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Strong approximations of bivariate uniform empirical processes

by

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ABSTRACT. – In 1975, Komlós, Major and Tusnády constructed a strong approximation of the uniform empirical process $\{\alpha_n(t), n \geq 1, t \in [0,1]\}$ by a Gaussian Kiefer process. We show that the global error bound provided by Komlós, Major and Tusnády may be improved by considering only local approximation. Moreover we provide explicit constants. We also prove a local refinement for Tusnády's Gaussian strong approximation of the bidimensional uniform empirical process. The main technical tool we use is a non asymptotic normal approximation of the hypergeometric distribution. © Elsevier, Paris

RÉSUMÉ. – En 1975, Komlós, Major et Tusnády ont réalisé l'approximation forte du processus empirique uniforme $\{\alpha_n(t), n \geq 1, t \in [0,1]\}$ par un processus gaussien de Kiefer. Nous montrons que la borne d'erreur globale donnée par Komlós, Major et Tusnády peut être améliorée si l'on ne considère que l'approximation locale. De plus nous donnons des constantes explicites. Nous établissons également une amélioration locale de l'approximation forte gaussienne du processus empirique uniforme bidimensionnel dûe à Tusnady. Le principal outil technique utilisé est l'approximation normale non asymptotique de la loi hypergéométrique. © Elsevier, Paris

1. INTRODUCTION

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{A}, P)$ with uniform distribution on $[0,1]$. In this paper we say that Ω is "rich enough" if there exists a variable on $(\Omega, \mathcal{A}, P)$, with uniform distribution on $[0,1]$ independent of the sequence $(X_i)_{i\geq 1}$. Let us denote by $\hat{F}_n$ the empirical distribution function associated with the n-sample $\hat{F}_n(t) = (\sum_{i=1}^n \mathbb{I}_{X_i \leq t})/n$ and by α_n the centered and normalized empirical process $\alpha_n(t) = \sqrt{n}(\hat{F}_n(t) - t)$ associated to $\hat{F}_n$. In 1975 Komlós, Major and Tusnády (KMT) proved the following deep and striking approximation theorem:

THEOREM 1.1. – *Suppose Ω rich enough. For any integer n there exists a Brownian bridge $B^{(n)}$ such that, for some absolute positive constants C, Λ, λ, the following inequality holds:*

$$P(\sup_{t\in[0,1]} \sqrt{n}|\alpha_n(t) - B^{(n)}(t)| \geq x + C\log n) \leq \Lambda \exp(-\lambda x), \qquad (1.1)$$

for all positive x.

This means that α_n may be uniformly approximated by a Gaussian process with rate $n^{-1/2}\log n$, substantially improving the previous approximation rate $n^{-1/4}(\log n)^{1/2}$ (loglog$n)^{1/4}$ provided by Brillinger (1969). It turns out that the $n^{-1/2}\log n$-rate is optimal (Komlós, Major and Tusnády (1975), see also Csörgő and Revesz (1981), page 140). By following KMT's ideas and by refining the Poissonization argument, Mason and Van Zwet (1987) prove a local refinement of Theorem 1.1. More precisely, they show that the $\log n$ factor in Inequality (1.1) may be replaced by $\log(na)$ when the deviation between α_n and the approximating Brownian bridge $B^{(n)}$ is uniformly controled on $[0,a]$ instead of $[0,1]$. Applications of this local approximation theorem can be found in Mason and Van Zwet (1987) and Mason (1988). Following the scheme suggested in Tusnady's dissertation (as described by Csörgő and Revesz (1981)), Bretagnolle and Massart (1989) provided explicit constants $C = 12$, $\Lambda = 2$, $\lambda = 1/6$ in Inequality (1.1). Although Theorem 1.1 or its refinements have many applications (see Csörgő and Revesz (1981), Csörgő and Horváth (1993)), Kiefer (1969) pointed out that in order to study almost sure properties of the bivariate process $\{\alpha_n(t), t \in [0,1], n \geq 1\}$ one needs information on the joint distribution of the approximating sequence $\{B^{(n)}(t), t \in [0,1], n \geq 1\}$. As a matter of fact, KMT proved (Komlós, Major and Tusnády (1975), Theorem 4) that this can be obtained with the cost of a possible loss of a $\log n$ factor in the rate. In this case $B^{(n)}(t) = K(n,t)/\sqrt{n}$ where K

is a bivariate Gaussian process: the so called Kiefer process. This means that there exists a sequence $\{B_j(t), t \in [0,1], j \geq 1\}$ of i.i.d. Brownian bridges such that $K(n,t) = \sum_{j=1}^{n} B_j(t)$. The corresponding error bound may be written as follows:

$$\mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq 1} |\sqrt{m}\alpha_m(t) - K(m,t)| \geq (x + C' \log(n)) \log(n)) \\ \leq \Lambda' \exp(-\lambda' x) \tag{1.2}$$

The aim of this paper is on the one hand to provide explicit constants in Inequality (1.2), on the other hand, to prove a local refinement of this inequality which has the same flavour as Mason and Van Zwet's above mentioned Theorem. The construction of the approximating Kiefer process involved in our results is the same as in Komlós, Major and Tusnády (1975). This means that the key step of the proof is to control the difference between the projection of the bivariate empirical process on the Haar basis and the corresponding Gaussian quantile approximation. This control is obtained via Lemma 2.5 below, which provides a non asymptotic normal approximation of the hypergeometric distribution. Apart from this crucial Lemma, we also use the theorems of Mason and Van Zwet (1987) and of Bretagnolle and Massart (1989). Following Tusnády (1977), we also study the problem of Gaussian strong approximation of the bidimensional uniform empirical process for some fixed sample size. As pointed out by Tusnády, this process is closely related to the bivariate process $\{\alpha_n(t),\ t \in [0,1],\ n \geq 1\}$ and one expects analogous strong approximation theorems. More precisely let $\hat{G}_n$ denote the empirical distribution function associated to a n sample with uniform distribution on the unit cube $[0,1] \times [0,1]$ and let $\beta_n(s,t) = \sqrt{n}(\hat{G}_n(s,t) - st)$ be the associated empirical bridge. Tusnády's theorem may be stated as follows: for all integer n there exists a continuous Gaussian process $D^{(n)}$ on $[0,1] \times [0,1]$ with $E(D^{(n)}(s,t)) = 0$ and $E(D^{(n)}(s,t)D^{(n)}(s',t')) = (s \wedge s')(t \wedge t') - ss'tt'$, such that:

$$\sup_{(s,t)\in[0,1]^2} |\beta_n(s,t) - D^{(n)}(s,t)| = \mathcal{O}(\frac{\log^2 n}{\sqrt{n}}) \qquad \text{in probability.}$$

Our technique allows us to prove a local refinement of this error bound:

$$\sup_{(s,t)\in[0,b]\times[0,a]} |\beta_n(s,t) - D^{(n)}(s,t)| = \mathcal{O}(\frac{\log^2(nab)}{\sqrt{n}}) \qquad \text{in probability.}$$

Organization of the paper. Theorems 2.2 and 2.3 are respectively proved in Sections 3 and 4. The crucial Lemma 2.5 is proved in the appendix. Another important lemma, Lemma 3.3, is proved in Section 3.4.

