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Ergodic theorems for surfaces with minimal random weights

by

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ABSTRACT. – The purpose of this paper is to formulate a higher dimensional analogue of the asymptotic shape theorem of first-passage percolation. The moment condition required depends on the curvature of the boundary: $p > \frac{3}{2}$ for the disk and $p > 2$ for the square. In first-passage percolation, there is a.s. convergence to the time constant even in an irrational direction if the passage times are integrable random variables. © Elsevier, Paris

RÉSUMÉ. – Dans [16], Kesten a généralisé plusieurs résultats de percolation du premier temps de passage. Ici, c'est le théorème de Cox et Durrett qui est démontré pour les surfaces minimales. La condition à imposer sur les moments pour obtenir p.s. la convergence uniforme dépend de la courbure du bord : $> \frac{3}{2}$ pour le disque et > 2 pour le carré. On montre aussi que si le temps de passage est intégrable, le temps de passage moyen converge p.s. même dans une direction irrationnelle. © Elsevier, Paris

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1. INTRODUCTION

The homogenization problem for partial differential equations with stationary random coefficients asks for the values of the effective parameters, for example, the electrical conductivity, or at least, bounds on their values. In the discrete case, one looks at the limit values of the effective parameters of a sequence of subgraphs of $\mathbb{Z}^d$, for example in [12].

In [11], however, $\mathbb{Z}^2$ is considered as an infinite electrical network; to each edge is assigned a random nonnegative number, the electrical conductivity of the edge, and the boundary conditions are replaced by giving the expected value of the potential differences in the horizontal and the vertical edges of the infinite electrical flow.

Then if the conductivities are stationary and bounded away from 0 and ∞, the average potential difference between the origin and a vertex of $\mathbb{Z}^2$ converges, in norm and a.s., in any given direction. Moreover the convergence is uniform with respect to direction (Kozlov [19] and [4]). To prove the a.s. convergence for higher dimensional networks, one needs stronger assumptions on the conductivities [19]: the ratios must be sufficiently close to 1.

However, in $\mathbb{Z}^3$, if the conductivities are stationary and bounded away from 0 and ∞, the average electrical flow through a triangle converges, in norm and a.s., for any given orientation of the triangle and the convergence is uniform with respect to the orientation (Depauw [9]). In this case, because of Kirchoff's laws, the electrical flow is the same through any orientable surface whose boundary is a given triangle; it is an additive process. Our first goal here is to give a subadditive version of this theorem.

The number assigned to each edge could be interpreted as its flow capacity, i.e. the maximal amount of liquid that can flow through the edge per unit time. Is it possible to obtain an ergodic theorem for the maximal flow through a given disk or square? This problem and related ones were considered by Kesten [16] under the assumption that the flow capacities are independent random variables. Here we will use a different notion of surface and of boundary which may very well coincide with those of [16] for minimal surfaces. But the interpretation as the maximal flow is not lost, it is still a higher-dimensional version of first-passage percolation, the shape theorem of Cox and Durrett [7], [18], [17], p. 1265 and [5] can be generalized to dimension three (i.e. the convergence is uniform with respect to direction), and the topological considerations are simple.

Finally, coming back to first-passage percolation in $\mathbb{Z}^2$, it is possible to define a path with the condition used to define a surface. Then the passage time between any two vertices of $\mathbb{Z}^2$ is defined and the average passage time from the origin converges a.s. in any given direction if $t(e)$ are stationary and integrable. This is proved in section 6. One should note that in an irrational direction, the process is not stationary. Actually the corresponding "additive" ergodic theorem, averages of the iterates of a $\mathbb{Z}^2$-action along a line with irrational slope, is not known.

As often as possible, the notations and definitions will be those of [16]. The faces of the unit cubes in $\mathbb{R}^3$ whose centers are vertices in $\mathbb{Z}^3$ and whose corners are in $\mathcal{L}^* = \mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, are called *plaquettes*. Each plaquette intersects a unique edge of $\mathbb{Z}^3$ and vice versa. To each edge e, or its corresponding plaquette π^*, is assigned a nonnegative number

$$t(e) = t(\pi^*).$$

Given a finite set of plaquettes E^*, define the weight of E^* as

$$V(E^*) = \sum_{\pi^* \in E^*} t(\pi^*).$$

As in [16], a path on $\mathbb{Z}^3$ is a sequence $(x_0, e_1, x_1, e_2 ... e_n, x_n)$ of vertices $x_0, x_1, ..., x_n$ alternating with edges $e_1, ..., e_n$ such that x_{i-1} and x_i are neighbors on $\mathbb{Z}^3$ and e_i is the edge between them, $1 \leq i \leq n$. If moreover $x_0 = x_n$, the path is called a closed path.

(1.1) DEFINITION. – *Let R be a closed convex plane region of $\mathbb{R}^3$ with a piecewise smooth boundary denoted by ∂R. ∂R is called a boundary for a set of plaquettes E^*, if any closed path in $\mathbb{Z}^3$ which crosses R p times from the negative side to the positive side and n times from the negative side to the positive side with $p - n \neq 0$, intersects E^* at least once.*

Whether ∂R is a boundary for E^* does not depend of the orientation chosen for R or for the paths. The relation between this notion of boundary and the one in [2] and [16] is discussed in section 5.

For a region R as in definition (1.1), define

$$\alpha(R) = \inf\{V(E^*) : \partial R \text{ is a boundary for the set of plaquettes } E^*\} \quad (1.2)$$

Now assume that $(t(e) : e \text{ an edge of } \mathbb{Z}^3)$ is a stationary ergodic sequence of nonnegative random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

First consider the case of a disk. For $r > 0$ and for $\vec{u}$ in $\mathbb{R}^3$, $|\vec{u}| = 1$, let $D(r, \vec{u})$ be the disk of radius r centered at the origin and perpendicular to $\vec{u}$.

$\vec{u} \in \mathbb{R}^3$ is called a *rational direction* if there is a real number M such that all three components of $M\vec{u}$ are rational numbers.

(1.3) THEOREM. – *If* $(t(e)\ :\ e$ *an edge of* $\mathbb{Z}^3)$ *is a stationary ergodic sequence of integrable nonnegative random variables, then for each* $\vec{u}$ *in* $\mathbb{R}^3, |\vec{u}| = 1$ *there is a real number* $\nu(\vec{u})$ *such that*

$$\lim_{r\to\infty} (\pi r^2)^{-1} \alpha(D(r,\vec{u})) = \nu(\vec{u}) \tag{1.4}$$

in L^1*-norm.* $\nu(\vec{u})$ *is continuous and the convergence is uniform with respect to direction. The limit also exists a.s. in any fixed rational direction* $\vec{u}$*.*

Moreover if the random variables $t(e)$ *have a finite moment of order* $p > 3/2$*, then the limit exists a.s. in all directions* $|\vec{u}| = 1$ *and the a.s. convergence is uniform with respect to direction.*

The proof that, a.s., the convergence in (1.4) is uniform with respect to direction relies on the following maximal inequality of Rubio de Francia [23], Theorem A:

Let σ be a compactly supported Borel measure in a hypersurface S of $\mathbb{R}^d$. Let $\mathcal{S}^*$ be the associated maximal operator defined by

$$(\mathcal{S}^* h)(x) = \sup_{r>0} \int_S h(x+ry)\,d\sigma(y). \tag{1.5}$$

If $\widehat{\sigma}(\xi)$, the Fourier transform of σ, verifies, for some $C > 0$,

$$|\widehat{\sigma}(\xi)| \le C|\xi|^{-a} \quad \text{with } a > 1/2 \tag{1.6}$$

then the maximal operator $\mathcal{S}^*$ is bounded in $L^p(\mathbb{R}^d)$ for all $p > p_a = (2a+1)/2a$.

Condition (1.6) is related to the curvature of the surface. In particular, if σ is the surface measure on a sphere in $\mathbb{R}^3$, (1.6) holds with $a = 1$ (see (2.1) and the following calculation). The L^p-boundedness of the maximal operator in this case was originally proved by Stein and Wainger [25].

More generally, by [21], if k principal curvatures of the surface are non-vanishing, $k \ge 1$, then $|\widehat{\sigma}(\xi)| \le C|\xi|^{-k/2}$. Therefore, if both principal curvatures of a surface in $\mathbb{R}^3$ are non-vanishing, that is the Gaussian curvature is non-vanishing, then $|\widehat{\sigma}(\xi)| \le C|\xi|^{-1}$. A discussion of the optimality of the moment condition p_a and examples of surfaces with different values of $a, 0 < a \le 1$, can be found in section 4.

There are various ways to generalize theorem (1.3) to surfaces verifying (1.6). The following theorem is one of them. Its proof is essentially the same as the proof of theorem (1.3) using for example the bijection $x \mapsto x/|x|$ from the surface S to the unit sphere.

(1.7) THEOREM. – *Let S be a surface defined by $\{y \in \mathbb{R}^3 : \Psi(y) = 0\}$ where $\Psi \in C^1(\mathbb{R}^3)$ and $\nabla\Psi \neq 0$ on S. Let σ be the surface measure on S. Assume that there are constants $C > 0$ and $a > 1/2$ such that the restriction of σ to any open neighborhood S' of S, $\sigma_{S'}$, verifies $|\widehat{\sigma}_{S'}(\xi)| \leq C|\xi|^{-a}$ for all $\xi \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$.*

Moreover assume that S is the boundary of compact convex region B of $\mathbb{R}^3$ containing the origin.

Let $D(r, \vec{u})$ be the intersection of the plane through the origin perpendicular to $\vec{u}$ with the region $\{x \in \mathbb{R}^3 : r^{-1}x \in B\}$.

If $(t(e) : e$ an edge of $\mathbb{Z}^3)$ is a stationary ergodic sequence of integrable nonnegative random variables, then for each $\vec{u}$ in $\mathbb{R}^3$, $|\vec{u}| = 1$ there is a real number $\nu(\vec{u})$ such that

$$\lim_{r \to \infty} r^{-2}\alpha(D(r, \vec{u})) = \nu(\vec{u}) \tag{1.8}$$

in L^1-norm. $\nu(\vec{u})$ is continuous and the convergence is uniform with respect to direction. The limit also exists a.s. in any fixed rational direction $\vec{u}$.

Moreover if the random variables $t(e)$ have a finite moment of order $p > (2a + 1)/2a$, then the limit exists a.s. in all directions $|\vec{u}| = 1$ and the a.s. convergence is uniform with respect to direction.

Next consider a flat boundary, for example the square. In this case, there is no analogous differentiation theorem. However, a different argument shows that the subadditive theorem still holds uniformly with respect to direction under a condition stronger than in the case of a disk: a finite moment of order $p > 2$.

For $r > 0$, $\mathbf{d}$ and $\mathbf{d}'$ two unit vectors such that $\mathbf{d} \cdot \mathbf{d}' = 0$, let $S(r, \mathbf{d}, \mathbf{d}')$ be the square centered at the origin with vertices $\pm r\sqrt{2}\mathbf{d}$ and $\pm r\sqrt{2}\mathbf{d}'$. The cross product of $\mathbf{d}$ and $\mathbf{d}'$, denoted by $\mathbf{d} \times \mathbf{d}'$, is a unit vector perpendicular to $S(r, \mathbf{d}, \mathbf{d}')$.

(1.9) THEOREM. – *If $(t(e) : e$ an edge of $\mathbb{Z}^3)$ is a stationary ergodic sequence of integrable nonnegative random variables, then for each pair of orthogonal unit vectors $\mathbf{d}, \mathbf{d}'$, there is a real number $\nu(\mathbf{d}, \mathbf{d}')$ such that*

$$\lim_{r \to \infty} (4r^2)^{-1}\alpha(S(r, \mathbf{d}, \mathbf{d}')) = \nu(\mathbf{d}, \mathbf{d}') \tag{1.10}$$

in L^1-norm. This convergence is uniform with respect to direction, $\nu(\mathbf{d}, \mathbf{d}')$ is continuous and the values of $\nu(\mathbf{d} \times \mathbf{d}')$ in (1.4) and of $\nu(\mathbf{d}, \mathbf{d}')$ in (1.10) are equal for all pairs of orthogonal unit vectors $\mathbf{d}, \mathbf{d}'$. The limit in (1.10) also exists a.s. for any fixed rational directions $\mathbf{d}, \mathbf{d}'$.

Moreover if the random variables $t(e)$ have a finite moment of order $p > 2$, then the limit exists a.s. in all directions and the a.s. convergence is uniform in $\mathbf{d}, \mathbf{d}'$.

