ANNALES DE L'I. H. P., SECTION B

SANJAR ASPANDIIAROV
ROUDOLF IASNOGORODSKI

Three-dimensional reflected driftless random walks in troughs : new asymptotic behavior

Annales de l'I. H. P., section B, tome 35, nº 1 (1999), p. 49-83
<http://www.numdam.org/item?id=AIHPB_1999__35_1_49_0>

© Gauthier-Villars, 1999, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (http://www.elsevier.com/locate/anihpb) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 35, n° 1, 1999, p. 49-83 *Probabilités et Statistiques*

Three-dimensional reflected driftless random walks in troughs: new asymptotic behavior

by

Sanjar ASPANDIIAROV

Université Paris V*,
UFR de Mathématiques et Informatique,
45, rue des Saints-Pères, 75270 Paris Cedex 06
e-mail address: aspandij@descartes.math-info.univ-paris5.fr

and

Roudolf IASNOGORODSKI

Université d'Orléans*,
MAPMO 1803, UFR Sciences BP 6759,
45067 Orléans Cedex 02
e-mail address: iasno@galois.labomath.univ-orleans.fr

ABSTRACT. – The present work deals with reflected random walks in the trough $Z_+^2 \times Z$ having a zero mean-drift in the interior of the domain. It is proved that in all non-critical cases the random walk is transient whereas it is recurrent in one critical subcase. The convergence problem for renormalized random walks is also studied. Two possible cases are separated. In the first one, so-called semimartingale case, the renormalized in a standard way random walks converge weakly to a semimartingale reflected Brownian motion described in [24], whereas in the second case the random walks exhibit a non-trivial behavior. Namely, one of its coordinates tends to infinity almost surely faster than $\sqrt{n}$. The obtained results are based on the estimates of the invariant measure of a driftless random walk in a wedge derived in [6]. © Elsevier, Paris

Key words: Limit theorems, recurrence classification, reflected random walks, semimartingale reflected Brownian motion.

**A.M.S. 1991 subject classification.* Primary 60 G 42, 60 J 10; Secondary 60 J 60.

RÉSUMÉ. – L'article est consacré à l'étude de marches aléatoires dans le quart d'espace $Z_+^2 \times Z$ avec drift nul à l'intérieur et réflection au bord du domaine. Nous montrons tout d'abord que cette marche est toujours transiente dans tous les cas non-critiques de réflection au bord. Nous nous intéressons ensuite au problème de convergence des marches renormalisées et montrons que deux situations sont possibles : le processus limite est une semimartingale continue décrite dans [24] ou bien la marche se comporte d'une facon inhabituelle, notamment, une des coordonnées tend vers l'infini presque sûrement avec la vitesse supérieure à $\sqrt{n}$. Pour l'obtenir on s'appuie sur des estimations de [6] pour la mesure invariante de la marche projetée à valeurs dans Z_+^2. © Elsevier, Paris

1. INTRODUCTION

Stochastic processes with boundary conditions in domains of R^d with smooth boundary such as random walks or diffusion processes have been studied quite extensively these last years. However, problems appearing in queuing networks theory, stochastic algorithms and some other applied fields made obvious the necessity of the study of stochastic processes with boundary reflection in non-smooth domains and with a discontinuous boundary reflection. Of particular interest here are the questions of existence and uniqueness of diffusion processes in such domains as well as those of recurrence classification of their discrete counterparts, i.e. driftless random walks.

One of first results dealing with these questions was obtained by S.R.S. Varadhan and R.J. Williams in [25], who found necessary and sufficient conditions for existence and uniqueness of a Brownian motion in a wedge with a constant (on each side of the wedge) oblique reflection from the boundary. It was shown later in [1] that any reflected Brownian motion in a wedge can be obtained as a weak limit of renormalized in a standard way reflected random walks with zero-drift in the interior of the wedge (see Theorem 3 in [1]). Moreover, in papers [2], [4], [5], [17] it has been proved that properties of reflected random walks and diffusion processes such as recurrence classification or passage-times are closely related to each other though not always identical. Namely, a recent study of two-dimensional reflected random walks brought into light some new phenomena for invariant measures having no analogues for reflected diffusions (see Remark 6, Theorem 7 in [6]).

The next important step in construction of reflected diffusions was initiated by J.M. Harrison, M. Reiman and R.J. Williams who came up with the notion of the so-called Semimartingale Reflecting Brownian Motion (SBRM). For the history of the subject and references we invite the reader to consult a recent survey by R.J. Williams [27]. Loosely speaking, these processes have a semimartingale decomposition and behave like diffusions with a constant drift and covariance matrix in the interior of the domain and their bounded variation parts increase in a prescribed direction when they hit the boundary in order to confine the processes in the domain. The existence and uniqueness of such processes in different polyhedron-like domains was then established in [8], [24]. Similarly to the two-dimensional case, for particular cases of reflection fields multidimensional SBRMs were shown to be approximations to renormalized reflected random walks (see e.g. [11], [19], [22]).

However, the obtained necessary and sufficient conditions under which SBRMs exist and unique or may be represented as a weak limit of discrete processes do not cover all possible reflection fields even in the simplest domains like n-dimensional troughs. The construction of reflected diffusions in dimensions greater than 2 with a "non-semimartingale" reflection on the boundary is an open theoretical problem which solution would also be of benefit for applications. Further discussion of the problem and its current state can be found in [26], [27].

As a first step in the study of non-semimartingale reflected diffusions we investigate recurrence and limiting properties of three-dimensional reflected random walks which is an interesting problem in itself. It is well known that even for the simplest case of unconstrained driftless random walks in R^n the transition from $n = 2$ to 3 brings a different recurrent behavior. Similarly, the addition of a boundary reflection even in the planar case may transform a null recurrent process in Z^2 into an ergodic (positive recurrent) or a transient one in Z^2_+ (see [2], [12]). Let us notice that in dimension 2 the boundary reflection does not much affect the limiting behavior of random walks (provided, of course, that they remain non-ergodic) in a sense that being renormalized in a standard $\sqrt{n}$ way they still converge. Moreover, this limiting behavior remains stable with respect to changes of the boundary reflection at the faces of co-dimension greater than 1 (i.e. the origin (0,0) in Z^2_+). That is whatever the reflection from the origin be, the limiting process remains the same. In the present work we are interested in driftless reflected random walks in $Z_+^2 \times Z$ whereas in the forthcoming paper [3] we will study the case Z_+^3. The questions to be addressed are the following:

1. What happens with a transient driftless random walk in Z^3 if one considers it only in the trough $Z_+^2 \times Z$ and adds a boundary reflection which is constant on each particular face ?
2. What will the asymptotic behavior of $Z_+^2 \times Z$-valued driftless random walk be? For instance, will it be diffusion (i.e. $\sqrt{n}$) one? If so, what are its limiting processes?
3. Would changes of the reflection from the edge of the trough $Z_+^2 \times Z$ affect this limiting behavior?

The answers obtained are valid for all non-critical boundary reflections and one critical subcase described in Definition 1. In non-critical case, it will be shown that whatever random walk in Z_+^2 be (i.e. transient, null or positive recurrent), the addition of the third unconstrained coordinate makes it always transient. We should mention here that even though the result looks very similar to that for unconstrained random walks in Z^3, it has a different nature. As for asymptotic behavior in the non-critical case, it will be proved in Theorem 2 that the following two situations are possible. In the first one, the $\sqrt{n}$-renormalized reflected random walks converge to the corresponding SBRM in the trough and the boundary reflection on the edge of the trough $Z_+^2 \times Z$ does not affect the limiting process. However, in the other, "non-semimartingale" case, the asymptotic behavior differs from the standard one. Namely, one has the almost sure drift of the process along the edge to $+\infty$ or $-\infty$ with the speed n^β, where $\beta > 1/2$. Furthermore, the direction of the drift depends heavily on the boundary conditions on the edge. It turns out that the main reason for this non-trivial behavior lies in ergodic properties of the "projected" random walk in Z_+^2 obtained in [6].

We also prove that contrary to the non-critical case, in one critical subcase the random walk is recurrent. Moreover, in this situation, even though $\alpha > 1$, the $\sqrt{n}$-renormalized reflected random walks still form a weakly relatively compact family. As we suspect, it can be further proved that these processes converge weakly to a continuous process which can be then called non-semimartingale diffusion process in $Z_+^2 \times Z$.

We believe the principal results of the paper provide further insight to the intricacies of multidimensional reflected diffusions.

2. NOTATION AND MAIN RESULTS

In the sequel G is the three-dimensional domain $G = R_+^2 \times R$. Its two faces and the edge are denoted by ∂G_1, ∂G_2 and ∂G_3, i.e.

$\partial G_1 = \{(z_1, z_2, z_3) \in G;\ z_1 = 0, z_2 \neq 0\}$ and $\partial G_2 = \{(z_1, z_2, z_3) \in G;\ z_2 = 0, z_1 \neq 0\}$. The interior of G is referred to as ∂G_0. For each $l = 0, 1, 2, 3$ let $\partial \widetilde{G}_l = \partial G_l \cap \{z_3 = 0\}$ and $\partial \widetilde{G} = \cup_{l=0,1,2,3} \partial \widetilde{G}_l$.

We will study an irreducible aperiodic discrete-time $Z_+^2 \times Z$-valued Markov chain $\{Z_n = (Z_1^n, Z_2^n, Z_3^n),\ n \geq 0\}$ with the following transition mechanism. The MC starting from the point $z = (z_1, z_2, z_3)$ of $Z_+^2 \times Z$ jumps to $(z_1 + i, z_2 + j, z_3 + k)$, $i, j \geq -1, k \in Z$ with transition probabilities $p_{i,j,k}^0$, (respectively $p_{i,j,k}^1$, $p_{i,j,k}^2, p_{i,j,k}^3$) according to $(z_1, z_2, z_3) \in \partial G_0$ (respectively ∂G_1, ∂G_2, ∂G_3) which satisfy the following moment conditions:

1. For any $l = 0, 1, 2, 3$, and for some $\delta > 0$,

$$\sum_{i,j,k} \left(|i|^{2+\delta} + |j|^{2+\delta} + |k|^{2+\delta}\right) p_{i,j,k}^l < \infty. \tag{1}$$

2. For any $i, j \geq -1, k \in Z$,

$$p_{-1,j,k}^1 = p_{i,-1,k}^2 = p_{i,-1,k}^3 = p_{-1,j,k}^3 = 0. \tag{2}$$

3. Boundedness from below of z_3-increments, i.e. there exists a positive k_0 such that for any $l = 0, 1, 2, 3$ and $i, j \geq -1$,

$$p_{i,j,k}^l = 0, \quad \text{if } k < -k_0. \tag{3}$$

4. a) Zero-drifts in the interior of the domain ∂G_0, i.e.

$$\sum_{i,j,k} i p_{i,j,k}^0 = \sum_{i,j,k} j p_{i,j,k}^0 = \sum_{i,j,k} k p_{i,j,k}^0 = 0. \tag{4}$$

b) For each $l = 0, 1, 2, 3$ let $C^l = (c_{i,j}^l)$, $l = 0, 1, 2, 3$ be the 3×3-dimensional covariance matrix of the jumps distribution from ∂G_l with elements

$$c_{11}^l = \sum_{i,j,k} i^2 p_{i,j,k}^l, \quad c_{12}^l = \sum_{i,j,k} ij p_{i,j,k}^l, \quad c_{13}^l = \sum_{i,j,k} ik p_{i,j,k}^l, \text{ etc.} \tag{5}$$

It is assumed that $c_{11}^0 c_{22}^0 - (c_{12}^0)^2 > 0$.

5. Non-degenerate reflection from the boundary $\partial G_1 \cup \partial G_2$, i.e.

$$\sum_{i,j,k} i p_{i,j,k}^1 \neq 0, \quad \sum_{i,j} j p_{i,j,k}^2 \neq 0, \quad \text{and} \quad \max\Big(\sum_{i,j,k} i p_{i,j,k}^3, \sum_{i,j,k} j p_{i,j,k}^3\Big) > 0. \tag{6}$$

Let us introduce a new discrete-time MC $\{\widetilde{Z}_n,\ n \geq 0\}$ obtained by projecting $\{Z_n,\ n \geq 0\}$ on the plane $\{z_3 = 0\}$, i.e. $(\widetilde{Z}^1_n, \widetilde{Z}^2_n) = (Z^1_n, Z^2_n)$,. Its transition probabilities from (z_1, z_2) of $\mathrm{Z}^2_+ \cap \partial\widetilde{G}_l, l = 0,1,2,3$ to $(z_1 + i, z_2 + j),\ i,j \geq -1$ are given by

$$\widetilde{p}^l_{i,j} = \sum_k p^l_{i,j,k}. \tag{7}$$

The vectors $\vec{P} = (p_1, p_2, p_3)$, $\vec{Q} = (q_1, q_2, q_3)$, $\vec{R} = (r_1, r_2, r_3)$, (resp. $\widetilde{P} = (p_1, p_2)$, $\widetilde{Q} = (q_1, q_2)$, $\widetilde{R} = (r_1, r_2)$) with

$$\begin{array}{lll} p_1 = \sum_{i,j,k} i p^1_{i,j,k}, & p_2 = \sum_{i,j,k} j p^1_{i,j,k}, & p_3 = \sum_{i,j,k} k p^1_{i,j,k}; \\ q_1 = \sum_{i,j,k} i p^2_{i,j,k}, & q_2 = \sum_{i,j,k} j p^2_{i,j,k}, & q_3 = \sum_{i,j,k} k p^2_{i,j,k}; \\ r_1 = \sum_{i,j,k} i p^3_{i,j,k}, & r_2 = \sum_{i,j,k} j p^3_{i,j,k}, & r_3 = \sum_{i,j,k} k p^3_{i,j,k} \end{array} \tag{8}$$

will be called the vectors of the boundary reflection of the MC Z (resp. $\widetilde{Z}$). Notice that the condition (01) simply means that the vectors $\vec{P}, \vec{Q}$ are non-tangent to the respective faces and $\widetilde{R}$ is non-zero. The 3×3-matrix which columns are formed by the coordinates of $\vec{P}, \vec{Q}, \vec{R}$ (resp. $\vec{P}, \vec{Q}, \vec{0}$) will be called the reflection matrix and denoted by S (resp. S^0) whereas the 2×2-matrix formed by $\widetilde{P}, \widetilde{Q}$ will be denoted by $\widetilde{S}$. The submatrix

$$\begin{pmatrix} c^0_{11} & c^0_{12} \\ c^0_{21} & c^0_{22} \end{pmatrix}$$

will be denoted by $\widetilde{C}^0$.

