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Large and moderate deviations for the local time of a recurrent Markov chain on \mathbb{Z}^2

by

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ABSTRACT. – Let (X_n) be a recurrent Markov chain on \mathbb{Z}^2 with $X_0 = (0, 0)$ such that for some constant C , $P[X_k = (0, 0)] \leq \frac{C}{k}$, and whose truncated Green function is slowly varying at infinity. Let L_n^0 denote the local time at zero of such a Markov chain. We prove various moderate and large deviation statements and limit laws for rescaled versions of L_n^0 , including functional versions of these. A version of Strassen's functional law of the iterated logarithm, recently discovered by E. Csáki, P. Révész and J. Rosen, can be derived as a corollary. © Elsevier, Paris

Key words: Local time, Markov chain, large deviations, Strassen's law.

RÉSUMÉ. – Soit (X_n) une chaîne de Markov récurrente sur \mathbb{Z}^2 , avec $X_0 = (0, 0)$, telle que pour une constante C , $P[X_k = (0, 0)] \leq \frac{C}{k}$, et telle que la fonction de Green est de variation lente à l'infini. Avec L_n^0 le temps local de (X_n) à zero, nous démontrons des résultats de grandes déviations

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et de déviations modérées pour certains changements d'échelle de L_n^0 , ainsi qu'une version fonctionnelle. Comme corollaire, on note un théorème du logarithme itéré fonctionnel de type Strassen, démontré récemment par E. Csáki, P. Révész, et J. Rosen. © Elsevier, Paris

1. INTRODUCTION AND STATEMENT OF RESULTS

Let (X_n) be a recurrent Markov chain on \mathbb{Z}^2 with $X_0 = (0, 0)$, and let $g(n) := \sum_{k=0}^n P[X_k = (0, 0)]$ be the truncated Green function. We can extend g to a continuous, increasing function $g(t), t \geq 0$. Since (X_n) is recurrent, $g(t) \rightarrow \infty$ for $t \rightarrow \infty$.

We will assume throughout that, for some positive constant C ,

$$P[X_k = (0, 0)] \leq \frac{C}{k}, \quad (1)$$

hence $g(n) \leq C \log n$. We will also assume throughout that

$$g \text{ is slowly varying at } \infty, \quad (2)$$

that is $g(tx)/g(t) \xrightarrow{t \rightarrow \infty} 1$ for any $x > 0$. Note that (1) is satisfied for symmetric random walks on \mathbb{Z}^2 , i.e. if $P[X_1 = (y, z)] = P[X_1 = -(y, z)]$, see [6], Proposition 2.14. Since our results depend only on (1) and (2), they might also apply to symmetric recurrent random walks on \mathbb{Z} in the domain of attraction of a Cauchy random variable.

We denote by L_n^0 the local time of X at $(0, 0)$, i.e. $L_n^0 := |\{0 \leq k \leq n : X_k = (0, 0)\}|$, and $L_0^0 = 0$. Let $\rho_0 = 0$, $\rho_k = \min\{j : j > \rho_{k-1}, X_j = (0, 0)\}$, $k = 1, 2, 3, \dots$. It is known, see [6], (and will follow from the proof of Theorem 1), that $L_n^0/g(n)$ converges in distribution to an exponential distribution, i.e.

$$P\left[\frac{L_n^0}{g(n)} \geq y\right] \xrightarrow{n \rightarrow \infty} e^{-y} \text{ for } y \geq 0. \quad (3)$$

Our goal is to investigate the fluctuations of L_n^0 , and associated functional laws.

THEOREM 1 (Moderate Deviations). – *Let $\psi(n)$ be a positive, non-decreasing function such that*

$$\gamma_n := \frac{n}{\psi(n)g(n)} \xrightarrow{n \rightarrow \infty} \infty.$$

Then $L_n^0/g(n)\psi(n)$ satisfies a large deviation principle with speed $\psi(n)g(n)/g(\gamma_n)$ and rate function y .

We refer to [2] for the definition of a large deviation principle. Here, it will be enough to show that

$$\frac{g(\gamma_n)}{\psi(n)g(n)} \log P \left[\frac{L_n^0}{g(n)\psi(n)} \geq y \right] \xrightarrow{n \rightarrow \infty} -y. \tag{4}$$

Theorem 1 is a moderate deviation principle since the speed can vary without changing the rate function. Further, the rate function does not depend on the distribution of ρ_1 .

The next theorem gives a large deviation principle for the distributions of L_n^0/n , with rate function which does depend on the distribution of ρ_1 .

THEOREM 2 (Large Deviations). – Let $\Lambda^*(y) = \sup_{\lambda \leq 0} (\lambda y - \log E[e^{\lambda \rho_1}])$ and

$$J(y) = \begin{cases} y\Lambda^*\left(\frac{1}{y}\right), & 0 < y \leq 1 \\ 0, & y = 0 \\ +\infty, & \text{otherwise} \end{cases}$$

Then the distributions of L_n^0/n satisfy a LDP with speed n and rate function J .