In section 4, there is an example of a sequence $(t(e) : e \,\text{an edge of}\, \mathbb{Z}^3)$ with finite moments of order $1 \leq p < 3/2$ and one with finite moment of order $1 \leq p < 2$ for the square, where the a.s. convergence is not uniform with respect to direction for the disk, and for the square, respectively. Some counterexamples for theorem (1.7) are given and the moment condition of theorem (1.9), a finite moment of order $p > 2$, appears as a limit condition (see remark (4.5)).

2. PROOF OF THEOREM (1.3)

The first part of the proof is to check that α is a subadditive process in the sense of Akcoglu and Krengel [1].

Let $\vec{u}$ be a fixed rational direction and let Π be the plane through the origin and perpendicular to $\vec{u}$. Let $R_1, ..., R_n$ be disjoint rectangles in Π such that $R = \cup_i R_i$ is also a rectangle. If, for $1 \leq i \leq n$, E_i^* is a set of plaquettes for which ∂R_i is a boundary, then $E^* = \cup_i E_i^*$ is a set of plaquettes for which ∂R is a boundary since if $0 \neq p - n = \sum_i (p_i - n_i)$ where p_i, n_i are the number of positive and negative crossings of R_i by a path γ, then $p_i - n_i \neq 0$ for some i and γ must intersect E^*. Therefore,

$$\alpha(R) \leq \sum_i \alpha(R_i).$$

An increasing sequence of disks in a fixed plane with rational direction $\vec{u}$ is a regular sequence in the sense of [20], definition 6.2.4. And since $\pi r^2/(\text{number of vertices of } \mathbb{Z}^3 \text{ in } D(r, \vec{u}))$ converges as $r \to \infty$, then by [1], theorem 2.5 (see also [20], theorem 6.2.9) the limit (1.4) exists a.s. and in L^1 in any fixed rational direction and it is nonrandom because $(t(e))$ is ergodic. One can also see that the convergence in L^1 is uniform with respect to direction and that ν is continuous.

The proof that the a.s. convergence is uniform with respect to direction will be done by contradiction. The first step is to prove the maximal inequality (2.4).

For $0 < \delta < 1$ and $\vec{u} \in \mathbb{R}^3, |\vec{u}| = 1$, define $S(\delta, \vec{u}) = \{x \in \mathbb{R}^3 : |x| = 1, |\vec{u} \cdot x| < \delta\}$. For $h \in \mathcal{C}^\infty(\mathbb{R}^3)$ with compact support, let

$$(\mathcal{S}(\delta, \vec{u}, r)h)(x) = \frac{1}{4\pi r^2 \delta} \int_{|y|=1} h(x + ry)\sigma_\delta(dy)$$

where σ_δ is the surface measure on $S(\delta, \vec{u})$.

In dimension three, it is simple to check that $\widehat{\sigma}_\delta$, the Fourier transform of σ_δ, satisfies, for all $\delta, 0 < \delta < 1$,

$$|\widehat{\sigma}_\delta(\xi)| \leq \min\{4, 4|\xi|^{-1}\} \quad \text{for all} \quad \xi \in \mathbb{R}^3 \setminus \{\mathbf{0}\}. \tag{2.1}$$

Indeed, if $\xi|\xi|^{-1} \notin S(\delta, \vec{u})$, then $\widehat{\sigma}_\delta(\xi) = \int_{|x|=1} e^{-2\pi i x \cdot \xi} \sigma_\delta(dx) = \int_0^{2\pi} \int_{\phi_1(\theta)}^{\phi_2(\theta)} e^{-2\pi i |\xi| \cos\phi} \sin\phi d\phi d\theta = \int_0^{2\pi} \frac{e^{2\pi i |\xi| t}}{2\pi i |\xi|}]_{-\cos\phi_1}^{-\cos\phi_2} d\theta$, hence $|\widehat{\sigma}_\delta(\xi)| \leq 2|\xi|^{-1}$. Similarly, if $\xi|\xi|^{-1} \in S(\delta, \vec{u})$, $|\widehat{\sigma}_\delta(\xi)| \leq 4|\xi|^{-1}$.

Since the measures σ_δ all have support on the unit sphere and satisfy (2.1), an examination of the proof of [23], Theorem A shows that for each $p > 3/2$, there is a constant $C_1 = C_1(p)$, independent of δ, such that

$$\| \sup_{r>0} \mathcal{S}(\delta, \vec{u}, r)h\|_p < C_1 \|h\|_p \quad \text{for all} \quad h \in L^p(\mathbb{R}^3). \tag{2.2}$$

For $f : \mathbb{Z}^3 \to \mathbb{R}$ with finite support, define

$$(\mathcal{A}(\delta, \vec{u}, r)f)(x) = \frac{1}{4\pi r^2 \delta} \sum f(x + y)$$

where the sum is over all $y \in \mathbb{Z}^3$ such that $|\vec{u} \cdot y| < \delta|y|$ and $r - 4 < |y| < r + 4$.

Then for $r > 9$,

$$\begin{aligned}(\mathcal{A}(\delta, \vec{u}, r)f)(x) &< \frac{2}{4\pi r^2 \delta} \int_{r-6}^{r+6} 4\pi\rho^2 \delta (\mathcal{S}(\delta, \vec{u}, \rho)|\widetilde{f}|)(x) d\rho \\ &< 15 \sup_{r>0} \mathcal{S}(\delta, \vec{u}, r)|\widetilde{f}|(x)\end{aligned}$$

where $\widetilde{f} : \mathbb{R}^3 \to \mathbb{R}$ is defined by $\widetilde{f}(z) = f(y)$ if $|z - y| < 1/2$ and $\widetilde{f}(z) = 0$ otherwise.

By (2.2), for $p > 3/2$, there is a constant C, independent of δ, such that

$$\| \sup_{r>9} \mathcal{A}(\delta, \vec{u}, r)f\|_{L^p(\mathbb{Z}^3)} < C\|f\|_{L^p(\mathbb{Z}^3)} \quad \text{for all} \quad f \in L^p(\mathbb{Z}^3). \tag{2.3}$$

where $L^p(\mathbb{Z}^3)$ is defined with respect to the counting measure.

Now let τ_y be the shift on $(\Omega, \mathcal{F}, P)$ identified with the product space; that is $t(e)(\tau_y \omega) = t(e - y)(\omega)$.

For $f \in L^p(\Omega, \mathcal{F}, P)$, define

$$(\mathcal{A}(\delta, \vec{u}, r) f)(\omega) = \frac{1}{4\pi r^2 \delta} \sum f(\tau_y \omega)$$

where the sum is over all $y \in \mathbb{Z}^3$ such that $|\vec{u} \cdot y| < \delta |y|$ and $r - 4 < |y| < r + 4$.

Then by the transfer principle ([27], [6]), there is a constant C, independent of δ and $\vec{u}$, $|\vec{u}| = 1$, such that,

$$\|\sup_{r>9} \mathcal{A}(\delta, \vec{u}, r) f\|_p < C \|f\|_p \quad \text{for all } f \in L^p(\Omega). \tag{2.4}$$

Suppose that for some $\varepsilon > 0$ and for all $\omega \in F$, a measurable subset of Ω such that $P(F) > 0$, there is a sequence $(r_k, \vec{u}_k)$ such that $r_k \to \infty$, the $\vec{u}_k$ are rational directions and for all k,

$$|(\pi r_k^2)^{-1} \alpha(D(r_k, \vec{u}_k)) - \nu(\vec{u}_k)| > \varepsilon. \tag{2.5}$$

To obtain a contradiction with (2.5), one can make the following choices. Since $p > 1$, it is possible to choose $\lambda > 0$ such that

$$\frac{C^p E(g^p)}{\lambda^p} \frac{90\lambda}{\varepsilon} < P(F) \quad \text{and} \quad 4Eg < \lambda.$$

where $g(\omega) = \sum t(e)$, the sum is over all the edges within 6 units of the origin and E is the expectation with respect to P.

Set $\delta = \varepsilon/(10\lambda)$.

Note that if $|\vec{u} \cdot \vec{v}| > 1 - \delta$, $|\vec{u}| = |\vec{v}| = 1$, then $|\nu(\vec{u}) - \nu(\vec{v})| < 4\delta Eg$ which is less than $\varepsilon/10$ for the choice made above.

Find n rational directions $\vec{v}_j$, $1 \le j \le n$ and $n \le 9/\delta$ such that $\{\vec{u} : |\vec{u} \cdot \vec{v}_j| > 1 - \delta, |\vec{u}| = 1\}$ is a covering of the unit sphere.

Let

$$A(\lambda, \delta, \vec{v}_j) = \{\omega : \sup_{r>9} \mathcal{A}(\delta, \vec{v}_j, r) g \le \lambda\} \tag{2.6}$$

$$P(\cup_j A^c(\lambda, \delta, \vec{v}_j)) < \frac{C^p E(g^p)}{\lambda^p} n < P(F) \frac{\varepsilon}{90\lambda} \frac{9}{\delta} < P(F). \tag{2.7}$$

Therefore $P(F \cap \cap_j A(\lambda, \delta, \vec{v}_j)) > 0$.

But for almost all ω in $F \cap \cap_j A(\lambda, \delta, \vec{v}_j)$ there is a number R such that for all $j, 1 \le j \le n$ and all $r > R$,

$$|(\pi r^2)^{-1} \alpha(D(r, \vec{v}_j)) - \nu(\vec{v}_j)| < \varepsilon/10.$$

Then for $r_k > R$,

$$\begin{aligned}|(\pi r_k^2)^{-1}\alpha(D(r_k,\vec{u}_k))-\nu(\vec{u}_k)| &\leq |(\pi r_k^2)^{-1}(\alpha(D(r_k,\vec{u}_k))-\alpha(D(r_k,\vec{v}_j)))| \\ &\quad + |(\pi r_k^2)^{-1}\alpha(D(r_k,\vec{v}_j)) - \nu(\vec{v}_j)| \\ &\quad + |\nu(\vec{v}_j) - \nu(\vec{u}_k)| \\ &< (\pi r_k^2)^{-1} 4\pi r_k^2 \delta\lambda + 2\varepsilon/10 \\ &< \varepsilon\end{aligned}$$

where $\vec{v}_j$ is a rational direction such that $|\vec{u}_k \cdot \vec{v}_j| > 1 - \delta$. This is in contradiction with (2.5).

3. PROOF OF THEOREM (1.9)

As in the proof of theorem (1.3), the norm and the a.s. convergence to a limit $\nu(\mathbf{d}, \mathbf{d}')$ in (1.10) follow from the multiparameter subadditive theorem [20], theorem 6.2.9. For any regular sequence, in the sense of [20], definition 6.2.4, in a fixed plane with rational direction $\vec{u}$, the limit is the same constant. A general expression for it could be given using [20], (6.2.4). Therefore if $\mathbf{d} \times \mathbf{d}' = \vec{u}$, then $\nu(\vec{u}) = \nu(\mathbf{d}, \mathbf{d}')$ since $\lim_{r\to\infty} \pi r^2/(\text{number of vertices in } D(r, \vec{u})) = \lim_{r\to\infty} 4r^2/(\text{number of vertices in } S(r, \mathbf{d}, \mathbf{d}'))$.

To prove that the a.s. convergence in (1.4) was uniform for all directions, the small gap between two disks was filled by the plaquettes near a spherical band. Similarly, to prove that the convergence in (1.10) is uniform with respect to direction, the gap between two close squares must be filled. Instead of a spherical band, we use the plaquettes near some pyramids (four for each side of the square) with base a rectangle. They are constructed in lemma (3.5).

Lemma (3.12) shows that there are enough pyramids with relatively small weights to fill the gap. The corners of the squares cause a new difficulty. It is dealt with by adding, again, plaquettes near a small spherical band.

At first, a.s., the uniform convergence is proved in a simpler case but where the existence of a finite moment of order > 2 is still needed. Then the additional arguments for the proof of theorem (1.9) will be given.

For an angle θ, $-\pi/4 \leq \theta \leq \pi/4$, let $\Delta(r, \theta)$ be the triangle with vertices $(0, 0, 0)$, $(r\cos\theta, r\cos\theta, r\sin\theta)$ and $(-r\cos\theta, r\cos\theta, r\sin\theta)$. Triangles in this family all have a common vertex at the origin and a side parallel to the x(1)-axis with endpoints in two fixed planes: $x(1) = x(2)$ and $x(1) = -x(2)$. This is why theorem (3.1) is easier to prove. We now state this first result.

(3.1) THEOREM. – *Let $(t(e) : e$ an edge of $\mathbb{Z}^3)$ be a stationary and ergodic sequence of nonnegative random variables with finite moment of order $p > 2$. Then for each $\theta, -\pi/4 \leq \theta \leq \pi/4$, there is a constant $\nu(\theta)$ such that*

$$\lim_{r \to \infty} (\operatorname{area}(\Delta(r, \theta)))^{-1} \alpha(\Delta(r, \theta)) = \nu(\theta) \quad a.s.$$

and the convergence is uniform with respect to θ in $[-\pi/4, \pi/4]$.