2. RESULTS

Throughout the paper we denote by ln the Neperian logarithm and by log the function $x \longrightarrow \ln(x \vee \mathrm{e})$. We recall the definition of a Brownian bridge $B^{(n)}$:

DEFINITION 2.1. – *A Brownian bridge $B^{(n)}$ is a continuous Gaussian process defined on $[0,1]$ such that $E(B^{(n)}(t)) = 0$ and $E(B^{(n)}(t)B^{(n)}(s)) = s \wedge t - st$.*

The following Theorem combines Theorem 1 of Bretagnolle and Massart (1989) and Theorem 1 of Mason and Van Zwet (1987).

THEOREM 2.1. – *Suppose Ω rich enough. For any integer n there exists a Brownian bridge $B^{(n)}$ such that for all positive x, the following inequalities hold:*

(i) For all $a \in [0,1]$

$$\mathbf{P}(\sup_{t\in[0,a]} \sqrt{n}|\alpha_n(t) - B^{(n)}(t)| \geq (x + C_1 \log(na)) \leq \Lambda_1 \exp(-\lambda_1 x)$$

where C_1, Λ_1, λ_1 are absolute positive constants,

(ii)

$$\mathbf{P}(\sup_{t\in[0,1]} \sqrt{n}|\alpha_n(t) - B^{(n)}(t)| \geq (x + 12\log(n)) \leq 2\exp(-x/6).$$

Note that (i) is slightly different from Theorem 1 of Mason and Van Zwet (1987) since we do not impose that $a \geq 1/n$. Propositions 3.7 and 3.8 and our definition of log allows us to take $a \geq 0$. Before stating our main results, it is useful to recall the definition of a Kiefer process.

DEFINITION. – *A Kiefer process K is a continuous Gaussian process defined on $\mathbb{R}^+ \times [0,1]$ such that $E(K(s,t)) = 0$ and $E(K(s,t)K(s',t')) = (s \wedge s')(t \wedge t' - tt')$.*

Remark. – If K is a Kiefer process, then $K(n,t), n \in \mathbb{N}^*, t \in [0,1]$ has the same distribution as $\sum_{k=1}^{n} B_k(t), n \in \mathbb{N}^*, t \in [0,1]$ where $(B_k)_{k\geq 1}$ is a sequence of independent Brownian bridges.

Theorem 2.2 below provides a local refinement of Inequality (1.2) as well as an evaluation of it's constants.

THEOREM 2.2. – *Suppose Ω rich enough. There exists a Kiefer process K such that for all positive x, the following inequalities hold:*

(i) For all $a \in [0,1]$

$$\mathbf{P}(\sup_{1\le m\le n}\sup_{0\le t\le a}|\sqrt{m}\alpha_m(t)-K(m,t)|\ge(x+C_2\log(na))\log(na))$$
$$\le \Lambda_2\exp(-\lambda_2 x) \tag{2.3}$$

where $C_2,\ \Lambda_2,\ \lambda_2$ *are absolute positive constants,*

(ii)

$$\mathbf{P}(\sup_{1\le m\le n}\sup_{0\le t\le 1}|\sqrt{m}\alpha_m(t)-K(m,t)|\ge(x+60\log(n))\log(n))$$
$$\le 0.67\exp(-x/30). \tag{2.4}$$

The corresponding local refinement of Tusnády's theorem for the bidimensional uniform empirical process may be stated as follows:

THEOREM 2.3. – *Suppose* Ω *rich enough. For all integer* n *there exists a continuous Gaussian process* $D^{(n)}$ *defined on* $[0,1]\times[0,1]$ *with* $E(D^{(n)}(s,t))=0$ *and* $E(D^{(n)}(s,t)D^{(n)}(s',t'))=(s\wedge s')(t\wedge t')-ss'tt'$ *such that for all positive* x *and for all* $a,b\in[0,1]$, *the following inequality holds:*

$$\mathbf{P}(\sup_{0\le s\le b,0\le t\le a}\sqrt{n}|\beta_n(s,t)-D^{(n)}(s,t)|\ge(x+C_3\log(nab))\log(nab))$$
$$\le \Lambda_3\exp(-\lambda_3 x) \tag{2.5}$$

where $C_3,\ \Lambda_3,\ \lambda_3$ *are absolute positive constants.*

To prove our results, we follow more or less the approach of Bretagnolle and Massart (1989). So it is useful to recall the proof of Theorem 2.1 (ii). Let $(x_i), i\in\mathbb{N}$ be a sequence of independent variables, uniformly distributed on $[0,1]$. The proof relies heavily on the following property: let I' be the left half of some given interval I, the conditional distribution of the number of x_i's belonging to I' given that the number of x_i's belonging to I is equal to n, is the distribution $\mathcal{B}(n,1/2)$. Then the key step of the proof is the following normal approximation of the symmetric binomial distribution, stated in Tusnády's dissertation and proved by Bretagnolle and Massart (1989). In this lemma, and throughout this paper, the generalized inverse of a cumulative distribution function F is defined by $F^{-1}(t)=\inf\{x;F(x)\ge t\}$.