Remarks

1. Comparing with Theorem 1, the large deviation principle holds for $\psi(n) = \frac{n}{g(n)}$. In this case, $\gamma_n = 1$ and Theorem 1 does not apply. Considering the proof of Theorem 2, it is easy to show that we have a LDP whenever $\gamma_n \xrightarrow{n \rightarrow \infty} \alpha$, $0 < \alpha < 1$.
2. Let $p_0 := P[X_1 = (0, 0)]$. Then we have $J(1) = -\log p_0$ if $p_0 > 0$ and $J(1) = \infty$ otherwise.
3. Let $L^0(\cdot)$ be the linear interpolation of L_n^0 between integer points. We believe (but have not checked the details) that the standard argument (see e.g. [2], Section 5.1) allows one to conclude that the distributions of $(\frac{L^0(nt)}{n})_{0 \leq t \leq 1}$ satisfy a large deviation principle (in $C[0, 1]$) with

rate function

$$\tilde{J}(f) = \begin{cases} \int_0^1 J(f'(s))ds, & f \text{ absolutely continuous with derivative } f' \\ +\infty, & \text{otherwise.} \end{cases}$$

As usual, we can derive an Erdős-Renyi law from the large deviation principle:

COROLLARY 1. – *Let $c > 0$ and $\eta_{n,j} := \frac{1}{c \log g(n)} (L_{j+\lfloor c \log g(n) \rfloor}^0 - L_j^0)$, $j = 0, 1, 2, \dots, n - \lfloor c \log g(n) \rfloor$. Then $\lim_{n \rightarrow \infty} \sup_{j=0,1,\dots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j} = d_c$, a.s., where $d_c = \inf \{y : J(y) \geq \frac{1}{c}\}$.*

For a random walk on \mathbb{Z} , this complements results of [5].

We next turn to the appropriate functional statements. Let $\psi(n)$ and γ_n be as in the statement of Theorem 1, and let $t(n, x)$ be a sequence of positive, increasing (in n, x) functions satisfying, for any $x \in]0, 1]$,

$$\lim_{n \rightarrow \infty} \frac{g\left(\frac{t(n,x)}{g(n)\psi(n)}\right)}{g(\gamma_n)} = x > 0. \tag{5}$$

For example, if $g(n) \sim C \log n$, and $\frac{\log \psi(n)}{\log n} \xrightarrow[n \rightarrow \infty]{} 0$, we can take $t(n, x) = n^x$. If $g(n) \sim C \log n$ and $\psi(n) = n^\beta$, ($0 < \beta < 1$), we can take $t(n, x) = n^{x(1-\beta)+\beta}$. If $g(n) \sim C \log_2 n$ and $\frac{\log \psi(n)}{\log n} \xrightarrow[n \rightarrow \infty]{} 0$, we can take $t(n, x) = e^{(\log n)^x}$ (here and throughout, $\log_k n$ denotes the k -th iterated logarithm function). If $g(n) \sim C \log_2 n$ and $\psi(n) = n^\beta$, ($0 < \beta < 1$), we can take $t(n, x) = n^\beta e^{(\log n)^x}$.

It is straightforward to check, using (5), that for $0 \leq x_1 < x_2 \leq 1$, we have

$$\frac{t(n, x_1)}{t(n, x_2)} \xrightarrow[n \rightarrow \infty]{} 0. \tag{6}$$

Let

$$\bar{L}_n(x) := \frac{L_{t(n,x)}^o}{g(n)\psi(n)}, \quad x \in [0, 1].$$

Note that $\bar{L}_n(x) \in M_+$, the space of non-negative Borel measures on $[0, 1]$. Equip M_+ with the topology of weak convergence. Our main functional statement is the following:

THEOREM 3 (Functional Moderate Deviations). – $\bar{L}_n(x)$ satisfies in M_+ a large deviation principle with speed $g(n)\psi(n)/g(\gamma_n)$ and rate function

$$I(m) = \begin{cases} \int_0^1 \frac{1}{x} m(dx), & \frac{1}{x} \in L_1(m) \\ \infty, & \text{otherwise.} \end{cases}$$

As in the one-dimensional case, we can deduce convergence in distribution from our large deviation bounds, taking $\psi(n) \equiv 1$.

THEOREM 4 (Functional Limit Law). – Let $t(n, x)$ be such that $g(t(n, x)) \sim xg(n)$, $x \in [0, 1]$. The distributions of $(\frac{L_{t(n,x)}^0}{g(n)})_{0 \leq x \leq 1}$ converge weakly to $\mu \in M_1(M_+)$, the distribution of the process $(Z_x)_{0 \leq x \leq 1}$ with increasing paths and independent increments given by

$$P[Z_{x_2} - Z_{x_1} \in B] = \frac{x_1}{x_2} \delta_o(B) + \left(1 - \frac{x_1}{x_2}\right) \int_B \frac{1}{x_2} e^{-\frac{1}{x_2}u} du, \quad (7)$$

for any $0 \leq x_1 < x_2 \leq 1$, B Borel subset of $[0, \infty[$.