For $x \in \mathbb{R}^3$, $0 < a \leq 1$ and $\rho > 0$, consider the cylinders

$$\begin{aligned} B_a(x, \rho) = \{y \in \mathbb{R}^3 : &|x(1) - y(1)| \leq \rho, \\ &|x(2) - y(2)|^2 + |x(3) - y(3)|^2 \leq a^2 \rho^2\} \end{aligned} \tag{3.2}$$

Let $\mathcal{A}_a^*$ be the associated maximal function. That is, for a stationary sequence of random variables $(f(x); x \in \mathbb{Z}^3)$,

$$\mathcal{A}_a^*(f, x)(\omega) = \sup_{\rho > 3/a} \mathcal{A}_{a,\rho}(f, x)(\omega)$$

where $\mathcal{A}_{a,\rho}(f, x)(\omega) = (\mathrm{m}(B_a(x, \rho)))^{-1} \sum f(y)(\omega)$ and where the sum is taken over all $y \in B_a(x, \rho) \cap \mathbb{Z}^3$. m is the counting measure on $\mathbb{Z}^3$.

Fix $0 < \zeta < \xi < 1$ and define $\mathcal{A}^*(f, x)(\omega) = \sup_{\zeta \leq a \leq \xi} \mathcal{A}_a^*(f, x)(\omega)$. The two basic properties are satisfied for this family of cylinders:

i) $B_a(x, \rho) \cap B_{a'}(y, \rho) \neq \emptyset$ implies $B_a(y, \rho) \subset B_\xi(x, 3\rho)$.

ii) $\mathrm{m}(B_\xi(x, 3\rho)) \leq 30(\frac{\xi}{\zeta})^2 \, \mathrm{m}(B_a(x, \rho))$.

Then by Vitali's covering lemma and by the transfer principle, the maximal function is weak-L^1:

(3.3) LEMMA. – *Let $0 < \zeta < \xi < 1$. For any stationary sequence $(f(x), x \in \mathbb{Z}^3)$ of integrable random variables*

$$P(\mathcal{A}^*(f, x) > \lambda) < 30 \xi^2 \zeta^{-2} \lambda^{-1} E|f|. \tag{3.4}$$

The construction needed will be based on the following lemma which extends the maximal inequality of [9], p. 58 to rectangles. The basic idea goes back to Sobolev [24].

(3.5) LEMMA. – *Let* $0 < a \leq 1$ *and let* $(t(e) : e$ *in* $\mathbb{Z}^3)$ *be a stationary and ergodic sequence of random variables. Let*

$$\alpha_a^*(t,x) = \sup(2a\rho^2)^{-1}\alpha(R(x,\rho))$$

where the sup *is taken over all rectangles* $R(x,\rho)$, $\rho > 3/a$, *with the axis of the cylinder* $B_a(x,\rho)$ *as one side and the opposite side on the surface of the cylinder.*

If the $(t(e))$ *have a finite moment of order* $p > 2$, *then*

$$\alpha_a^*(t,x) \leq c_1(\mathcal{A}_a^*(g^p,x))^{1/p} \quad a.s.$$

where $g(y) = \max_{e=[x,y]} t(e)$.

The constant c_1 *depends only on the lattice* $\mathbb{Z}^3$, *in particular, it does not depend on the value of* a.

Proof of lemma (3.5). – Take $\rho > 3/a$ and let $R(0,\rho)$ be a rectangle with one side on the axis of the cylinder $B_a(0,\rho)$ and the opposite side on the surface of the cylinder.

For each integer n, $0 \leq n < a\rho$, let E_n^* be a set of plaquettes such that

i) $\partial R(0,\rho)$ is a boundary for E_n^*.

ii) the center of any plaquette of E_n^* is within 3 units of (the interior of) one of the four *triangles* forming the four-sided pyramid of height n and base $R(0,\rho)$.

iii) the tops of these pyramids all lie on the same side of $R(0,\rho)$.

It follows that all the vertices of the edges corresponding to the plaquettes of E_n^* are in $B_a(0,4\rho)$.

Figure 1 represents one pyramid over a rectangle $R(x,\rho)$ inside a cylinder $B_a(x,\rho)$.

Write $E_n^* = G_n^* \cup H_n^*$ where H_n^* contains all the plaquettes whose centers are within 9 units of $\partial R(0,\rho)$ and $G_n^* = E_n^* \setminus H_n^*$.

Then since $\alpha(R(0,\rho)) \leq V(E_n^*)$ for all $0 \leq n < a\rho$, $\alpha(R(0,\rho)) \leq (a\rho)^{-1}\sum_n V(E_n^*)$ and

$$(2a\rho^2)^{-1}\alpha(R(0,\rho)) \leq (2a^2\rho^3)^{-1}\sum_n V(G_n^*) + (2a^2\rho^3)^{-1}\sum_n V(H_n^*) \quad (3.6)$$

As in [9], p. 52, $((2a\rho^2)^{-1}V(H_n^*))^2 \leq (a\rho)^{-2}((2\rho)^{-1}V(H_n^*))^2 \leq c(a^2\rho^3)^{-1}\sum g^2(y)$ where the sum is taken over $y \in B_a(0,4\rho) \cap \mathbb{Z}^3$ and where c does not depend on a.

$$\sup_{R(0,\rho)} (2a^2\rho^3)^{-1}\sum_n V(H_n^*) \leq c(\mathcal{A}^*(g^p,0))^{1/p} \quad \text{for all} \quad p \geq 2 \quad (3.7)$$

where c does not depend on a.

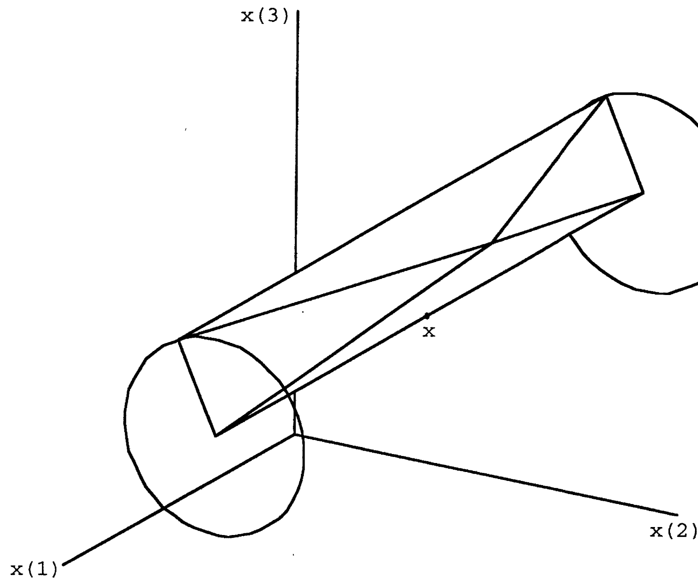

Figure 1

To show the maximal inequality for the G_n^*, first write

$$(2a^2\rho^3)^{-1}\sum_n V(G_n^*) = (2a^2\rho^3)^{-1}\sum_{\pi^*\in G^*} t(\pi^*)\,\mathrm{m}\{n : \pi^* \in G_n^*\}$$
$$\le (2a^2\rho^3)^{-1}\sum_{y\in G} g(y)w(y) \tag{3.8}$$

where $G_n = \{y : [y,z] \in G_n^* \text{ for some } z\}$, $G^* = \bigcup G_n^*$, $G = \bigcup G_n$ and $w(y) = 6\,\mathrm{m}\{n : y \in G_n\}$ and m is the counting measure on $\mathbb{Z}$.

To estimate $w(y)$, consider two cases. If y is within 4 units of a triangle whose side belonging to $R(0,\rho)$ is of length $a\rho$, then

$$w(y) \le \frac{8\sqrt{\rho^2 + a^2\rho^2}}{d_1(y)/2} \le 32\frac{\rho}{d_1(y)} \tag{3.9}$$

where $d_1(y)$ is the distance between y and the side of $R(0,\rho)$ of length $a\rho$ closest to y. Similarly, if y is within 4 units of a triangle whose side belonging to $R(0,\rho)$ is of length 2ρ, then

$$w(y) \le \frac{8\sqrt{2a^2\rho^2}}{d_2(y)/2} \le 32\frac{a\rho}{d_2(y)} \tag{3.10}$$

where $d_2(y)$ is the distance between y and the side of $R(0,\rho)$ of length 2ρ closest to y.

Use (3.9), (3.10) and Hölder's inequality (with $p^{-1} + q^{-1} = 1$) to estimate (3.8):

$$(2a^2\rho^3)^{-1}\sum_n V(G_n^*) \le (2a^2\rho^3)^{-1}\sum_{B\cap\mathbb{Z}^3} 32\rho(d_1(y)^{-1} + ad_2(y)^{-1})g(y)$$
$$\le c(2a^2\rho^2)^{-1}(\sum g^p(y))^{1/p}((\sum d_1(y)^{-q})^{1/q} + a(\sum d_2(y)^{-q})^{1/q})$$
$$\le c((2\pi a^2\rho^3)^{-1}\sum g^p(y))^{1/p}(a^2\rho^3)^{1/p}(a^2\rho^2)^{-1}$$
$$((\sum d_1(y)^{-q})^{1/q} + a(\sum d_2(y)^{-q})^{1/q})$$

Using cylindrical coordinates, we find that

$$\sum d_1(y)^{-q} \le 2a\rho \arctan a \int_0^{\sqrt{2}\rho} r^{-q+1}dr \le c_q a^2\rho^{3-q}$$

if $1 \le q < 2$, and $(a^2\rho^3)^{1/p}(a^2\rho^2)^{-1}(a^2\rho^{3-q})^{1/q} = 1$.

Similarly, $\sum d_2(y)^{-q} \le 2\rho\int_0^{\pi/4}\int_0^{\sqrt{2}a\rho} r^{-q+1}dr \le c_q\rho(a\rho)^{2-q}$ if $1 \le q < 2$, and $(a^2\rho^3)^{1/p}(a^2\rho^2)^{-1}a(a^{2-q}\rho^{3-q})^{1/q} = 1$. •

Under the assumptions of lemma (3.3) and lemma (3.5), we obtain that there is a constant c_2 such that for any $0 < \zeta < \xi < 1$ and any $x \in \mathbb{Z}^3$,

$$P(\sup_{\zeta\le a\le\xi} \alpha_a^*(t,x) > \lambda) \le c_2\xi^2\zeta^{-2}\lambda^{-p}Eg^p \quad \text{for all } \lambda > 0. \tag{3.11}$$

An angle θ is called a *rational direction* if the normal to the triangle $\Delta(r,\theta)$, $(0, -\sin\theta, \cos\theta)$, is a rational direction.

Let θ, $-\pi/4 \le \theta \le \pi/4$ be a fixed rational direction. Let $z = (0, r\cos\theta, r\sin\theta)$ be the middle point of the side of $\Delta(r,\theta)$ parallel to the x(1)-axis. For $\rho > 0$, let $C(\rho,\delta)$ be the conical volume

$$C(\rho,\delta) = \{x \in \mathbb{R}^3 : x(1)^2 + x(2)^2 \le (\delta x(3))^2, 0 \le x(3) \le \rho\}.$$

There is a rotation ϕ in $O(3)$ such that $\phi(0,0,1) = z$. Put $C(\rho,\theta,\delta) = \phi(C(\rho,\delta))$, that is, $C(\rho,\theta,\delta)$ is a cone such that the origin and z are on its axis.

Since the ergodic theorem holds in $C(\rho,\theta,\delta)$, if A is a measurable subset of Ω, then $\mathrm{m}(\{y \in C(\rho,\theta,\delta) \cap \mathbb{Z}^3, \tau_y\omega \in A\})/\,\mathrm{m}(C(\rho,\theta,\delta))$ converges to $P(A)$ a.s. when $\rho \to \infty$, where τ_y, as in section 2, is the shift on Ω.

This implies the following simple fact:

Suppose $\mathrm{m}(\{y \in C(\rho,\theta,\delta) \cap \mathbb{Z}^3, \tau_y\omega \in A\}) > (1-\eta)\,\mathrm{m}(C(\rho,\theta,\delta))$ *for all* $\rho > \rho_0 = \rho_0(A,\theta,\delta)$ *then there is* $\rho_1 > \rho_0$ *such that if* $\rho > \rho_1$ *and if* $\rho' \geq (1+4\eta)\rho$ *then there is at least one* y_0 *in* $\mathbb{Z}^3 \cap (C(\rho',\theta,\delta) \setminus C(\rho,\theta,\delta))$ *such that* $\tau_{y_0}\omega \in A$.

