A_a$ where a is some sequence of type $\mathcal{E}$. The following assertions are equivalent:*

1. $M_0^J \cap B_a = \{0\}$
2. $J \in G(A_a)$
3. $\hat{J}$ *does not vanish on* $\mathbb{U}_{r_a}$.

Proof of $1 \Rightarrow 3$

Let $z \in \mathbb{U}_{r_a}$ verifying $\hat{J}(z) = 0$. We have

$$\forall k \in \mathbb{Z}^d \sum_{n \in \mathbb{Z}^d} J(n) z^{n+k} = 0$$

Taking real and imaginary parts, we see that the sequences $u := (\Re e(z^n))_n$ and $v := (\Im m(z^n))_n$ belong to M_0^J. Since $z \neq 0$, one of them does not identically vanish, u for example. It remains to prove that $u \in B_a$. On the one hand, we have

$$|u_n| \leq |z^n| \leq r_a^{|n|}.$$

On the other hand, by the definition of r_a, $a_{|n|} \geq r_a^{|n|}$. Then $u \in B_a$.

Proof of 2 $\Rightarrow$ 1

Let us first prove

LEMMA 3. – 1. $\forall c \in A_a, \forall d \in B_a \quad c * d \in B_a$, *with*

$$||c * d||_{B_a} \leq ||c||_{A_a} ||d||_{B_a}$$

2. $\forall c, c' \in A_a, d \in B_a \;\; c * (c' * d) = (c * c') * d$

Proof of the lemma:

$$\begin{aligned}|(c * d)_n| &\leq \sum_k |c_k| \; |d_{n-k}| \leq \sum_k |c_k| ||d||_{B_a} a_{n-k} \\ &\leq \sum_k |c_k| ||d||_{B_a} a_{|n|} a_{|k|} \leq a_{|n|} ||c||_{A_a} ||d||_{B_a}\end{aligned}$$

Then $c * d \in B_a$, with $||(c * d)||_{B_a} \leq ||c||_{A_a} ||d||_{B_a}$.

First, we will show that the double indexed sequence $(c'_k c_l d_{n-k-l})_{k,l}$ is summable.

$$\begin{aligned}\sum_{k,l} |c'_k c_l d_{n-k-l}| &= \sum_k |c'_k| \sum_l |c_l d_{n-k-l}| \\ &\leq \sum_k |c'_k| a_{n-k} ||c||_{A_a} ||d||_{B_a} \leq a_{|n|} ||c'||_{A_a} ||c||_{A_a} ||d||_{B_a}\end{aligned}$$

Then, we shall write

$$(c * (c' * d))_n = \sum_k c_k (c' * d)_{n-k} = \sum_k c_k \sum_l c'_l d_{(n-k)-l} = \sum_{k,l} c_k c'_l d_{n-k-l}$$

Thus, we set $p = k + l$ (summation by bundles). Hence

$$\begin{aligned}(c * (c' * d))_n &= \sum_{p,l} c_{p-l} c'_l d_{n-p} = \sum_p [\sum_l c_{p-l} c'_l] d_{n-p} \\ &= \sum_p (c * c')_p d_{n-p} = ((c * c') * d)_n \quad \blacksquare\end{aligned}$$

We can now prove the implication 2 $\Rightarrow$ 1:

Let $u \in M_0^J \cap B_a$ and $J \in G(A_a)$: since J is even, we can write $J * u = 0$. But this implies that:

$$u = (J^{-1} * J) * u = J^{-1} * (J * u) = J^{-1} * 0 = 0.$$

Proof of $3 \Rightarrow 2$

This will follow from Lemma 4 and from the results of spectral theory mentioned above.

LEMMA 4.

$$\Delta = \{\chi_z; z \in \mathbb{U}_{r_a}\}$$

Proof. – We have already shown that the χ_z defined by (4) are characters. Let us show the converse: so, let $\chi \in \Delta$ and $c \in A_a$. We denote by $(e_1, ..., e_d)$ the canonical basis of $\mathbb{C}^d$. For each $n \in \mathbb{Z}^d$, we define δ_n as the sequence indexed by $\mathbb{Z}^d$ composed by vanishing components, except the n^{th} which equals 1.

We can represent c by a strongly convergent series:

$$c = \sum_{n \in \mathbb{Z}^d} c_n \delta_n = \sum_{n \in \mathbb{Z}^d} c_n \delta_{e_1}^{*n_1} * ... * \delta_{e_d}^{*n_d},$$

the second equality being given by the identity $\delta_n * \delta_p = \delta_{n+p}$.

If we set $z = (\chi(\delta_{e_1}), .., \chi(\delta_{e_d}))$, the continuity of χ allows to write

$$\chi(c) = \sum_{n \in \mathbb{Z}^d} c_n z^n = \hat{c}(z)$$

It remains to prove that $z \in \mathbb{U}_{r_a}$. We have, for $1 \le k \le d$,

$$|\chi(\delta_{e_k})| \le \rho(\delta_{e_k}) = \inf_{j \ge 1} \|\delta_{e_k}^{*j}\|_{A_a}^{1/j} = \inf_{j \ge 1} \|\delta_{je_k}\|_{A_a}^{1/j} = \inf_{j \ge 1} a_j^{1/j} = r_a$$

(Here the spectral radius is computed with the help of the spectral radius formula.)

In the same way, $|\chi(\delta_{-e_k})| \le r_a$. Then $\chi(\delta_{-e_k}) = \chi(\delta_{e_k}^{-1}) = \chi(\delta_{e_k})^{-1}$. Thus, we have

$$\frac{1}{r_a} \le |\chi(\delta_{e_k})| \le r_a \tag{5}$$

Moreover, this last inequality shows that $r_a \ge 1$, as announced at the beginning of this subsection.

It is now clear that $\chi = \chi_z$, for some z in $\mathbb{U}_{r_a}$. So the lemma is proved. ■

Remark. – The formula $c_n = \int_{\mathbb{U}} \frac{\hat{c}(z)}{z^n} dz$ shows that the mapping $c \mapsto \hat{c}$ is injective. Lemma 4 allows to identify $\hat{c}$ with the map $\chi \mapsto \chi(c)$. The so called Gelfand's transform is then an algebraic isomorphism: we say that A_a is a semi-simple algebra. This will be very important later, because it allows us to deal with functions rather than with sequences.

We can now prove the last implication: $\hat{J}$ does not vanish on $\mathbb{U}_{r_a}$ means that for all $z \in \mathbb{U}_{r_a}$ $\chi_z(J) \neq 0$. But from Lemma 4, this means that for all χ in Δ, $\chi(J) \neq 0$: so 0 does not belong to the spectrum of J and J is invertible. ■

COROLLARY 1. – *0 is the only bounded sequence in M_0^J if and only if $\hat{J}$ does not vanish on $\mathbb{U}$.*

Proof. – It suffices to apply Theorem 1 to the constant sequence $a_n = 1$. ■

This result has already been proved by Georgii ([7], chapter 13), using the well-known theorem of Wiener about the functions of the class A. (This theorem could itself be easily deduced from Lemma 4.)