LEMMA 2.4. – *Let* Y *be a standard normal random variable, let* Φ *be the cumulative distribution function of* Y *and let* Ψ_n *be the cumulative* *distribution function of the binomial distribution $\mathcal{B}(n, 1/2)$. Then the following inequalities hold:*

1. $\mid \Psi_n^{-1} \circ \Phi(Y) - (n/2) \mid \leq 1 + (\sqrt{n}/2) \mid Y \mid$
2. $\mid \Psi_n^{-1} \circ \Phi(Y) - (n/2) - (\sqrt{n}/2)Y \mid \leq 1 + Y^2/8.$

For the bivariate empirical process $\{\alpha_n(t), n \geq 1, t \in [0,1]\}$, or for the bidimensional empirical process $\{\beta_n(s,t), s \in [0,1], t \in [0,1]\}$, the corresponding property can be described as follows. Let $(x_i), i \in \mathbb{N}$, be a sequence of independent variables uniformly distributed on $[0,1] \times [0,1]$. Let R be a rectangle of $[0,1] \times [0,1]$. Let us denote by X, n_1, n_2, n the number of x_i's belonging respectively to the north west quarter of R, to the north half of R, to the west half of R and to R. Then given n, n_1, n_2, X has hypergeometric distribution $\mathcal{H}(n, n_1, n_2)$. Lemma 2.5 below provides a normal approximation result for the hypergeometric distribution. This lemma will play the same role in the proof of Theorems 2.2 and 2.3, as Lemma 2.4 plays in the proof of Theorem 2.2 (ii). Due to the skewness of the hypergeometric distribution the statement of Lemma 2.5 involves a corrective term which does not appear in Lemma 2.4. We will show that this term can not be avoided (Section E in Appendix).

LEMMA 2.5. – *Let Y be a standard normal random variable, let Φ be the cumulative distribution function of Y and let Φ_{n,n_1,n_2} be the cumulative distribution function of the hypergeometric distribution $\mathcal{H}(n, n_1, n_2)$. We set $p = n_1/n$, $p' = n_2/n$, $q = 1-p$, $q' = 1-p'$. We denote by $m := npp'$ the mean of $\mathcal{H}(n, n_1, n_2)$ and we denote by $\sigma^2 := npp'qq'$ the approximation of the variance of $\mathcal{H}(n, n_1, n_2)$. We define $\delta = p - q$ and $\delta' = p' - q'$. Then for all $\eta > 0$ such that $|\delta\delta'| \leq 1 - \eta$ the following inequalities hold:*

1. $\mid \Phi_{n,np,np'}^{-1} \circ \Phi(Y) - m \mid \leq a + \sigma \mid Y \mid + b \mid \delta\delta' \mid Y^2$
2. $\mid \Phi_{n,np,np'}^{-1} \circ \Phi(Y) - m - \sigma Y \mid \leq c + dY^2$

where a, b, c and d are positive constants which depend only on η. Moreover $b \geq 1/6$ as soon as $|\delta\delta'| \geq \rho > 0$. Indeed, if $b < 1/6$, there exists some values of Y such that Inequality 1. is violated.

To evaluate the constants in Theorem 2.2, one needs an evaluation of constants a, b, c, d of Lemma 2.5 for $|\delta\delta'| \leq 1/8$. We obtain the following result:

A special case of Lemma 2.5. – *If $|\delta\delta'| \leq 1/8$ and $npp'qq' \geq 4.5$, we get:*

1. $\mid \Phi_{n,np,np'}^{-1} \circ \Phi(Y) - m \mid \leq 3 + \sigma \mid Y \mid + (7/5)^2 \mid \delta\delta' \mid Y^2$
2. $\mid \Phi_{n,np,np'}^{-1} \circ \Phi(Y) - m - \sigma Y \mid \leq 3 + 0.41Y^2.$

3. PROOF OF THEOREM 2.2

The proof of this theorem requires two steps: a construction of processes and a control of the error approximation between the processes. In Section 3.1 we consider a sequence of independent Wiener processes $(W_i)_{i\geq 1}$ defined on $[0,1]$. (We recall that the Wiener process W_i is a continuous Gaussian process such that $E(W_i(t)) = 0$ and $E(W_i(t)W_i(s)) = s \wedge t$.) For all integer $N \geq 0$, given $\{W_i(t); 2^N < i \leq 2^{N+1}; t \in [0,1]\}$, we construct a vector $U^N \in \mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ with the same distribution as:

$$A^N = ((\mathbb{1}_{X_i \in](k-1)2^{-N}, k2^{-N}]}; 1 \leq k \leq 2^N); 2^N < i \leq 2^{N+1}).$$

In Section 3.2 we establish inequalities (2.3) and (2.4) of Theorem 2.2 for the sequence $(\hat{F}_m(.), K(m,.))_{m\geq 1}$ where K is the Kiefer process defined below. By Skorohod's Theorem (1976) there exists a sequence $(\tilde{W}_i)_{i\geq 1}$, $\tilde{W}_i : (\Omega, \mathcal{A}, P) \longrightarrow \mathcal{C}([0,1])$, with the same distribution as $(W_i)_{i\geq 1}$ and such that $(A^N, (\tilde{W}_i)_{2^N < i \leq 2^{N+1}}); N \geq 0)$ has the same distribution as $(U^N, (W_i)_{2^N < i \leq 2^{N+1}}); N \geq 0)$. Thus we define K on $\mathbb{N}^* \times [0,1]$ by:

$$K(m,t) = \sum_{i=1}^{m} B_i(t)$$

where $B_i(t) = \tilde{W}_i(t) - t\tilde{W}_i(1)$ is the Brownian bridge associated to $\tilde{W}_i$.

3.1. Construction of U^N

Distribution of A^N. – We recall that a vector Z of $\mathbb{R}^k$ has multinomial distribution $\mathcal{M}_k(n, p_1, \ldots, p_k)$, $n \in \mathbb{N}^*$, $p_i \in [0,1]$, $\sum_{i=1}^k p_i = 1$ if

$$P(Z = (n_1, \ldots, n_k)) = \frac{n!}{n_1! \ldots n_k!} p_1^{n_1} \ldots p_k^{n_k}$$

for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$ with $\sum_{i=1}^k n_i = n$. Clearly, the distribution of the vector A^N is characterized by the two points below:

1) Each line $\mathcal{X}_i = (\mathbb{1}_{X_i \in](k-1)2^{-N}, k2^{-N}]}; 1 \leq k \leq 2^N)$ has multinomial distribution $\mathcal{M}_{2^N}(1, 2^{-N}, \ldots, 2^{-N})$.

2) The vectors $\mathcal{X}_{2^N+1}, \ldots, \mathcal{X}_{2^{N+1}}$ are mutually independent.

In order to construct U^N satisfying 1) and 2) above, first we find the conditional distribution of A^N given a filtration $\mathcal{F}$. Next we construct a vector U^N such that the conditional distributions of A^N and U^N are equal.