For $\lambda > 0$, let $A_\lambda = \left\{\sup_{\zeta \leq a \leq \xi} \alpha_a^*(t,\mathbf{0}) \leq \lambda\right\}$. Then by inequality (3.11),

$$P(A_\lambda) > 1 - c_3\xi^2\zeta^{-2}\lambda^{-p} \quad \text{for all} \quad \lambda > 0.$$

Combine this inequality with both observations above to obtain a final lemma.

(3.12) LEMMA. – *Given* $0 < \zeta < \xi < 1$, $\lambda > 0$ *and* $\delta > 0$, *for almost all* ω *there is a real number* $\overline{\rho}$ *such that if* $\rho > \overline{\rho}$ *and if* $\rho' \geq (1+4\eta)\rho$, *where* $\eta = 2c_3\xi^2\zeta^{-2}\lambda^{-p}$, *then there is at least one* $x \in \mathbb{Z}^3 \cap (C(\rho',\theta,\delta) \setminus C(\rho,\theta,\delta))$ *such that* $\alpha_a^*(t,x) \leq \lambda$ *for all* $\zeta \leq a \leq \xi$.

The construction will also include plaquettes near two triangles, $I_i(r,\theta,\delta), i = 1,2$, described below. Let $\vec{u} = (0,-\sin\theta,\cos\theta)$ and let $I(r,\theta,\delta)$ be the triangle with vertices

$$(0,0,0), \quad (0, r\cos\theta, r\sin\theta) \pm \delta r\vec{u}.$$

This triangle lies in the plane $x(1) = 0$. $I_1(r,\theta,\delta)$ will be the triangle in the plane $x(1) = x(2)$ whose projection in the plane $x(1) = 0$ is $I(r,\theta,\delta)$ and $I_2(r,\theta,\delta)$, the triangle in the plane $x(1) = -x(2)$ whose projection in the plane $x(1) = 0$ is $I(r,\theta,\delta)$. The vertices of I_1, for example, are $r(\cos\theta,\cos\theta,\sin\theta) \pm \delta r(-\sin\theta,-\sin\theta,\cos\theta)$ and, for $i = 1,2$,

$$\mathrm{area}(I_i(r,\theta,\delta)) < c_4\delta r^2. \tag{3.13}$$

Let $I_i(r,\theta,\delta)^*, i = 1,2$, be the set of all plaquettes with a vertex within 3 units of (the interior of) $I_i(r,\theta,\delta)$.

Let $f(x) = \sum_{|x-y|\leq 5} g(y)$ (g was defined in lemma (3.5)) and let $F_i(r,\theta,\delta) = \sum f(x)$ where the sum is taken over all vertices of $\mathbb{Z}^3$ that are inside the triangle $I_i(r,\theta,\delta)$. Note that since the triangle $I_i(r,\theta,\delta)$ lies in the plane $x(1) = x(2)$ or $x(1) = -x(2)$, $\mathrm{area}(I_i(r,\theta,\delta)/\,\mathrm{m}(I_i(r,\theta,\delta)) \to \sqrt{2}$ as $r \to \infty$. The ergodic theorem holds for F_i (see [20], section 6.2) and therefore

$$(\mathrm{m}(I_i(r,\theta,\delta)))^{-1}F_i(r,\theta,\delta) \to Ef(\mathbf{0}) \text{ as } r \to \infty \text{ a.s.}$$

Since $V(I_i(r,\theta,\delta)^*) \leq F_i(r,\theta,\delta)$, since $\mathrm{m}(I_i(r,\theta,\delta)) \leq c_5\delta r^2$ for all $r > 1/\delta$ and since $Ef(\mathbf{0}) \leq c_6 Eg$, for some constant c_6, we obtain that there is a constant c_7 such that for all $-\pi/4 \leq \theta \leq \pi/4$ and $0 < \delta < 1$,

$$V(I_i(r,\theta,\delta)^*) < c_7\delta r^2 Eg \text{ a.s.} \tag{3.14}$$

for all r sufficiently large.

Proof of theorem (3.1). – It will be proved by contradiction. Suppose there are $F \in \mathcal{F}, P(F) > 0$, and ε, $0 < \varepsilon < 1/2$ such that for all $\omega \in F$, there is a sequence $(r_k, \theta_k; k = 1, 2, ...)$ where $r_k \to \infty$ and θ_k are rational directions such that

$$|(\mathrm{area}(\Delta(r_k,\theta_k)))^{-1}\alpha(\Delta(r_k,\theta_k)) - \nu(\theta_k)| > \varepsilon. \tag{3.15}$$

There is a set Ω', $P(\Omega') = 1$, such that for all $\omega \in \Omega'$ and all rational directions θ: (3.12) and (3.14) hold for a dense and countable set of values for the other parameters: the relations between them are given in (3.16) and (3.17), and, $\lim_{r\to\infty}(\mathrm{area}(\Delta(r,\theta)))^{-1}\alpha(\Delta(r,\theta)) = \nu(\theta)$.

Now choose one $\omega \in \Omega' \cap F$ and set

$$\xi = \varepsilon/(1000\lambda), \qquad \delta_1 = 10\xi \text{ and } \delta_2 = \zeta = \xi/20. \tag{3.16}$$

For these choices, $\eta = 2c_3\xi^2\zeta^{-2}\lambda^{-p} = 800c_3\lambda^{-p}$.

Since $p > 1$, it is possible to take λ large enough so that the following inequalities hold:

$$\delta_2 + (1+4\eta)(1+4\zeta)(1+\delta_2) - 1 \leq 2\xi, \qquad (1+4\eta)(1+4\zeta) \leq 2 \tag{3.17}$$

$$c_7\delta_1(2r_k)^2 Eg < \varepsilon/40. \tag{3.18}$$

There must be a subsequence of (θ_k) which converges to some angle θ_0. And since ν is continuous, $\nu(\theta_k)$ converges to $\nu(\theta_0)$ along the same subsequence. So assume that for ω, there is a sequence (r_k, θ_k) such that $\theta_k \to \theta_0$ and such that (3.15) holds. If θ_0 is a rational direction, let $\theta = \theta_0$, otherwise choose a rational direction θ such that

$$\left|\frac{\cos\theta}{\cos\theta_0} - 1\right| \leq \frac{\varepsilon}{30\bar{\nu}}, \quad |\theta - \theta_0| < \frac{\delta_2}{10} \text{ and } |\nu(\theta_0) - \nu(\theta)| < \frac{\varepsilon}{10} \tag{3.19}$$

where $\bar{\nu} = 1 + \sup_{-\pi\leq 4\theta\leq\pi} \nu(\theta)$.

Since $\omega \in \Omega'$, by (3.12) and (3.14), there is a real number $\tilde{\rho}$ such that if $r > \tilde{\rho}$:

$$\begin{aligned}&\text{there is at least one vertex } x \text{ in } C((1+4\eta)(1+4\zeta)r,\theta,\delta_2)\backslash C((1+4\zeta)r,\theta,\delta_2)\\ &\qquad \text{such that } \alpha_a^*(t,x) \le \lambda \text{ for all } \zeta \le a \le \xi,\\ &V(I_i(2r,\theta,\delta_1)^*) < c_7 \delta r^2 Eg \text{ and,}\\ &|(\text{area}(\Delta(r,\theta)))^{-1}\alpha(\Delta(r,\theta)) - \nu(\theta)| < \varepsilon/10. \end{aligned}\tag{3.20}$$

Choose k large enough so that

$$r_k > \tilde{\rho},\quad \left|\frac{\cos\theta_0}{\cos\theta_k} - 1\right| < \frac{\varepsilon}{30\bar{\nu}},\quad |\theta_k - \theta_0| < \frac{\delta_2}{10} \text{ and } |\nu(\theta_k) - \nu(\theta_0)| < \frac{\varepsilon}{10}.$$

Let $\Delta(r_k,\theta_k)^*$ be a set of plaquettes for which $\partial\Delta(r_k,\theta_k)$ is a boundary and such that

$$|\alpha(\Delta(r_k,\theta_k)) - V(\Delta(r_k,\theta_k)^*)| < \frac{\varepsilon}{20}\,\text{area}(\Delta(r_k,\theta_k)).\tag{3.21}$$

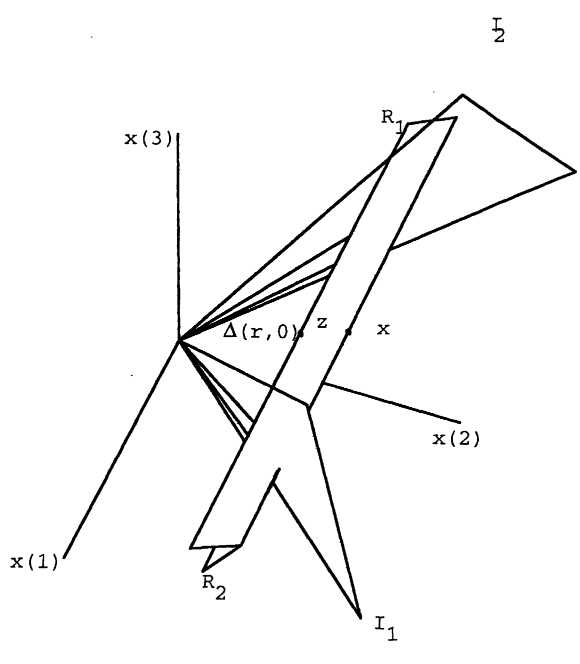

Figure 2

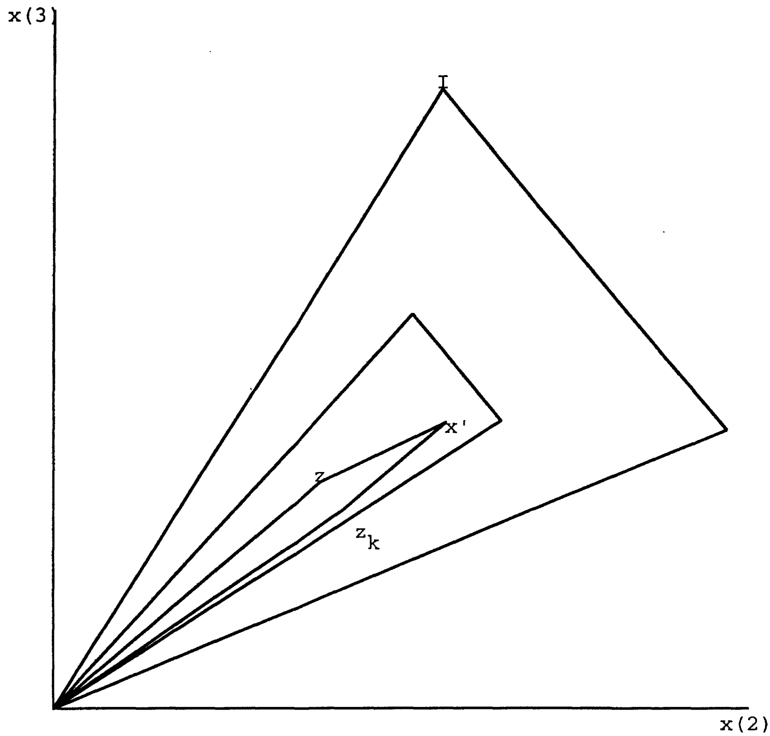

Figure 3

Consider two rectangles R_1 and R_2 whose longest sides are of length $4r_k$ and are parallel to the side of $\Delta(r_k, \theta)$ containing the point $z = (0, r_k \cos\theta, r_k \sin\theta)$. z is on one of the longest side of R_1. $z_k = (0, r_k \cos\theta_k, r_k \sin\theta_k)$ is on one of the longest side of R_2. The opposite side in both rectangles is a common side with middle point x. Figure 2 shows $\Delta(r_k, \theta)$, R_1 and the triangles $I_i(2r, \theta, \delta_1)$. $\Delta(r_k, \theta_k)$ and R_2 lie under $\Delta(r_k, \theta)$ and R_1. Figure 3 is the projection of figure 2 in the plane $x(1) = 0$ (x' is the projection of x). The projection of the cone $C((1 + 4\eta)(1 + 4\zeta)r, \theta, \delta_2)$ is also drawn.

The two shortest sides of R_i are of length $2a_i r_k$. By inequality (3.17),

$$\zeta < ((1 + 4\zeta) - 1)/2 \leq a_1 \leq ((1 + 4\eta)(1 + 4\zeta)(1 + \delta_2) - 1)/2 \leq \xi.$$

Since $|\theta_k - \theta| < \delta_2/5$, $|z_k - z| \leq \delta_2 r_k$, and by inequality (3.17),

$$\zeta < a_2 \leq (\delta_2 + (1 + 4\eta)(1 + 4\zeta)(1 + \delta_2) - 1)/2.$$

Therefore $\alpha^*_{a_i}(t, x) \leq \lambda$ for each $i = 1, 2$.