Let Φ be any isomorphism of R^2 transforming the quadratic form $c^0_{11}x^2 + 2c^0_{12}xy + c^0_{22}y^2$ into $x^2 + y^2$. For instance, such Φ can be defined as follows

$$\begin{cases} u = a(bx - ry) \\ v = ya\sqrt{1 - r^2} \end{cases}, \tag{9}$$

where

$$a = \sqrt{\frac{c^0_{11}}{c^0_{11}c^0_{22} - (c^0_{12})^2}}, \quad b = \sqrt{\frac{c^0_{22}}{c^0_{11}}}, \quad r = \frac{c^0_{12}}{\sqrt{c^0_{11}c^0_{22}}}$$

Then $H = \Phi(\widetilde{G})$ is the wedge of angle $\xi = \arccos(-r)$. Let α_1, α_2 be the angles that $\Phi(\widetilde{P}), \Phi(\widetilde{Q})$ make with the inward normals to the corresponding sides of the wedge H positive angles being toward the corner (the exact expression for α in terms of $\widetilde{p}^{l}_{i,j}$ is given in [1]). Set

$$\alpha = \frac{\alpha_1 + \alpha_2}{\xi},$$

where

As was proved in [4], [5], [2], the parameter α fully determines the recurrence classification of the Markov chain $\{\widetilde{Z}_n,\ n \geq 0\}$. Namely, $\widetilde{Z}$ is recurrent, if and only if $\alpha \geq 0$ and is positive recurrent if and only if $\alpha > 2$. In what follows, we will also need to distinguish the case $\alpha < (\geq) 1$. These conditions can be expressed in terms of the original transition probabilities.

Remark 1.

$$\begin{aligned}
&\alpha = 1\ (resp. < 1)\ \ if\ and\ only\ if\ \ det\,(\widetilde{S}) = p_1q_2 - p_2q_1 = 0\ (resp. > 0)\\
&\alpha = 0\ (resp. < 0)\ \ if\ and\ only\ if\ \ - c^0_{11}\frac{q_2}{q_1} - c^0_{22}\frac{p_1}{p_2} + 2c^0_{12} = 0\ (resp. < 0)\\
&\alpha = 2\ (resp. > 2)\ \ if\ and\ only\ if\ \ p_1q_2 - p_2q_1 < 0\\
&\qquad\qquad\qquad and - c^0_{11}\frac{p_2}{p_1} - c^0_{22}\frac{q_1}{q_2} + 2c^0_{12} = 0\ (resp. > 0).
\end{aligned}$$

DEFINITION. – *The critical case is the case when either a)* $\alpha = 1$ *or b)* $\alpha > 1$ *and* $det\,(S) = 0$.

Our main results are formulated as follows.

THEOREM 1. – *(Recurrence classification (non-critical case)).*
The MC $\{Z_n,\ n \geq 0\}$ *is transient.*

THEOREM 2. – *(Stability theorem (non-critical case)).*
Let the initial value Z_0 *be a constant vector with probability* 1.

1 *If* $\alpha < 1$, *then the processes* $\left\{n^{-\frac{1}{2}} Z_{[n\cdot]},\ n \geq 1\right\}$ *converge weakly in the Skorokhod topology to W, where* $\{W_t,\ t \geq 0\}$ *is an SBRM in G starting from (0,0,0) with the reflection matrix* S^0, *zero-drifts and the covariance matrix* C^0 *(for definition of W see [24]).*

2 *If* $\alpha \in (1,2)$, *then the processes* $\left\{n^{-\frac{1}{2}}(Z^1_{[n\cdot]}, Z^2_{[n\cdot]}),\ n \geq 1\right\}$ *converge weakly in the Skorokhod topology to* $\widetilde{W}$, *where* $\{\Phi(\widetilde{W}_t),\ t \geq 0\}$ *is a (non-semimartingale) Brownian motion with oblique reflection* $\Phi(\widetilde{P}), \Phi(\widetilde{Q})$ *in the wedge H starting from (0,0) (a definition of* $\widetilde{W}$

can be found in [25]). However, the third coordinate of Z has a different asymptotic behavior. Namely, for any $\epsilon \neq 0$,

$$\frac{Z_n^3}{n^{\frac{\alpha}{2}+\epsilon}} \xrightarrow{\text{a.s.}} \begin{cases} +\infty, & if\ det\ (S) < 0,\ and\ \epsilon < 0, \\ -\infty & if\ det\ (S) > 0,\ and\ \epsilon < 0, \\ 0, & if\ det\ (S) \neq 0,\ and\ \epsilon > 0. \end{cases} \tag{10}$$

3 *If $\alpha = 2$, then $\left\{n^{-\frac{1}{2}}(Z^1_{[n\,\cdot]}, Z^2_{[n\,\cdot]}),\ n \geq 1\right\}$ converge weakly to $(0,0)$ and (10) holds.*

4 *If $\alpha > 2$, then*

$$\left(Z_n^1, Z_n^2\right) \xrightarrow{w} (a,b), \quad \frac{Z_n^3}{n} \xrightarrow{\text{a.s.}} c, \tag{11}$$

where the random vector (a,b) is distributed according to the stationary distribution π of the MC $\{\widetilde{Z}_n,\ n \geq 0\}$ and the constant c is equal to

$$c = \frac{\pi(\partial \widetilde{G}_3)\text{det }(S)}{\text{det }(\widetilde{S})}. \tag{12}$$

As for critical cases, we were able to obtain reasonable answers only in the critical subcase b).

THEOREM 3. – *(Critical subcase b)).*
If $\alpha > 1$ and $\det(S) = 0$, *then:*
1. The MC Z is recurrent.
2. The family of renormalized MC $\left\{n^{-\frac{1}{2}} Z_{[n\,\cdot]},\ n \geq 1\right\}$ is weakly relatively compact.

CONJECTURE. – We suspect that:
1. In the critical subcase a) the MC Z is transient.
2. In the critical subcase b) the processes $\left\{n^{-\frac{1}{2}} Z_{[n\,\cdot]},\ n \geq 1\right\}$ converge weakly to a continuous three-dimensional process in G, which might be then called non-semimartingale Brownian motion.

Before starting the proof let us introduce some more notation.

DEFINITION 2. – *For each $n \geq 0$, let $\mathcal{F}_n$ and $\widetilde{\mathcal{F}}_n$ be the filtrations $\sigma\{Z_1, \ldots, Z_n\}$ and $\sigma\{(Z_1^1, Z_1^2), \ldots, (Z_n^1, Z_n^2)\}$. For each $i \geq 1$, let us define $Y_i = Z_i - Z_{i-1}$ and the $\mathcal{F}$-martingale $\{M_k,\ k \geq 0\}$,*

$$M_k = \sum_{i=1}^{k} [Y_i - E(Y_i | Y_i, \ldots, Y_{i-1})] = \sum_{i=1}^{k} [Y_i - E(Y_i | Z_{i-1})].$$

Remark 2.

$$E(Y_i|Z_{i-1}) = E(Y_i|\widetilde{Z}_{i-1}) = \vec{P}1_{(\widetilde{Z}_{i-1}\in\partial\widetilde{G}_1)}+\vec{Q}1_{(\widetilde{Z}_{i-1}\in\partial\widetilde{G}_2)}+\vec{R}1_{(\widetilde{Z}_{i-1}\in\partial\widetilde{G}_3)}.$$

Notation. – For each $j = 1, 2, 3$, π_n^j denotes the total time spent by the process Z (or $\widetilde{Z}$) on the boundary ∂G_j (or $\partial\widetilde{G}_j$) up to the time n, i.e

$$\pi_n^j = \sum_{i=0}^{n} 1_{(Z_i\in\partial G_j)} \equiv \sum_{i=0}^{n} 1_{(\widetilde{Z}_i\in\partial\widetilde{G}_j)}. \tag{13}$$

Throughout the paper we adopt the usual convention $\sum_{i=1}^{0} a_i \equiv 0$. The convergence in probability will be denoted by $\xrightarrow{P}$.

The proof of the main results varies according to the values of α.

3. PROOF OF THE MAIN RESULTS

3.1 Preliminaries

3.1.1. *One local limit theorem for sum of independent random variables*

We start with one local limit theorem for sum of independent non-degenerate random variables with values in Z.

DEFINITION 3 ([20], Chapter 7.1). – *Random variables* $\{X_n, n \geq 0\}$ *form k-sequence, if the set of all distinct distributions of the variables* X_n *contains* k *elements.*

Before enunciating the main result of this section let us discuss some properties of characteristic functions of integer-valued random variables.

LEMMA 1. – *Let* X *be a r.v. (not necessarily Z-valued) with characteristic function* ϕ. *Suppose for some* $t_0 \in$ R, $|\phi(t_0)| = 1$. *Then for any integer* $m > 0$, *such that* $E|X|^m < \infty$,

$$\frac{d^m\phi(t_0)}{dt^m} = i^m E(X^m)\phi(t_0). \tag{14}$$

If $m \geq 2$, *then there are a positive* $\epsilon = \epsilon_{t_0}$ *and a continuous at* t_0 *function* $\delta_{t_0}(t)$ *such that* $\delta_{t_0}(t_0) = 0$ *and for any* $t \in [t_0 - \epsilon, t_0 + \epsilon]$,

$$\begin{aligned}\log(\phi(t)) =& i\,arg\,(\phi(t_0)) + iE(X)(t - t_0)\\ &- \frac{Var(X)}{2}(t - t_0)^2 + Var(X)(t - t_0)^2\delta_{t_0}(t).\end{aligned} \tag{15}$$

Moreover, if $t_0 \neq 0$, then there is $a \in \mathrm{R}$ such that with probability 1, the random variable X takes values on the lattice $\{a + \frac{2\pi}{t_0}k,\ k \in \mathrm{Z}\}$.

Proof of Lemma 1. – We first prove (14). If $t_0 = 0$, then the assertion is a well-known property of characteristic functions. Let now $t_0 \neq 0$. By assumptions there is $a \in \mathrm{R}$ such that $\phi(t_0) = \mathrm{e}^{it_0 a}$, or, equivalently, $E(\mathrm{e}^{it_0(X-a)}) = 1$. Hence, $E(\cos(t_0(X-a))) = 1$. Since $\cos(x) \leq 1$, then with probability 1, $\cos(t_0(X-a)) = 1$. Consequently, with probability 1, $\sin(t_0(X-a)) = 0$ and $P(X \in \{a + dk,\ k \in \mathrm{Z}\}) = 1$ with $d = \frac{2\pi}{t_0}$. Also $\mathrm{e}^{it_0 X} = \mathrm{e}^{it_0 a} = \phi(t_0)$. Then, $\frac{d^m \phi(t_0)}{dt^m} = i^m E(X^m \mathrm{e}^{it_0 X}) = i^m E(X^m)\phi(t_0)$.

As for (15), in both cases $t_0 = 0$ and $t_0 \neq 0$ it follows from the Taylor's formula of the second order for the function $\log(\phi)$ in ϵ-neigbourhood of t_0 with sufficiently small ϵ for which in this neigbourhood of t_0 $|\phi(t) - \phi(t_0)| < 1$. ■

As consequence we obtain one useful property of Z-valued random variables X. For any such X there are some $a \in \mathrm{Z}, d \in \mathrm{N},\ d > 0$ such that $P(X \in \{a + dk,\ k \in \mathrm{Z}\}) = 1$. Although such a and d are not unique, one can still choose a maximal positive d for which this property is verified. In fact, take any x_1, x_2 belonging to the lattice of values of X, i.e. $x_1 = a + dk_1, x_2 = a + dk_2$ for some $k_1, k_2 \in \mathrm{Z}$. Then $d = \frac{x_1 - x_2}{k_1 - k_2}$ which shows that d belongs to the set $\{\frac{|x_1 - x_2|}{k},\ k \in \mathrm{Z}\}$. This allows us to choose uniquely the maximal d.

DEFINITION 4. – *Let X be a Z-valued random variable. Maximal span of the variable X is the maximal number $d \in \mathrm{N}, d > 0$ for which $P(X \in \{a + dk,\ k \in \mathrm{Z}\}) = 1$ with some $a \in Z$.*

The last lemma easily yields the following result.