Definition of the filtration $\mathcal{F}$. – For $m_1, m_2 \in \mathbb{N} \cap [0, 2^N]$ we define the $\mathbb{R}^{2^N}$ vector $\zeta_{]m_1,m_2]}$ as the vector with $m_1 + 1$-th to m_2-th coordinates equal to 1 and the rest of the coordinates equal to 0. We denote by $< \ | \ >$ the usual scalar product and by $a \otimes b$ the vector $(a_1b_1, a_1b_2, \ldots, a_1b_{2^N}, a_2b_1, \ldots, a_{2^N}b_{2^N})$. We define the vectors $(e_{j,k})$; $0 \leq j \leq N$; $0 \leq k \leq 2^{N-j} - 1$ by:

$$e_{j,k} = \zeta_{]k2^j,(k+1)2^j]}$$

The $\mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ vector $e_{i,l} \otimes e_{j,k}$ may be interpreted as the indicator of the rectangle $]l2^i, (l+1)2^i] \times]k2^j, (k+1)2^j]$ and the quantity $< A^N | e_{i,l} \otimes e_{j,k} >$ is equal to $\sum_{s=2^N+l2^i+1}^{(l+1)2^i} \mathbb{I}_{X_s \in]k2^{j-N},(k+1)2^{j-N}]}$. In the sequel, we denote $A_{j,k}^{i,l} :=< A^N | e_{i,l} \otimes e_{j,k} >$.

We define the σ-fields $\mathcal{F}_{i-1,j}$, $i = 0, \ldots, N$, $j = 0, \ldots N$ by:

$$\begin{aligned}
\mathcal{F}_{N,j} &= \sigma\{A_{j,k}^{N,0} \ ; \ k = 0, \ldots, 2^{N-j} - 1\} \\
\mathcal{F}_{i,j} &= \sigma\{A_{0,k}^{i+1,l} \ ; \ l = 0, \ldots, 2^{N-(i+1)} \ ; \ k = 0, \ldots, 2^N - 1, \\
&\qquad A_{j,k}^{i,l} \ ; \ l = 0, \ldots, 2^{N-i} \ ; \ k = 0, \ldots, 2^{N-j} - 1\} \qquad \text{for } i \leq N-1.
\end{aligned}$$

These σ-fields form a decreasing filtration with respect to the order $\prec$ defined by:

$$(i_1, j_1) \prec (i_2, j_2) \Leftrightarrow \begin{cases} li_1 < i_2 \\ \text{or} \\ i_1 = i_2 \text{ and } j_1 < j_2. \end{cases}$$

Conditional characterization of the distribution of A^N. – We recall that a variable X has binomial distribution $\mathcal{B}(n, p)$, $n \in \mathbb{N}^*$, $p \in [0, 1]$ if

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \leq k \leq n.$$

We recall that a variable X has hypergeometric distribution $\mathcal{H}(n, n_1, n_2)$, $n \in \mathbb{N}^*$, $n_1, n_2 \in \mathbb{N} \cap [0, n]$ if

$$\mathbf{P}(X = k) = \frac{\binom{n_1}{k}\binom{n-n_1}{n_2-k}}{\binom{n}{n_2}}, \qquad \max(0, n_1+n_2-n) \leq k \leq \min(n_1, n_2).$$

The following proposition gives a conditional characterization of A^N.

PROPOSITION 3.1. – *1) For $j = 1, \dots, N$ the variables $A^{N,0}_{j-1,2k}$, $k = 0, \dots, 2^{N-j} - 1$, are conditionally to $\mathcal{F}_{N,j}$ independent and*

$$\mathcal{L}(A^{N,0}_{j-1,2k}/\mathcal{F}_{N,j}) = \mathcal{L}(A^{N,0}_{j-1,2k}/A^{N,0}_{j,k}) = \mathcal{B}(A^{N,0}_{j,k}, 1/2).$$

For $j = N$ we have $A^{N,0}_{N,0} = 2^N$.

2) For $i = 1, \dots, N$ and $j = 1, \dots, N$ the variables $A^{i-1,2l}_{j-1,2k}$, $l = 0, \dots, 2^{N-i} - 1$, $k = 0, \dots, 2^{N-j} - 1$ are conditionally to $\mathcal{F}_{i-1,j}$ independent and

$$\begin{aligned}\mathcal{L}(A^{i-1,2l}_{j-1,2k}/\mathcal{F}_{i-1,j}) &= \mathcal{L}(A^{i-1,2l}_{j-1,2k}/A^{i,l}_{j,k}, A^{i,l}_{j-1,2k}, A^{i-1,2l}_{j,k}) \\ &= \mathcal{H}(A^{i,l}_{j,k}, A^{i,l}_{j-1,2k}, A^{i-1,2l}_{j,k}).\end{aligned}$$

For $i = 0, \dots, N-1$ and $j = N$ we have $A^{i,l}_{N,0} = 2^i$, $l = 0, \dots, 2^{N-i} - 1$.

Remark. – the relations $A^{I,l}_{J-1,2k} + A^{I,l}_{J-1,2k+1} = A^{I,l}_{J,k}$ and $A^{I-1,2l}_{J,k} + A^{I-1,2l+1}_{J,k} = A^{I,l}_{J,k}$ give the distribution of A^N.

Instead of Proposition 3.1 we prove the following more general statement which can be used for the both bidimensional constructions (Theorems 2.2 and 2.3).

PROPOSITION 3.2. – *Let n be an even integer. Let $B := \{1, 2, \dots, |B|\}$, $|B|$ even and let B_1, B_2 be such that $B = B_1 \cup B_2$, $|B_1| = |B_2| = |B|/2$. Let A be a set of indices. Suppose that $A = I_1 \cup \dots \cup I_b \cup \dots \cup I_{|B|}$ where $|I_b| = |A|/|B|$ for all $b \in B$. Let $(\vec{R}', \vec{R}'') \in I\!R^{|A|} \times I\!R^{|A|}$ be a multinomial vector $(\vec{R}', \vec{R}'') \sim \mathcal{M}_{2|A|}(n, 1/(2|A|), \dots, 1/(2|A|))$. We set $\vec{R} := \vec{R}' + \vec{R}''$. Let $\vec{T}'$ and $\vec{T}''$ be independent vectors with the same multinomial distribution $\mathcal{M}_{|A|}(n/2, 1/|A|, \dots, 1/|A|)$. We set $\vec{T} := \vec{T}' + \vec{T}''$. (Clearly $\vec{R}$ and $\vec{T}$ are multinomial vector $\mathcal{M}_{|A|}(n, 1/|A|, \dots, 1/|A|)$.) We set $\vec{D}'_{B_1} = \{\sum_{a \in I_b} R'_a; b \in B_1\}$, $\vec{D}''_{B_1} = \{\sum_{a \in I_b} R''_a; b \in B_1\}$, $\vec{C}'_{B_1} = \{\sum_{a \in I_b} T'_a; b \in B_1\}$, $\vec{C}''_{B_1} = \{\sum_{a \in I_b} T''_a; b \in B_1\}$ and $\vec{D}'_{B_2}, \vec{D}''_{B_2}, \vec{C}'_{B_2}, \vec{C}''_{B_2}$ in the same way. We have:*