Then, for each i, there is a set of plaquettes R_i^* such that ∂R_i is a boundary for R_i^* and $V(R_i^*) \leq \lambda$ area(R_i). Moreover, we can assume that all the plaquettes of R_i^* are as in i)-iii) of the proof of lemma (3.5) and lie inside $B_{a_i}(x, 8r_k)$. Figure 2 would be complete with the addition of two pyramids like the one shown in figure 1.

The projection of E_i^* in the plane $x(1) = 0$ must lie inside the disk centered at x' of radius $8a_i r_k$. It lies inside the triangle $I(2r_k, \theta, \delta_1)$ since $8a_i r_k \leq 8\xi r_k \leq \delta_1(1+\zeta)r_k$.

The common side of R_1 and of R_2 which contains x intersects $I_i^*(2r_k, \theta, \delta_1)$, $i = 1, 2$ since $2r_k \geq (1+4\eta)(1+4\zeta)r_k \cos\theta$.

The other side of R_2 which contains z_k intersects $I_i^*(2r_k, \theta, \delta_1)$, $i = 1, 2$, since $|\tan(\theta_k - \theta)| < 2|\theta_k - \theta| < \delta_1$.

These last three remarks imply that $\partial\Delta(r_k, \theta)$ is a boundary for the set of plaquettes $\Delta(r_k, \theta_k)^* \cup R_1^* \cup R_2^* \cup I_1(2r_k, \theta, \delta_1)^* \cup I_2(2r_k, \theta, \delta_1)^*$ and therefore

$$\begin{aligned}\alpha(\Delta(r_k, \theta)) &\leq V(\Delta(r_k, \theta_k)^*) + V(R_1^*) + V(R_2^*) \\ &\quad + V(I_1(2r_k, \theta, \delta_1)^*) + V(I_2(2r_k, \theta, \delta_1)^*) \\ &< \alpha(\Delta(r_k, \theta_k)) + \frac{\varepsilon}{40} r_k^2 \cos\theta_k + 2\lambda(8\xi r_k^2) + 2c_7\delta_1(2r_k)^2 Eg\end{aligned}$$

$$\begin{aligned}(\text{area}(\Delta(r_k, \theta_k)))^{-1}\alpha(\Delta(r_k, \theta_k)) &> \left(\nu(\theta) - \frac{\varepsilon}{10}\right)\frac{\cos\theta}{\cos\theta_k} - \frac{\varepsilon}{20} - \frac{80}{1000}\varepsilon - \frac{\varepsilon}{20} \\ &> \nu(\theta_0) - \varepsilon/2 \\ &> \nu(\theta_k) - \varepsilon \qquad (3.22)\end{aligned}$$

Similarly, $\partial\Delta(r_k, \theta_k)$ is a boundary for the set of plaquettes $\Delta(r_k, \theta)^* \cup R_1^* \cup R_2^* \cup I_1(2r_k, \theta, \delta_1)^* \cup I_2(2r_k, \theta, \delta_1)^*$ where $\Delta(r_k, \theta)^*$ is a set of plaquettes such that $|\alpha(\Delta(r_k, \theta_k)) - V(\Delta(r_k, \theta_k)^*)| < (\varepsilon/20)\,\text{area}(\Delta(r_k, \theta_k))$ and therefore

$$(\text{area}(\Delta(r_k, \theta_k)))^{-1}\alpha(\Delta(r_k, \theta_k)) < \nu(\theta_k) + \varepsilon. \qquad (3.23)$$

The proof of theorem (3.1) is complete. •

Proof of theorem (1.9). – The easiest way to obtain theorem (1.9) is probably to improve slightly lemma (3.5) and then to combine the above arguments with those of section 2.

For a, $0 < a < 1$, $\rho > 0$ and for $\mathbf{b}$ be a unit vector of $\mathbb{R}^3$, let $B_{a,\mathbf{b}}(0, \rho)$ be the cylinder $\phi(B_a(0, \rho))$ where $B_a(0, \rho)$ is the cylinder defined in (3.2) and ϕ is a rotation in $O(3)$ such that $\phi(1,0,0) = \mathbf{b}$. For $x \in \mathbb{R}^3$, let $B_{a,\mathbf{b}}(x,\rho) = x + B_{a,\mathbf{b}}(0,\rho)$.Let $\mathcal{A}^*_{a,\mathbf{b}}$ be the associated maximal function.

For $0 < \zeta < \xi < 1$, let

$$\alpha^*_{a,\mathbf{b}}(t,x) = \sup \alpha(R(x,\rho))$$

where the sup is taken over all rectangles $R(x,\rho)$ with one side on the axis of $B_{a,\mathbf{b}}(x,\rho)$ and the opposite side on the surface of the cylinder and with $a\rho > 3$.

For any fixed unit vector $\mathbf{b}_0$, using lemma (3.5),

$$\begin{aligned}&\sup_{|\mathbf{b}\cdot\mathbf{b}_0|>1-\zeta^2/2} \sup_{\zeta<a<\xi} \alpha^*_{a,\mathbf{b}}(t,x)\\ &\leq \sup_{|\mathbf{b}\cdot\mathbf{b}_0|>1-\zeta^2/2} \sup_{\zeta<a<\xi} c_1\big(\mathcal{A}^*_{a,\mathbf{b}}(g^p,x)\big)^{1/p}\\ &\leq \sup_{\zeta<a<3\xi} c_1\big(\mathcal{A}^*_{a,\mathbf{b}_0}(g^p,x)\big)^{1/p}\end{aligned}$$

This implies that

$$P(\sup_{a,\mathbf{b}} \alpha^*_{a,\mathbf{b}}(t,x) > \lambda) \leq c_2(3\xi/\zeta)^2 \frac{E(g^p)}{\lambda^p} \tag{3.24}$$

where the sup is taken over $\zeta < a < \xi$ and $|\mathbf{b}\cdot\mathbf{b}_0| > 1-\zeta^2/2$. The value of c_2 is independent of ζ, ξ, λ and $\mathbf{b}_0$.

The proof will be done by contradiction. Suppose that for some $F \in \mathcal{F}$, $P(F) > 0$ and some $\varepsilon > 0$, for all $\omega \in F$, there is a sequence $(r_k, \mathbf{d}_k, \mathbf{d}'_k)$, $k \geq 1$, $\mathbf{d}_k, \mathbf{d}'_k$ are rational directions, $r_k \to \infty$ and such that

$$|(4r_k^2)^{-1}\alpha(S(r_k,\mathbf{d}_k,\mathbf{d}'_k)) - \nu(\mathbf{d}_k,\mathbf{d}'_k)| > 10\varepsilon \quad \text{for all } k = 1,2,...$$

First choose λ_1 such that $\dfrac{C^p E(g^p)}{\lambda_1^p}\dfrac{90\lambda_1}{\varepsilon} < P(F)$. Then set $\delta_1 = \varepsilon/(10\lambda_1)$.

Let $\widetilde{F} = F \cap \cap_j A(\lambda_1, \delta_1, \vec{v}_j)$ be defined for these values as in (2.6). $P(\widetilde{F}) > 0$ by (2.7).

Choose $\omega \in \widetilde{F}$ such that the ergodic theorems used hold at least for rational values of the parameters.

Set $\xi = \varepsilon/(1000\lambda_2)$, $\zeta = \delta_2 = \xi/20$, $\eta = 2c_2(3\xi/\zeta)^2 E(g^p)\lambda_2^{-p}$.

Then the value of λ_2 is chosen large enough so that (3.17) and the following two inequalities hold for these values:

$$\zeta < 1/10, \qquad \delta_2 < \delta_1/10. \tag{3.25}$$

Assume that $\mathbf{d}_k \to \mathbf{d}_0$ and $\mathbf{d}'_k \to \mathbf{d}'_0$ as $k \to \infty$.

Let z_{km}, $1 \leq m \leq 4$ be the middle points of the sides of $S(r_k, \mathbf{d}_k, \mathbf{d}'_k)$ numbered in such a way that $z_{km}/r_k \to z_{0m}$ as $k \to \infty$. Note that the middle points of the sides of $S(r, \mathbf{d}_0, \mathbf{d}'_0)$ are given by rz_{0m} for all $r > 0$. Let $\mathbf{b}_{km}$, $1 \leq m \leq 4$, be unit vectors parallel to the side of $S(r_k, \mathbf{d}_k, \mathbf{d}'_k)$ containing the point z_{km}, that is, a numbering of $(\pm\mathbf{d}_k \pm \mathbf{d}'_k)/\sqrt{2}$.

Choose rational directions $\mathbf{d}, \mathbf{d}'$ such that:

i) $|\nu(\mathbf{d}_k, \mathbf{d}'_k) - \nu(\mathbf{d}, \mathbf{d}')| < \varepsilon$ for all k sufficiently large.

ii) There is a disk centered at $\vec{v}_j$ which contains both $\vec{u}_0 = \mathbf{d}_0 \times \mathbf{d}'_0$ and $\vec{u} = \mathbf{d} \times \mathbf{d}'$; that is, $|\vec{u}_0 \cdot \vec{v}_j| > 1 - \delta_1$ and $|\vec{u} \cdot \vec{v}_j| > 1 - \delta_1$.

iii) There is a numbering z_m, $\mathbf{b}_m$, $1 \leq m \leq 4$ of the middle points of the sides of $S(1, \mathbf{d}, \mathbf{d}')$ and of the unit vectors parallel to its sides such that

$$|\mathbf{b}_{km} \cdot \mathbf{b}_m| > 1 - \zeta^2/2 \quad \text{and} \quad |z_{km} - r_k z_m| < \delta_2 r_k \tag{3.26}$$

for $1 \leq m \leq 4$ and for all k sufficiently large.

For $1 \leq m \leq 4$, let $C_m(\rho, \delta_2)$ be a rotation of the conical volume $C(\rho, \delta_2)$ around the origin such that the origin and z_m are on the axis.

For all r large enough, by lemma (3.12),

iv) for each $1 \leq m \leq 4$, there is at least one vertex x_m in $C_m((1 + 4\eta)(1 + 4\zeta)r, \delta_2) \setminus C_m((1 + 4\zeta)r, \delta_2)$ such that $\alpha^*_{a,\mathbf{b}}(t, x_m) \leq \lambda_2$ for all a, $\zeta \leq a \leq \xi$ and all $\mathbf{b}$, $|\mathbf{b} \cdot \mathbf{b}_m| > 1 - \zeta^2/2$.
and by the subadditive ergodic theorem,

v) $|(4r^2)^{-1}\alpha(S(r, \mathbf{d}, \mathbf{d}')) - \nu(\mathbf{d}, \mathbf{d}')| < \varepsilon$.

Choose k large enough so that inequalities i)-iii) hold and iv) -v) hold for $r = r_k$.

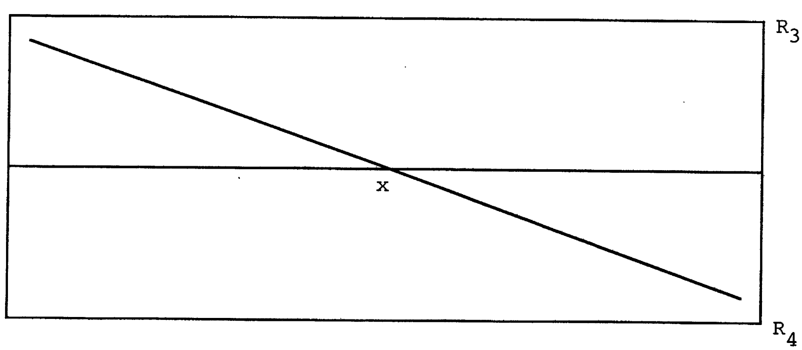

Figure 4

For each $1 \leq m \leq 4$, construct four rectangles R_{im}, $1 \leq i \leq 4$. The longest side of each rectangle is $4r_k$. The shortest sides are of length $2a_{im}r_k$.

The longest sides of R_{1m} are parallel to $\mathbf{b}_m$ and the longest sides of R_{2m} are parallel to $\mathbf{b}_{km}$. $r_k z_m$ is on one of the longest side of R_{1m}. z_{km} is on one of the longest side of R_{2m}. The opposite sides intersect at their middle point x_m. The shortest sides of R_{im}, $i = 1, 2$, are of length $2a_{im}r_k$.