COROLLARY 1. – *Let X be a Z-valued random variable with maximal span d. Then $|\phi(t_0)| = 1$ if and only if $t_0 = \frac{2\pi k}{d}$ for some $k \in \mathrm{Z}$.*

Remark 3. – *It is immediate by the last corollary that one necessary and sufficient condition for the existence of $t_0 \neq 0, t_0 \in [-\pi, \pi]$ such that $|\phi(t_0)| = 1$ is that the maximal span d is greater or equal to two.*

PROPOSITION 1. – *Let $\{X_n,\ n \geq 0\}$ be a k-sequence of independent square integrable random variables taking values in Z. Denote by S the set of possible distributions of the variables X_n. For any $n \geq 1$, set $A_n = \sum_{i=1}^{n} E(X_i)$ and $B_n = \sum_{i=1}^{n} Var(X_i)$. Then*

1. *There exists a positive constant C depending only on S such that for any $n \geq 1$,*

$$\sup_{x\in Z} P\left(\sum_{i=1}^{n} X_i = x\right) \leq \frac{C}{\sqrt{B_n}}. \tag{16}$$

2. *Suppose additionally that the maximal spans of all distributions from S are equal to 1. Then there exists a positive function $C = C(n)$ depending only on S such that $C(n) \to 0$ as $n \to \infty$ and*

$$\sup_{x\in Z} \left| \sqrt{B_n} P\left(\sum_{i=1}^{n} X_i = x\right) - N\left(\frac{x - A_n}{\sqrt{B_n}}\right)\right| \leq C(n), \quad n \geq 1, \tag{17}$$

where N is the density of the standard gaussian variable.

Remark 4. – *The interest of this theorem lies in fact that the estimates obtained are uniform on numbers of random variables having particular distributions from S among n random variables $X_1, \ldots, X_n$.*

Proof. – Let $P_1, \ldots, P_k$ be the set of possible distinct distributions of the variables X, $\bar{X}_1, \ldots, \bar{X}_k$ be random variables with respective distributions $P_1, \ldots, P_k$ and $\phi_1, \ldots, \phi_k$ be their characteristic functions. Then for any $n \geq 1$, the characteristic function $f_n(t)$ of the random variable $S_n = \frac{\sum_{i=1}^{n}(X_i - EX_i)}{\sqrt{B_n}}$ is equal to

$$f_n(t) = \exp\left(-\frac{itA_n}{\sqrt{B_n}}\right)\Pi_{j=1}^{k}\left(\phi_j\left(\frac{t}{\sqrt{B_n}}\right)\right)^{n_j}, \tag{18}$$

where n_j is the number of the random variables X_i which has the distribution P_j, i.e. $n_j = \sum_{i=1}^{n} 1_{(P_{X_i}=P_j)}$.

For all $x \in Z$ and any $n \geq 1$, denote $p_n(x) = \sqrt{B_n}P(\sum_{i=1}^{n} X_i = x)$. Then by the inversion formula,

$$p_n(x) = \frac{1}{2\pi}\int_{-\pi\sqrt{B_n}}^{\pi\sqrt{B_n}} e^{-\frac{it(x-A_n)}{\sqrt{B_n}}} f_n(t)dt. \tag{19}$$

We would like to consider separately two subsets of the interval $[-\pi, \pi]$: a union of small neigbourhoods of points t_0, where some of the functions $|\phi_j|$ are equal to 1 and its complement. For each $j = 1, \ldots, k$, let T_j be the set of zeros of the function $|\phi_j(t)| - 1$ on the interval $[-\pi, \pi]$. Set $T = \cup_{j=1}^{k} T_j$. Notice that card$(T) < \infty$.

For each $s \in T$, let us denote by ϵ_s any number satisfying the assertion of Lemma 1 with $t_0 = s$. For any $t \in T$, define $J_t^1 = \{j = 1, \ldots, k; |\phi_j(t)| = 1\}$ and $J_t^2 = \{j = 1, \ldots, k; |\phi_j(t)| \neq 1\}$. Let us consider any $t \in T$ and any $j \in \{1, \ldots, k\}$. If $j \in J_t^1$, then by Lemma 1 it follows that for any $u \in [t - \epsilon_t, t + \epsilon_t]$,

$$|\phi_j(u)| = \left| e^{-\frac{Var(X_j)(u-t)^2}{2} + Var(X_j)(u-t)^2 \delta_{j,t}(u-t)} \right|, \tag{20}$$

with some continuous at 0 function $\delta_{j,t}$ satisfying $\delta_{j,t}(0) = 0$. Suppose $j \in J_t^2$. Since the absolute value of the function $h_j(u) = \phi_j(u) \exp(\frac{Var(X_j)(u-t)^2}{2})$ is less than 1 for $u = t$, then there exists $\epsilon < \epsilon_t$ such that $|h_j(u)| < 1$ for any $u \in [t - \epsilon, t + \epsilon]$. That is,

$$|\phi_j(u)| \leq e^{-\frac{Var(X_j)(u-t)^2}{2}}. \tag{21}$$

In other words for all $j \in J_t^2$, the bound (20) holds with function $\delta_{j,t} \equiv 0$. In this way we define $\delta_{j,t}$ for any index j and any $t \in T$.

Recall that T is finite. Therefore one can choose a sufficiently small ϵ such that $\epsilon \leq \min_{t \in T}(\epsilon_t)$ and $\epsilon \leq 0.5 \min_{t_1, t_2 \in T, t_1 \neq t_2}(|t_1 - t_2|)$ and for any $t \in T$ in ϵ-neigbourhood of t the inequalities (21)–(20) hold. Set $V = \cup_{t \in T}(t - \epsilon, t + \epsilon) \cap [-\pi, \pi]$ and $V^c = [-\pi, \pi] \setminus V$. For any $n \geq 1$ define sets $V_n = \{x\sqrt{B_n};\ x \in V\}$ and $V_n^c = [-\pi\sqrt{B_n}, \pi\sqrt{B_n}] \setminus V_n$. Notice that since the functions ϕ_j are continuous, then $b_\epsilon \equiv \sup_{j=1,\ldots,k} \sup_{u \in V^c}(|\phi_j(u)|) < 1$. The proofs of two parts are almost the same so that we prove in detail the first assertion and sketch out the proof of the second one.

1) Let us split the integral in (19)

$$p_n(x) = \frac{1}{2\pi}\left(\int_{V_n} + \int_{V_n^c}\right) = I + II. \tag{22}$$

By (18) and (19) and our choice of ϵ (see conditions (21)-(20) it follows that

$$|I| \leq \frac{1}{2\pi} \sum_{t \in T} \int_{-\epsilon\sqrt{B_n}}^{\epsilon\sqrt{B_n}} e^{-\frac{1}{2}u^2} \left| e^{\frac{u^2}{B_n} \sum n_j Var(X_j) \delta_{j,t}(\frac{u}{\sqrt{B_n}})} \right| du, \tag{23}$$

where for each $t \in T$, $\delta_{j,t}$ are continuous functions at 0 such that $\delta_{j,t}(0) = 0$ (recall that by definition $\delta_{j,t} \equiv 0$ for all $j \in J_t^2$). For any $\epsilon > 0$, set

$$\Delta(\epsilon) = \sup_{j=1,\ldots,k} \sup_{t \in T} \sup_{u \in [-\epsilon, \epsilon]} \{|\delta_{j,t}(u)|\} \qquad (24).$$

By definition and continuity of $\delta_{j,t}$ at 0 it follows that $\Delta(\epsilon) \to 0$ as $\epsilon \to 0$. Recalling the inequality $|e^a| \leq e^{|a|}$,

$$|I| \leq \frac{1}{2\pi} \sum_{t\in T} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} e^{u^2\Delta(\epsilon)} du \leq \frac{\operatorname{card}(T)}{\sqrt{2\pi(1-2\Delta(\epsilon))}}. \tag{25}$$

On the other hand, since $\sum_j n_j = n$,

$$|II| \leq \int_{V_n^c} \leq \max_{j=1,\ldots,k} \{\sqrt{Var(\bar{X}_j)}\}(b_\epsilon)^n \sqrt{n} \to 0, \quad \text{as } n \to \infty. \tag{26}$$

The estimates (22)-(26) permit to conclude the proof of the first assertion by choosing a sufficiently small ϵ.

2) The second assertion is proved in a similar way. By using the expression for the characteristic function of the standard normal distribution we get from (19) that

$$\begin{aligned} &\sqrt{B_n} P\left(\sum_{i=1}^{n} X_i = x\right) - N\left(\frac{x-A_n}{\sqrt{B_n}}\right) \\ &= \frac{1}{2\pi} \int_{-\pi\sqrt{B_n}}^{\pi\sqrt{B_n}} e^{-\frac{it(x-A_n)}{\sqrt{B_n}}} (f_n(t) - e^{-\frac{1}{2}t^2}) dt - 2\int_{\pi\sqrt{B_n}}^{\infty} e^{-\frac{1}{2}t^2} dt. \end{aligned} \tag{27}$$

Since the maximal spans of $P_1, \ldots, P_k$ are equal to 1, then $|\phi_1|, \ldots, |\phi_k|$ are equal to 1 only at 0. That is $T = \{0\}$. Set $\epsilon = \epsilon_0$, $V = [-\epsilon, \epsilon]$, $V_n = [-\epsilon\sqrt{B_n}, \epsilon\sqrt{B_n}]$ and $V_n^c = [-\pi\sqrt{B_n}, \pi\sqrt{B_n}] \setminus V_n$. Split the integral in (27),

$$\frac{1}{2\pi} \int_{-\pi\sqrt{B_n}}^{\pi\sqrt{B_n}} e^{-\frac{it(x-A_n)}{\sqrt{B_n}}} (f_n(t) - e^{-\frac{1}{2}t^2}) dt = \frac{1}{2\pi}\int_{V_n} + \frac{1}{2\pi}\int_{V_n^c} = I + II. \tag{28}$$

From (18) it follows that

$$|I| \leq \frac{1}{2\pi} \sum_{t\in T} \int_{-\epsilon\sqrt{B_n}}^{\epsilon\sqrt{B_n}} e^{-\frac{1}{2}u^2} \left| e^{\frac{u^2}{B_n}\sum n_j Var(X_j)\delta_{j,t}(\frac{u}{\sqrt{B_n}})} - 1 \right| du, \tag{29}$$

where the functions $\delta_{j,t}$ were defined in Lemma 1. Therefore, by the inequality $|e^a - 1| \leq e^{|a|} - 1$ valid for any $a \in C$,

$$|I| \leq \frac{\sqrt{B_n}}{2\pi} \sum_{t\in T} \int_{-\epsilon}^{\epsilon} e^{-\frac{B_n}{2}u^2} (e^{u^2\Delta(\epsilon)B_n} - 1) du \leq \frac{\operatorname{card}(T)}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{1-2\Delta(\epsilon)}} - 1 \right), \tag{30}$$

with $\Delta(\epsilon)$ defined in (24) The estimation of the second integral in (28) is straightforward. Namely,

$$|II| \leq \frac{1}{2\pi}\int_{\epsilon\sqrt{B_n}}^{\pi\sqrt{B_n}} |f_n(t)|dt + \frac{1}{2\pi}\int_{-\pi\sqrt{B_n}}^{-\epsilon\sqrt{B_n}} |f_n(t)|dt + \frac{1}{\pi}\int_{\epsilon\sqrt{B_n}}^{\pi\sqrt{B_n}} e^{-\frac{1}{2}t^2}dt. \tag{31}$$

Immediately, the third integral here tends to 0. Using the arguments which led to (26) (recall that outside the interval $[-\epsilon, \epsilon]$ all functions $|\phi_j|$ are different from 1) the last two remaining integrals can be shown to tend to 0 as $n \to \infty$. This and (27)-(30) conclude the proof of the proposition. ■

3.1.2 *Asymptotics for the time spent on the axis*

As initial step in the proof, let us describe the asymptotic behavior of the time spent by Z on the axis ∂G_3. This description will play an important role in proofs of the main results in both cases $\alpha > 1$ and $\alpha < 1$. The first one is

PROPOSITION 2. – *Suppose $\alpha \geq 0$. Then, as $n \to \infty$,*

$$\frac{\pi_n^3}{n^\beta} \xrightarrow{\text{a.s.}} \begin{cases} +\infty & \beta < \frac{\alpha}{2} \wedge 1; \\ 0, & \beta > \frac{\alpha}{2} \wedge 1. \end{cases} \tag{32}$$

If $\alpha = 2$, then

$$\frac{\pi_n^3}{n} \xrightarrow{\text{a.s.}} 0. \tag{33}$$

Finally, if $\alpha > 2$, then

$$\frac{\pi_n^3}{n} \xrightarrow{\text{a.s.}} \pi(\partial\widetilde{G}_3). \tag{34}$$

Proof. – 1. If $\alpha > 2$, then (32), (34) are trivial consequences of positive recurrence of the MC $\widetilde{Z}$ and follow from the ergodic theorem for such MC (see Theorem 2 in [7], p.92).

2. Let now $\alpha \in [0, 2]$. Observe that $n \in \left[\sum_{i=1}^{\pi_n^3} \tau_i, \sum_{i=1}^{\pi_n^3+1} \tau_i\right)$, where τ_k are the amounts of time elapsed between successive hittings of $\partial\widetilde{G}_3$ by the process $\widetilde{Z}$, i.e.

$$\eta_0 = 0, \ \eta_1 = \inf\{s \geq 0; \ \widetilde{Z}_s \in \partial\widetilde{G}_3\},$$
$$\eta_k = \inf\{s > \eta_{k-1}; \ \widetilde{Z}_s \in \partial\widetilde{G}_3\}; \ \tau_k = \eta_k - \eta_{k-1}.$$

Obviously, since the Markov chain $\{\widetilde{Z}_n, \ n \geq 0\}$ is recurrent, then $\pi_n^3 \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. Therefore, in order to check (32), (33) it suffices to prove

LEMMA 2. – *Suppose* $\alpha \in [0,2]$. *Then*

$$\frac{n}{(\sum_{i=1}^{n}\tau_k)^{\beta}} \xrightarrow{\text{a.s.}} \begin{cases} +\infty, & \beta < \frac{\alpha}{2}; \\ 0, & \beta > \frac{\alpha}{2}. \end{cases} \tag{35}$$

Moreover, if $\alpha = 2$, *then* $\frac{n}{\sum_{i=1}^{n}\tau_k} \xrightarrow{\text{a.s.}} 0$.