$$\begin{aligned}P(\vec{D}'_{B_1} = \vec{u}'_{B_1}, \vec{D}'_{B_2} = \vec{u}'_{B_2}, \vec{D}''_{B_1} = \vec{u}''_{B_1}, \vec{D}''_{B_2} = \vec{u}''_{B_2}, \vec{R} = \vec{v}) = P(\vec{R} = \vec{v}) \\ \times \prod_{c \in B_1} P_{\mathcal{B}(m_c, 1/2)}(m'_c) \prod_{c \in B_1} P_{\mathcal{H}(m_c, u_c, m'_c)}(u'_c)\end{aligned} \tag{3.6}$$

and

$$P(\vec{C}'_{B_1} = \vec{u}'_{B_1}, \vec{C}'_{B_2} = \vec{u}'_{B_2}, \vec{C}''_{B_1} = \vec{u}''_{B_1}, \vec{C}''_{B_2} = \vec{u}''_{B_2}, \vec{T} = \vec{v})$$
$$= \frac{P(\vec{T} = \vec{v}) \mathbb{1}_{\sum_{c \in B_1} m'_c = n/2}}{P_{\mathcal{B}(n,1/2)}(n/2)} \times \prod_{c \in B_1} P_{\mathcal{B}(m_c,1/2)}(m'_c) \prod_{c \in B_1} P_{\mathcal{H}(m_c,u_c,m'_c)}(u'_c) \tag{3.7}$$

where u'_c, u_c, m'_c, m_c are respectively the c-th coordinates of the vectors $\vec{u}'_{B_1}$, $\vec{u}'_{B_1} + \vec{u}''_{B_1}$, $\vec{u}'_{B_1} + \vec{u}'_{B_2}$, $\vec{u}'_{B_1} + \vec{u}'_{B_2} + \vec{u}''_{B_1} + \vec{u}''_{B_2}$ and with the notation $P_{\mathcal{V}}(k) := P(X = k)$ when X has distribution $\mathcal{V}$.

Proof of Proposition 3.2. – Let us denote respectively by v_a, v'_a, v''_a the a-th coordinate of $\vec{v}$, $\vec{v}'$, $\vec{v}''$. Proof of (3.6):

$$P(\vec{R}' = \vec{v}', \vec{R}'' = \vec{v}'') = P(\vec{R} = \vec{v}) \prod_{a \in A} P_{\mathcal{B}(v_a,1/2)}(v'_a) \tag{3.8}$$

Thus the variables R'_a, $a \in A$, are conditionally to $\vec{R}$ independent and $\sum_{a \in I_b} R'_a \sim \mathcal{B}(\sum_{a \in A} R_a, 1/2)$. We obtain:

$$P(\vec{D}'_{B_1} = \vec{u}'_{B_1}, \vec{D}'_{B_2} = \vec{u}'_{B_2}, \vec{D}''_{B_1} = \vec{u}''_{B_1}, \vec{D}''_{B_2} = \vec{u}''_{B_2}, \vec{R} = \vec{v})$$
$$= P(\vec{R} = \vec{v}) \times \prod_{c \in B_1} P_{\mathcal{B}(u_c,1/2)}(u'_c) \prod_{c \in B_1} P_{\mathcal{B}(m_c - u_c,1/2)}(m'_c - u'_c)$$
$$= P(\vec{R} = \vec{v}) \times \prod_{c \in B_1} P_{\mathcal{B}(m_c,1/2)}(m'_c) \prod_{c \in \mathcal{B}_1} P_{\mathcal{H}(m_c,u_c,m'_c)}(u'_c).$$

Proof of (3.7):

$$P(\vec{T}' = \vec{v}', \vec{T}'' = \vec{v}'') = \frac{P(\vec{T} = \vec{v}) \mathbb{1}_{\sum_{a \in A} v'_a = n/2}}{P_{\mathcal{B}(n,1/2)}(n/2)} \prod_{a \in A} P_{\mathcal{B}(v_a,1/2)}(v'_a)$$

Here we use a factorization to obtain:

$$P(\vec{C}'_{B_1} = \vec{u}'_{B_1}, \vec{C}'_{B_2} = \vec{u}'_{B_2}, \vec{C}''_{B_1} = \vec{u}''_{B_1}, \vec{C}''_{B_2} = \vec{u}''_{B_2}, \vec{T} = \vec{r})$$
$$= \frac{P(\vec{T} = \vec{v}) \mathbb{1}_{\sum_{c \in B_1} m'_c = n/2}}{P_{\mathcal{B}(n,1/2)}(n/2)} \times \prod_{c \in B_1} \Big(\sum_{\substack{v'_a ; a \in I(c) \\ \sum_{a \in I(c)} v'_a = u'_c}} P_{\mathcal{B}(v_a,1/2)}(v'_a) \Big)$$
$$\times \prod_{c \in B_2} \Big(\sum_{\substack{v'_a ; a \in I(c) \\ \sum_{a \in I(c)} v'_a = m'_c - u'_c}} P_{\mathcal{B}(v_a,1/2)}(v'_a) \Big).$$

(Since $|B_1| = |B_2|$, we can write u'_c, m'_c for $c \in B_2$.)
We use:

$$\sum_{\substack{v'_a; a \in I(c) \\ \sum_{a\in I(c)} v'_a = u'_c}} P_{\mathcal{B}(v_a,1/2)}(v'_a) = P_{\mathcal{B}(u_c,1/2)}(u'_c)$$

and conclude as previously. □

Remark. – The construction used in the proof of Theorem 2.3 is justified by (3.6). The construction used in the proof of Theorem 2.2 is justified by (3.7) but this justification is not immediate. Therefore we detail the proof of Proposition 3.1 below.