R_{3m}, R_{4m} and R_{1m} have a common side of length $4r_k$; it is the side of R_{1m} which contains the point x_m. R_{3m} and R_{4m} are in a plane which contains x_m and the longest side of R_{2m} going through x_m (if $\mathbf{b}_m$ and $\mathbf{b}_{km}$ are parallel, R_{3m} and R_{4m} are superfluous). Set the length of their shortest sides by $a_{3m} = a_{4m} = \zeta$. Figure 4 shows R_{3m}, R_{4m}, one side of R_{1m} (the other side is on the boundary of $S(r_k, \mathbf{d}, \mathbf{d}'))$ and one side of R_{2m} (the other side is on the boundary of $S(r_k, \mathbf{d}_k, \mathbf{d}'_k))$.

As in the previous proof, by inequality (3.17), $2\zeta \le a_{1m} \le \xi$. By (3.17) and (3.26), $a_{2m} \le \xi$. The lower bound requires a longer calculation. First note that since $|z_{km}| = |r_k z_m| = r_k$, a_{2m} is minimum if z_{km} coincide with $r_k z_m$ and x_m is at a distance $\delta_2(1 + 4\zeta)r_k$ of the axis of the cone. Let φ_m be the angle between $\mathbf{b}_m$ and $\mathbf{b}_{km}$, then one finds that

$$a_{2m} \ge (4\zeta|\cos\varphi_m| - \delta_2(1+4\zeta|\sin\varphi_m|))/2 \ge (4\zeta(1-\zeta^2/2) - \zeta^2(1+4\zeta))/2 > \zeta.$$

By (3.26), $2r_k|\sin\varphi_m| \le 2r_k\zeta$. Hence the longest side of R_{2m} containing x_m is completely inside the rectangle formed by R_{3m} and R_{4m} (as it is drawn in figure 4).

Choose R^*_{3m} and R^*_{4m} such that $V(R^*_{im}) \le \lambda_2(8\xi r_k^2), i = 3, 4$ and such that all the plaquettes, except a few which are close to the boundary, lie on the same side as the origin.

Let $D(r_k, \vec{v}_j)^*$ be the set of all plaquettes which lie within 4 units of $\{y \in \mathbb{R}^3 : |y \cdot \vec{v}_j| < \delta_1|y|, |y| = r_k\}$, then $V(D(r_k, \vec{v}_j)^*) < 4\pi r_k^2 \delta_1 \lambda_1$.

If $\partial S(r_k, \mathbf{d}_k, \mathbf{d}'_k)$ is a boundary for a set of plaquettes E^*_k and if $\partial S(r_k, \mathbf{d}, \mathbf{d}')$ is a boundary for a set of plaquettes E^*, then, since $\delta_2 < \delta_1/10$, $\partial S(r_k, \mathbf{d}_k, \mathbf{d}'_k)$ is a boundary for $E^* \cup D(r_k, \vec{v}_j)^* \cup_{im} R^*_{im}$ and $\partial S(r_k, \mathbf{d}, \mathbf{d}')$ is a boundary for $E^*_k \cup D(r_k, \vec{v}_j)^* \cup_{im} R^*_{im}$. Therefore,

$$\begin{aligned}
&|(4r_k^2)^{-1}\alpha(S(r_k, \mathbf{d}_k, \mathbf{d}'_k)) - \nu(\mathbf{d}_k, \mathbf{d}'_k)| \\
&\qquad < (4r_k^2)^{-1}|\alpha(S(r_k, \mathbf{d}_k, \mathbf{d}'_k)) - \alpha(S(r_k, \mathbf{d}, \mathbf{d}'))| \\
&\qquad\quad + |(4r_k^2)^{-1}\alpha(S(r_k, \mathbf{d}, \mathbf{d}')) - \nu(\mathbf{d}, \mathbf{d}')| + |\nu(\mathbf{d}, \mathbf{d}') - \nu(\mathbf{d}_k, \mathbf{d}'_k)| \\
&\qquad < (4r_k^2)^{-1}\big(4\pi r_k^2 \delta_1 \lambda_1 + 16\lambda_2(8\xi r_k^2)\big) + 2\varepsilon \\
&\qquad < 10\varepsilon \qquad \bullet
\end{aligned}$$

(3.27) REMARK. – It is possible to prove theorem (1.9) under a slightly weaker moment condition: $t(e)$ belongs to the Lorentz space $L(2, 1)$. Two other examples of differentiation theorems where this sharp integrability condition occurs are mentioned in [26], VI.7.10. The only changes in the proof would occur in lemma (3.5) where rearrangements would be used in Hölder's inequality and the L^p−norm would be replaced by the Lorentz norm.

(3.28) REMARK. – For each p, $1 < p < \infty$, there is a strong type maximal inequality corresponding to the weak type maximal inequality (3.4):

$$E|\sup_{0<a\leq\xi} \mathcal{A}_a^*(f,x)|^p \leq c(p)E|f|^p.$$

With this inequality, it is possible to prove theorem (3.1) without introducing ζ (unless we only assume that $t(e)$ is in $L(2,1)$), but ζ is still needed to prove theorem (1.9) (even if we assume that $E(t(e)^p) < \infty$ for some $p > 2$).

4. SOME COUNTEREXAMPLES

Often, counterexamples in ergodic theory are constructed with Rohlin towers ([8], for example). Here as well the constructions could be done with three dimensional Rohlin towers. Instead, as in Häggström and Meester [13], the random variables $t(e)$ are defined from a sequence of i.i.d. random variables.

In the examples described below, it does not seem possible to make different choices in (4.1), (4.2) or even (4.6) and obtain a counterexample with $t(e)$ having a finite moment of order $p = 3/2$ for the disks or $p = 2$ for the squares. Actually one would expect that if $(t(e) : e$ an edge of $\mathbb{Z}^3)$ is an independent sequence of random variables then theorem (1.3) holds if the moment of order $p = 3/2$ is finite. This would be analogous to the results in first-passage percolation.

However, the convergence might not be uniform with respect to direction for an arbitrary stationary sequence $(t(e), e$ an edge of $\mathbb{Z}^3)$ without a finite moment of order $p > 3/2$. The corresponding differentiation theorem does not hold for $p = 3/2$; for example (see also [26 p.472]), the maximal function $\mathcal{S}^*f$ of $f(x) = |x|^{-2} (\log|x|)^{-1} \mathbb{1}_{(|x|<1/e)}$ for the spherical averages, defined in (1.5), is infinite a.s. and f belongs to $L^{3/2}(\mathbb{R}^3)$.

Before describing the counterexamples for theorems (1.7) and (1.9), we complete the remarks in [23, p.397] on the optimality of the moment condition $p_a = (2a + 1)/2a$ in [23, theorem A].

Three basic counterexamples for differentiation theorems.

Let χ denote the characteristic function of the unit ball in $\mathbb{R}^3$. Consider the following three functions: $f_0(x,y,z) = (x^2+y^2+z^2)^{-\gamma/2}\chi$, $f_1(x,y,z) = (y^2+z^2)^{-\gamma/2}\chi$ and $f_2(x,y,z) = |z|^{-\gamma}\chi$. It is easy to check that $f_i \in L^p(\mathbb{R}^3)$ if and only if $\gamma p < 3 - i$, $i = 0, 1, 2$.

Then we will give examples of surfaces satisfying the hypothesis of theorem (1.7) whose Fourier transforms have different rates of decrease at infinity: $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-a}$. For $0 < a \leq 1/2$, f_2 gives the counterexample with the widest range of p: $1 \leq p < 1/a$. For $1/2 \leq a \leq 3/4$, it is f_1: $1 \leq p < 4a/(4a-1)$. For $3/4 \leq a \leq 1$, it is f_0: $1 \leq p < 3/2$. Note that

$$\frac{1}{a} = \frac{4a}{4a-1} = \frac{2a+1}{2a} = 2 \quad \text{when } a = 1/2.$$

1. $0 < a \leq 1/2$. Consider the surface $S : |x|^b + y^b + z^b = 1$ where b is such that $2 \leq b < \infty$.

The Gaussian curvature vanishes at 6 different points of S. The rate of decrease is smallest in a neighborhood of one of these points and when ξ is perpendicular to the surface (see [21]). The calculations to find the exponent a can be done using, for example, [3], section 8.4. We obtain that $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-2/b}$.

Then it is simple to check that $\mathcal{S}^* f_2 = +\infty$ in a region of positive Lebesgue measure if $\gamma > 2/b$. Therefore, take γ such that $p < 1/\gamma < b/2$ to obtain a counterexample for each $p < 2/b (= 1/a)$.

2. $1/2 \leq a \leq 3/4$. Consider the surface $S : |x|^b + y^2 + z^2 = 1$, $2 \leq b \leq 3$. One principal curvature vanishes at the points of $x = 0, y^2 + z^2 = 1$. As in the preceding example, it is possible to calculate that the Fourier transform verifies $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-b/2(b-1)}$.

Then it is simple to check that $\mathcal{S}^* f_1 = +\infty$ in a set of positive measure if $\gamma > (b+1)/b$. Therefore take γ such that $p < 2/\gamma < 2b/(b+1)$ to obtain a counterexample for each $p < 4a/(4a-1)$, since $a = b/2(b-1)$.

3. $3/4 \leq a \leq 1$. Consider the same surface as in 2. with $3 \leq b < \infty$. Then $\mathcal{S}^* f_0 = +\infty$ in a set of positive measure if $\gamma > 2$. Therefore there is a counterexample for each $p < 3/2$.

An example of a stationary ergodic sequence of random variables $(t(e)$: e an edge of $\mathbb{Z}^3)$ with finite moments of order p for all $1 \leq p < 3/2$ such that a.s. the convergence in (1.8) is not uniform in $|\vec{u}| = 1$ for any surface S as in theorem (1.7).

Choose θ, $0 < \theta < 1/8$ and set

$$p_n = \theta^n, n = 1, 2, ..., p_0 = 1 - \theta/(1-\theta) \text{ and } t_n = \theta^{-2n/3}, n = 0, 1, ... \tag{4.1}$$

Let $(Z(x) : x \in \mathbb{Z}^3)$ be a sequence of independent identically distributed random variables with distribution

$$P(Z(x) = n) = p_n, \quad n = 0, 1, 2...$$

Let $(t(e) : e$ an edge of $\mathbb{Z}^3)$ be a sequence of stationary random variables defined by

$$t(e) = \max\{t_n : n = Z(x) \text{ for some vertex } x \in \mathbb{Z}^3 \text{ within 3 units of } e\}.$$

Note that if $Z(x) = n$ then $t(e) \geq t_n$ for all edges within 3 units of x.

Then for all p, $1 \leq p < 3/2$, and for any edge e,

$$E(t(e)^p) \leq 1 + \sum_1^\infty t_n^p(1 - (1 - p_n)^{30}) < \infty$$

since for each p, if $0 < \beta < 1 - 2p/3$ then $t_n^p(1 - (1 - p_n)^{30}) < \theta^{\beta n}$ for all n sufficiently large. In particular $\mu = E(t(e))$ is finite.

To check that for almost all ω the convergence is not uniform with respect to direction, we use Borel-Cantelli.

$$\text{Let} \quad q_0 = \inf\{|y| : y \in S\} \quad \text{and} \quad q_n = q_0\theta^{-n/3}/\sqrt{2\mu}.$$

For $n = 2, 3, ...,$ define

$$A_n = \{\omega : \text{ there is a vertex } x, q_{n-1} < |x| < q_n, Z(x) = n\}.$$

If $\omega \in A_n$ for infinitely many n, then there is a sequence of vertices $(x_k, k = 1, 2...)$ and a sequence $(n(k), k = 1, 2...)$ such that $Z(x_k) = n(k)$ and $q_{n(k)-1} < |x_k| < q_{n(k)}$.

For each x_k, there is a plane region $D(r_k, \vec{u}_k)$ with x_k on its boundary. And since $r_k^{-1}x_k$ is on S, $q_0 < r_k^{-1}|x_k|$. Therefore,

$$r_k^{-2}\alpha(D(r_k, \vec{u}_k)) \geq r_k^{-2}t_{n(k)} \geq (q_0/|x_k|)^2 t_{n(k)} \geq q_0^2 q_{n(k)}^{-2} t_{n(k)} \geq 2\mu \geq \nu(\vec{u})$$

for all $|\vec{u}| = 1$. Hence the convergence to $\nu(\vec{u})$ cannot be uniform in $|\vec{u}| = 1$.

It remains to check that $P(A_n \quad i.o.) = 1$. $(A_n, n = 2, 3, ...)$ is an independent sequence of events and $P(A_n) \geq 1 - (1 - p_n)^{m_n}$ where m_n is the number of vertices of $\mathbb{Z}^3$ such that $q_{n-1} < |x| < q_n$. $m_n \geq q_n^3 - q_{n-1}^3 = q_0^3(1 - \theta)\theta^{-n}$ for all n sufficiently large. Hence $P(A_n) > 1 - \exp(-q_0^3\theta(1 - \theta)/2)$ for all n sufficiently large.