Proof. – We start with the simplest case $\alpha = 2$. In this case, it follows from the non-ergodicity of the MC $\widetilde{Z}$ that the random variables τ_k are not integrable. Hence, by a version of the strong large numbers law for non-integrable i.i.d. random variables $\frac{\sum_{i=1}^{n}\tau_k}{n} \xrightarrow{\text{a.s.}} \infty$, as was to be proved. We now prove the statement (35). Recalling Theorem 2 from [21][1] we immediately see that

$$\limsup_{n\to\infty} n^{-1/\beta}\textstyle\sum_{i=1}^{n}\tau_k = 0, \quad \text{if} \quad \beta < \underline{\delta};$$
$$\liminf_{n\to\infty} n^{-1/\beta}\textstyle\sum_{i=1}^{n}\tau_k = +\infty, \quad \text{if} \quad \beta > \overline{\delta},$$

where

$$\underline{\delta} = \sup\{p;\ \limsup_{x\to\infty} x^p h(x) = 0\}; \quad \overline{\delta} = \sup\{p;\ \liminf_{x\to\infty} x^p h(x) = 0\};$$

$$h(x) = P_{(0,0)}(\tau_1 > x) + x^{-2}E_{(0,0)}(\tau_1^2 1_{(\tau_1 \le x)}) + x^{-1}E_{(0,0)}(\tau_1 1_{(\tau_1 \le x)}).$$

We claim that $\underline{\delta} = \overline{\delta} = \frac{\alpha}{2}$. In fact, as follows from Theorem 3 in [4], there exist constants $A, B, \widetilde{A}, \widetilde{C}$ such that the following bound holds for all $x \ge A$ and $|\widetilde{z}| > \widetilde{C}\widetilde{A}$,

$$P_{\widetilde{z}}(\widetilde{T}_{\widetilde{A}} \ge x) \ge \begin{cases} Bx^{-\frac{\alpha}{2}}, & \text{if } \alpha > 0; \\ \frac{B}{\log(x)}, & \text{if } \alpha = 0. \end{cases} \tag{36}$$

where the random time $\widetilde{T}_{\widetilde{A}}$ is defined as $\widetilde{T}_{\widetilde{A}} = \inf\{n \ge 0;\ |\widetilde{Z}_n| \le \widetilde{A}\}$. By irreducibility of the MC $\widetilde{Z}$, there exist $\widetilde{z}_0 \in \mathrm{Z}_+^2$, $k_0 \in Z_+, \kappa > 0$ such that $|\widetilde{z}_0| > \widetilde{C}\widetilde{A}$ and starting from the origin, the MC $\widetilde{Z}$ hits $\widetilde{z}_0$ at time k_0 without returning to the origin up to k_0 with probability greater or equal to κ. Let us fix them. Then, since $P_{\widetilde{z}_0}$-a.s. $\widetilde{T}_{\widetilde{A}} < \widetilde{T}_0$, (36) and the strong Markov property of $\widetilde{Z}$ imply that there exists a positive constant C such that for all $x \ge A$,

$$P_{(0,0)}(\tau_1 > x) > \kappa P_{\widetilde{z}_0}(\widetilde{T}_{\widetilde{A}} \ge x) > \begin{cases} Cx^{-\frac{\alpha}{2}}, & \text{if } \alpha > 0; \\ C\log^{-1}(x), & \text{if } \alpha = 0 \end{cases} \tag{37}$$

[1] This reference was kindly pointed out to us by Jean Bertoin.

On the other hand, Theorem 6 in [5] ensures that the constant A can be chosen in such a way that for some positive constants D, ε and for all $x \geq A$,

$$P_{(0,0)}(\tau_1 > x) < \begin{cases} Dx^{-\frac{\alpha}{2}} \log^{1+\varepsilon}(x), & \text{if } \alpha > 0; \\ D \log^{-1}(x) \log_2^{1+\varepsilon}(x), & \text{if } \alpha = 0. \end{cases} \tag{38}$$

Since $h(x) \geq P_{(0,0)}(\tau_1 > x)$, then the lower bound (37) implies that $\underline{\delta} \leq \overline{\delta} \leq \frac{\alpha}{2}$.

Moreover, the upper bound (38) and some easy calculations show that there exist positive constants c_1, c_2 such that for all sufficiently large x,

$$h(x) \leq \begin{cases} c_1 x^{-\frac{\alpha}{2}} \log^{1+\varepsilon}(x) + c_2 x^{-1}, & \text{if } \alpha > 0; \\ c_1 \log^{-1}(x) \log_2^{1+\varepsilon}(x), & \text{if } \alpha = 0. \end{cases}$$

(here we use an easy inequality $E(\tau^p 1_{(\tau \leq x)}) \leq p \int_0^{x^{1/p}} u^{p-1} P(\tau > u) du$, for any $p > 0$). Since $\alpha \in [0, 2]$, the last upper bounds imply that $\frac{\alpha}{2} \leq \underline{\delta}$, which concludes the proof of Lemma 2 and Proposition 2.

3.1.3 *Limit theorem for renormalized two-dimensional reflected random walks in the orthant*

We now need to state one modification of the main results (Theorem 1 and Remark 4) in [1]. For any $n \geq 1$, let $\widetilde{G}_n = \{(z_1/\sqrt{n}, z_2/\sqrt{n}), (z_1, z_2) \in \mathbf{Z}_+^2\}$ and let $\{\widetilde{z}_n\}$ be any converging sequence from $\widetilde{G}$ such that for any n, $\widetilde{z}_n \in \widetilde{G}_n$. Denote its limit by $\widetilde{z}$. Let $\{\{\widetilde{Z}_k^n, k \geq 0\}, n \geq 1\}$ be any family of $\widetilde{G}$-valued Markov chains with the same transition mechanism as $\{\widetilde{Z}_k, k \geq 0\}$ such that $\widetilde{Z}_0^n = \widetilde{z}_n \sqrt{n}$. Fix such $\{\widetilde{z}_n\}$ and $\{\widetilde{Z}^n\}$ and set

$$X_t^n = \frac{\widetilde{Z}_{[nt]}^n}{\sqrt{n}}, \quad t \geq 0, \quad n \geq 1. \tag{39}$$

PROPOSITION 3. – *Let $\{\{X_t^n, t \geq 0\}, n \geq 1\}$ be the above defined family of $\widetilde{G}$-valued Markov chains. If $\alpha < 2$ (resp. $\alpha \geq 2$), then X^n converge weakly to $\widetilde{W}$, where $\{\Phi(\widetilde{W}_t), t \geq 0\}$ is a (non-semimartingale) Brownian motion with oblique reflection $\Phi(\widetilde{P}), \Phi(\widetilde{Q})$ in the wedge H starting from (0,0) (a definition of $\widetilde{W}$ can be found in [25]).*

The only difference between our setting and that of [1] consists in the fact that in [1] the process $\widetilde{Z}$ was allowed to jump only at its neighbors, i.e. $\widetilde{p}_{i,j}^{\,l}$ were equal to 0 if either of $i, j > 1$. This difference complicates the verification of the continuity of paths of the limiting process but not very much. Namely, in the present situation, the continuity follows from Theorem 3.10.2 in [10], Lemma 5.5.1 in [16] and Lindeberg's condition: for any $\epsilon > 0$, $t \geq 0$, $\frac{1}{n}\sum_{k=1}^{[nt]} E\left(| \triangle \widetilde{Z}_k^n|^2 1_{(|\triangle \widetilde{Z}_k^n|>\epsilon\sqrt{n})} \mid \mathcal{F}_{k-1}\right) \xrightarrow{P} 0$, which in turn follows from the moment condition (1) and the construction of $\widetilde{Z}_k^n$. Apart from that the proof in [1] remains unchanged and covers this more general setting.

3.2. Non-semimartingale case $\alpha > 1$

Proof of Theorem 2. – Theorem 3 in [2] yields that the Markov chain $\{\widetilde{Z}_n, n \geq 0\}$ is null recurrent (resp. positive recurrent), if $\alpha \in (1,2]$ (resp. $\alpha > 2$) and has an unique (up to a positive multiplicative constant, if $\alpha \in (1,2]$) stationary distribution π. Recall that by definition $(Z_n^1, Z_n^2) = (\widetilde{Z}_n^1, \widetilde{Z}_n^2)$. Hence, the convergence of the first two coordinates (renormalized by $\sqrt{n}$, if $\alpha \in (1,2]$) is an immediate consequence of the positive recurrence of the MC $\widetilde{Z}$ in the case $\alpha > 2$ and Proposition 3 in the complementary case.

Let us prove (10)-(12). The key idea of the proof consists in the following simple but powerful observation. By Theorem 7 in [6], it follows that even though $\pi(\widetilde{G})$ might be infinite (which happens in the case $\alpha \in (1,2]$), the total measure of the boundary

$\pi(\partial\widetilde{G})$ is always finite if $\alpha > 1$. Hence, by the generalized ergodic theorem (Theorem 1 in [7], p. 91) for any $j = 1, 2$,

$$\frac{\pi_n^j}{\pi_n^3} \xrightarrow{\text{a.s.}} \frac{\pi(\partial\widetilde{G}_j)}{\pi(\partial\widetilde{G}_3)}, \quad \text{as } n \to \infty. \tag{40}$$

We are now prepared to study the asymptotic behavior of Z. For any $n \geq 0$,

$$Z_n = Z_0 + M_n + \sum_{i=1}^{n} E(Y_i|Z_{i-1}). \tag{41}$$

(recall that Y_i were defined in Definition 2). Since there exists a positive constant c such that $E(|Y_i|^2|Z_{i-1}) < c$ a.s. for all i, then the quadratic variation of the martingale M is of the order $O(n)$. The stability theorem for martingales (see, e.g. [18], Prop. VII-2-4) implies that for any $\varepsilon > 0$,

$$\frac{M_n}{n^{1/2+\varepsilon}} \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \to \infty. \tag{42}$$

Let us fix an arbitrary $\beta \in (\frac{1}{2}, \frac{\alpha}{2})$. Then from (40)–(42), Remark 2 and Proposition 2,

$$\frac{Z_n}{\pi_n^3} = \frac{Z_0 + M_n}{n^\beta}\frac{n^\beta}{\pi_n^3} + \frac{\sum_{i=1}^n E(Y_i|Z_{i-1})}{\pi_n^3} \xrightarrow{\text{a.s.}} \vec{P}\frac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + \vec{Q}\frac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} + \vec{R},$$
$$\text{as } n \longrightarrow \infty. \tag{43}$$

Recalling the weak convergence of $\left(\frac{Z_n^1}{\sqrt{n}}, \frac{Z_n^2}{\sqrt{n}}\right)$, $\frac{1}{\pi_n^3}(Z_n^1, Z_n^2) = \frac{\sqrt{n}}{\pi_n^3}\left(\frac{Z_n^1}{\sqrt{n}}, \frac{Z_n^2}{\sqrt{n}}\right) \xrightarrow{w} (0,0)$. From (43) it follows that this sequence converges almost surely to $(0,0)$. Therefore,

$$\begin{cases} p_1\dfrac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + q_1\dfrac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} = -r_1, \\ p_2\dfrac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + q_2\dfrac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} = -r_2. \end{cases} \tag{44}$$

Substituting the obtained values of $\frac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)}, \frac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)}$ one gets that

$$p_3\tfrac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + q_3\tfrac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} + r_3 = \frac{\det\ (S)}{\det\ (\widetilde{S})}.^2 \tag{45}$$

By (43) for all $\beta \geq 0$,

$$\lim_{n\to\infty}\frac{Z_n}{n^\beta} = \left(\vec{P}\frac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + \vec{Q}\frac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} + \vec{R}\right)\lim_{n\to\infty}\frac{\pi_n^3}{n^\beta}.$$

Hence,

$$\lim_{n\to\infty}\frac{Z_n^3}{n^\beta} = \frac{\det\ (S)}{\det\ (\widetilde{S})}\lim_{n\to\infty}\frac{\pi_n^3}{n^\beta}.$$

Then (10)-(12) follow from Remark 2, (34) in the case $\alpha > 2$ and (32). ■

Proof of Theorem 1. – The desired assertion follows immediately from convergence statement of Theorem 2. ■

By the similar token we get the following result.

---

[2] We would like to mention here that in the case $\alpha > 2$ the equivalence $p_3\frac{\pi(\partial\widetilde{G}_1)}{\pi(\partial\widetilde{G}_3)} + q_3\frac{\pi(\partial\widetilde{G}_2)}{\pi(\partial\widetilde{G}_3)} + r_3 > 0$ iff $det\,(S) < 0$ was first noticed by M. Menshikov who obtained it by a different method.

Remark 5. – If $\det(S) > 0$ and $\alpha > 1$, then for any $z^0 = (z_1^0, z_2^0, z_3^0)$ such that $z_3^0 > 0$, the first hitting-time τ_0 of the plane $\{z_3 = 0\}$ is finite with P_{z^0}-probability 1.

Proof of Theorem. – 1. Let us first prove the recurrence. We need to prove that starting from an arbitrary initial state z the process spends with P_z-probability 1 an infinite time in z. Since the MC $\{Z_n,\ n \geq 0\}$ is irreducible and aperiodic, then it suffices to prove that for some initial state $z = (0, 0, z_3)$ with an arbitrary z_3 the MC spends an infinite time in z.