Proof of Proposition 3.1. – Part 1) is the result used for the unidimensional construction (Theorem 2.1). Let us notice that this result follows from (3.8) by taking

$$\vec{R}' = (A^{N,0}_{j-1,2k}; k = 0, \dots, 2^{N-j} - 1)$$
$$\vec{R}'' = (A^{N,0}_{j-1,2k+1}; k = 0, \dots, 2^{N-j} - 1).$$

To prove Part 2) we set

$$\vec{T}'_l = (A^{i-1,2l}_{0,k}; k = 0, \dots, 2^N - 1)$$
$$\vec{T}''_l = (A^{i-1,2l+1}_{0,k}; k = 0, \dots, 2^N - 1)$$
$$\vec{C}'_{B_1,l} = (A^{i-1,2l}_{j-1,2k}; k = 0, \dots, 2^{N-j} - 1)$$
$$\vec{C}'_{B_2,l} = (A^{i-1,2l}_{j-1,2k+1}; k = 0, \dots, 2^{N-j} - 1)$$

for $l = 0, \dots, 2^{N-i} - 1$. We get $\mathcal{F}_{i-1,j} = \sigma(\vec{T}_l, \vec{C}'_{B_1,l} + \vec{C}'_{B_2,l}; l = 0, \dots, 2^{N-i} - 1)$. Since the 2^{N-i} σ-fields: $\sigma(\vec{T}'_l, \vec{T}''_l)$, $l = 0, \dots, 2^{N-i} - 1$, are mutually independent and since the variables $\vec{T}_l, \vec{C}'_{B_1,l}, \vec{C}'_{B_2,l}$ are $\sigma(\vec{T}'_l, \vec{T}''_l)$-mesurable we have:

$$\mathcal{L}(\vec{C}'_{B_1,l}; l = 0, \dots, 2^{N-i} - 1/\mathcal{F}_{i-1,j})$$
$$= \otimes_{l=0}^{2^{N-i}-1} \mathcal{L}(\vec{C}'_{B_1,l}/\sigma(\vec{T}_l, \vec{C}'_{B_1,l} + \vec{C}'_{B_2,l}))$$

and we can apply (3.7). □

Construction of U^N. – Given the sequence $\{W_{2^N+i}(t)\ ;\ 1 \le i \le 2^N; t \in [0,1]\}$ we have to construct a vector $U^N \in \mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ such that $\mathcal{L}(U^N) = \mathcal{L}(A^N)$. As previously we denote $U^{i,l}_{j,k} :=< U^N | e_{i,l} \otimes e_{j,k} >$.

We construct $((U_{j,k}^{i,l}; k = 0, \dots, 2^{N-j} - 1); l = 0, \dots, 2^{N-i} - 1)$ following the order defined previously. Thus, the pair (i, j) takes successive values:

$$\begin{array}{ll} & (N,N) \prec (N,N-1) \prec \dots \prec (N,0) \\ \prec & (N-1,N) \prec (N-1,N-1) \prec \dots \prec (N-1,0) \\ \prec & \dots \\ \prec & (0,N) \prec (0,N-1) \prec \dots \prec (0,0) \end{array} \tag{3.9}$$

Let $\Phi, \Psi_n, \Phi_{n,n_1,n_2}$ be the cumulative distribution functions of the distributions $\mathcal{N}(0,1)$, $\mathcal{B}(n,p)$, $\mathcal{H}(n,n_1,n_2)$. Let $Z_{j-1,2k}^{N,0}$ and $Z_{j-1,2k}^{i-1,2l}$, $i = 1, \dots, N$, $l = 0, \dots, 2^{N-i} - 1$, $j = 1, \dots, N$, $k = 0, \dots, 2^{N-j} - 1$ be some independent random variables with uniform distribution on $[0,1]$. From Proposition 3.1 we obtain that the vector U^N defined by:

$$\begin{cases} U_{N,0}^{N,0} = 2^N \\ U_{j-1,2k}^{N,0} = \Psi^{-1}_{U_{j,k}^{N,0}}(Z_{j-1,2k}^{N,0}) \\ U_{j-1,2k+1}^{N,0} = U_{j,k}^{N,0} - U_{j-1,2k}^{N,0} \\ 0 \leq k \leq 2^{N-j} - 1 \end{cases}$$

for $j = N, \dots, 1$ (i.e. for the first line of the array (3.9)) and by:

$$\begin{cases} U_{N,0}^{i-1,2l} = 2^{i-1} \\ U_{j-1,2k}^{i-1,2l} = \Phi^{-1}_{U_{j,k}^{i,l}, U_{j-1,2k}^{i,l}, U_{j,k}^{i-1,2l}}(Z_{j-1,2k}^{i-1,2l}) \\ U_{j-1,2k}^{i-1,2l+1} = U_{j-1,2k}^{i,l} - U_{j-1,2k}^{i-1,2l} \\ U_{j-1,2k+1}^{i-1,2l} = U_{j,k}^{i-1,2l} - U_{j-1,2k}^{i-1,2l} \\ U_{j-1,2k+1}^{i-1,2l+1} = U_{j-1,2k+1}^{i,l} - U_{j-1,2k+1}^{i-1,2l} \\ 0 \leq k \leq 2^{N-j} - 1 \quad 0 \leq l \leq 2^{N-i} - 1 \end{cases}$$

when $(i-1, j-1)$ describes the other lines of the array (3.9), has the same distribution as A^N. Then it is enough to construct the sequence $(Z_{k,l}^{i,l})$. We denote by W^N the $\mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ vector defined by:

$$W^N = ((W_{i+2^N}(k2^{-N}) - W_{i+2^N}((k-1)2^{-N}); \\ k = 1, \dots, 2^N);\ i = 1, \dots, 2^N).$$

Let $\tilde{e}_{j,k}$ be the $\mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ vectors defined by:

$$\tilde{e}_{j,k} = e_{j-1,2k} - \frac{e_{j,k}}{2} \qquad \text{for } 1 \leq j \leq N \; ; \; 0 \leq k \leq 2^{N-j} - 1.$$

Then $\tilde{\mathcal{B}} = \{\tilde{e}_{j,k} \; ; \; 1 \leq j \leq N \; ; \; 0 \leq k \leq 2^{N-j} - 1\} \cup e_{N,0}$ is an orthogonal basis of $\mathbb{R}^{2^N}$ and $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{B}}$ is an orthogonal basis of $\mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$. Thus the variables $< W^N | e_{N,0} \otimes \tilde{e}_{j,k} >$, $< W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} >$, $1 \leq i \leq N$, $0 \leq l \leq 2^{N-i} - 1$, $1 \leq j \leq N$, $0 \leq k \leq 2^{N-j} - 1$ are independent Gaussian variables with expectation 0, $\mathrm{Var}(< W^N | e_{N,0} \otimes \tilde{e}_{j,k} >) = 2^j/4$ and $\mathrm{Var}(< W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} >) = 2^{i+j-N}/16$. Hence we take:

$$\begin{aligned} Z^{N,0}_{j-1,2k} &= \Phi((\frac{2^j}{4})^{-1/2} < W^N | e_{N,0} \otimes \tilde{e}_{j,k} >) \\ & \qquad \text{for } j = N, \ldots, 1 \; , \; k = 0, \ldots, 2^{N-j} - 1 \\ Z^{i-1,2l}_{j-1,2k} &= \Phi((\frac{2^{i+j-N}}{16})^{-1/2} < W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} >) \\ & \qquad \text{for } (i-1, j-1) = (N,N) \prec \ldots \prec (0,0) \; , \\ & \qquad l = 0, \ldots, 2^{N-i} - 1 \; , \; k = 0, \ldots, 2^{N-j} - 1. \end{aligned}$$