For each surface $S_b : |x|^b + y^2 + z^2 = 1$, $b \geq 2$, *an example of a stationary ergodic sequence of random variables* $(t(e) : e$ *is an edge of* $\mathbb{Z}^3)$ *with finite moments of order* $1 \leq p < 2b/(b+1)$ *such that the a.s. convergence in (1.8) is not uniform with respect to direction.*

Choose γ such that $0 < \gamma < (1+b)/3b$. Then set $l = b((1+b)/3b - \gamma)$.

It is possible to find a value for θ, $0 < \theta < 1/8$, which verifies (4.4) below. Then set

$$p_n = \theta^n, \quad t_n = \theta^{-\gamma n}, n = 1, 2, ..., \quad p_0 = 1 - \theta/(1-\theta), \quad t_0 = \theta \quad \text{and}$$
$$l_n = \theta^{-ln}, \quad q_n = \theta^{-n/3}, \quad n = 0, 1, ... \tag{4.2}$$

Let $(Z(x) : x \in \mathbb{Z}^3)$ be a sequence of independent identically distributed random variables with distribution

$$P(Z(x) = n) = p_n, \quad n = 0, 1, 2...$$

Let $(t(e) : e$ an edge of $\mathbb{Z}^3)$ be a sequence of stationary random variables defined by

$$\begin{aligned} t(e) = \max\{t_n : n = Z(x) \text{ for some vertex } & x \in \mathbb{Z}^3 \\ \text{within } l_n \text{units of the segment } & (y(1) + q_n s, y(2), y(3)), \\ -1 \leq s \leq 1, \text{ where } & y \text{ is the middle point of } e\}. \end{aligned}$$

There is a constant $c_8 > 1$, independent of θ, such that for any $y \in \mathbb{R}^3$, the number of vertices $x \in \mathbb{Z}^3$ within l_n units of the segment $(y(1) + q_n s, y(2), y(3))$, $-1 \leq s \leq 1$ is less than $c_8 q_n l_n^2$ for all $n \geq 1$. Hence, for all $n \geq 1$, and for any edge e,

$$P(t(e) \geq t_n) \leq 1 - (1 - \theta^n/(1-\theta))^{c_8 q_n l_n^2}.$$

And if $0 < \beta < \frac{2}{3} - 2l - \gamma p$, then for all $n \geq 1$,

$$\begin{aligned} -\log(1 - \theta^{(\gamma p + \beta)n}) \geq \theta^{(\gamma p+\beta)n} & > \theta^{(-2l+2/3)n} \\ & \geq -\left(\frac{1-\theta}{2}\right)\theta^{-2ln - n/3} \log(1 - \theta^n/(1-\theta)) \\ & > -(7/16c_8)\, c_8 q_n l_n^2 \log(1 - \theta^n/(1-\theta)). \end{aligned}$$

Therefore there is a constant $c_9 > 1$ such that

$$1 - c_9 \theta^{(\gamma p+\beta)n} < \left(1 - \theta^{(\gamma p+\beta)n}\right)^{c_9} < (1 - \theta^n/(1-\theta))^{c_8 q_n l_n^2}.$$

$$\text{Hence} \quad P(t(e) \geq t_n) \leq c_9 \theta^{(\gamma p+\beta)n}.$$

Therefore

$$E(t(e)^p) = \sum_{n=0}^{\infty} t_n^p P(t(e) = t_n) \leq \theta^p\left(1 - \frac{\theta}{1-\theta}\right) + \sum_{n=1}^{\infty} \theta^{-\gamma n p} P(t(e) \geq t_n)$$

$$\leq \theta^p + c_9 \sum_{n=1}^{\infty} \theta^{\beta n} = \theta^p + c_9(\theta/(1-\theta^\beta)) < \infty$$

$$\text{for all } p \geq 1 \text{ such that } \quad \gamma p < \frac{2}{3} - 2l. \tag{4.3}$$

In particular, $Et(e) < \theta + c_9(\theta/(1-\theta^\beta))$ where β is such that $0 < \beta < 2/3 - 2l - \gamma$. Choose θ, $0 < \theta < 1/8$ such that, for this value of β,

$$\theta + c_9(\theta/(1-\theta^\beta)) < 1/2. \tag{4.4}$$

For $n = 2, 3, \ldots$ define

$$A_n = \{\omega : \text{ there is a vertex }, q_{n-1} < |x| < q_n, \\ Z(x) = n, x(2)^2 + x(3)^2 \geq x(1)^2\}.$$

As in the preceding example, one checks that $P(A_n \quad i.o.) = 1$. Therefore, for almost all ω there is a sequence of vertices $(x_k, k = 1, 2\ldots)$ and a sequence $(n(k), k = 1, 2\ldots)$ such that $q_{n(k)-1} < |x_k| < q_{n(k)}$, $Z(x_k) = n$, $x_k(2)^2 + x_k(3)^2 \geq x_k(1)^2$.

For each x_k, take r_k and $\vec{u}_k$, $|\vec{u}_k| = 1$ such that $(0, x_k(2), x_k(3))$ is on the boundary of $D(r_k, \vec{u}_k)$. Then $r_k^2 = x_k(2)^2 + x_k(3)^2$. And if $\partial D(r_k, \vec{u}_k)$ is a boundary for a set of plaquettes E^* then the number of plaquettes of E^* within l_n units of the segment $(x_k(1) + q_n s, x_k(2), x_k(3))$, $-1 \leq s \leq 1$,

$$\geq r_k \left(1 - \left(\frac{r_k - l_{n(k)}}{r_k}\right)^2\right)^{1/b} \geq r_k \left(l_{n(k)}/r_k\right)^{1/b}$$

if $l_{n(k)}/r_k < 1/2$. This condition is verified for all k sufficiently large if $l < 2/3$, and a fortiori if (4.3) holds. Therefore

$$r_k^{-2}\alpha(D(r_k, \vec{u}_k)) \geq r_k^{-2} r_k \left(\frac{l_{n(k)}}{r_k}\right)^{1/b} t_{n(k)} \geq \frac{t_{n(k)}}{q_{n(k)}} \left(\frac{l_{n(k)}}{q_{n(k)}}\right)^{1/b} = 1 > \nu(\vec{u})$$

for all $|\vec{u}| = 1$. The convergence cannot be uniform.

The last calculation is to find the values of p which verify (4.3) for some γ and l as in (4.2) and for some $b \geq 2$.

$$1 \leq p < \gamma^{-1}(-2l+2/3) = \frac{2b(1-3l)}{1+b-3l} = 2b - \frac{2b^2}{1+b-3l} < 2b - \frac{2b^2}{1+b} = \frac{2b}{b+1}.$$

(4.5) REMARK. – As $b \to \infty$, the "limit shape" of S_b is a cylinder and the intersection of this cylinder with a plane containing the x(1)-axis is a rectangle. Moreover, as $b \to \infty$, $2b/(b+1) \to 2$. This coincide with the moment condition of theorem (1.9).

An example of a stationary ergodic sequence of random variables $(t(e) :$ *e an edge of* $\mathbb{Z}^3)$ *with finite moments of order p for all* $1 \leq p < 2$ *such that a.s. the convergence in (1.10) is not uniform with respect to direction.*

Choose θ, $0 < \theta < 1/8$ and set

$$t_n = \theta^{-n/3}, n = 0, 1, \ldots \quad p_n = \theta^n, n = 1, 2, \ldots \text{ and } p_0 = 1 - \theta/(1-\theta). \tag{4.6}$$

Let $(Z(x) : x \in \mathbb{Z}^3)$ be a sequence of independent identically distributed random variables with distribution:

$$P(Z(x) = n) = p_n, \quad n = 0, 1, 2\ldots$$

Let $(t(e) : e \text{ an edge of } \mathbb{Z}^3)$ be a sequence of stationary random variables defined by:

$$\begin{aligned} t(e) = \max\{t_n : n = Z(x) \quad & \text{for some vertex} \in \mathbb{Z}^3 \\ & \text{within 3 units of the segment} \\ & (y(1) + q_n s, y(2), y(3)),\ -1 \leq s \leq 1, \\ & \text{where } y \text{ is the middle point of } e\}. \end{aligned}$$

$$E(t(e)^p) \leq \sum_n t_n^p P(t(e) \geq t_n) \leq \theta^{-np/3}\left(1 - (1 - \theta^n/(1-\theta))^{30t_n}\right) < \infty$$

since for each p, $1 \leq p < 2$, if $0 < \beta < (2-p)/3$, then for some constant c_{10} and for all n sufficiently large,

$$\theta^{-np/3}\left(1 - (1 - \theta^n/(1-\theta))^{30t_n}\right) < c_{10}\theta^{\beta n}/(1-\theta).$$

In particular, there is $0 < \mu < \infty$ such that for any edge e, $Et(e) \leq \mu$.

For $n = 2, 3, \ldots$ define

$$A_n = \{\omega : \text{there is a vertex}\, x,\ t_n < 6\sqrt{\mu}|x| < t_{n+1},\\ Z(x) = n,\ x(1)^2 + x(3)^2 \leq x(2)^2\}.$$

As in the previous examples, by Borel-Cantelli lemma, $P(A_n \text{ i.o.}) = 1$. Therefore for almost all ω, there is a sequence $(x_k, k = 2, 3 \ldots)$ such that $|x_k| \to \infty$ and

$$\frac{1}{4r_k^2}\alpha(S_k) \geq \frac{1}{4r_k^2}t_k^2 \geq \frac{t_k^2}{4|x_k|^2} \geq 2\mu$$

where $r_k^2 = x_k(2)^2 + x_k(3)^2$ and S_k is the square with vertices $(\pm r_k, x_k(2), x_k(3))$ and $(\pm r_k, -x_k(2), -x_k(3))$. Hence the convergence to ν cannot be uniform.

5. REMARKS ON THE DEFINITION OF BOUNDARY

In [16] and in [2] , the boundary of a set of plaquettes E^* was defined as the set of edges in $\mathcal{L}^*$ which belong to an odd number of plaquettes. In [16], one also says that E^* separates $-\infty$ from $+\infty$ over a rectangle $S = [k_1, k_2] \times [l_1, l_2]$ if there is no path on $\mathbb{Z}^3$ in $(S \times \mathbb{Z}) \setminus E$ from $S \times \{+N\}$ to $S \times \{-N\}$ for all sufficiently large $N > 0$ where E is the set of edges associated to E^*.

(5.1) LEMMA. – *Let R be a rectangle $[-\frac{1}{2}, k + \frac{1}{2}) \times [-\frac{1}{2}, l + \frac{1}{2}) \times \{\frac{1}{2}\}$ and let E^* be a set of plaquettes.*

i) If the perimeter of R, ∂R, is the boundary of E^* in the sense of [16] *or* [2] *then it is a boundary for E^* in the sense of definition (1.1).*

ii) *If ∂R is a boundary for E^* in the sense of definition (1.1) then E^* separates $-\infty$ from $+\infty$ over R.*

Proof. – i) From [16, lemma 3.2] which still holds for closed paths: for any closed oriented path γ in $\mathbb{Z}^3$, (number of intersections of γ with R) – (number of intersections of γ with E^*)
is an even integer.

ii) holds because if for some $N > 0$, there is a path γ from $R \times \{+N\}$ to $R \times \{-N\}$, in $R \times \mathbb{Z}$, it must intersect R an odd number of times and if N is large enough, it is possible to join both ends by a path that doesn't intersect E^*. But since ∂R is a boundary for E^*, this closed path must intersect E^* at least once and the intersection must be with γ. •

Another possibility would be to say that the perimeter of a convex plane region R in $\mathbb{R}^3$, with a piecewise smooth boundary, denoted by ∂R, is a boundary for a set of plaquettes E^*, if any closed path in $\mathbb{Z}^3$ which crosses R an odd number of times intersects E^* at least once. With this definition of boundary, α is still a subadditive process for which theorems (1.7) and (1.9) hold (with possibly smaller values for $\nu(\vec{u})$).

However, this definition does not eliminate some non-orientable surfaces for which the interpretation as the minimal flow would be lost (see [2]).

6. FIRST-PASSAGE PERCOLATION

In the two-dimensional lattice Z^2, it makes sense to define a path between two vertices as in definition (1.1). We could say that a set of edges E of $\mathbb{Z}^2$ is a path between two points x and y of $\mathbb{R}^2$ if any closed path in $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})$ that crosses the segment from x to y an odd number of times intersects E at least once. Then a connected set E of edges of $\mathbb{Z}^2$ such that there is an edge of E adjacent to $\hat{x}$ and there is an edge of E adjacent to $\hat{y}$ (where $\hat{x}$ is the notation for the vertex of $\mathbb{Z}^2$ which is the closest to x and in case of a tie take the one in the square containing part of the segment from x to y, and consider open segments in the first definition) would be a path from x to y.