Let us fix any such initial state z. Since det $(S) = 0$ and $\alpha \neq 1$, then there exists a vector $\vec{v} = (v_1, v_2, v_3)$ such that $v_3 \neq 0$ and

$$(\vec{v}, \vec{P}) = (\vec{v}, \vec{Q}) = (\vec{v}, \vec{R}) = 0. \tag{46}$$

Hence, the process U defined for all $n \geq 0$ by $U_n = (\vec{v}, Z_n)$ is a martingale. Furthermore, it is a square-integrable martingale. Its quadratic variation A_n is expressed in terms of the covariance matrices C^i of the distribution of the vector $Z_{n+1} - Z_n$ conditioned on $\{Z_n \in \partial G_i\}$ and satisfies for any $n \geq 0$ the following upper bound

$$\begin{aligned} A_n &= \sum\nolimits_{k=1}^{n} E((U_k - U_{k-1})^2 | \mathcal{F}_{k-1}) \\ &= \sum\nolimits_{i=1}^{3} (C^i\vec{v},\ \vec{v})\pi_{\partial G_i}(n) + (C^0\vec{v}, \vec{v})\pi_{\partial G_0}(n), \\ &\leq \max_{i=0,1,2,3} (C^i\vec{v},\ \vec{v})\, n, \end{aligned} \tag{47}$$

Let τ be the first return-time to the axis ∂G_3. Since $\alpha > 1$, then by Theorem 6 in [AI2], $E_z(\sqrt{\tau})$ is finite. Hence, by (47), $E_z(\sqrt{A_\tau})$ is also finite. By one classical result on square-integrable martingales (see Proposition VII-2-3 b) in [18]), τ is a regular stopping time for the martingale $\{U_n,\ n \geq 0\}$ and $E_z(\vec{v}, Z_\tau) = (\vec{v}, z) = v_3 z_3$. But by the definition of τ, $E_z(\vec{v}, Z_\tau) = v_3\, E_z(Z_\tau^3)$ and, hence, $E_z(Z_\tau^3) = z_3$. Introduce the successive hitting times of ∂G_3 by the MC Z, i.e.

$$\tau_0 = 0, \quad \tau_n = \inf\{k > \tau_{n-1},\ Z_k \in \partial G_3\}, \quad n \geq 1$$

and the random walk $\{Z_{\tau_n}^3,\ n \geq 0\}$. Increments of this random walk are i.i.d. random variables with zero mean, since by the strong Markov property

$$E\Big(E(Z_{\tau_{n+1}}^3 - Z_{\tau_n}^3 | \mathcal{F}_{\tau_n})\Big) = E\big(E_{Z_{\tau_n}}(Z_\tau^3) - Z_{\tau_n}^3\big) = 0.$$

By a well-known result (see e.g. [Revuz], Prop. 3.5.5), it then follows that the random walk is recurrent. Hence, $Z^3_{\tau_n} = z_3$ for an infinite number of n's. Therefore, $Z_{\tau_n} = (0, 0, z_3)$ for an infinite number of n's and so is Z_n. This implies the desired recurrence.

2. Let us now handle the second assertion of the theorem. As is known from [1], the first two coordinates of the process $\{\frac{Z_{[n\cdot]}}{\sqrt{n}},\ n \geq 0\}$ converge. Therefore, by a well-known result of weak convergence, in order to prove the weak relative compactness assertion, it is sufficient to prove that the third coordinate $\frac{Z^3_{[n\cdot]}}{\sqrt{n}}$ converges as $n \to \infty$.

Let $\vec{v}$ with $v_3 \neq 0$ be a vector satisfying the property (46). We introduce the sequence of processes V^n by setting for each $n \geq 0$ and $t \geq 0$, $V^n_t = (\vec{v}, \frac{Z_{[nt]}}{\sqrt{n}})$. For each $k, n \geq 1$, let us set $U^n_k = \frac{(\vec{v}, Z_k - Z_{k-1})}{\sqrt{n}}$. Then, for each $n \geq 0$ and $t \geq 0$, $V^n_t = \sum_{k=1}^{[nt]} U^n_k$. We will now prove that the processes V^n converge to a driftless diffusion process. By one variant of a well-known central limit theorem for martingales (see e.g. Theorem 3.33 in [14], Chapter VIII]) it suffices to verify the following two conditions:

$$\text{For any } \epsilon > 0,\ t \geq 0, \sum_{k=1}^{[nt]} E\Big(|U^n_k|^2 1_{(|U^n_k| > \epsilon)} \,|\, \mathcal{F}_{k-1}\Big) \xrightarrow{P} 0.$$
$$\text{For any } t \geq 0 \text{ and } i, j = 1, 2, 3, \sum_{k=1}^{[nt]} E\big(|U^n_k|^2 \,|\, \mathcal{F}_{k-1}\big) \xrightarrow{P} ct, \quad (48)$$

for some positive constant c. The first condition is satisfied since

$$\begin{aligned}
&\sum_{k=1}^{[nt]} E\Big(|U^n_k|^2 1_{(|U^n_k| > \epsilon)} \,|\, \mathcal{F}_{k-1}\Big) \\
&= \frac{1}{n} \sum_{l=0}^{3} \sum_{k=1}^{[nt]} E\big(|Z_k - Z_{k-1}|^2 1_{(Z_{k-1} \in \partial G_l)} 1_{(|Z_k - Z_{k-1}| > \epsilon\sqrt{n})} \,|\, \mathcal{F}_{k-1}\big) \\
&\leq t \max_{l=0,1,2,3} E\big(|Y^l|^2 1_{(|Y^l| > \epsilon\sqrt{n})}\big) \to 0,
\end{aligned}$$

where Y^l is a vector with distribution equal to that of Y_k conditioned on $Z_{k-1} \in \partial G_l$. Notice that convergence to 0 follows from the square integrability of Y^l. Check now the second condition.

$$\begin{aligned}
\sum_{k=1}^{[nt]} E\big(|U^n_k|^2 \,|\, \mathcal{F}_{k-1}\big) &= \frac{1}{n} \sum_{l=0}^{3} (C^l \vec{v}, \vec{v}) \sum_{k=1}^{[nt]} 1_{(\widetilde{Z}_{k-1} \in \partial \widetilde{G}_l)} \\
&= \sum_{l=0}^{3} (C^l \vec{v}, \vec{v}) \frac{\pi^l_{[nt]}}{n} \qquad (49)
\end{aligned}$$

where C^l are the covariance matrices of the transition mechanism defined in (5). We now separate two cases.

a) Let $\alpha \in (1,2]$. Then, by Proposition 2 and (40) we have that for all $l = 1,2,3$,

$$\frac{\pi^l_{[nt]}}{n} = \frac{\pi^l_{[nt]}}{\pi^3_{[nt]}} \frac{\pi^3_{[nt]}}{n} \xrightarrow{P} 0.$$

Hence, $\frac{\pi^0_{[nt]}}{n} \xrightarrow{P} t$ and $\sum_{k=1}^{[nt]} E(|U^n_k|^2 \mid \mathcal{F}_{k-1}) \xrightarrow{P} (C^0\vec{v},\vec{v})t$, as $n \to \infty$. b) If $\alpha > 2$, then by ergodic theorem for any $l = 0,1,2,3$, $\frac{\pi^l_{[nt]}}{n} \xrightarrow{P} \widetilde{\pi}(\partial\widetilde{G}_l)t$, and, consequently, $\sum_{k=1}^{[nt]} E(|U^n_k|^2 \mid \mathcal{F}_{k-1}) \xrightarrow{P} \sum_{l=0}^{3}(C^l\vec{v},\vec{v})\widetilde{\pi}(\partial\widetilde{G}_l)t$.

This terminates the verification of the second condition from (48). Therefore, the processes V^n converge. But the first two coordinates of the processes $\{\frac{Z_{[n\cdot]}}{\sqrt{n}},\ n \geq 0\}$ converge. Since $v_3 \neq 0$, then so do the third coordinates. ■

As was mentioned earlier, the crucial role in the proof of all these results was played by asymptotic equivalence of the times spent on the boundary (see (40)). However, this argument can not be applied in the case $\alpha = 1$ because of the next result which shows that times spent by the process Z on the boundaries ∂G_1 and ∂G_2 on the time interval $[0,n]$ are of the same order as $n \to \infty$ and are asymptotically greater than that spent on ∂G_3. More precisely,

Remark 6. – If $\alpha = 1$, then there exists a positive constant γ such that

$$\sup_n |E_{(0,0)}(\pi^1_{\tau\wedge n}) - \gamma E_{(0,0)}(\pi^2_{\tau\wedge n})| < \infty. \tag{50}$$

Moreover, $\lim_{n\to\infty} \frac{E_{(0,0)}(\pi^3_n)}{E_{(0,0)}(\pi^1_n)} = 0$.

Proof of Remark 6. – Here we will only prove the first part of the remark. The proof of the second assertion relies upon purely analytic methods and will be given elsewhere.

Since $\alpha = 1$, then the vectors of boundary reflection $\widetilde{P}$ and $\widetilde{Q}$ for the two-dimensional MC $\widetilde{Z}$ are collinear. Let $\gamma > 0$ be such that $\widetilde{P} = -\gamma\widetilde{Q}$. We fix any vector $\widetilde{v}_0$ orthogonal to $\widetilde{P}$. These vectors $(\widetilde{v}_0, \widetilde{P})$ form then a basis of R^2.

Let τ be the first return time to the origin by the MC $\widetilde{Z}$. Then for any $n \geq 1$ and any two-dimensional vector $\widetilde{v}$ we have that

$$\begin{aligned}(\widetilde{v}, E_{(0,0)}(Z_{n\wedge\tau} - Z_1)) &= (\widetilde{v},\widetilde{P})E_{(0,0)}(\pi^1_{\tau\wedge n}) + (\widetilde{v},\widetilde{Q})E_{(0,0)}(\pi^2_{\tau\wedge n}) \\ &= (\widetilde{v},\widetilde{P})(E_{(0,0)}(\pi^1_{\tau\wedge n}) - \gamma E_{(0,0)}(\pi^2_{\tau\wedge n})).\end{aligned} \tag{51}$$

Applying this to the vector $\widetilde{v} = \widetilde{v}_0$ we obtain that $\left(\widetilde{v}_0, E_{(0,0)}(Z_{n\wedge\tau} - Z_1)\right) = 0$ which shows that for any $n \geq 1$ there is a constant δ_n such that $E_{(0,0)}(Z_{n\wedge\tau} - Z_1) = \delta_n \widetilde{P}$. Consequently, $\left(\widetilde{P}, E_{(0,0)}(Z_{n\wedge\tau} - Z_1)\right)) = \delta_n |\widetilde{P}|^2$. On the other hand, setting in (51) $\widetilde{v} = \widetilde{P}$, we see that $\left(\widetilde{P}, E_{(0,0)}(Z_{n\wedge\tau} - Z_1)\right) = |\widetilde{P}|^2 (E_{(0,0)}(\pi^1_{\tau\wedge n}) - \gamma E_{(0,0)}(\pi^2_{\tau\wedge n}))$. Combining these two equalities we see that for any $n \geq 1$, $E_{(0,0)}(\pi^1_{\tau\wedge n}) - \gamma E_{(0,0)}(\pi^2_{\tau\wedge n}) = \delta_n$.

Notice that for any $n \geq 1$, $\delta_n \widetilde{P} \geq -E_{(0,0)}(Z_1)$ (here the inequality should be understood in the coordinate sense). Since $\alpha = 1$, then one of the coordinates of $\widetilde{P}$ is positive and another one is negative. Hence, $\sup_{n\geq 1}(|\delta_n|) < \infty$ and

$$\sup_{n\geq 1} |E_{(0,0)}(\pi^1_{\tau\wedge n}) - \gamma E_{(0,0)}(\pi^2_{\tau\wedge n})| = \sup_{n\geq 1} |\delta_n| < \infty. \blacksquare$$

3.3. Semimartingale case $\alpha < 1$.

Contrary to the previous case the proofs of Theorems 1 and 2 are completely independent.