Let us remark that for $j = N, N-1, \ldots, 0$ successively, the construction of $(U^{N,0}_{j,k}; \; k = 0, \ldots, 2^{N-j} - 1)$ is the same as the construction made by Komlós, Major and Tusnády (1975-Theorem 3), Mason and Van Zwet (1987) and Bretagnolle and Massart (1989). The behaviour of the deviation $| < U^N - W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > |$ is explained by Lemma 3.3. The first part gives a control of $| < U^N - W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > |$ on an event which avoids large deviations and allows to control the probability of such an event. The second part gives an exponential bound of the error probability. This lemma is proved in Section 3.4.

LEMMA 3.3. – *Given W^N, let U^N be constructed as in Section 3.1 and let $\xi^{i,l}_{j,k}$ be the standard normal variable defined by $< W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} >$:*

$$\xi^{i,l}_{j,k} = (\frac{2^{i+j-N}}{16})^{-1/2} < W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > .$$

1. Let ϵ be a real number of $]0,1[$ and let $\Theta^{i,l}_{j,k}(\epsilon)$ be the event:

$$\begin{aligned} \Theta^{i,l}_{j,k}(\epsilon) = & \{|U^{i-1,2l}_{j,k} - 2^{i-1+j-N}| \leq \epsilon 2^{i-1+j-N}\} \\ & \cap \{|U^{i-1,2l+1}_{j,k} - 2^{i-1+j-N}| \leq \epsilon 2^{i-1+j-N}\} \\ & \cap \{|U^{i,l}_{j-1,2k} - 2^{i+j-1-N}| \leq \epsilon 2^{i+j-1-N}\} \\ & \cap \{|U^{i,l}_{j-1,2k+1} - 2^{i+j-1-N}| \leq \epsilon 2^{i+j-1-N}\}. \end{aligned}$$

We have

$$\mathbf{P}((\Theta_{j,k}^{i,l}(\epsilon))^C) \leq 8\exp(-2^{i+j-N-1}h(\epsilon))$$

where $h(t) = (1+t)\ln(1+t) - t$. *Moreover,*

(a) *For* $\epsilon \leq 0.35$ *and* $i+j-N \geq 8$ *on the event* $\Theta_{j,k}^{i,l}(\epsilon)$ *we have*

$$\begin{aligned} &| < U^N - W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > | \\ &\quad \leq 3 + 0.6039(\xi_{j,k}^{i,l})^2 + 1.628 \times 2^{-(i+j-N)} | < U^N | \tilde{e}_{i,l} \otimes e_{j,k} > |^2 \\ &\quad + 0.874210 \times 2^{-(i+j-N)} | < U^N | e_{i,l} \otimes \tilde{e}_{j,k} > |^2 \\ &\quad + 0.032332 \times 2^{-(i+j-N)} | < U^N | e_{i,l} \otimes e_{j,k} > -2^{i+j-N} |^2. \end{aligned}$$

(b) *For each pair* (i,l) *there exists a sequence of independent standard normal variables* $({\xi^*}_{j,k}^{i,l}), j = 1, \ldots, N,\ k = 0, \ldots, 2^{N-j} - 1$, *such that we have*

$$| < U^N | e_{i,l} \otimes \tilde{e}_{j,k} > | \leq 1 + \sqrt{\frac{1+\epsilon}{4}}\, 2^{(i+j-N)/2} |{\xi^*}_{j,k}^{i,l}| \text{ on } \Theta_{j,k}^{i,l}(\epsilon).$$

For each pair (j,k) *there exists a sequence of independent standard normal variables* $({\xi^{**}}_{j,k}^{i,l}), i = 1, \ldots, N,\ l = 0, \ldots, 2^{N-i} - 1$ *such that, for* $i+j-N \geq 2\ln(6/(1-\epsilon))/\ln(2)$, *we have*

$$| < U^N | \tilde{e}_{i,l} \otimes e_{j,k} > | \leq 3 + \sqrt{\frac{1+\epsilon}{4}}\, 2^{(i+j-N)/2} |{\xi^{**}}_{j,k}^{i,l}| \text{ on } \Theta_{j,k}^{i,l}(\epsilon).$$

2. *There exists universal positive constants* D, μ *such that for all positive real* t *we have:*

$$\mathbf{P}(| < U^N - W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > | \geq t) \leq D\exp(-\mu t).$$

Remark. – the joint distribution of these three sequences ξ, ξ^*, ξ^{**} is unavailable. Nevertheless it is obvious that they are not independent.

3.2. Proof of Inequalities (2.3) and (2.4)

We recall that $\log(x) = \ln(x \vee e)$ and $K(m,t) = \sum_{i=1}^m B_i(t)$, $B_i(t) = \tilde{W}_i(t) - t\tilde{W}_i(1)$, where $((A^N, (\tilde{W}_i)_{2^N < i \leq 2^{N+1}}), N \geq 0)$ has the same distribution as $((U^N, (W_i)_{2^N < i \leq 2^{N+1}}), N \geq 0)$. Let us denote by $\mathcal{P}_n(a)$ the following probability:

$$\mathcal{P}_n(a) = \mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t) - \sum_{i=1}^m B_i(t)| \geq (x + C_2\log(na))\log(na)).$$