Actually, in this way one can define a path between any two points x, y of $\mathbb{R}^d$, as it was done in [10, p.173] for example.

The quantities defined for surfaces will now be given their original interpretation. To each edge e between two vertices x, y, $|x-y|=1$, of $\mathbb{Z}^d$ is assigned a nonnegative number $t(e)=t(x,y)$. Given a set E^* of edges in $\mathbb{Z}^d$, the passage time of E^* is defined as

$$T(E^*)=\sum_{e\in E^*} t(e).$$

For x, y two vertices of $\mathbb{R}^d$, define

$$T(x,y)=\inf\{T(E^*) : E^* \text{ is a path from } x \text{ to } y\}.$$

Let $\mathbf{b}$ in $\mathbb{R}^d$ be such that all its coordinates are rational numbers with smallest common denominator M. Then if $(t(e) : e \text{ an edge of } \mathbb{Z}^d)$ is a stationary sequence of integrable random variables, $(T(mM\mathbf{b}, nM\mathbf{b}) : m,n\in\mathbb{N})$ is a stationary subadditive process and it follows from Kingman's subadditive theorem (see [18]) that $\frac{1}{n|\mathbf{b}|}T(0,n\mathbf{b})$ converges a.s. (and in L_1) as $n\to\infty$ to $\inf_n(n|\mathbf{b}|)^{-1}\int T(0,n\mathbf{b})dP$.

However let $\mathbf{b}$ in $\mathbb{R}^d$ be such that, say, its first coordinate is 1 and one of the other coordinates is irrational. Then the subadditive process in this direction is not stationary. But the L_1-convergence is uniform with respect to direction and therefore we can obtain the L_1- convergence of $\frac{1}{n|\mathbf{b}|}T(0, n\mathbf{b})$ by using the convergence in the rational directions. The a.s. convergence in the rational directions might not be uniform if we just assume that the random variables $(t(e))$ are integrable. Another difficulty is that a first step in most proof of the subadditive theorem is the convergence in the additive case. But here, the ergodic theorem is not known for a sequence such as $\widehat{n\mathbf{b}}$. The a.s. convergence of the average first-passage time in the irrational directions can still be proved using the ideas of the fairly recent proof of the ergodic theorem given in [15] (see also [14]).

(6.1) THEOREM. – *Let $(t(e) : e$ an edge of $\mathbb{Z}^d)$ be a stationary ergodic sequence of nonnegative integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. Fix $\mathbf{b}$ in $\mathbb{R}^d$, $|\mathbf{b}| = 1$. Then $\dfrac{1}{n}T(0, n\mathbf{b})$ converges a.s. as $n \to \infty$ to a constant $\nu(\mathbf{b}) = \inf_n \dfrac{1}{n}\displaystyle\int T(0, n\mathbf{b})dP$.*

Proof. – Let $A_n(\omega) = n^{-1}T(0, n\mathbf{b})$ and let

$$\underline{A}(\omega) = \liminf_n A_n(\omega), \qquad \bar{A}(\omega) = \limsup_n A_n(\omega).$$

Also let $g(\omega) = \displaystyle\sum_{|x|<6\sqrt{d}} t(0, x)$.

First check that $\underline{A}$ and $\bar{A}$ are nonrandom. Let $z = (1, 0, ..., 0)$, then

$$|A_n(\omega) - A_n(\tau_z\omega)| \leq n^{-1}(g(\omega) + g(\tau_{\widehat{n\mathbf{b}}}(\omega))$$

which converges a.s. to 0 as $n \to \infty$. And since the sequence $(t(e))$ is ergodic, $\underline{A}(\omega) = \underline{A}$ and $\bar{A}(\omega) = \bar{A}$.

Next show that $\nu(\mathbf{b}) \leq \underline{A}$.

Choose $\varepsilon > 0$ and an integer $k \geq \sqrt{d}$.

Define $\sigma(\omega) = \min(n \geq k : A_n(\omega) < \underline{A} + \varepsilon)$.

For an integer $M > k$, define

$$n(\omega) = \begin{cases} \sigma(\omega) & \text{if} \quad \sigma(\omega) \leq M \\ 1 & \text{if} \quad \sigma(\omega) > M \end{cases}$$

Define $\omega_0 = \omega$, $n_i = n(\omega_i)$, $i \geq 0$, $N_i = n_0 + ... + n_{i-1}$, $i \geq 1$ and $\omega_i = \tau_{\widehat{N_i\mathbf{b}}}(\omega)$, $i \geq 1$.

Let $i(N) = \max(i : N_i \leq N)$. Then, for any real $B > 0$,

$$T(0, N\mathbf{b}) \leq \sum_{i=0, n_i \neq 1}^{i(N)} (n_i A_{n_i}(\omega_i) + B)$$
$$+ \sum_{i=0}^{i(N)} (1_{(g>B)}(\omega_i) + 1_{(\sigma>M)}(\omega_i)) g(\omega_i) + \sum_{j=M-N}^{N} g(\tau_{\widehat{j\mathbf{b}}}\omega)$$

$$\int T(0, N\mathbf{b})\, dP \leq N(\underline{A} + \varepsilon) + \frac{N}{k} B$$
$$+ N \int (1_{(g>B)} + 1_{(\sigma>M)})(\omega) g(\omega)\, dP + M \int g(\omega)\, dP.$$

Divide by N and let $N \to \infty$,

$$\nu(\mathbf{b}) \leq \underline{A} + \varepsilon + \frac{B}{k} + \int (1_{(g>B)}(\omega) + 1_{(\sigma>M)}(\omega)) g(\omega)\, dP.$$

Let $M \to \infty$, then $k \to \infty$, then $B \to \infty$ and finally $\varepsilon \to 0$,

$$\nu(\mathbf{b}) \leq \underline{A}.$$

Finally show that $\nu(\mathbf{b}) \geq \bar{A}$.
Fix an integer $k \geq 1$. For $n, (j-1)k \leq n \leq jk$, $|T(0, n\mathbf{b}) - T(0, jk\mathbf{b})| \leq \sum_{i=(j-1)k}^{jk} g(\tau_{\widehat{i\mathbf{b}}}\omega)$, and therefore,

$$\limsup_n n^{-1} T(0, n\mathbf{b}) = \limsup_n (nk)^{-1} T(0, nk\mathbf{b})$$

Choose $\varepsilon > 0$ and an integer $k \geq 1$.

Define $\sigma(\omega) = \min(n \geq 2 : A_{nk}(\omega) > \bar{A} - \varepsilon)$.

For an integer $M > 2$, define

$$n(\omega) = \begin{cases} \sigma(\omega) & \text{if} \quad \sigma(\omega) \leq M \\ 1 & \text{if} \quad \sigma(\omega) > M \end{cases}$$

Define $\omega_0 = \omega$, $n_i = n(\omega_i)$, $i \geq 0$, $N_i = n_0 + \ldots + n_{i-1}$, $i \geq 1$ and $\omega_i = \tau_{\widehat{N_i k\mathbf{b}}}(\omega)$, $i \geq 1$.

Let $f_k(\omega) = T(0, k\mathbf{b}) + g(\omega)$ and $i(N) = \max(i : N_i \leq N)$.

Then, since each passage time on the right is less than the corresponding sum on the left,

$$\sum_{i=0}^{N} f_k(\tau_{\widehat{ik\mathbf{b}}}\omega) \geq \sum_{i=0, n_i \neq 1}^{i(N)} n_i k A_{n_i k}(\omega_i)$$

$$\geq (N - M)k(\bar{A} - \varepsilon) - \sum_{i=0}^{i(N)} \bar{A} 1_{(\sigma > M)}(\omega_i)$$

$$\int f_k \, dP \geq \frac{(N - M)k}{N}(\bar{A} - \varepsilon) - \bar{A}P(\sigma > M)$$

Let $N \to \infty$, then $M \to \infty$, to obtain $\frac{1}{k}\int f_k \, dP \geq \bar{A} - \varepsilon$. Let $\varepsilon \to 0$ and $k \to \infty$,

$$\nu(\mathbf{b}) = \lim \frac{1}{k} \int T(0, k\mathbf{b}) \, dP \geq \bar{A}.$$

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REFERENCES

[1] M. A. AKCOGLU and U. KRENGEL, Ergodic theorems for superadditive processes, *J. Reine Angew. Math.*, Vol. **323**, 1981, pp. 53-67.

[2] M. AIZENMAN, J. T. CHAYES, L. CHAYES, J. FRÖLICH and L. RUSSO, On a sharp transition from area law to perimeter law in a system of random surfaces, *Comm. Math. Phys.*, Vol. **92**, 1983, pp. 19-69.

[3] N. BLEISTEIN and R. A. HANDELSMAN, Asymptotic Expansions of Integrals, *Dover Publications*, 1975.

[4] D. BOIVIN, Weak convergence for reversible random walks in a random environment, *Ann. Probab.*, Vol. **21**, 1993, pp. 1427-1440.

[5] D. BOIVIN, First-passage percolation: the stationary case, *Probab. Th. Rel. Fields*, Vol. **86**, 1990, pp. 491-499.

[6] A. CALDERON, Ergodic theory and translation invariant operators, *Proc. Nat. Acad. Sci. USA*, Vol. **59**, 1968, pp. 349-353.

[7] J. T. COX and R. DURRETT, Some limit theorems for percolation processes with necessary and sufficient conditions, *Ann. Probab.*, Vol. **9**, 1981, pp. 583-603.

[8] A. DEL JUNCO and J. ROSENBLATT, Counterexamples in Ergodic Theory and Number Theory, *Math. Ann.*, Vol. **245**, 1979, pp. 185-197.

[9] J. DEPAUW, Thèse de doctorat, *Université de Bretagne Occidentale*, 1994.

[10] R. DURRETT, Lecture Notes on Particule Systems and Percolation, *Wads-worth & Brooks/Cole*, 1988.

[11] K. GOLDEN and G. PAPANICOLAOU, Bounds for effective parameters of heterogeneous media by analytic continuation, *Comm. Math. Phys.*, Vol. **90**, 1983, pp. 473-491.

[12] G. GRIMMETT and H. KESTEN, First-passage percolation, network flows and electrical resistances, *Z. Wahrsch. verw. Gebiete*, Vol. **66**, 1984, pp. 335-366.

[13] O. HAGGSTROM and R. MEESTER, Asymptotic shapes for stationary first passage percolation, *Ann. Probab.*, Vol. **23**, 1995, pp. 1511-1522.

[14] Y. KAMAE, A simple proof of the ergodic theorem using non-standard analysis, *Israel J. Math.*, Vol. **42**, 1982, pp. 284-290.

[15] M. KEANE, Ergodic theory and subshifts of finite type. In *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford University Press, 1991.

[16] H. KESTEN, Surfaces with minimal random weights and maximal flows: A higher dimensional version of first-passage percolation, *Illinois J. Math.*, Vol. **31**, 1987, pp. 99-166.

[17] H. KESTEN, Percolation theory and first-passage percolation, *Ann. Probab.*, Vol. **15**, 1987, pp. 1231-1271.

[18] H. KESTEN, Aspects of first-passage percolation, *Lecture Notes in Math.*, Vol. **1180**, Springer, New York, 1986, pp. 125-264.

[19] S. M. KOZLOV, The method of averaging and walks in inhomogeneous environments, *Russian Math. Surveys*, Vol. **40**, 1985, pp. 73-145.

[20] U. KRENGEL, Ergodic Theorems, *de Gruyter Studies in Mathematics 6*, de Gruyter, Berlin, 1985.

[21] W. LITTMAN, Fourier transforms of surface-carried measures and differentiability of surface averages, *Bull. of the AMS*, Vol. **69**, 1963, pp. 766-770.

[22] D. RICHARDSON, Random growth in a tesselation, *Proc. Cambridge Philos. Soc.*, Vol. **74**, 1973, pp. 515-528.

[23] J. L. RUBIO DE FRANCIA, Maximal functions and Fourier transforms, *Duke Math. J.*, Vol. **53**, 1986, pp. 395-404.

[24] S. L. SOBOLEV, Applications of Functional Analysis in Mathematical Physics. Translations of Mathematical Monographs, *American Mathematical Society*, Vol. **7**, 1963.

[25] E. M. STEIN and S. WAINGER, Problems in harmonic analysis related to curvature, *Bulletin of the AMS*, Vol. **84**, 1978, pp. 1239-1295.

[26] E. M. STEIN, Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, *Princeton University Press*, 1993.

[27] N. WIENER, The ergodic theorem, *Duke Math. J.*, Vol. **5**, 1939, pp. 1-18.

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