Proof of Theorem 1. – (case $\alpha < 1$).
Since in the case $\alpha < 0$, the MC $\widetilde{Z}$ is transient, then the original MC Z is also transient. We therefore left with the case $\alpha \in [0, 1)$. Let us denote by $p^n_{(0,0)}$ and $\widetilde{p}^n_{(0,0)}$ the n-step return probabilities to the origins:

$$p^n_{(0,0,0)} = P(Z_n = (0,0,0) | Z_0 = (0,0,0))$$
$$\text{and} \quad \widetilde{p}^n_{(0,0)} = P(\widetilde{Z}_n = (0,0) | \widetilde{Z}_0 = (0,0)).$$

We will use a well-known criteria of transience of countable irreducible aperiodic MC which ensures that

$$Z \text{ is transient} \quad \text{iff} \quad \sum_{n=1}^{\infty} p^n_{(0,0,0)} < \infty.$$

We now investigate the asymptotic behavior of $p^n_{(0,0,0)}$ as $n \to \infty$. Suppose that with probability 1, $Z_0 = (0,0,0)$. Set $V = \{(i,j) \in \mathbb{Z}^2_+;\ \max(i,j) \leq 1, \widetilde{p}^l_{-i,-j} > 0 \text{ for some } l = 0,1,2\}$ (the probabilities $\widetilde{p}^l_{-i,-j}$ were defined in (7)). Since the Markov chains $\{Z_n,\ n \geq 0\}$ and $\{\widetilde{Z}_n,\ n \geq 0\}$ are irreducible, the set V is non-empty. Then

$$p^n_{(0,0,0)} \leq P(\widetilde{Z}_{n-1} \in V, Z^3_n = 0) = E\left(1_{(\widetilde{Z}_{n-1} \in V)} P(Z^3_n = 0 \,|\, \widetilde{\mathcal{F}}_{n-1})\right). \tag{52}$$

It is easy to see that the conditional distribution of Z_n^3 on σ-algebra $\widetilde{\mathcal{F}}_{n-1}$ is the same as the distribution of the sum

$$\sum_{i=1}^{n-\pi_{n-1}^1-\pi_{n-1}^2-\pi_{n-1}^3} \xi_i^0 + \sum_{j=1}^{3} \sum_{i=1}^{\pi_{n-1}^j} \xi_i^j,$$

where the random variables ξ_i^j are independent and for each $j = 0, 1, 2, 3$, ξ_i^j have the same distribution as that of Y^3 on $\{\widetilde{Z}_{i-1} \in \partial\widetilde{G}_j\}$ (recall that π_n^j stands for the total time spent by the process $\widetilde{Z}$ on the boundary $\partial\widetilde{G}_j$ up to the time n). By Remark 2

$$E(\xi^0) = 0, E(\xi^1) = p_3, E(\xi^2) = q_3, E(\xi^3) = r_3.$$

We drop the subindex i in ξ_i^j since for each fixed j the random variables are identically distributed. Let us denote by $v_j^2 = \mathrm{Var}(\xi^j)$. These variances are finite because of the moment condition (1). For any $n \geq 1$, set

$$\begin{aligned} A_n &= p_3\pi_{n-1}^1 + q_3\pi_{n-1}^2 + r_3\pi_{n-1}^3, \\ B_n &= v_0^2(n - \pi_{n-1}^1 - \pi_{n-1}^2 - \pi_{n-1}^3) + v_1^2\pi_{n-1}^1 + v_2^2\pi_{n-1}^2 + v_3^2\pi_{n-1}^3. \end{aligned}$$

By Proposition 1 applied to random variables ξ_i^j, there exists a positive constant c such that uniformly on π_n^j, $j = 0, 1, 2, 3$ for any $n \geq 1$,

$$P(Z_n^3 = 0 \,|\, \widetilde{\mathcal{F}}_{n-1}) \leq \frac{c}{\sqrt{B_n}} \leq \frac{c}{\min_j\{\sqrt{\mathrm{Var}(\xi^j)}\}\sqrt{n}}.$$

This and (52) show that there exists a positive constant c_1 such that for all $n \geq 1$,

$$p_{(0,0,0)}^n \leq \tfrac{c_1 P(\widetilde{Z}_{n-1} \in V \,|\, \widetilde{Z}_0 = (0,0))}{\sqrt{n}}. \tag{53}$$

By the time-homogeneity of the transition mechanism,

$$\begin{aligned} \widetilde{p}_{(0,0)}^n &\geq \textstyle\sum_{l=0}^3 \sum_{(i,j)\in V\cap\partial\widetilde{G}_l} P(\widetilde{Z}_{n-1}=(i,j) \,|\, \widetilde{Z}_0=(0,0))\widetilde{p}_{-i,-j}^1 \\ &\geq \big(\min\{\widetilde{p}_{-i,-j}; (i,j) \in V\}\big)P(\widetilde{Z}_{n-1} \in V \,|\, \widetilde{Z}_0 = (0,0)). \end{aligned}$$

which together with (53) imply that there exists a positive constant c_2 such that

$$p_{(0,0,0)}^n \leq c_2\widetilde{p}_{(0,0)}^n/\sqrt{n}. \tag{54}$$

Let us next study the convergence of the sum $\sum_{n=0}^{\infty} \widetilde{p}_{(0,0)}^{n}/\sqrt{n}$. Obviously,

$$\sum_{n=0}^{\infty} \widetilde{p}_{(0,0)}^{n}/\sqrt{n} < \infty \quad \text{iff} \quad \sum_{n=0}^{\infty} s_n/n^{\frac{3}{2}} < \infty, \tag{55}$$

where $s_n = \sum_{k=0}^{n} \widetilde{p}_{(0,0)}^{k}$.

The following lemma will provides an estimate leading to certain bounds for s_n as $n \to \infty$ from (37), (38). It can be also used in a different context and regarded as a generalized tauberian theorem (cf. [13], Ch. 13, Theorem 5).

DEFINITION 5. – *A positive function f defined on $[0, \infty)$ is called subexponential, if there exists a positive p such that* $\limsup_{x \to \infty} x^{-p} f(x) < \infty$.

LEMMA 3. – *Let $(p_k)_{k=0}^{\infty}$ be a sequence of positive reals, $s_n = \sum_{k=0}^{n} p_k$ and let f be the generating function of the sequence p_k on $[0, 1)$ defined by $f(x) = \sum_{k=0}^{\infty} p_k x^k$. Suppose there exist positive constants c_1, c_2 and a subexponential function L_1 such that for all sufficiently small $a > 0$,*

$$\frac{c_1}{a^\beta L_1(\frac{1}{a})} \leq f(e^{-a}) \leq \frac{c_2}{a^\beta}. \tag{56}$$

Then there exist positive constants c_3, c_4 such that for all sufficiently large n,

1. *(Upper bound).* $s_n \leq c_3 n^\beta$.
2. *(Lower bound).* For any positive sequence a_n such that as $n \to \infty$, $a_n = o(1)$, $na_n \to \infty$, and

$$L_1\left(\frac{1}{a_n}\right) = o\Big((na_n)^{-\beta} e^{na_n}\Big)$$

the following inequality holds:

$$s_n \geq c_4 a_n^{-\beta} L_1^{-1}\left(\frac{1}{a_n}\right). \tag{57}$$

Remark 7. – Sequences a_n with the properties described in the second assertion of the last lemma exist because of the subexponentiality of L_1. For instance, one can take $a_n = n^{-\gamma}$ with $\gamma \in (0, 1)$.

Remark 8. – As we shall see, the proof of the upper bound for s_n needs only the upper bound in (56).

Proof. – The proof of the upper bound is elementary. Namely, by definition

$$s_n \leq e\sum_{k=0}^{n} p_k e^{-\frac{k}{n}} \leq ef(e^{-\frac{1}{n}}) \leq ec_2 n^\beta,$$

as asserted. Let us now deal with the lower bound. First of all, observe that the just proved upper bound implies that for all $a > 0$,

$$\begin{aligned} f(e^{-a}) &= (1-e^{-a})\sum\nolimits_{k=0}^{\infty} s_k e^{-ak} \leq s_n\left(1-e^{-a(n+1)}\right) \\ &\quad + (1-e^{-a})\sum\nolimits_{k=n+1}^{\infty} s_k e^{-ak} \\ &\leq s_n\left(1-e^{-a(n+1)}\right) + 3c_2(1-e^{-a})\sum\nolimits_{k=n+1}^{\infty} k^\beta e^{-ak}. \quad (58) \end{aligned}$$

Observe also that for all (n, a) such that $n\,a > \beta$ the function $g(x) = x^\beta e^{-ax}$ decreases on $[n, \infty)$ and, hence,

$$\sum_{k=n+1}^{\infty} k^\beta e^{-ak} \leq \int_n^\infty u^\beta e^{-au} du = a^{-\beta-1}\int_{na}^\infty u^\beta e^{-u} du. \quad (59)$$

Let us denote the last integral in the RHS of (59) by $\gamma(\beta, na)$. Notice that, as $x \to \infty$,

$$\int_x^\infty u^\beta e^{-u} du = O\left(x^\beta e^{-x}\right). \quad (60)$$

Let now a_n be any sequence satisfying the conditions of the lemma. Hence, from (56)-(58) it follows that for all sufficiently large n, $n\,a_n > \beta$ and

$$\begin{aligned} s_n &\geq f(e^{-a_n}) - 3c_2(1-e^{-a_n})a_n^{-\beta-1}\gamma(\beta, na_n) \\ &\geq c_1 a_n^{-\beta} L_1^{-1}\left(\frac{1}{a_n}\right) - 3c_2(1-e^{-a_n})a_n^{-\beta-1}\gamma(\beta, na_n). \end{aligned} \quad (61)$$

Then, recalling the estimate (60) with $x = na_n$ and the properties of a_n, we arrive at the desired lower bound. ■

Remark 9. – We will need the following assertion which is a straightforward modification of the upper bound in the last lemma:
If for some positive constant c and for all sufficiently small a, $f(e^{-a}) \leq c\log(1/a)$, then there exists a positive constant c_1 such that for all sufficiently large n, $s_n \leq c_1\log(n)$.

In order to apply the last two results to the sequence $p_k = \widetilde{p}^{k}_{(0,0)}$ bounds for the generating function of this sequence are needed. Recall the following classical formula:

$$f(e^{-a}) \equiv \sum_{k=0}^{\infty} \widetilde{p}^{k}_{(0,0)} e^{-ak} = \frac{1}{1 - E(e^{-a\tau})}, \tag{62}$$

where τ is the first return time to the origin $(0,0)$ of the MC $\widetilde{Z}$ starting from the origin. By (37), there exists a positive constant B such that for all $a > 0$,

$$1 - E(e^{-a\tau}) = a \int_0^{\infty} e^{-ax} P(\tau > x) dx > \begin{cases} B a^{\frac{\alpha}{2}}, & \text{if } \alpha > 0; \\ B \log^{-1}(1/a), & \text{if } \alpha = 0. \end{cases}$$

Hence, from (62), the upper bound in the assertion of the last lemma with $\beta = \frac{\alpha}{2}$ and the last remark we get that for sufficiently large n, $s_n < cn^{\frac{\alpha}{2}} \log(n)$ and

$$\sum_{n=0}^{\infty} s_n n^{-\frac{3}{2}} < \widetilde{c} \sum_{n=0}^{\infty} n^{\frac{\alpha-}{2}} \log(n) < \infty,$$

since $\alpha < 1$. (12) concludes the proof of Theorem 1. ■

Remark 10. – Similarly, one can prove that a driftless random walk in the half-space $\mathrm{Z}^2 \times \mathrm{Z}_+$ with any constant reflection on the plane $\{z_3 = 0\}$ is transient.

3.4. Proof of Theorem 2 ($\alpha < 1$).

The proof is broken into many steps. The starting point is decomposition (1) of $Z_{[nt]}$. The first task is to prove the tightness of the family $\left\{\left(\frac{\pi^1_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^2_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^3_{[n\cdot]}}{\sqrt{n}}\right), n \geq 1\right\}$ which in turn implies the tightness of the family $\left\{\frac{\sum_{i=1}^{[n\cdot]} E(Y_i|Z_{i-1})}{\sqrt{n}}, n \geq 1\right\}$. Next, the weak convergence of the martingales $\left\{\frac{M_{[n\cdot]}}{\sqrt{n}}, n \geq 1\right\}$ to a diffusion process will be proved. This will then give us the tightness of the family $\left\{\frac{Z_{[nt]}}{\sqrt{n}}, n \geq 1\right\}$. An identification of the limit process will possible because of Proposition 3 and one uniqueness result for SBRMs proved by L. Taylor and R.J. Williams in [24]. Let us now proceed.

Let $\widetilde{z} \in \widetilde{G}$. Let us fix any sequence $\{\widetilde{z}_n\}$ converging to $\widetilde{z}$ such that for any $n \geq 1$, $\widetilde{z}_n \in \widetilde{G}_n$. Let us also fix any family of $\widetilde{G}$-valued Markov chains $\{\{\widetilde{Z}^n_k,\ k \geq 0\},\ n \geq 1\}$ with the same transition mechanism as $\{\widetilde{Z}_k,\ k \geq 0\}$ such that $\widetilde{Z}^n_0 = \widetilde{z}_n\sqrt{n}$. For each $n \geq 1$, let X^n be the stochastic process defined by (39).

Notation. – For all $j = 1, 2, 3$ let us denote by $\sigma^{j,n}_t = \frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]} 1_{(\widetilde{Z}^n_k \in \partial\widetilde{G}_j)}$. We will omit the subscript 1 in $\sigma^{j,n}_1$.

The following result will be crucial for the proof of the relative compactness of the family of R^3_+-valued processes $\{\Sigma^n_\cdot = (\sigma^{1,n}_\cdot, \sigma^{2,n}_\cdot, \sigma^{3,n}_\cdot),\ n \geq 1\}$.

LEMMA. – *There exists a positive constant c, such that for any $t > 0$ and for any initial values $\widetilde{Z}^n_0 \in Z^2_+$, we have*

$$\limsup_{n\to\infty} E\left[\sigma^{1,n}_t + \sigma^{2,n}_t + \sigma^{3,n}_t\right] \leq c\sqrt{t}. \tag{63}$$

Proof. – For the sake of typography, throughout the proof we will omit the superscript n in X^n and $\widetilde{Z}^n$. Let us first prove the lemma in the case $t = 1$. We define positive functions f_1, f_2 on $\widetilde{G}$ by

$$f_1(z_1, z_2) = 1 - e^{-z_1}, f_2(z_1, z_2) = 1 - e^{-z_2} \tag{64}$$

and apply to them the martingale property of P. Then, for each $n \geq 1$,

$$E\left[f_i(X_1) - f_i(X_0) - \sum_{k=0}^{n} L^n f_i(\widetilde{Z}_k)\right] = 0, \quad \text{for } i = 1, 2, \tag{65}$$

where L^n is the infinitesimal operator of the process $\frac{\widetilde{Z}_\cdot}{\sqrt{n}}$ defined on the set of bounded functions by

$$L^n f(z) = \sum_{i,j\geq -1} \widetilde{p}^{\,l}_{i,j}\left[f\left(z + \frac{(i,j)}{\sqrt{n}}\right) - f(z)\right], \quad \text{if } z \in \partial\widetilde{G}_l,\ l = 0, 1, 2, 3. \tag{66}$$

Obviously, $f_i \in C^2_b(\widetilde{G})$. Then, the moment conditions (1)-(4) on the increments of the MC Z (and, hence, $\widetilde{Z}$, $\widetilde{Z}^{(n)}$) and the Taylor's formula applied to $L^n f_i(\cdot)$ yield that there exist positive constants $c_{i,1}$ (which do not depend on the initial values Z_0) such that for all sufficiently large n,

$$\begin{aligned} E[f_i(X_1) - f_i(X_0)] \geq E\Bigg[&\frac{1}{\sqrt{n}}\sum_{k=0}^{n} 1_{(\widetilde{Z}_k\in\partial\widetilde{G}_1)}(\nabla f_i(\widetilde{Z}_k), \widetilde{P}) \\ &+\frac{1}{\sqrt{n}}\sum_{k=0}^{n} 1_{(\widetilde{Z}_k\in\partial\widetilde{G}_2)}(\nabla f_i(\widetilde{Z}_k), \widetilde{Q}) + \\ &+ \frac{1}{\sqrt{n}}\sum_{k=0}^{n} 1_{(\widetilde{Z}_k\in\partial\widetilde{G}_3)}(\nabla f_i(\widetilde{Z}_k), \widetilde{R}) - c_{i,1}\Bigg], \qquad i = 1, 2. \end{aligned} \tag{67}$$

Let us now estimate $E[f_i(X_1) - f_i(X_0)]$.