J. Bertoin kindly pointed out to us that in fact the process $(Z_x)_{0 \leq x \leq 1}$ in Theorem 4 is a pure jump process which can be constructed from an inhomogeneous Poisson point process. Indeed, one may construct a Poisson point process $N(x, z)$ on $[0, 1] \times \mathbb{R}_+$ with intensity $n(x, z) dx dz = x^{-2} \exp(-z/x) dx dz$ and define $Y_x = \int_0^\infty z dz N(x, z)$. Obviously, $(Y_x)_{0 \leq x \leq 1}$ possesses increasing paths and independent increments. Moreover, it is not hard to check, using the identity valid for any $\alpha, \beta > 0$,

$$\lim_{\epsilon \rightarrow 0} \left(\int_\epsilon^\infty \frac{e^{-\alpha z}}{z} dz - \int_\epsilon^\infty \frac{e^{-\beta z}}{z} dz \right) = \log \beta - \log \alpha,$$

that for any $\lambda \geq 0$,

$$E(\exp(-\lambda(Y_{x+y} - Y_x))) = \frac{1 + \lambda x}{1 + \lambda(x + y)} = E(\exp(-\lambda(Z_{x+y} - Z_x))),$$

proving that the processes $(Z_x)_{0 \leq x \leq 1}$ and $(Y_x)_{0 \leq x \leq 1}$ have the same law.

We close this section by mentioning that the functional moderate deviations of Theorem 3 are strong enough to derive by standard arguments the following Strassen law of the iterated logarithm presented in [1], Theorem 5. Obtaining such a derivation was actually the original motivation

for this work. Since the arguments are standard, see [3], Theorem 1.4.1, we do not provide a proof.

THEOREM 5 (E. Csáki, P. Révész and J. Rosen). – *Let $t(n, x)$ be such that $g(t(n, x)) \sim xg(n)$, $x \in [0, 1]$. The set $(\frac{L_{t(n,x)}^0}{g(n)\log_2 g(n)})_{0 \leq x \leq 1}$, n large enough, is relatively compact in M_+ with limit points K , where $K = \{m : I(m) \leq 1\}$.*

2. PROOFS

We begin by stating some simple bounds on $g(n)$.

LEMMA 1. – *We have*

$$\lim_{n \rightarrow \infty} \frac{g(n)}{g(ng(n))} = 1, \quad (8)$$

and

$$\lim_{n \rightarrow \infty} \frac{g(n)}{g(n/g(n))} = 1. \quad (9)$$

Proof of Lemma 1. – We have

$$\begin{aligned} g(ng(n)) - g(n) &\leq \sum_{j=n}^{\lceil ng(n) \rceil} P[X_j = (0, 0)] \\ &\leq C \sum_{j=n}^{\lceil ng(n) \rceil} \frac{1}{j} \leq C' \log g(n), \end{aligned}$$

where C' is some (fixed, depending on C) constant. The limit (8) follows by dividing by $g(ng(n))$ and using the monotonicity of $g(\cdot)$. The proof of (9) is analogous. \square

Lemma 1 is needed for the following crucial estimate for the tail of the distribution of the excursion ρ_1 . For a more precise statement, which we do not need here, see [6].

PROPOSITION 1

$$P[\rho_1 > n] \leq \frac{1}{g(n)}$$

and

$$P[\rho_1 > n] \sim \frac{1}{g(n)}$$

i.e. $g(n)P[\rho_1 > n] \xrightarrow{n \rightarrow \infty} 1$.

Proof of Proposition 1

1. A last exit decomposition gives

$$\sum_{k=0}^n P[X_k = (0, 0)] P[L_{n-k}^0 = 0] = 1.$$

Since $P[L_{n-k}^0 = 0] \geq P[L_n^0 = 0], k = 0, 1, \dots, n$, this implies $g(n)P[L_n^0 = 0] \leq 1$, hence

$$P[\rho_1 > n] = P[L_n^0 = 0] \leq \frac{1}{g(n)}.$$

2. In the same way,

$$1 \leq \sum_{j=0}^k P[X_j = (0, 0)]P[L_{n-k}^0 = 0] + \sum_{j=k+1}^n P[X_j = (0, 0)]$$

hence $1 \leq g(k)P[L_{n-k}^0 = 0] + g(n) - g(k)$, so

$$g(k)P[L_{n-k}^0 = 0] \geq 1 - (g(n) - g(k)). \tag{10}$$

Choose $k = k(n) = \lfloor n - \frac{n}{g(n)} \rfloor$, and note that, for some $C', C'' > 0$,

$$\begin{aligned} g(n) - g(k) &= \sum_{j=k}^n P[X_j = (0, 0)] \leq C \sum_{j=k}^n \frac{1}{j} \\ &\leq C'(\log n - \log k) \leq C'' \log\left(1 - \frac{1}{g(n)}\right) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

This, together with (9) of Lemma 1, yields the proposition. □

Proof of Theorem 1. – We begin with a quick proof of the lower bound in (4). Let Y_1, Y_2, \dots be i.i.d. with the same distribution as ρ_1 . Then

$$\begin{aligned} P[L_n^0 \geq \psi(n)g(n)y] &\geq P\left[\sum_{i=1}^{\lceil g(n)\psi(n)y \rceil} Y_i \leq n\right] \\ &\geq P\left[\max_{1 \leq i \leq \lceil g(n)\psi(n)y \rceil} Y_i \leq \frac{n}{\lceil g(n)\psi(n)y \rceil}\right] \\ &= \left(1 - P\left[\rho_1 > \frac{n}{\lceil g(n)\psi(n)y \rceil}\right]\right)^{\lceil g(n)\psi(n)y \rceil} \end{aligned}$$

Now apply Proposition 1 and the fact that $g(\cdot)$ is slowly varying to get

$$\liminf_{n \rightarrow \infty} \frac{g\left(\frac{n}{\psi(n)g(n)}\right)}{g(n)\psi(n)} \log P[L_n^0 \geq \psi(n)g(n)y] \geq -y.$$