COROLLARY 2. – *Let J be a potential such that $\sum_{n\in\mathbb{Z}^d} |J(n)||n|^\alpha < +\infty$ for some $\alpha > 0$ The following assertions are equivalent*

1. $M_0^J \cap \{(u_n)_{n\in\mathbb{Z}^d} : \sup_{n\in\mathbb{Z}^d} \frac{|u_n|}{1+|n|^\alpha} < +\infty\} = \{0\}$
2. *$\hat{J}$ does not vanish on $\mathbb{U}$.*

Proof. – We verify that the sequence $a_n = 1 + 2^\alpha n^\alpha$ is of type $\mathcal{E}$ with $r_a = 1$, and apply theorem 1. ■

We say that a sequence $(u_n)_{n\in\mathbb{Z}^d}$ is fastly decreasing when, for each polynomial P, the sequence $(u_n P(n))_{n\in\mathbb{Z}^d}$ is bounded. It happens if and only if the map $\theta \mapsto \hat{u}(e^{i\theta})$ is C^∞. We say that a sequence $(u_n)_{n\in\mathbb{Z}^d}$ is slowly increasing if there exists a polynomial P such that for every $n \in \mathbb{Z}^d$, $|u_n| \leq P(n)$.

COROLLARY 3. – *When J is a fastly decreasing potential and $\hat{J}$ is strictly positive on $\mathbb{U}$, then M_0^J does not contain any slowly increasing sequence, except 0.*

Proof. – It is a consequence of the previous corollary. ■

This result has already be proved by Dobrushin [5], using the theory of distributions.

To simplify the notations, we now denote by A_α (resp. B_α) the sets A_a (resp. B_a), where a is the reference sequence of exponent α defined by formula (3).

DEFINITION. – We say that J is exponentially decreasing if it verifies one of the following equivalent conditions:

1. $J \in A_a$ for some sequence a of type $\mathcal{E}$ such that $r_a > 1$.
2. $J \in A_\eta$ for some $\eta > 1$.
3. There exists $K > 0$ and $0 \leq \alpha < 1$ such that $\forall n \in \mathbb{Z}^d$, $|J(n)| \leq K\alpha^n$.

COROLLARY 4. – *Let J be an exponentially decreasing potential such that $\hat{J}$ is strictly positive on $\mathbb{U}$.*

Then there exists $\alpha > 1$ such that $M_0^J \cap B_\alpha = \{0\}$.

Proof. – Let $r > 1$ such that $J \in A_r$. We want to prove that there exists $\alpha > 1$ such that $\hat{J}$ does not vanish on $\mathbb{U}_\alpha$. Let us suppose that

$$\forall n \geq n_0, \ \exists z_n \in \mathbb{U}_{1+1/n}, \ \hat{J}(z_n) = 0.$$

The sequence $(z_n)_{n \geq n_0}$ is bounded, so it admits a limit point z: clearly, we have $z \in \mathbb{U}$ and, since $\hat{J}$ is continuous, $\hat{J}(z) = 0$: this is a contradiction.

So, there exists $\alpha \in]1, r[$ such that $\hat{J}$ does not vanish on $\mathbb{U}_\alpha$. Since $J \in A_r$, *a fortiori* $J \in A_\alpha$. Then, Theorem 1 gives the desired result. ■

We now want to impose supplementary conditions on the support of μ. Given a sequence $(a_n)_n$ of type $\mathcal{E}$, we define

$$\mathcal{P}_a(\Omega) = \{\mu \in \mathcal{P}(\Omega) \quad | \quad \mu(B_a) = 1\};$$

and for $\alpha \geq 1$,

$$\mathcal{P}_\alpha(\Omega) = \{\mu \in \mathcal{P}(\Omega) \quad | \quad \mu(B_\alpha) = 1\}.$$

THEOREM 2. – *Let $J \in A_a$ where a is some sequence of type $\mathcal{E}$ verifying*

$$\exists K > 0 \quad \forall n \geq 1 \quad a_n \geq K\sqrt{\ln|n|} \tag{6}$$

We suppose that $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$.

The following assertions are equivalent:

1. $M_0^J \cap B_a = \{0\}$
2. $J \in G(A_a)$
3. *$\hat{J}$ does not vanish on $\mathbb{U}_{r_a}$.*
4. $|\mathfrak{G}_{J,h}^\beta \cap \mathcal{P}_a(\Omega)| = 1$.

Proof. – Let us suppose that 1. , 2. and 3. –which are equivalent– are checked. Since $\hat{J}$ does not vanish on $\mathbb{U}$, which is compact, we see that $\frac{1}{\hat{J}}$ is bounded, and therefore integrable. Propositions 1 and 2 imply that the stationary Gaussian measure with mean $\frac{h}{\hat{J}(1)}$ and with $\frac{1}{\hat{J}}$ as spectral density belongs to $\mathfrak{G}^{\beta}_{J,h}$. Let $(Y_i)_{i\in\mathbb{Z}^d}$ be a random variable whose law under P is this Gaussian measure.

Let σ^2 be its variance and $L \in \mathbb{R}$ be such that $\frac{L^2K^2}{2\sigma^2} \geq d+1$. For $|i| \geq 1$, we have

$$P(|Y_i| \geq La_{|i|}) \leq \frac{\sigma}{\sqrt{2\pi}La_{|i|}} e^{-\frac{L^2a^2_{|i|}}{2\sigma^2}} \leq \frac{\sigma}{\sqrt{2\pi}La_{|i|}} \frac{1}{|i|^{d+1}}$$

From Borel-Cantelli's lemma, we conclude that $P(|Y_i| \geq La_{|i|} \text{ i.o.}) = 0$. So $(Y_i)_{i\in\mathbb{Z}^d} \in B_a$ almost surely.

We have just proved that $|\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)| \geq 1$. Now let us consider $\mu \in \mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)$. By proposition 2 and the remark which follows, we can find random variables $(Z_i), (Y_i)$ with Z and Y independent such that $(Z_i)_{i\in\mathbb{Z}^d} \in M_0^J$ almost surely, (Y_i) is as before, and the law of $(Y_i + Z_i)_{i\in\mathbb{Z}^d}$ is μ. $Z+Y \in B_a$ a.s and $Y \in B_a$ a.s. , so $Z \in B_a$ a.s. We have $Z \in M_0^J \cap B_a$ a.s, and $M_0^J \cap B_a = \{0\}$. Then, $Z = 0$ a.s. This proves that $|\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)| = 1$.