As in (67), let us apply the martingale property and the Taylor's formula to the functions f_i^2. Then, using the properties $f_i^2 \in C_b^2(\widetilde{G})$, $\nabla f_i^2 = (0,0)$ on $\partial\widetilde{G}_3$ and the moment conditions (1)-(4),

$$\begin{aligned}
E[f_i(X_1)] &\leq \left(E[f_i^2(X_1)]\right)^{\frac{1}{2}} \\
&\leq \left[f_i^2(X_0) + E\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n}(1_{(\widetilde{Z}_k \in \partial\widetilde{G}_1)}(\nabla f_i^2(\widetilde{Z}_k), \widetilde{P})\right.\right. \\
&\left.\left. +1_{(\widetilde{Z}_k \in \partial\widetilde{G}_2)}(\nabla f_i^2(\widetilde{Z}_k), \widetilde{Q})\right) + c_{i,2}\right]^{\frac{1}{2}}.
\end{aligned} \tag{68}$$

Trivially, on $\partial\widetilde{G}_1$,

$$\begin{aligned}
&(\nabla f_1(z)\widetilde{P}) = p_1, &&(\nabla f_1^2(z)\widetilde{P}) = 0, \\
&(\nabla f_2(z)\widetilde{P}) = p_2 e^{-z_2}, &&(\nabla f_2^2(z)\widetilde{P}) = 2p_2 e^{-z_2}(1 - e^{-z_2}),
\end{aligned} \tag{69}$$

and on $\partial\widetilde{G}_1$

$$\begin{aligned}
&(\nabla f_1(z)\widetilde{Q}) = q_1 e^{-z_1}, &&(\nabla f_1^2(z)\widetilde{Q}) = 2q_1 e^{-z_1}(1 - e^{-z_1}), \\
&(\nabla f_2(z)\widetilde{Q}) = q_2, &&(\nabla f_2^2(z)\widetilde{Q}) = 0.
\end{aligned} \tag{70}$$

The following cases will now be separated.

1. Let $p_2 \leq 0, q_1 \leq 0$. Then, from (67)-(70) and the triangle inequality we get that for all sufficiently large n,

$$\begin{cases} p_1\sigma^{1,n} - |q_1|\sigma^{2,n} + r_1\sigma^{3,n} \leq c_{1,1} + \sqrt{c_{1,2}} = c_3, \\ -|p_2|\sigma^{1,n} + q_2\sigma^{2,n} + r_2\sigma^{3,n} \leq c_{2,1} + \sqrt{c_{2,2}} = c_4. \end{cases} \tag{71}$$

By Remark 1, $c_5 = \det(\widetilde{S}) > 0$. By definition (8)), $r_1, r_2, p_1, q_2 \geq 0$. Moreover, $p_1 > 0$, $q_2 > 0$ and $\max(r_1, r_2) > 0$ because of (6). Suppose, for example, $r_1 > 0$. Then the last bounds imply that for all sufficiently large n,

$$\begin{aligned}
&\sigma^{1,n} \leq \frac{q_2c_3 + |q_1|c_4}{c_5}; \quad \sigma^{2,n} \leq \frac{p_1c_4 + |p_2|c_3}{c_5}; \quad \sigma^{3,n} \\
&\leq \frac{c_3c_5 + |q_1|(p_1c_4 + |p_2|c_3)}{r_1c_5},
\end{aligned} \tag{72}$$

as was to be proved.

2. Let $p_2 > 0, q_1 \leq 0$. Denote $\rho_n = \left(p_2 E\left[\frac{1}{\sqrt{n}}\sum_{k=0}^n 1_{(\widetilde{Z}_k \in \partial\widetilde{G}_1)} e^{-\widetilde{Z}_k^2}\right]\right)^{\frac{1}{2}}$ (here $\widetilde{Z}_k^2$ is the second coordinate of $\widetilde{Z}_k$). Again (67)-(70) and the triangle inequality imply that for all sufficiently large n,

$$\begin{cases} p_1\sigma^{1,n} - |q_1|\sigma^{2,n} + r_1\sigma^{3,n} \leq c_3, \\ \rho_n^2 + q_2\sigma^{2,n} + r_2\sigma^{3,n} \\ \quad \leq \left(2p_2 E\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n} 1_{(\widetilde{Z}_k \in \partial\widetilde{G}_1)} e^{-\widetilde{Z}_k^2}(1 - e^{-\widetilde{Z}_k^2})\right]\right)^{\frac{1}{2}} + \sqrt{c_{2,2}} \\ \quad \leq \sqrt{2}\rho_n + \sqrt{c_{2,2}}. \end{cases} \tag{73}$$

Hence, $\rho_n^2 - \sqrt{2}\rho_n - \sqrt{c_{2,2}} \leq 0$ and, therefore, there exists a positive constant c_6 such that $\rho_n < c_6$. Suppose, for example, $r_1 > 0$. This and the second bound in (73) implies that there exists a positive constant c_7 such that $\sigma^{2,n} < c_7$. Finally, the first bound in (73) concludes the proof.

3. The proof in the cases $p_2 \leq 0, q_1 > 0$ or $p_2 > 0, q_1 > 0$ is carried out in the similar manner and is left to the reader.

Notice finally that the constants appearing in these arguments do not depend on the initial values of $\widetilde{Z}$. This enables us to get (bounest) for an arbitrary t by passing to the limit over the particular subsequence $A_n = [nt]$. ∎

LEMMA 5. – *Let $\widetilde{Z}_0^n = \widetilde{z}_n$ be such that for any n, $\widetilde{z}_n \in G_n$, and $\widetilde{z}_n \to \widetilde{z}$ with some $\widetilde{z} \in \widetilde{G}$. The family of the processes $\{\Sigma^n,\ n \geq 1\}$ on $D_{\mathrm{R}_+^3}$ is tight. Moreover, any weak limit of this family is a R_+^3-valued process with continuous paths starting from $(0,0,0)$.*

Proof. – Since R_+^3 is separable and the trajectories of Σ^n have jumps of size at most $3/\sqrt{n}$, then any weak limit has continuous paths.

Let us prove the tightness by verifying the following sufficient conditions of relative compactness ([10], Theorem 3.7.2, Prop. 3.8.3). For each fixed $\epsilon \geq 0$ and $n \geq 1$, define inductively:

$$\tau_0^{\epsilon,n} = 0, \quad \tau_{k+1}^{\epsilon,n} = \inf\{s \geq \tau_k^{\epsilon,n};\ |\Sigma^n(s) - \Sigma^n(\tau_k^{\epsilon,n})| \geq \epsilon\}.$$

We simply denote $\tau_1^{\epsilon,n}$ by $\tau(\epsilon, n)$. The following two statements need to be verified:

1. The compact containment condition: for each $\delta \in (0,1)$ and $t > 0$ there exists $\epsilon > 0$ such that

$$\sup_n P(\tau(\epsilon, n) \leq t) < \delta, \tag{74}$$

2. For any $T > 0$ and for any $\epsilon > 0$,

$$\lim_{t \to 0} \sup_n \sup_{k \geq 0} P\left(\tau_k^{\epsilon,n} < T, \tau_{k+1}^{\epsilon,n} - \tau_k^{\epsilon,n} < t\right) = 0. \tag{75}$$

Both of the statements (74) and (75) are easy consequences of the strong Markov property, the Chebyshev's inequality and Lemma 4. Let us prove, for instance, (75). In fact, since the processes $\sigma^{j,n}$ are positive and non-decreasing, then we immediately have from Lemma 4 that for any $T > 0$, $\epsilon > 0$ and $t > 0$,

$$\begin{aligned} &P\left(\tau_k^{\epsilon,n} < T, \tau_{k+1}^{\epsilon,n} - \tau_k^{\epsilon,n} < t\right) \\ &\leq E\left[1_{(\tau_k^{\epsilon,n} < T)} P\left(\max_{j=1,2,3} (\sigma^{j,n}(\tau_k^{\epsilon,n} + t) - \sigma^{j,n}(\tau_k^{\epsilon,n})) \geq \frac{\epsilon}{3} \mid \widetilde{\mathcal{F}}_{\tau_k^{\epsilon,n}}\right)\right] \\ &\leq 3c\sqrt{t}/\epsilon. \end{aligned}$$

Passing to the limit as $t \to 0$ gets (75). ∎

COROLLARY 2. – *Let Z_0 be a constant vector a.s. Then $\left\{\left(\frac{\pi^1_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^2_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^3_{[n\cdot]}}{\sqrt{n}}\right), n \geq 1\right\}$ is a tight family. Moreover, $\frac{\pi^3_{[n\cdot]}}{\sqrt{n}}$ converges weakly to 0 and any weak limit of the family $\left\{\left(\frac{\pi^1_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^2_{[n\cdot]}}{\sqrt{n}}\right), n \geq 1\right\}$ is a R^2_+-valued process with continuous paths starting from $(0, 0)$.*

Proof. – The tightness and the continuity of any weak limit point follows from the last lemma. Let us now prove that $\frac{\pi^3_{[n\cdot]}}{\sqrt{n}} \xrightarrow{P} 0$.

If $\alpha \geq 0$, then the convergence immediately follows from

Proposition 2, whereas in the case $\alpha < 0$, it is an easy consequence of the transience of the MC $\widetilde{Z}$. In fact, if $\alpha < 0$, then for any initial value of $\widetilde{Z}_0$ the probability of not hitting of $\partial\widetilde{G}_3$ is positive. It is easy then to see from the strong Markov property that $\pi^3_{[n\cdot]} \xrightarrow{P} \pi^3_\infty$, where π^3_∞ has the following distribution:

$$P(\pi^3_\infty = 0) = 1 - p_{\widetilde{Z}_0}; \quad P(\pi^3_\infty = k) = p_{(0,0)}(1 - p_{(0,0)})^{k-1},$$

where for any $\widetilde{z}$, $p_{\widetilde{z}}$ is the probability of not hitting of $\partial\widetilde{G}_3$ starting from $\widetilde{z}$. ∎

The next step in the proof of Theorem 2 is

LEMMA 6. – *The family of the processes $\left\{\frac{M_{[n\cdot]}}{\sqrt{n}}, n \geq 1\right\}$ converge weakly in D_{R^3} to a 3-dimensional diffusion process $\{B_t, t \geq 0\}$ with the covariance matrix $C^0 t$ and zero-drifts.*

Proof. – The proof is similar to that of Remark relcom and will only be sketched out. Let us denote by $U_k^n = \frac{Y_k - E(Y_k|Z_{k-1})}{\sqrt{n}}$.

Then for any n, t $\frac{M_{[n\cdot]}}{\sqrt{n}} = \sum_{k=1}^{[nt]} U_k^n$. By the multidimensional variant of the central limit theorem for martingales (see e.g. Theorem 3.33 in [14], Chapter VIII]) in order to prove the lemma it suffices to verify the following two conditions:

1. For any $\epsilon > 0$, $t \geq 0$, $\sum_{k=1}^{[nt]} E\Big(|U_k^n|^2 1_{(|U_k^n|>\epsilon)} \mid \mathcal{F}_{k-1}\Big) \xrightarrow{P} 0$.
2. For any $t \geq 0$ and $i, j = 1, 2, 3$, $\sum_{k=1}^{[nt]} E\Big(U_k^{n,i} U_k^{n,j} \mid \mathcal{F}_{k-1}\Big) \xrightarrow{P} c_{i,j}^0 t$.

The first condition is verified as in the proof of Lemma 3 using the moment condition (1). The second condition is an immediate consequence of Corollary 2. In fact,

$$\sum_{k=1}^{[nt]} E\Big(U_k^{n,i} U_k^{n,j} \mid \mathcal{F}_{k-1}\Big) = \frac{1}{n} \sum_{k=1}^{[nt]} \left[\sum_{l=0}^{3} c_{i,j}^l 1_{(Z_{k-1} \in \partial G_l)}\right]$$

and by Corollary 2 for each $l = 1, 2, 3$, $\frac{1}{n} \sum_{k=1}^{[nt]} 1_{(Z_{k-1} \in \partial G_l)} = \frac{\pi_{[nt]}^l}{n} \xrightarrow{P} 0$. ■

We have now all necessary ingredients to prove Theorem 2. First of all, Corollary 2, Lemma 6 and one well-known result on the tightness in Skorokhod topology (see, e.g. [14], Chapter VI, Corollary 3.33) imply that the families $\left\{\left(\frac{M_{[n\cdot]}}{\sqrt{n}}, \Sigma_\cdot^n\right),\ n \geq 1\right\}$ and $\left\{\frac{Z_{[n\cdot]}}{\sqrt{n}},\ n \geq 1\right\}$ are also tight. Let us choose any subsequence n_k providing the weak convergence. Let us denote the limiting processes by $(B, \lambda^1, \lambda^2, 0)$ and W respectively. Then, for any $t \geq 0$,

$$W_t = B_t + \vec{P}\lambda_t^1 + \vec{Q}\lambda_t^2. \tag{76}$$

Notice that the process W is a semimartingale since the processes λ^1, λ^2 are increasing and B is a martingale.