We next turn to the proof of the upper bound. We follow the standard strategy to apply Chebycheff's inequality and to optimize over the parameter. Due to Chebycheff's inequality,

$$\begin{aligned} P[L_n^0 \geq g(n)\psi(n)y] &\leq P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)y \rfloor} Y_i \leq n\right] \\ &\leq E[e^{-\lambda_n Y_1}]^{\lfloor g(n)\psi(n)y \rfloor} e^{\lambda_n n} \end{aligned} \quad (11)$$

for each $\lambda_n > 0$. Recall $\gamma_n = \frac{n}{\psi(n)g(n)}$. Taking logarithms and dividing by $\frac{g(n)\psi(n)}{g(\gamma_n)}$, (11) yields

$$\begin{aligned} &\frac{g(\gamma_n)}{g(n)\psi(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \\ &\leq g(\gamma_n)y \frac{\lfloor g(n)\psi(n)y \rfloor}{g(n)\psi(n)y} \log E[e^{-\lambda_n Y_1}] + \frac{g(\gamma_n)\lambda_n n}{\psi(n)g(n)} \end{aligned} \quad (12)$$

Next we show that for each $\delta > 0$, and $C_n > 0$ large enough, we have

$$\log E[e^{-\lambda_n Y_1}] \leq \frac{1-\delta}{g(C_n)} (e^{-\lambda_n C_n} - 1). \quad (13)$$

Indeed, observe that

$$\begin{aligned} \log E[e^{-\lambda_n Y_1}] &= \log E[e^{-\lambda_n \rho_1}] \leq E[e^{-\lambda_n \rho_1}] - 1 \\ &\leq e^{-\lambda_n C_n} P[\rho_1 \geq C_n] + P[\rho_1 < C_n] - 1 \\ &= P[\rho_1 \geq C_n] (e^{-\lambda_n C_n} - 1) \leq \frac{1-\delta}{g(C_n)} (e^{-\lambda_n C_n} - 1) \end{aligned}$$

where we used Proposition 1 in the last inequality.

Substituting this estimate in (12), we get

$$\begin{aligned} &\frac{g(\gamma_n)}{\psi(n)g(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \\ &\leq y(1-\delta) \frac{g(\gamma_n)}{g(C_n)} (e^{-\lambda_n C_n} - 1) + \frac{g(\gamma_n)\gamma_n}{C_n} \lambda_n C_n. \end{aligned} \quad (14)$$

Choose $C_n = K\gamma_n g(\gamma_n)$, $\lambda_n = \frac{K'}{C_n}$ with $K, K' > 0$. Then the r.h.s. of (14) is

$$y(1 - \delta) \frac{g(\gamma_n)}{g(K\gamma_n g(\gamma_n))} (e^{-K'} - 1) + \frac{g(\gamma_n)}{Kg(\gamma_n)} K'. \tag{15}$$

Due to Lemma 1 and the fact that $g(\cdot)$ is slowly varying, $\frac{g(\gamma_n)}{g(K\gamma_n g(\gamma_n))} \xrightarrow{n \rightarrow \infty} 1$. Hence (14) and (15) yield

$$\limsup_{n \rightarrow \infty} \frac{g(\gamma_n)}{\psi(n)g(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \leq y(1 - \delta)(e^{-K'} - 1) + \frac{K'}{K}$$

and the upper bound follows by letting $\delta \rightarrow 0$, $K' \rightarrow \infty$, $\frac{K'}{K} \rightarrow 0$. \square

Remark. – In particular, taking in the proof of the upper and the lower bound $\psi(n) \equiv 1$, we have

$$\frac{g(\frac{n}{g(n)})}{g(n)} \log P \left[\frac{L_n^0}{g(n)} \geq y \right] \xrightarrow{n \rightarrow \infty} -y.$$

Together with (9) in Lemma 1, this implies that for $y \geq 0$,

$$P \left[\frac{L_n^0}{g(n)} \geq y \right] \xrightarrow{n \rightarrow \infty} e^{-y},$$

as noted in (3).

Proof of Theorem 2. – Note first that $P[L_n^0 \geq ny] = 0$ if $y > 1$. As in the proof of Theorem 1, we have

$$P \left[\sum_{i=1}^{\lceil ny \rceil} Y_i \leq n \right] \leq P[L_n^0 \geq ny] \leq P \left[\sum_{i=1}^{\lfloor ny \rfloor} Y_i \leq n \right].$$

But

$$P \left[\sum_{i=1}^{\lfloor ny \rfloor} Y_i \leq n \right] \leq P \left[\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i \leq \frac{1}{y} \right]$$

so we ask about large deviations of the arithmetic mean of a sequence of i.i.d. random variables. Cramér’s theorem (see [2], Theorem 2.2.3) implies that the distributions of $\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i$ (or $\frac{1}{\lceil ny \rceil} \sum_{i=1}^{\lceil ny \rceil} Y_i$) satisfy a LDP with

speed $\lfloor ny \rfloor$ (or $\lceil ny \rceil$) and rate function Λ^* . Note that $Y_1 \geq 0$, $E[Y_1] = \infty$ hence $\Lambda^*(y) \rightarrow 0$ for $y \rightarrow \infty$. Since we have

$$\frac{1}{n} \log P \left[\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i \leq \frac{1}{y} \right] = \frac{\lfloor ny \rfloor}{n} \frac{1}{\lfloor ny \rfloor} \log P \left[\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i \leq \frac{1}{y} \right]$$

and $\frac{\lfloor ny \rfloor}{n} \xrightarrow[n \rightarrow \infty]{} y$, the claim follows. \square

In order to prove Corollary 1, we need the following preliminary proposition.