Now, we will suppose that $|\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)| = 1$. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a realization of this measure. Let $x \in M_0^J \cap B_a$. Using Proposition 2, one can see that the law of $(X_i + x_i)_{i\in\mathbb{Z}^d}$ belongs to $\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)$. But $|\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)| = 1$, so we get $x = 0$. So we have proved that $M_h^J \cap B_a = \{0\}$. ■

Remark. – By proposition 1, Condition 4. implies $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$. Then, for a sequence a verifying (6), a potential $J \in A_a$ is such that $|\mathfrak{G}^{\beta}_{J,h} \cap \mathcal{P}_a(\Omega)| = 1$ if and only if $\hat{J}$ does not vanish on $\mathbb{U}_{r_a}$ and $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$. ■

1.3 A connected set of parameters for which there is existence and uniqueness

The aim of this subsection is to get informations about the structure of the set of potentials for which there is a unique associated Gibbs measure (within a certain class of probability measures). We will prove that, under suitable symmetry assumptions, a potential J for which there is uniqueness can be continuously perturbed until any pair-interaction vanishes (*i.e* until $J(k) = 0$ for each $k \neq 0$) and such that along this perturbation the uniqueness of the corresponding Gibbs measure is preserved.

Let Γ be the group generated by the orthogonal symmetries with axis e_i, $1 \leq i \leq d$. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$. We say that a potential J is Γ-invariant if it verifies

$$\forall \Theta \in \Gamma \quad J \circ \Theta = J. \tag{7}$$

Remember that a potential is necessary even. Then, it is easy to see that for $d = 1$, every potential is Γ-invariant. In many cases, the potential has a natural symmetry which makes it Γ-invariant. (For example, if $J(n)$ only depends on a l^p norm of n.)

Let $\alpha > 1$. We denote by $\mathcal{S}_\alpha$ the set of potentials J which belongs to A_α and are Γ-invariant. We let

$$\mathcal{S}_\alpha^+ = \{J \in \mathcal{S}_\alpha \quad | \quad \hat{J}(\mathbb{U}) \subset \mathbb{R}^+ \text{ and } 0 \notin \hat{J}(\mathbb{U}_\alpha)\}.$$

By theorem 2, for each $\beta > 0$ and $h \in \mathbb{R}$, we have

$$\mathcal{S}_\alpha^+ = \{J \in \mathcal{S}_\alpha \quad | \quad |\mathfrak{G}_{J,h}^\beta \cap \mathcal{P}_\alpha(\Omega)| = 1\},$$

endowed with the topology inherited from A_α. We will show the following theorem

THEOREM 3. – *$\mathcal{S}_\alpha^+$ is an open connected subset of $\mathcal{S}_\alpha$.*

Proof. – We will need some lemmas. Two of them are only stated, since their proofs can be found in [8] (They are given for $d = 1$, but they can be easily generalized.)

We let

$$\hat{A}_a = \{\hat{x}; x \in A_a\}$$

Let $J \in \mathcal{S}_\alpha^+$. Our method is to exhibit a well chosen $B \in A_\alpha$ such that $\exp(B) = J$ and consider on $[0, 1]$ the map $\gamma : t \mapsto \exp(tB)$. Remember that A_α is a Banach algebra with the convolution as multiplication. Then $\exp(B)$ is well defined by the absolutely convergent series:

$$\exp(B) = \sum_{k=0}^{+\infty} \frac{B^{*k}}{k!}.$$

LEMMA 5.

$$\forall B \in A_a \quad \exp(\hat{B}) = \widehat{\exp(B)} \text{ on } \mathbb{U}_{r_a}.$$

Proof. – Let $P_n(X) = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \ldots + \frac{X^n}{n!}$. Since $B \mapsto \hat{B}$ is an algebraic homomorphism, we have

$$\forall B \in A_a \quad P_n(\hat{B}) = \widehat{P_n(B)}$$

For each $z \in \mathbb{U}_{r_a}$, it means $P_n(\hat{B}(z)) = \widehat{P_n(B)}(z)$. By the definition of exp (in A_a and in $\mathbb{C}$) and the continuity of $B \mapsto \hat{B}(z)$, we get the desired result. ■

LEMMA 6. – *Let* $f : \mathbb{U}_\alpha \to \mathbb{C}$ *such that*

1. f *is continuous on* $\mathbb{U}_\alpha$.
2. f *is holomorphic on the interior of* $\mathbb{U}_\alpha$.
3. *The map* $g(z) = f(\alpha z)$ *belong to* $\hat{A}_1$.
4. $\forall z \in \mathbb{U}_\alpha \forall (\epsilon_1, \ldots, \epsilon_d) \in \{-1, +1\}^d \quad f(z_1^{\epsilon_1}, \ldots, z_d^{\epsilon_d}) = f(z)$.
5. $f(\mathbb{U}) \subset \mathbb{R}$.

Then there exists $a \in \mathcal{S}_\alpha \quad f = \hat{a}$.

Proof: by 2., there exists an unique sequence $(a_n)_{n \in \mathbb{Z}^d}$ such that for each z in the interior of $\mathbb{U}_\alpha$

$$f(z) = \sum_{n \in \mathbb{Z}^d} a_n z^n.$$

The uniqueness of this expansion implies (using assumption 4) that a is Γ-invariant and (using assumption 5), that a is real-valued. For each $n \in \mathbb{Z}^d$ and $\frac{1}{\alpha} < r < \alpha$, we have

$$r^{s(n)} a_n = \frac{1}{(2\pi)^d} \int_{[-\pi, +\pi]^d} f(re^{i\theta_1}, \ldots, re^{i\theta_d}) e^{-i(\theta_1 n_1 + \ldots + \theta_d n_d)} d\theta_1 \ldots d\theta_d, \tag{8}$$

where $s_n = \sum_{1 \leq i \leq d} n_i$. Since f is uniformly continuous on $\mathbb{U}_\alpha$, one can make r tend to α and the formula (8) remains true for $r = \alpha$. We have found the Fourier expansion of g. By 3., we get

$$\sum_{n \in \mathbb{Z}^d} \alpha^{s(n)} |a_n| < +\infty.$$

Moreover, we have

$$|a_n| \alpha^{|n|} \leq \sum_{p \in \Gamma(n)} |a_p| \alpha^{s(p)},$$

where $\Gamma(n) = \{\Theta(n); \Theta \in \Gamma\}$. Then

$$\sum_{n\in\mathbb{Z}^d} |a_n|\alpha^{|n|} \leq 2^d \sum_{n\in\mathbb{Z}^d} \alpha^{s(n)}|a_n| < +\infty.$$

We have proved that $a \in \mathcal{S}_\alpha$ and $f = \hat{a}$. ■

LEMMA 7. – *Let f be a map defined on $\mathbb{U}$ such that, for each $z \in \mathbb{U}$, there exists a neighborhood W of z and $g \in \hat{A}_1$ verifying $f = g$ on W.*

Then $f \in \hat{A}_1$.