On the other hand, from Theorems 1, 2 in [9] and Proposition 3 it follows that the first two coordinates of the processes $\frac{Z_{[n\cdot]}}{\sqrt{n}}$, i.e. $\frac{\widetilde{Z}_{[n\cdot]}}{\sqrt{n}}$ converge weakly to a semimartingale $\widetilde{W}$ with the following decomposition for any $t \geq 0$,

$$\begin{pmatrix} \widetilde{W}_t^1 \\ \widetilde{W}_t^2 \end{pmatrix} = \widetilde{B}_t + \widetilde{S}^0 \begin{pmatrix} L_t^1 \\ L_t^2 \end{pmatrix}. \tag{77}$$

where the process $\widetilde{B}$ is a 2-dimensional diffusion process with covariance matrix $\widetilde{C}^0$ and zero-drifts starting from $(0, 0)$ and the processes L^1, L^2 are continuous $\mathbb{R}_+$-valued non-increasing processes such that L^i can increase only when $\widetilde{W}$ is on the face $\partial\widetilde{G}_i$. Then $\widetilde{W} \equiv (W^1, W^2)$. Moreover, since both processes are semimartingales, their triplets of semimartingale characteristics should coincide. In particular, their martingale terms $\widetilde{B}$ and (B^1, B^2) should coincide. Since $\det(\widetilde{S}^0) \neq 0$, we also get $(\lambda^1, \lambda^2) \equiv (L^1, L^2)$. Finally, from (76) for any $t \geq 0$,

$$W_t = B_t + \vec{P}L_t^1 + \vec{Q}L_t^2, \tag{78}$$

where the process B is a 3-dimensional diffusion process with covariance matrix C^0 starting from $(0, 0, 0)$ and the processes L^1, L^2 are continuous $\mathbb{R}_+$-valued non-increasing processes such that L^i can increase only when W is on the face ∂G_i. But by Theorem 3.4 of [24] such semimartinglale reflected diffusion W is unique. This terminates the proof of Theorem 2. ∎

The last result of this section completes Remark 5 and Theorem 1 in the case $\alpha > 1$.

Remark 11. – If $\alpha < 1$, then for any $z^0 = (z_1^0, z_2^0, z_3^0)$ such that $z_3^0 > 0$, the first hitting-time τ_0 of the plane $\{z_3 = 0\}$ is finite with P_{z^0}-probability 1.

Proof. – Let $z^0 = (z_1^0, z_2^0, z_3^0)$ such that $z_3^0 > 0$. The proof consists in proving the following stronger claim: there exists a subsequence (n_k) and a positive constant c such that

$$\liminf_{k\to\infty} \frac{Z_{n_k}^3}{\sqrt{2n_k \log_2 n_k}} \leq -c. \tag{79}$$

We start by recalling decomposition (1) for the third coordinate of Z, i.e.

$$Z_n^3 = Z_0^3 + M_n^3 + \sum_{i=1}^{n} E(Y_i^3 | Z_{i-1}) = Z_0^3 + M_n^3 + p_3\pi_n^1 + q_3\pi_n^2 + r_3\pi_n^3. \tag{80}$$

Let us first study the asymptotic behavior of M which is a square-integrable martingale w.r.t. to the standard filtration generated by Z. Define the process A by $A_n = \sum_{i=1}^n E((M_i - M_{i-1})^2 | Z_{i-1})$. Then

$$A_n = \sum_{j=0}^{3} v_j^2 \pi_n^j,$$

where, as above, v_j^2 is the conditional variance of $Y_i^3 = Z_i^3 - Z_{i-1}^3$ on $\{\widetilde{Z}_{i-1} \in \partial\widetilde{G}_j\}$. By definition, it follows that for any $n \geq 0$,

$$n \min_{j=0,1,2,3} (v_j^2) \leq A_n \leq n \max_{j=0,1,2,3} (v_j^2). \tag{81}$$

Obviously, $A_n \to \infty$ almost surely as $n \to \infty$. Since the processes $\left\{\left(\frac{\pi^1_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^2_{[n\cdot]}}{\sqrt{n}}, \frac{\pi^3_{[n\cdot]}}{\sqrt{n}}\right), n \geq 1\right\}$ converge weakly by Corollary 2, then there exists a subsequence n_k such that $n_k \to \infty$ as $k \to \infty$ and for any $j = 0, 1, 2, 3$,

$$\frac{\pi^j_{n_k}}{\sqrt{2n_k \log_2 n_k}} \xrightarrow{\text{a.s.}} 0. \tag{82}$$

The result which will allow us to study the behavior of M is the following iterated logarithm law type result for square-integrable martingales.

LEMMA 7 (Theorem 3.2 of [15]). – *Let $\{M_n, n \geq 0\}$ be a square-integrable martingale w.r.t. a filtration $(\mathcal{F}_n)$. For any $n \geq 1$, set $X_n = M_n - M_{n-1}$ and*

$$A_n = \sum_{i=1}^{n} E(X_i^2 | \mathcal{F}_{i-1}).$$

Define random function $\{M(t), t \geq 0\}$ by

$$M(t) = M_n \quad on\ t \in [A_n, A_{n+1}).$$

Suppose $A_n \to \infty$ as $n \to \infty$ and for some fixed $\beta \geq 0$, almost surely

$$\sum_{k=1}^{n} \frac{(\log_2 A_k)^\beta}{A_k} E\left(X_k^2 1_{(X_k^2 > A_k (\log A_k)^{-1} (\log_2 A_k)^{-2(\beta+1)})} | \mathcal{F}_{k-1}\right) < \infty. \tag{83}$$

Then, redefining M if necessary on a new probability space, there exists a Brownian motion W such that almost surely, as $t \to \infty$,

$$|M(t) - W(t)| = o\left(\sqrt{t} (\log_2 t)^{\frac{1-\beta}{2}}\right). \tag{84}$$

In particular, as $n \to \infty$,

$$M_n = W(A_n) + o\left(\sqrt{A_n} (\log_2 A_n)^{\frac{1-\beta}{2}}\right). \tag{85}$$

Let us now verify that the martingale M and the process A satisfy the condition (83) of the last lemma for some positive β. Fix any such β. Since the variables Y_i^3 admit finite moments of the order $2 + \delta$ and these moments are uniformly bounded in i, then the increments $X_i = M_i - M_{i-1}$ also admit finite moments of the order $2 + \delta$ which are also uniformly bounded in i. By Chebyshev inequality it then follows that there exists a positive constant $C > 0$ such that for any $B > 0$,

$$E(X_k^2 1_{(X_k^2 > B)} | \mathcal{F}_{k-1}) \leq C B^{-\frac{\delta}{2}}. \tag{86}$$

Substituting B by $\frac{A_n}{\log A_n (\log_2 A_n)^{2(\beta+1)}}$ one sees from the upper bound on A from (81) that the condition (83) is satisfied.

Let us choose any positive $\beta < 1$ and set for any $n \geq 0$, $\phi(n) = \sqrt{2n \log_2 n}$. Then by (80), (81) and (85) it follows that almost surely,

$$\begin{aligned} \liminf_{k\to\infty} \frac{Z_{n_k}^3}{\phi(n_k)} &= \liminf_{k\to\infty} \frac{M_{n_k}^3 + p_3 \pi_{n_k}^1 + q_3 \pi_{n_k}^2 + r_3 \pi_{n_k}^3}{\phi(n_k)} \\ &= \liminf_{k\to\infty} \frac{W(A_{n_k})}{\phi(n_k)}, \end{aligned} \tag{87}$$

for any subsequence n_k for which (82) holds. Notice that the process Y defined for any $m \geq 0$ by $Y_m = W(A_{n_m})$ is a Gaussian square-integrable martingale with quadratic variation

$$\sum_{k=1}^{m} E((Y_k - Y_{k-1})^2 | \mathcal{F}_{n_{k-1}}) = \sum_{k=1}^{m} E(A_{n_k} - A_{n_{k-1}} | \mathcal{F}_{n_{k-1}})$$

bounded from below by $n_m \min_{j=0,1,2,3}(v_j^2)$ and from above by $n_m \max_{j=0,1,2,3}(v_j^2)$. This fact and the iterated logarithm law for gaussian martingales (see [18], page 154) imply that the last limit in (87) is less than a negative constant. This terminates the proof of (79) and the remark. ∎

REFERENCES

[1] S. Aspandiiarov, On the convergence of 2-dimensional Markov chains in quadrants with boundary reflection, *Stoch. and Stochastic Reports.*, Vol. **53**, 1995, pp. 275–303.

[2] S. Aspandiiarov, R. Iasnogorodski and M. Menshikov, Passage-time moments for non-negative stochastic processes and an application to reflected random walks in a quadrant. *Ann. Prob.*, Vol. **26**, 1996, pp. 932–960.

[3] S. Aspandiiarov, R. Iasnogorodski and M. Menshikov, Reflected random walks in a three-dimensional orthant (in preparation).

[4] S. Aspandiiarov and R. Iasnogorodski, Tails of passage-times for non-negative stochastic processes and an application to stochastic processes with boundary reflection in a wedge. *Stoch. Proc. Appl.*, Vol. **66**, 1997 pp. 115–145.

[5] S. Aspandiiarov and Iasnogorodski, General criteria of integrability of functions of passage-times for non-negative stochastic processes and their applications. *Theory of Prob. and its Appl.* (to appear).

[6] S. Aspandiiarov and Iasnogorodski, Asymptotic behavior of stationary distributions for countable Markov chains with some applications. *Bernoulli* (to appear).

[7] Kai Lai CHUNG, Markov Chains with Stationary Transition Probabilities. Springer, Berlin, 1967.

[8] J. DAI and R. J. WILLIAMS, Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedrons. *Theory of Prob. and its Appl.*, Vol. **40**, 1996, pp. 1–25.

[9] R. D. De BLASSIE, Explicit semimartingale representation of Brownian motion in a wedge. *Stoch. Proc. Appl.*, Vol. **34**, 1990 pp. 67-97.

[10] S. N. ETHIER, T. G. KURTZ, Markov Processes. Wiley, Chichester, 1986.

[11] J. M. HARRISON, R. J. WILLIAMS, Brownian models of open queuing networks with homogeneous customer populations. *Stochastics and Stoch. Reports.*, Vol. **22**, 1987, pp. 77-115.

[12] G. FAYOLLE, V. MALYSHEV and M. MENSHIKOV, Random walks in a quarter plane with zero drifts. *Ann. Inst. H. Poincare.*, Vol. **28**, 1992, pp. 179-195.

[13] W. FELLER, An Introduction to Probability Theory and its Applications. Wiley, 2nd edn, New York, 1971.

[14] J. JACOD and A. N. SHIRYAEV, Limit Theorems for Stochastic Processes. Springer, Berlin, 1987.

[15] N. C. JAIN, K. JOGDEO and W. F. STOUT, Upper and lower functions for martingales and mixing processes. *Ann. Prob.*, Vol. **3**, 1975, pp. 119-145.

[16] R. Sh. LIPTZER, A. N. SHIRYAEV, Theory of Martingales. Kluwer, Dordrecht, 1989.

[17] M. MENSHIKOV and R. J. WILLIAMS, Passage-time moments for continuous non-negative stochastic processes and applications. *Adv. Appl. Prob.* (to appear).

[18] J. NEVEU, Discrete-Parameter Martingales. North-Holland, Amsterdam, 1975.

[19] W. P. PETERSEN, Diffusion approximations for networks of queues with multiple customer types. *Math. Oper. Res.*, Vol. **9**, 1991, pp. 90-118.

[20] V. PETROV, Sums of Independent Random Variables. Springer-Verlag, 1975.

[21] W. PRUITT, The growth of random walks and Lévy processes. *Ann. Prob.*, Vol. **9**, 1981, pp. 948-956.

[22] M. I. REIMAN, Open queuing networks in heavy traffic. *Math. Oper. Res.*, Vol. **9**, 1984, pp. 441-458.

[23] D. REVUZ, Markov Chains. North-Holland, Amsterdam, 1975.

[24] L. M. TAYLOR and R. J. WILLIAMS, Existence and uniqueness of SRBM in an orthant. *Prob. Theory and Rel. Fields.*, Vol. **96**, 1993, pp. 283-318.

[25] S. R. S. VARADHAN, R. J. WILLIAMS, Brownian motion in a wedge with oblique reflection. *Comm. Pure and Applied Math.*, Vol. **38**, 1985, pp. 405-443.

[26] R. J. WILLIAMS, On the approximation of queuing networks in heavy traffic. *Stochastic Networks: Theory and Applications,* eds. F. P. Kelly, S. Zachary and I. Ziedins, Clarendon Press, Oxford, 1996, pp. 35-56.

[27] R. J. WILLIAMS, Semimartingale reflecting Brownian motion in the orthant. *The IMA Volumes in Mathematics and its Applications.*, Vol. **71**, Springer, Berlin, 1995.

[28] R. J. WILLIAMS, Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahr. verw. Geb.*, Vol. **69**, 1985 pp; 161-176.

(Manuscript received October 2, 1996;
revised May 26, 1998.)