PROPOSITION 2. – *Let $\psi(n) \rightarrow 0$, $\psi(n)g(n) \rightarrow \infty$. Then, for each $x > 0$, $\frac{1}{\psi(n)} P \left[\frac{L_n^0}{g(n)\psi(n)} \leq x \right] \xrightarrow[n \rightarrow \infty]{} x$.*

Proof of Proposition 2

1. We have

$$\begin{aligned} P[L_n^0 \leq g(n)\psi(n)x] &\leq P \left[\sum_{j=1}^{\lceil g(n)\psi(n)x \rceil} Y_j \geq n \right] \\ &\leq P \left[\max_{1 \leq j \leq \lceil g(n)\psi(n)x \rceil} Y_j \geq \frac{n}{\lceil g(n)\psi(n)x \rceil} \right] \\ &= 1 - \left(1 - P \left[Y_1 \geq \frac{n}{\lceil g(n)\psi(n)x \rceil} \right] \right)^{\lceil g(n)\psi(n)x \rceil} \\ &\leq 1 - \left(1 - \frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x \rceil}\right)} \right)^{\lceil g(n)\psi(n)x \rceil} \end{aligned}$$

where we used Proposition 1 in the last inequality. Since $1 - z \leq -\log z$, the last term is

$$\leq -\lceil g(n)\psi(n)x \rceil \log \left(1 - \frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x \rceil}\right)} \right).$$

Hence

$$\begin{aligned} \frac{1}{\psi(n)} P \left[\frac{L_n^0}{g(n)\psi(n)} \leq x \right] \\ \leq -\frac{\lceil g(n)\psi(n)x \rceil}{g(n)\psi(n)} \log \left(1 - \frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x \rceil}\right)} \right)^{g(n)} \end{aligned} \quad (16)$$

Provided that

$$\frac{g(n)}{g(\frac{n}{g(n)\psi(n)})} \xrightarrow{n \rightarrow \infty} 1, \tag{17}$$

(16) implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{\psi(n)} P \left[\frac{L_n^0}{g(n)\psi(n)} \leq x \right] \leq x. \tag{18}$$

But (17) holds true since

$$g(n) \geq g(\frac{n}{g(n)\psi(n)}) \geq g(\frac{n}{g(n)})$$

and $\frac{g(n)}{g(n/g(n))} \xrightarrow{n \rightarrow \infty} 1$ due to Lemma 1.

2.

$$\begin{aligned} P[L_n^0 \leq g(n)\psi(n)x] &\geq P \left[\sum_{j=1}^{\lfloor g(n)\psi(n)x \rfloor} Y_j \geq n \right] \\ &\geq P \left[\max_{1 \leq j \leq \lfloor g(n)\psi(n)x \rfloor} Y_j \geq n \right] \\ &= 1 - (1 - P[Y_1 \geq n])^{\lfloor g(n)\psi(n)x \rfloor}. \end{aligned}$$

Now we use the inequality $1 - z \geq -z \log z$ ($0 < z < 1$) with $z = (1 - P[Y_1 \geq n])^{\lfloor g(n)\psi(n)x \rfloor}$ to get

$$\begin{aligned} P[L_n^0 \leq g(n)\psi(n)x] &\geq -\frac{\lfloor g(n)\psi(n)x \rfloor}{g(n)x} \log(1 - P[Y_1 \geq n])^{g(n)x} \\ &\cdot (1 - P[Y_1 \geq n])^{\lfloor g(n)\psi(n)x \rfloor}. \end{aligned} \tag{19}$$

Proposition 1 implies that

$$(1 - P[Y_1 \geq n])^{g(n)x} \xrightarrow{n \rightarrow \infty} e^{-x}$$

and therefore

$$(1 - P[Y_1 \geq n])^{\lfloor g(n)\psi(n)x \rfloor} \xrightarrow{n \rightarrow \infty} 1.$$

We conclude from (19) that

$$\liminf_{n \rightarrow \infty} \frac{1}{\psi(n)} P \left[\frac{L_n^0}{g(n)\psi(n)} \geq x \right] \geq x.$$

□

Proof of Corollary 1

1. Let $d \in \mathbb{R}, J(d) > \frac{1}{c}$, choose $\delta > 0$ such that $J(d) - \delta > \frac{1}{c}$, and fix any $d' > d$. We show that

$$P\left[\sup_{j=0,1,\dots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j} \geq d' \text{ for infinitely many } n\right] = 0. \quad (20)$$

Let $\psi(n) = (\log g(n))^\gamma$ where $\gamma > 1$. Since we can take the sup in $\sup_{j=0,1,\dots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j}$ over those j with $X_j = (0, 0)$ only, without changing the value, and since $\eta_{n,j}$ has the same distribution as $\eta_{n,0}$ for those j , we have

$$\begin{aligned} P\left[\sup_{j=0,1,\dots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j} \geq d\right] \\ \leq P[L_n^0 \geq g(n)\psi(n)] + \psi(n)g(n)P[\eta_{n,0} \geq d]. \end{aligned} \quad (21)$$

Now we have to estimate the terms on the r.h.s. of (21):

$$P[L_n^0 \geq g(n)\psi(n)] \leq e^{-\psi(n)(1-\delta)} \quad (22)$$

for n big enough, due to Theorem 1 and

$$P[\eta_{n,0} \geq d] \leq e^{-c \log g(n)(J(d)-\delta)} \quad (23)$$

for n big enough, due to Theorem 2.