Proof. – See [8]. It is to note that the first proof of this result was given by Wiener. ■

LEMMA 8. – *For each $f \in \hat{A}_1$, $\epsilon > 0$ and $z \in \mathbb{U}$ there exists a neighborhood W of z and $a \in A_1$ such that $\|a\|_{A_1} < \epsilon$ and $\hat{a} = f$ on W.*

Proof. – See [8]. ■

LEMMA 9. – *Let f be a continuous map on $\mathbb{U}$ and $g \in \hat{A}_1$ such that* $\exp(f) = g$.

Then $f \in \hat{A}_1$.

Proof. – Let $z_0 \in \mathbb{U}$. Let $\epsilon > 0$ be such that for each $z \in B(z_0, \epsilon)$

$$|g(z) - g(z_0)| < \frac{|g(z_0)|}{2}.$$

Let ln be a determination of the logarithm in $B(g(z_0), \frac{|g(z_0)|}{2})$ such that

$$\ln(g(z_0)) = f(z_0).$$

Since f is continuous, we have for each $z \in B(z_0, \epsilon)$, $f(z) = \ln(g(z))$. By lemma 8, there exists $\delta < \epsilon$ and $a \in A_1$ with $\|a - g(z_0)e\|_{A_1} < \frac{|g(z_0)|}{2}$ and $g = \hat{a}$ on $B(z_0, \delta)$. There exists a sequence $(b_n)_{n\geq 0}$ such that

$$\ln z = \sum_{n\geq 0} b_n(z - g(z_0))^n \text{ for } z \in B(g(z_0), \frac{|g(z_0)|}{2}).$$

Then, for each $z \in B(z_0, \delta)$

$$f(z) = \ln(g(z)) = \ln(\hat{a}(z)) = \sum_{n\geq 0} b_n(\hat{a}(z) - g(z_0))^n$$

Since the radius of convergence of the power series $\sum b_n x^n$ is strictly greater than $\|a - g(z_0)e\|_{A_1}$, we can we define

$$b = \sum_{n \geq 0} b_n (a - g(z_0)e)^n.$$

We have $f(z) = \hat{b}(z)$ for $z \in B(z_0, \delta)$. Then, by Lemma 7, $f \in \hat{A}_1$. ■

We now explain the different steps in the proof of the theorem. Lemma 5 make us able to handle functions rather than sequences. Then, once we will have found a logarithm to $\hat{J}$, we will use Lemma 6 to check it is in the right space. But, since $\mathbb{U}_\alpha$ is not simply connected, we must use a trick to define a logarithm of $\hat{J}$. In fact, we will lift $\hat{J}$ to replace $\mathbb{U}_\alpha$ by a convex set.

We denote by E_α the range of the annulus $\{z \in \mathbb{C},\ \ \frac{1}{\alpha} \leq |z| \leq \alpha\}$ by the map $z \mapsto z + \frac{1}{z}$. An easy computation shows that E_α is a full closed ellipse:

$$E_\alpha = \left\{ z = x + iy;\quad (x, y) \in \mathbb{R}^2,\quad \frac{x^2}{(\alpha + \frac{1}{\alpha})^2} + \frac{y^2}{(\alpha - \frac{1}{\alpha})^2} \leq 1 \right\}.$$

Since J is Γ-invariant, we can define a map ϕ on E_α^d by

$$\phi\left(z_1 + \frac{1}{z_1}, \ldots, z_d + \frac{1}{z_d} \right) = \hat{J}(z).$$

Since $\mathbb{U}_\alpha$ is compact, using the sequence criterium for continuity, we see that ϕ is continuous. By Theorem 2, $\hat{J}$ does not vanish, so neither does ϕ. Since E_α^d is convex, there exist a continuous $\Phi : E_\alpha^d \to \mathbb{C}$ such that $\phi = \exp(\Phi)$. We define Ψ on $\mathbb{U}_\alpha$ by

$$\Psi(z) = \Phi\left(z_1 + \frac{1}{z_1}, \ldots, z_d + \frac{1}{z_d} \right). \tag{9}$$

Then we have $\hat{J} = \exp(\Psi)$. By Theorem 2, $\hat{J}(1, \ldots, 1) \in \mathbb{R}_*^+$. By adding a constant to Ψ, we can assume that $\Psi(1, \ldots, 1) \in \mathbb{R}$. Since $\hat{J}(\mathbb{U}) \subset \mathbb{R}^+$, we have $\Psi(\mathbb{U}) \subset \mathbb{R} + 2i\pi\mathbb{Z}$. But $\mathbb{U}$ is connected and Ψ is continuous, so $\Psi(\mathbb{U}) \subset \mathbb{R}$.

Since Ψ is locally a logarithm of $\hat{J}$, Ψ is holomorphic in the interior of $\mathbb{U}_\alpha$.

Lemma 9 implies that $z \mapsto \Psi(\alpha z)$ belongs to $\hat{A}_1$.

(9) implies that Ψ verifies the condition 4. of Lemma 6. Then, we can apply Lemma 6 and we get $B \in \mathcal{S}_\alpha$ such that $\Psi = \hat{B}$.

Now, we set, for $t \in [0,1]$, $\gamma(t) = \exp(tB)$. γ is continuous, with $\gamma(0) = e$.

By Lemma 5

$$\widehat{\gamma(1)} = \widehat{\exp(B)} = \exp(\hat{B}) = \exp(\Psi) = \hat{J}.$$

Since A_α is semi-simple, it implies $\gamma(1) = J$. It remains to prove that $\gamma([0,1]) \subset \mathcal{S}_\alpha^+$.

For $t \in [0,1]$, since $\mathcal{S}_\alpha$ is a closed sub-algebra of A_α, we have $\exp(tB) \in \mathcal{S}_\alpha$. Moreover $\exp(tB)$ is already invertible, and $\widehat{\exp(tB)}(\mathbb{U}) = \exp(t\Psi(\mathbb{U})) \subset \mathbb{R}^+$, because $\Psi(\mathbb{U}) \subset \mathbb{R}$. Then, by Theorem 2, it follows that $\gamma(t) \in \mathcal{S}_\alpha^+$.

Since every point can be linked to e, we have proved that $\mathcal{S}_\alpha^+$ is arcwise connected.

It is easy to see that $J \mapsto m(J) = \inf\{\hat{J}(z); z \in \mathbb{U}\}$ is continuous.

So $\mathcal{S}_\alpha^+ = \mathcal{S}_\alpha \cap (G(A_\alpha) \cap m^{-1}(]0,+\infty[))$ is a open subset of $\mathcal{S}_\alpha$. ■

Remark. – Using Proposition 2, it would be easy to prove that, to this map $t \mapsto \gamma(t)$ drawed in $\mathcal{S}_\alpha^+$, corresponds a continuous flow with values in $\mathfrak{G}_{J,h}^\beta \cap \mathcal{P}_\alpha(\Omega)$. ■

1.4 Phase transition for $d = 1$

When the dimension of the lattice is 1, it is sometimes possible to give more precise details about phase transition. The following theorem shows it.

In all this section 1.4, $d = 1$.

THEOREM 4. – *Let us suppose that $J \in A_\alpha$, with $\alpha > 1$ and that J does not degenerate, i.e. $\exists k \neq 0$ $J(k) \neq 0$. If moreover $\hat{J}$ does not vanish on the circle of radius α $\{z \in \mathbb{C},\ |z| = \alpha\}$ or, equivalently, on the circle of radius $\frac{1}{\alpha}$, then $M_0^J \cap B_\alpha$ is a finite-dimensional linear space, whose dimension is exactly the number of roots of $\hat{J}$ (counted with their multiplicity) in the annulus $\mathbb{U}_\alpha$.*

We will need the following lemma:

LEMMA 10. – *We suppose $u \in A_\alpha$. Then, if z is an interior point of $\mathbb{U}_\alpha$ satisfying $\hat{u}(z) = 0$, there exists $y \in A_\alpha$ such that $u = (\delta_{e_1} - z\delta_0) * y$.*

Proof of the lemma. – We define a sequence $x := (x_n)_{n \in \mathbb{Z}}$ by

$$x_n = \frac{1}{z^n} \sum_{k=-\infty}^{n} u_k z^k$$

The assumption $\hat{u}(z) = 0$ allows to have the representation

$$x_n = -\frac{1}{z^n} \sum_{k=n+1}^{+\infty} u_k z^k.$$

We want to show that $x \in A_\alpha$. We will prove $\sum_{k\leq 0} |x_k|\alpha^{-k} < +\infty$ using the last representation and $|z| > \frac{1}{\alpha}$. Similarly, we could prove $\sum_{k\geq 0} |x_k|\alpha^k < +\infty$ using the first representation and $|z| < \alpha$.

Let us show that the sequence $S_n = \sum_{k=0}^n |x_{-k}|\alpha^k$ is bounded. For its purpose, we will use a discrete integration by parts – sometimes called a Abel's transform–: we have

$$|x_{-k}|\alpha^k = x_{-k}z^{-k} \quad \epsilon_{-k}(\alpha z)^k$$

where $\epsilon_k = +1$ when $x_k > 0$, and $\epsilon_k = -1$ otherwise. Then, we set

$$T_n = \sum_{k=0}^n \epsilon_{-k}(\alpha z)^k \text{ and } T_{-1} = 0.$$

Thus, we have

$$\begin{aligned} S_n &= \sum_{k=0}^n x_{-k}z^{-k}(T_k - T_{k-1}) \\ &= x_{-n}z^{-n}T_n - \sum_{k=0}^{n-1}[x_{-k}z^{-k} - x_{-(k+1)}z^{-(k+1)}]T_k. \end{aligned}$$

Using the definition of x, we get

$$S_n = T_n \sum_{k=-\infty}^{-n} u_k z^k - \sum_{k=0}^{n-1} u_{-k}z^{-k}T_k.$$

We easily verify that

$$\forall n \geq 0 \quad |T_n| \leq \frac{\alpha|z|}{\alpha|z| - 1}(\alpha|z|)^n.$$

We deduce

$$S_n \leq \frac{\alpha|z|}{\alpha|z| - 1}\left[\sum_{k=-\infty}^{-n} \alpha^n|z|^{n+k} + \sum_{k=0}^{n-1} |u_{-k}|\alpha^k\right].$$

Since $|z| > \frac{1}{a}$, $n+k \leq 0$ implies $|z|^{n+k} \leq \alpha^{-(n+k)}$, we finally get

$$\forall n \geq 0 \quad S_n \leq \frac{\alpha|z|}{\alpha|z|-1} \sum_{k=-\infty}^{0} |u_k|\alpha^{-k}$$

Thus, $x \in A_\alpha$. So,we set $y = -\frac{1}{z}x$ and y is the desired element. ■

Proof of the theorem. – The equivalence between the lack of roots on $|z| = \alpha$ and on $|z| = \frac{1}{\alpha}$ comes from the identity $\hat{J}(z) = \hat{J}(\frac{1}{z})$.

Since $\hat{J}$ is holomorphic on the interior of $\mathbb{U}_\alpha$, the theorem of isolated roots implies that $\hat{J}$ can only have a finite number of roots, necessary with finite multiplicities. Let us note them z_i, with multiplicity d_i.

Applying several times Lemma 5, we find $Y \in A_\alpha$, such that

$$J = \prod_i (\delta_{e_1} - z_i\delta_0)^{*d_i} * Y$$

Since $\widehat{u*v} = \hat{u}\hat{v}$, we see that $\hat{Y}$ does not vanish on $\mathbb{U}_\alpha$. So by theorem 1, Y is invertible. Then, $J * u = 0$ is equivalent to

$$\prod_i (\delta_{e_1} - z_i\delta_0)^{*d_i} * u = 0$$

The solutions in $\mathbb{C}$ of this linear recurrence equation are well-known: It is a linear space on $\mathbb{C}$ with $\sum \alpha_i$ as dimension (see for example [1], page 654). The solutions can be written as $u_n = \sum_i P_i(n)z_i^n$ where P_i is a polynomial with coefficients in $\mathbb{C}$ whose degree is strictly smaller than d_i.

Nevertheless, we are looking for solutions in R: since J is a real valued function, when z is a root of multiplicity k of $\hat{J}$, so does $\overline{z}$. We conclude that the desired sequences are of the form

$$\begin{aligned} u_n = &\sum_{z_i \in \mathbb{R}} P_i(n)z_i^n \\ &+ \sum_{\Im m(z_i)>0} |z_i|^n[\cos(n\arg(z_i))Q_i(n) + \sin(n\arg(z_i))R_i(n)] \end{aligned}$$

where P_i, Q_i, R_i are real polynomials whose degrees are strictly smaller than d_i.

It defines a real linear space of dimension $\sum_i d_i$. ■

2. THE EXPONENTIAL INTERACTION: AN EXACTLY SOLVABLE MODEL

In all this section, $d = 1$. We study here in more details a particular case of Gibbs measures introduced in the first section, *i.e* the Gibbs measures on $\mathbb{R}^{\mathbb{Z}}$ associated to the potential defined by

$$\begin{cases} \Phi_{\{i\}}(\omega) = \beta\left(\dfrac{K}{2}\omega_i^2 + h\omega_i\right) \\ \Phi_{\{i,j\}}(\omega) = \epsilon\beta c^{|i-j|}\omega_i\omega_j \end{cases} \tag{10}$$

where $\beta, K > 0, h \in \mathbb{R}, \epsilon \in \{+1,-1\}$ and $0 < |c| < 1$.

This corresponds to the interaction coefficients J defined by

$$\begin{cases} J(0) = K \\ J(k) = \epsilon c^{|k|} \quad \text{for } k \in \mathbb{Z}\backslash\{0\} \end{cases}$$

For this potential, we will denote $\mathfrak{G}^{\beta}_{J,h}$ by $\mathfrak{G}(\beta, K, \epsilon, c, h)$. In this section, we draw the phase diagram, *i.e.* we determinate for which values of the parameters we have existence of a Gibbs measure or phase transition.