Let $\lambda > 1, n_0 = 0$ and $n_k = \lceil g^{-1}(\lambda^k) \rceil, k = 1, 2, \dots$. Then we see from (22) and (23), applying the Borel-Cantelli lemma, that

$$P\left[\sup_{j=0,1,\dots,n_k-\lfloor c \log g(n_k) \rfloor} \eta_{n_k,j} \geq d \text{ for infinitely many } k\right] = 0.$$

In other words, we have proved (20) along the subsequence (n_k) with d replacing d' . Let $n_k \leq n \leq n_{k+1}$ and observe that, for $j = 0, 1, \dots, n - \lfloor c \log g(n) \rfloor$,

$$\begin{aligned} \eta_{n,j} &\leq \eta_{n_{k+1},j} \frac{\log g(n_{k+1})}{\log g(n)} \leq \eta_{n_{k+1},j} \frac{\log g(n_{k+1})}{\log g(n_k)} \\ &\leq \eta_{n_{k+1},j} \frac{k+1}{k} \end{aligned}$$

For k big enough, $\eta_{n_{k+1},j} < d$ implies $\eta_{n,j} < d'$. This completes the proof of (20).

2. Let $d \in \mathbb{R}, J(d) < \frac{1}{c}$. Choose $\delta > 0$ and $\lambda > 1$ such that $\lambda(J(d) + \delta) < \frac{1}{c}$. We will construct a subsequence n_k such that

$$P\left[\sup_{0 \leq j \leq n_k - \lfloor c \log g(n_k) \rfloor} \eta_{n_k, j} < d \text{ for infinitely many } k\right] = 0. \quad (24)$$

Fixing n , let $j_0^n := 0, j_m^n := \inf\{j : j > j_{m-1}^n + \lfloor c \log g(n) \rfloor, X_j = (0, 0)\}$, $M^n := M^n(\omega) = \max\{m : j_m^n \leq n\}$ and $J^n := \{j_0^n, \dots, j_{M^n-1}^n\}$. Then $(\eta_{n, j})_{j \in J^n}$ are i.i.d. with the same distribution as $\eta_{n, 0}$. Let $\psi(n)$, to be determined below, satisfy the assumptions of Proposition 2. We have

$$\begin{aligned} &P\left[\sup_{0 \leq j \leq n - \lfloor c \log g(n) \rfloor} \eta_{n, j} < d\right] \\ &\leq P\left[M^n < \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor}\right] + P[\eta_{n, 0} < d] \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor}. \end{aligned} \quad (25)$$

But, for each $\tilde{\delta} > 0$, and all n large enough,

$$P\left[M^n < \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor}\right] \leq P[L_n^0 < \lfloor g(n)\psi(n) \rfloor] \leq (1 + \tilde{\delta})\psi(n) \quad (26)$$

for n large enough, where we used Proposition 2 in the last inequality. Turning now to the second term in (25), we first note that, by Theorem 2, for all n large enough,

$$P[\eta_{n, 0} \geq d] \geq e^{-c \log g(n)(J(d) + \delta)} \geq e^{-\beta \log g(n)}$$

for n large enough, where $\beta := c(J(d) + \delta) < 1$. Hence

$$\begin{aligned} P[\eta_{n, 0} < d] \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor} &\leq (1 - e^{-\beta \log g(n)}) \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor} \\ &\leq e^{-\frac{(1-\delta)\psi(n)g(n)^{1-\beta}}{c \log g(n)}} \end{aligned} \quad (27)$$

for n large enough. Considering (26) and (27), it remains to specify a subsequence (n_k) and a positive function $\psi(\cdot)$ such that $\psi(n) \xrightarrow[n \rightarrow \infty]{} 0$, $\psi(n)g(n) \xrightarrow[n \rightarrow \infty]{} \infty$ and

$$\sum_k \psi(n_k) < \infty \quad (28)$$

$$\sum_k e^{-\frac{(1-\delta)\psi(n_k)g(n_k)^{1-\beta}}{c \log g(n_k)}} < \infty \quad (29)$$

Then, (24) follows from (25), (26) and (27) together with the Borel-Cantelli lemma. We finish the proof by observing that (28) and (29)

are satisfied for $n_k = g^{-1}(2^k)$ and $\psi(n) = \log g(n)/g(n)^\gamma$ where $0 < \gamma < 1 - \beta$. □