2.1 Existence of a Gibbs measure in $\mathfrak{G}(\beta, K, \epsilon, c, h)$

2.1.1. Values for which $\hat{J}$ is non negative

From Proposition 1.1, a necessary condition for the existence of a Gibbs measure is the positivity of $\hat{J}$. Let us compute $\hat{J}$.

$$\hat{J}(z) = K + \epsilon \sum_{k\in\mathbb{Z}\backslash\{0\}} c^{|k|}z^k = K + \epsilon c\left(\frac{z}{1-cz} + \frac{z^{-1}}{1-cz^{-1}}\right)$$

Or equivalently

$$\hat{J}(e^{i\theta}) = K + 2\epsilon c\frac{\cos\theta - c}{-2c\cos\theta + 1 + c^2}.$$

A necessary and sufficient condition to have $\hat{J} \geq 0$ is then

$$K \geq K_c^{\epsilon} = \sup_{\theta\in[0,2\pi]} -2\epsilon c\frac{\cos\theta - c}{-2c\cos\theta + 1 + c^2}$$

Since $t \mapsto \frac{t-c}{-2ct+1+c^2}$ is non decreasing $(1 - c^2 > 0)$, we have

$$\frac{1}{1-c} \geq \frac{\cos\theta - c}{-2c\cos\theta + 1 + c^2} \geq -\frac{1}{1+c};$$

the first (resp. the second) inequality is an equality if and only if $\theta = 0$ (resp. $\theta = \pi$).

By a separate analysis for the different values of ϵ and c, we get finally

$$K_c^\epsilon = \frac{2|c|}{1+\epsilon|c|}.$$

2.1.2 Integrability of $\hat{J}^{-1}$

Another necessary condition for the existence of a Gibbs measure in Proposition 1 is the integrability of $\hat{J}^{-1}$ with respect to the Haar measure (condition (2)).

If $K > K_c^\epsilon$, we have for each $z \in \mathbb{U}$ $\hat{J}(z) > 0$, and then $z \mapsto \hat{J}(z)^{-1}$ is a continuous function, therefore it is integrable.

If $K = K_c^\epsilon$, $\hat{J}^{-1}$ is not integrable: we will prove it only for $\epsilon = +1$ and $c > 0$, since the other cases can be proved similarly.

$$\hat{J}(e^{i\theta}) = 2c\left(\frac{1}{1+c} + \frac{\cos\theta - c}{-2c\cos\theta + 1 + c^2}\right)$$

Hence

$$\hat{J}(e^{i\theta})^{-1} = \frac{1}{(1-c)2c} \quad \frac{1+c^2-2c\cos\theta}{1+\cos\theta}$$

and

$$\int_{\mathbb{U}} \hat{J}(z)^{-1}dz = \int_0^{2\pi} \frac{1}{(1-c)4\pi c} \quad \frac{1+c^2-2c\cos\theta}{1+\cos\theta}\, d\theta = +\infty$$

since $\frac{1+c^2-2c\cos\theta}{1+\cos\theta} \sim_\pi \frac{2(1+c)^2}{(\theta-\pi)^2}$.

2.1.3 Domain of existence of Gibbs measure

THEOREM 5. – *Associated to the potential defined by* (10), *there exists a Gibbs measure in* $\mathfrak{G}(\beta, K, \epsilon, c, h)$ *if and only if the self potential is large enough, that is*

$$K > K_c^\epsilon = \frac{2|c|}{1+\epsilon|c|}.$$

Proof. – First and second conditions of Proposition 1 are fulfilled if and only if $K > K_c^\epsilon$. Then, the condition 3 of Proposition 1 is automatically true, because we get $\hat{J}(1) > 0$ and we exhibit the constant sequence equal to $\frac{h}{\hat{J}(1)}$ as an element of M_h^J. ■

2.2 Phase transition

Under the assumption of existence, there exists a phase transition if and only if the set M_h^J contains more than one sequence, or, by linearity, if and only if $M_0^J \neq \{0\}$. Then, we now describe M_0^J.

LEMMA 11. – *Every sequence u in M_0^J satisfies the recurrence equation:*

$$(K-\epsilon)u_{n+1} + \left(-K\left(c+\frac{1}{c}\right)+\epsilon 2c\right)u_n + (K-\epsilon)u_{n-1} = 0$$

Proof. – By the definition of M_0^J, we have (eventually after a reindexation).

$$\begin{cases} J(0)u_n & + & \sum_{i=1}^{+\infty} J(i)u_{n+i} & + & \sum_{i=1}^{+\infty} J(i)u_{n-i} & = & 0 \\ J(0)u_{n+1} & + & \sum_{i=2}^{+\infty} J(i-1)u_{n+i} & + & \sum_{i=0}^{+\infty} J(i+1)u_{n-i} & = & 0 \\ J(0)u_{n-1} & + & \sum_{i=0}^{+\infty} J(i+1)u_{n+i} & + & \sum_{i=2}^{+\infty} J(i-1)u_{n-i} & = & 0 \end{cases}$$

We now multiply the left hand side respectively by $c+\frac{1}{c}, -1, -1$ and sum. Since

$$\forall i \geq 2,\ \left(c+\frac{1}{c}\right)J(i) - J(i+1) - J(i-1) = 0$$

we get

$$J(0)\left[\left(c+\frac{1}{c}\right)u_n - u_{n+1} - u_{n-1}\right] + (u_{n+1}+u_{n-1})J(1)\left(c+\frac{1}{c}\right) \\ -[J(1)u_n + J(2)u_{n-1}] - [J(1)u_n + J(2)u_{n+1}] = 0$$

Computing $J(0), J(1), J(2)$, we get the desired result. ■

LEMMA 12. – *Let $S \in \mathbb{R}$ such that $|S| > 2$. Let c be the exponential parameter defined in* (10). *We set*

$$E_c = \{u = (u_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |c^{|n|}u_n| < +\infty\}$$

$$R_S = \{u = (u_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \forall n \in \mathbb{Z},\ u_{n+1} - Su_n + u_{n-1} = 0\}$$

1. *The following assertions are equivalent*

(a) $E_c \cap R_S \neq \{0\}$
(b) $v \in E_c \cap R_S$ *where* $v_n = \gamma^n$ *and* γ *is a (real) root of* $x^2 - Sx + 1 = 0$
(c) $R_S \subset E_c$
(d) $|S| < |c + \frac{1}{c}|$.

2. *When these assumptions are satisfied, we have*

$$\forall u \in R_S,\ \forall k \in \mathbb{Z},\ \sum_{n \in \mathbb{Z}^*} c^{|n|} u_{k+n} = \frac{S - 2c}{-S + c + c^{-1}} u_k$$

Proof.

$(a) \Rightarrow (b)$ Every sequence in $E_c \cap R_S$ different from the 0 sequence can be written as $u_n = A\gamma^n + B\gamma^{-n}$, where $(A, B) \neq (0, 0)$.
When $A = 0$ or $B = 0$, the result is immediate. Otherwise, if we suppose for example $\gamma > 1$, we get $u_n \sim_{+\infty} A\gamma^n$ and $u_n \sim_{-\infty} B\gamma^{-n}$. Then $\sum_{n>0} c^n \gamma^n$ and $\sum_{n<0} c^{-n} \gamma^n$ are both convergent series: it proves that $v \in E_c$.

$(b) \Rightarrow (c)$ It is immediate, because every sequence in R_S can be written as $u_n = A\gamma^n + B\gamma^{-n}$.

$(c) \Rightarrow (a)$ Immediate, because $R_S \neq \{0\}$.

$(b) \Rightarrow (d)$ (b) implies $\sum_{n \geq 1} [|\gamma|^n + |\gamma|^{-n}] |c|^n < +\infty$

$$\iff \sum_{n \geq 1} |\gamma c|^n + \sum_{n \geq 1} \left| \frac{c}{\gamma} \right|^n < +\infty.$$

Thus $(b) \Leftrightarrow\ |c| < |\gamma| < \frac{1}{|c|}$. By considering the map $x \mapsto x + \frac{1}{x}$, it is easy to see that $|c| < |\gamma| < \frac{1}{|c|}$ is equivalent to $|\gamma + \gamma^{-1}| < |c + c^{-1}|$. But $\gamma + \gamma^{-1} = S$, so $(b) \Leftrightarrow\ |S| < |c + \frac{1}{c}|$.

To show the second part of the lemma, it suffices, by linearity, to prove it for $v_n = \gamma^n$, where γ is a root of $x^2 - Sx + 1 = 0$.

$$\begin{aligned}\sum_{n \geq 1} c^n [\gamma^{k+n} + \gamma^{k-n}] &= \gamma^k \sum_{n \geq 1} (c\gamma)^n + \left(\frac{c}{\gamma}\right)^n \\ &= \gamma^k \left[\frac{c\gamma}{1 - c\gamma} + \frac{c/\gamma}{1c/\gamma}\right] \\ &= \gamma^k \left[\frac{S - 2c}{-S + c + c^{-1}}\right]\end{aligned}$$

(Of course, we use the fact that $\gamma + \gamma^{-1} = S$.) ∎

Let us return now to the initial problem of phase transition. Let $K > K_c^\epsilon$.

We apply the results of Lemma 11: when $K = 1$ and $\epsilon = 1$, we have $M_0^J = \{0\}$.

Otherwise, $K - \epsilon \neq 0$. Thus, we set

$$S = \frac{K(c+\frac{1}{c}) - \epsilon 2c}{K - \epsilon}$$

Let us suppose that $|S| > 2$.

We get $M_0^J \subset R_S$, where $M_0^J = M_0^J \cap E_c \subset R_S \cap E_c$.

Using Lemma 7 1., we have $M_0^J \neq \{0\} \Rightarrow |S| < |c + \frac{1}{c}|$.

Conversely, if $|S| < |c+\frac{1}{c}|$, since $S = \frac{K(c+\frac{1}{c})-\epsilon 2c}{K-\epsilon} \Leftrightarrow K = -\epsilon\frac{S-2c}{-S+c+c^{-1}}$, the second part of Lemma 7 implies

$$|S| < \left|c + \frac{1}{c}\right| \Rightarrow R_S \subset M_0^J.$$

This implies $M_0^J \neq \{0\}$.

We have now to determinate when $|S| < |c + \frac{1}{c}|$ and to verify that $K > K_c^\epsilon$ implies $|S| > 2$. To this aim, let us study the map $f(K)$

$$K \mapsto \frac{K(c+\frac{1}{c}) - \epsilon 2c}{K - \epsilon}$$

It is a rational function whose discriminant is $-\frac{\epsilon}{c}(1 - c^2)$. We separate four different cases.

1. $\epsilon = +1 \quad c > 0$

K	0		$K_c^\epsilon = \frac{2c}{1+c}$		$\frac{3c^2+1}{2(1+c^2)}$			1			$+\infty$
f	$2c$	$\searrow$	-2	$\searrow$	$-(c+\frac{1}{c})$	$\searrow$	$-\infty$	$\Vert$	$+\infty$	$\searrow$	$c+\frac{1}{c}$

2. $\epsilon = +1 \quad c < 0$

K	0		$K_c^\epsilon = \frac{-2c}{1-c}$		$\frac{3c^2+1}{2(1+c^2)}$			1			$+\infty$
f	$2c$	$\nearrow$	2	$\nearrow$	$-(c+\frac{1}{c})$	$\nearrow$	$+\infty$	$\Vert$	$-\infty$	$\nearrow$	$c+\frac{1}{c}$

3. $\epsilon = -1 \quad c > 0$

K	0		$K_c^\epsilon = \frac{2c}{1-c}$		$+\infty$
f	$2c$	$\nearrow$	2	$\nearrow$	$(c+\frac{1}{c})$

K	0		$K_c^\epsilon = \frac{-2c}{1+c}$		$+\infty$
f	$2c$	$\searrow$	-2	$\searrow$	$(c+\frac{1}{c})$

4. $\epsilon = -1 \; c < 0$

It is then clear that for $K > K_c^\epsilon$, $|S| > 2$.

For $\epsilon = -1$, we have $|S| < |c + \frac{1}{c}|$ for all $K > K_c^\epsilon$.

For $\epsilon = +1$, on a $|S| < |c + \frac{1}{c}|$ if and only if $K < \frac{3c^2+1}{2(1+c^2)}$.

Then, we have proved

THEOREM 6. – 1. *For $\epsilon = -1$, the set $\mathfrak{G}(\beta, K, \epsilon, c, h)$ is non-empty if and only if $K > K_c^{-1} = \frac{2|c|}{1-|c|}$. Then, there exists a phase transition, since the set M_0^J is a two-dimensional linear space.*

2. *For $\epsilon = +1$, the set $\mathfrak{G}(\beta, K, \epsilon, c, h)$ is non-empty if and only if $K > K_c^1 = \frac{2|c|}{1+|c|}$. Then, there exists a phase transition if and only if $K < \frac{3c^2+1}{2(1+c^2)}$, in which case M_0^J is a two-dimensional linear space.*