Proof of Theorem 3. – We begin by proving a finite distribution result, from which the required LDP will follow by standard projective limits arguments. Note first that for $0 = x_0 < x_1 < x_2 < \dots < x_k \leq 1$, and $0 = a_0 \leq a_1 \leq a_2 \leq \dots \leq a_k < \infty$, and with Y_i as in the proof of Theorem 1,

$$\begin{aligned} &P[\bar{L}_n(x_1) \geq a_1, \bar{L}_n(x_2) \geq a_2, \dots, \bar{L}_n(x_k) \geq a_k] \\ &\leq P \left[\sum_{i=1}^{\lfloor g(n)\psi(n)a_1 \rfloor} Y_i \leq t(n, x_1), \dots, \sum_{i=1}^{\lfloor g(n)\psi(n)a_k \rfloor} Y_i \leq t(n, x_k) \right] \\ &\leq P \left[\sum_{i=1}^{\lfloor g(n)\psi(n)a_1 \rfloor} Y_i \leq t(n, x_1), \sum_{i=\lfloor g(n)\psi(n)a_1 \rfloor + 1}^{\lfloor g(n)\psi(n)a_2 \rfloor} Y_i \leq t(n, x_2), \dots, \right. \\ &\quad \left. \sum_{i=\lfloor g(n)\psi(n)a_{k-1} \rfloor + 1}^{\lfloor g(n)\psi(n)a_k \rfloor} Y_i \leq t(n, x_k) \right] \\ &= \prod_{j=1}^k P \left[\sum_{i=\lfloor g(n)\psi(n)a_{j-1} \rfloor + 1}^{\lfloor g(n)\psi(n)a_j \rfloor} Y_i \leq t(n, x_j) \right]. \end{aligned}$$

Write $g(n)\psi(n) = g(t(n, x_j))\bar{\psi}_j(t(n, x_j))$, then for any $\delta > 0$ and n large enough,

$$\begin{aligned} &P[\bar{L}_n(x_1) \geq a_1, \dots, \bar{L}_n(x_k) \geq a_k] \\ &\leq \prod_{j=1}^k P \left[\sum_{i=\lfloor g(t(n, x_j))\bar{\psi}_j(t(n, x_j))a_{j-1} \rfloor + 1}^{\lfloor g(t(n, x_j))\bar{\psi}_j(t(n, x_j))a_j \rfloor} Y_i \leq t(n, x_j) \right] \\ &\leq \prod_{j=1}^k P \left[\sum_{i=1}^{\lfloor g(t(n, x_j))\bar{\psi}_j(t(n, x_j))(a_j - a_{j-1}) \rfloor - 1} Y_i \leq t(n, x_j) \right] \\ &\leq \prod_{j=1}^k \exp \left(-(a_j - a_{j-1}) \frac{\bar{\psi}_j(t(n, x_j))g(t(n, x_j))}{g(\frac{t(n, x_j)}{\bar{\psi}_j(t(n, x_j))g(t(n, x_j))})} (1 - \delta) \right) \\ &= \prod_{j=1}^k \exp \left(-(a_j - a_{j-1}) \frac{\psi(n)g(n)}{g(\frac{t(n, x_j)}{\psi(n)g(n)})} (1 - \delta) \right) \end{aligned}$$

where the last inequality holds for n large enough and follows from the proof of the upper bound in Theorem 1. Therefore, using the assumption (5),

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\bar{L}_n(x_1) \geq a_1, \dots, \bar{L}_n(x_k) \geq a_k] \\ & \leq - \sum_{j=1}^k (a_j - a_{j-1}) \frac{(1 - \delta)}{x_j}. \end{aligned}$$

Taking now $\delta \rightarrow 0$ yields

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\bar{L}_n(x_1) \geq a_1, \dots, \bar{L}_n(x_k) \geq a_k] \\ & \leq - \sum_{j=1}^k \frac{(a_j - a_{j-1})}{x_j}, \end{aligned} \tag{30}$$

proving a finite dimensional upper bound.

We next turn to a complementary lower bound. We first show that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\bar{L}_n(x_1) \geq a_1, \dots, \bar{L}_n(x_k) \geq a_k] \\ & \geq - \sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}. \end{aligned} \tag{31}$$

Indeed, assume w.l.o.g. $a_{j-1} < a_j, j = 1, 2, \dots, k$. We have, setting $\varphi_{n,j} := \lceil g(n)\psi(n)a_j \rceil$,

$$\begin{aligned} & P \left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j, j = 1, 2, \dots, k \right] \\ & \geq P \left[\sum_{i=1}^{\varphi_{n,1}} Y_i \leq t(n, x_1), \sum_{i=\varphi_{n,1}+1}^{\varphi_{n,2}} Y_i \leq t(n, x_2) - t(n, x_1), \dots \right. \\ & \quad \left. \sum_{i=\varphi_{n,k-1}+1}^{\varphi_{n,k}} Y_i \leq t(n, x_k) - t(n, x_{k-1}) \right] \\ & \geq \prod_{j=1}^k P \left[\sum_{i=\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_i \leq t(n, x_j) - t(n, x_{j-1}) \right]. \end{aligned} \tag{32}$$

Observe that for $j = 1, 2, \dots, n$

$$\begin{aligned}
 & P \left[\sum_{i=\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_i \leq t(n, x_j) - t(n, x_{j-1}) \right] \\
 & \geq P \left[\max_{\varphi_{n,j-1}+1 \leq i \leq \varphi_{n,j}} Y_i \leq \frac{t(n, x_j) - t(n, x_{j-1})}{\varphi_{n,j} - \varphi_{n,j-1} - 1} \right] \\
 & \geq P \left[\max_{\varphi_{n,j-1}+1 \leq i \leq \varphi_{n,j}} Y_i \leq \frac{t(n, x_j) - t(n, x_{j-1})}{\varphi_{n,j}} \right] \\
 & \geq \left(1 - \frac{1}{g\left(\frac{t(n, x_j) - t(n, x_{j-1})}{\varphi_{n,j}}\right)} \right)^{\varphi_{n,j} - \varphi_{n,j-1} - 1} \tag{33}
 \end{aligned}$$

where the last inequality is due to Proposition 1. Note that due to (5) and (6),

$$\frac{g(\gamma_n)}{g\left(\frac{t(n, x_j) - t(n, x_{j-1})}{[g(n)\psi(n)a_j]}\right)} \xrightarrow{n \rightarrow \infty} \frac{1}{x_j} \tag{34}$$

(31) now follows from (32), (33) and (34).