Remarks. – For $\alpha < \frac{1}{c}$, we have $J \in A_\alpha$, so Theorem 4 gives the description of $M_0^J \cap B_\alpha$. But, by Lemma 12, every element in M_J^0 belongs to B_γ, where γ is the largest root of $x^2 - Sx + 1 = 0$. But, by Lemma 12 $|S| < c + \frac{1}{c}$, so $|\gamma| < \frac{1}{c}$. Then, we have

$$M_0^J = \cup_{\alpha < \frac{1}{c}} (M_0^J \cap B_\alpha).$$

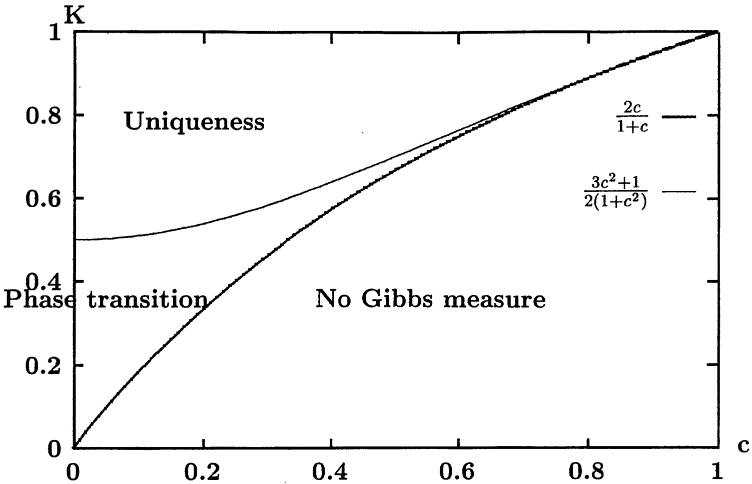
This example shows that sometimes, the fact to restrict us to the study of the subsets $M_0^J \cap B_\alpha$ is not a restriction at all.

We can remark here that the phase diagramm does neither depend on β –it is a general fact in the case of the quadratic Hamiltonian– nor on h. ■

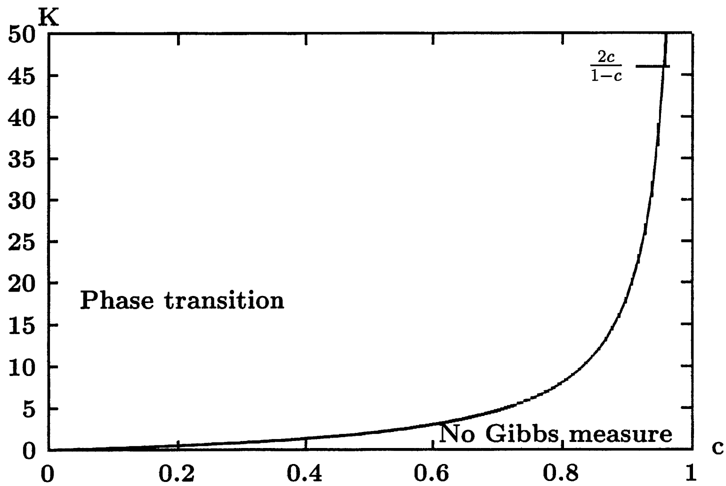
3. CONCLUDING REMARKS ON THE INSTABILITY OF UNIQUENESS OF GIBBS MEASURES

The instability of the uniqueness of phase under small perturbations on the potential is one of the difficulties for the study of unbounded potentials with infinite range.

Let us give a simple example in our framework. In section 2, we have exhibited non trivial cases of uniqueness of phase for a potential with infinite range. Since the potential decreases exponentially, we could expect that, after truncation at a large radius,we would get a potential with the



Phase diagram for $\epsilon = 1$ and $c \in]0, 1[$.



Phase diagram for $\epsilon = -1$ and $c \in]0, 1[$.

same property of uniqueness. But theorem 2 shows it is false: if we denote by J^R the truncated potential, $z^R \widehat{J^R}$ is a non constant polynomial, so $\widehat{J^R}$ does have a root which belongs to the critical annulus $\mathbb{U}_{r_a}$ for large r_a: since J^R has finite range, it belongs to every A_a. The trouble is that the roots of $\widehat{J^R}$ went to infinity when R tends to infinity and do not stay in the annulus $\mathbb{U}_{r_a}$. How explain this discontinuity phenomenon? The first reaction is drastic: As mathematician physicist, we can eliminate from the set of Gibbs measures the elements which have no "physical reality". It is the idea beside Corollaries 3 and 5; in this more restrictive sense of Gibbs measure, we recover the stability of the uniqueness of phase.

A mathematical explanation for this discontinuity phenomenon is that the set

$$\{(u_n)_{n\in\mathbb{Z}^d} : \forall k \in \mathbb{Z}^d \sum_{n\in\mathbb{Z}^d} |J(n)u_{k+n}| < +\infty\}$$

in which we seek *a priori* harmonic functions is not a normed space: that makes its topology harder to describe.

Nevertheless, we want to state a result which shows the stability of uniqueness in a weak sense.

THEOREM 7. – *Let* $J \in A_a$ *such that* $M_0^J \cap B_a = \{0\}$.
Then, there exists $\epsilon > 0$ *such that*

$$\forall K \in A_a,\ ||J - K||_{A_a} < \epsilon \Longrightarrow M_0^K \cap B_a = \{0\}.$$

Proof. – It is an easy consequence of theorem 1 and the fact that $G(A_a)$ is open. ■

It shows that small restrictions for the Gibbs measures, *i.e.* the choice of a certain class of measures, can allow to preserve the stability of the uniqueness.

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REFERENCES

[1] J.M. ARNAUDIÈS and H. FRAYSSE, Cours de mathématiques - 1 Algèbre, Dunod, 1987.
[2] G. BENFATTO, E. PRESUTTI and M. PULVIRENTI, DLR measures for one-dimensional harmonic systems, *Z. Wahrsch. Verw. Geb.*, Vol. **41**, 1978, pp. 305–312.
[3] B. CHABAT, *Introduction à l'analyse complexe*, Mir, 1990.
[4] S. C. CHAY, On quasi-markov random fields, *J. of Multivariate Analysis*, Vol. **2**, 1972, pp. 14–76.
[5] R.L. DOBRUSHIN, Gaussian random fields - Gibbsian point of view. In R.L. Dobrushin and Ya.G. Sinai, editors, *Multicomponent random systems*, volume 6 of *Advances in Probability and Related Topics*, Dekker, 1980.

[6] O. GARET, Mesures de Gibbs gaussiennes et dynamiques aléatoires associées sur $\mathbb{R}^{\mathbb{Z}^d}$, Thèse de doctorat, Lille, 1998.
[7] H.-O. GEORGII, Gibbs Measures and Phase Transitions, Walter de Gruyter, 1988.
[8] J.P KAHANE and R. SALEM, Ensembles parfaits et séries trigonométriques, Hermann, 1994.
[9] H.-R. KÜNSCH, Gaussian markov random fields, *J. Fac. of Science of Univ. of Tokyo*, Vol. **26**, 1979, pp. 53–73.
[10] H.-R. KÜNSCH, Reellwertige Zufallsfelder auf einem Gitter: Interpolationsprobleme, Variationsprinzip und statistische Analyse, PhD thesis, Zürich, 1980.
[11] G.K. PEDERSEN, Analysis Now, Springer-Verlag, 1995.
[12] Yu. A. ROSANOV, On Gaussian fields with given conditional distributions, *Th. of Prob. Appl.*, Vol. **12**, 1967, pp. 433–443.

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