In the second step, we prove that, for $0 < \delta < \min\{a_j - a_{j-1}, j = 1, 2, \dots, k\}$ we have

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \\
 & \quad \times \log P[\bar{L}_n(x_1) \in (a_1 - \delta, a_1 + \delta), \dots, \bar{L}_n(x_k) \in (a_k - \delta, a_k + \delta)] \\
 & \geq - \sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}. \tag{35}
 \end{aligned}$$

To prove (35), observe that

$$\begin{aligned}
 & P \left[\frac{L_{t(n, x_j)}^o}{\psi(n)g(n)} \in (a_j - \delta, a_j + \delta), j = 1, 2, \dots, k \right] \\
 & \geq P \left[\frac{L_{t(n, x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j = 1, 2, \dots, k \right] \\
 & \quad - \sum_{\ell=1}^k P \left[\frac{L_{t(n, x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L_{t(n, x_\ell)}^o}{g(n)\psi(n)} \geq a_\ell + \delta \right].
 \end{aligned}$$

Since

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P \left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j = 1, 2, \dots, k \right] \\ & \geq -\frac{a_1 - \delta}{x_1} - \sum_{j=2}^k \frac{a_j - a_{j-1}}{x_j} \geq -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j} \end{aligned}$$

due to the first step, it is enough to show that for $\ell = 1, 2, \dots, k$ we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P \left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L_{t(n,x_\ell)}^o}{g(n)\psi(n)} \geq a_\ell + \delta \right] \\ & < -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}. \end{aligned}$$

But, using the upper bound (30), we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P \left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L_{t(n,x_\ell)}^o}{g(n)\psi(n)} \geq a_\ell + \delta \right] \\ & \leq -\sum_{j=1}^{\ell-1} \frac{a_j - a_{j-1}}{x_j} - \frac{a_\ell + 2\delta - a_{\ell-1}}{x_\ell} - \frac{a_{\ell+1} - a_\ell - 2\delta}{x_{\ell+1}} - \sum_{j=\ell+2}^k \frac{a_j - a_{j-1}}{x_j} \\ & < -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j} \end{aligned}$$

where we used $\frac{2\delta}{x_\ell} - \frac{2\delta}{x_{\ell+1}} > 0$ in the last inequality. This completes the proof of the lower bound.

It now follows from (30) and (35) that for $0 < x_1 < \dots < x_k < 1$, the random vector $\{\bar{L}_n(x_j)\}_{j=1}^k$ satisfies in \mathbb{R}^k the LDP with good rate function

$$I_k(y_1, \dots, y_k) = \sum_{j=1}^k \frac{(y_j - y_{j-1})}{x_j},$$

where $y_0 := 0$. By [2], Thm 4.6.1 (see Section 5.1 in [2] for a similar argument), we have that the random monotone function $\{\bar{L}_n(x)\}_{x \in [0,1]}$ satisfies the LDP in $M_+^\omega([0,1])$ (with $M_+^\omega([0,1])$ denoting $M_+([0,1])$ equipped with the topology of pointwise convergence) with good rate function

$$I_X(m) = \sup_{0=x_0 < x_1 < \dots < x_k < 1} \sum_{i=1}^k \frac{m(x_i) - m(x_{i-1})}{x_i}.$$

It then follows by monotone convergence that

$$I_\chi(m) = I(m) = \int_0^1 \frac{m(dx)}{x}.$$

Finally, note that the topology in $M_+^\omega([0, 1])$ is stronger than the topology in $M_+([0, 1])$, which concludes the proof of the theorem by an application of [2], Corollary 4.2.6. \square

Proof of Theorem 4. – Let $0 = a_0 < a_1 < \dots < a_k \leq 1$ as before. Recall that with $\psi(n) \equiv 1$, (30) and (31) imply that

$$P\left(\frac{L_{t(n, x_j)}^0}{g(n)} \geq a_j, j = 1, 2, \dots, k\right) \xrightarrow{n \rightarrow \infty} \exp\left(-\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}\right).$$

But sets of the form $A = \{f : f(x_j) \geq a_j, j = 1, 2, \dots, k\}$ generate the Borel σ -field on M_+ , hence in order to prove convergence of the finite-dimensional marginals of $\frac{L_{t(n, \cdot)}^0}{g(n)}$ to those of Z_x , we only have to check that

$$P[Z_{x_j} \geq a_j, j = 1, 2, \dots, k] = \exp\left(-\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}\right),$$

which follows from an explicit computation using (7). Tightness of the distributions of $\frac{L_{t(n, \cdot)}^0}{g(n)}$ is immediate from Prohorov's theorem. \square

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