To obtain Inequality (2.3) of Theorem 2.2 we have to prove the existence of positive constants Λ_2, λ_2 such that

$$\sup_{a\in[0,1]} \mathcal{P}_n(a) \leq \Lambda_2 \exp(-\lambda_2 x)$$

and to obtain Inequality (2.4) of Theorem 2.2 we have to prove that

$$\mathcal{P}_n(1) \leq 0.67 \exp(-x/30).$$

This follows from Propositions 3.4, 3.5 and 3.6. Inequality of Proposition 3.4 is due to Kolmogorov (Brownian part) and to Dvoretzky, Kiefer and Wolfowitz (1956). The constants for the empirical bridge are due to Massart (1990). Propositions 3.5 and 3.6 are proved in Section 3.3. Proposition 3.6 is the key to Theorem 2.2, but with this proposition we can only control

$$\sup_{2^N+1\leq m\leq 2^{N+1}} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t) - \sum_{i=1}^{m} B_i(t)|$$

when $2^N a$ is large enough. When we can not apply Proposition 3.6, we systematically use the following bound:

$$\begin{aligned} P(\sup_m \sup_t |\sqrt{m}\alpha_m(t) - \sum_{i=1}^{m} B_i(t)| > x) \leq\ & P(\sup_m \sup_t |\sqrt{m}\alpha_m(t)| > \lambda x) \\ & + P(\sup_m \sup_t |\sum_{i=1}^{m} B_i(t)| > (1-\lambda)x) \end{aligned} \tag{3.10}$$

with $0 < \lambda < 1$, and we control $\sup_m \sup_t |Z_m(t)|$ directly, where $Z_m \in \{\sqrt{m}\alpha_m, \sum_{i=1}^m B_i\}$. This is the subject of Propositions 3.4 and 3.5.

PROPOSITION 3.4. – *Let Z_m be one of the processes $\sqrt{m}\alpha_m$ or $\sum_{i=1}^m B_i$.*

$$\mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq 1} |Z_m(t)| \geq x) \leq 2n \exp(-2x^2/n).$$

PROPOSITION 3.5. – *For all $a \in [0,1]$ we have*

$$\mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t)| \geq x) \leq 2\exp(1)\exp(-\frac{x^2(1-a)}{2na + (2x(1-a)/3)})$$

$$\textit{and } \mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} |\sum_{i=1}^{m} B_i(t)| \geq x) \leq 2\exp(1)\exp(-\frac{x^2(1-a)}{2na}).$$

Remark 1. – Propositions 3.4 and 3.5 hold for any sequence of independent Brownian bridges $(B_i)_{i\geq 1}$ (not only for $B_i(t) = \tilde{W}_i(t) - t\tilde{W}_i(1)$).

PROPOSITION 3.6. – *Suppose Ω rich enough. For the sequence of independent Brownian bridges $(B_i)_{i\geq 1}$, defined by $B_i(t) = \tilde{W}_i(t) - t\tilde{W}_i(1)$ we have:*

(i) There exists absolute positive constants C_0, Λ_0, λ_0 such that:

$$\mathbf{P}\Big(\sup_{2^N+1\leq m\leq 2^{N+1}} \sup_{0\leq t\leq a} |(\sqrt{m}\alpha_m(t) - \sqrt{2^N}\alpha_{2^N}(t)) - (\sum_{i=1}^{m} B_i(t) - \sum_{i=1}^{2^N} B_i(t))|$$
$$\geq (x + C_0\log(2^N a))\log(2^N a)\Big) \leq \Lambda_0 \exp(-\lambda_0 x)$$

for all positive x, for all integers N and for all $a \in [0,1]$ satisfying $x + C_0\log(2^N a) \leq (2^N a)/12$.

(ii) For all positive x and for all integers N satisfying $x + 42\log(2^N) \leq (2^N)/12$ the following inequality holds:

$$\mathbf{P}\Big(\sup_{2^N+1\leq m\leq 2^{N+1}} \sup_{0\leq t\leq 1}$$
$$|(\sqrt{m}\alpha_m(t) - \sqrt{2^N}\alpha_{2^N}(t)) - (\sum_{i=1}^{m} B_i(t) - \sum_{i=1}^{2^N} B_i(t))|$$
$$\geq (x + 42\log(2^N))\log(2^N)\Big) \leq 0.0017\exp(-x/21).$$

Control of $\mathcal{P}_n(a)$. – We choose $C_2 = D_0 + (D_1/\ln 2)$, where $D_0 = 42$ and $D_1 = 12$ for $a = 1$, and $D_0 = C_0 \vee 42$ and $D_1 = C_1 \vee 12$ in the general case (C_0 is the constant of Proposition 3.6 *(i)*, C_1 is the constant of Theorem 2.1 *(i)*). Throughout this proof, we systematically take $\lambda = 1/2$ when we use Inequality (3.10). Propositions 3.4 and 3.5 give a control of the right side of Inequality (3.10).

If $x + D_0\log(na) > na/12$, then $x + C_2\log(na) > na/12$. When $a \in [0,1[$, Proposition 3.4 (for $a > 1/2$) and Proposition 3.5 (for $a \leq 1/2$) give the result. When $a = 1$ and $n \geq 3$, Proposition 3.4 gives $\mathcal{P}_n(1) \leq 0.5\exp(-x/24)$. When $a = 1$ and $n \leq 2$ we remark that $\sup_{1\leq m\leq n}\sup_{0\leq t\leq 1}|\sqrt{m}\alpha_m(t)| \leq 2$. Thus Proposition 3.4 gives the result.

If $x + D_0\log(na) \leq na/12$, we remark that $na \geq 4206$ (because $D_0 \geq 42$), and in this case we have $\log na = \ln na$. Let N_0 be the integer such that $2^{N_0} < n \leq 2^{N_0+1}$ ($N_0 \geq 12$). Let A_0 be the integer such that $2^{A_0-N_0-1} < a \leq 2^{A_0-N_0}$ ($12 \leq A_0 \leq N_0$). We set $j_0 = N_0 - A_0 + 3$ ($3 \leq j_0 \leq N_0$). Notice that in the case $a = 1$ we have $A_0 = N_0$ and $j_0 = 3$.

For all integers $m > 2^{j_0}$ we define $\pi(m)$ as the greatest integer of the grid $\{2^l; l \geq j_0\}$ with $\pi(m) < m$. Then we have

$$\mathcal{P}_n(a) \leq \mathcal{P}^{j_0}(a) + \mathcal{P}^1_{N_0}(a) + \mathcal{P}^2_{N_0}(a)$$

where

$$\begin{aligned}
\mathcal{P}^{j_0}(a) = \mathbf{P}(&\sup_{1\leq m\leq 2^{j_0}} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t) - \sum_{i=1}^{m} B_i(t)| \\
&> (x + C_2\log(na))\log(na)) \\
\mathcal{P}^1_{N_0}(a) = \mathbf{P}(&\sup_{2^{j_0}<m\leq 2^{N_0+1}} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t) - \sqrt{\pi(m)}\alpha_{\pi(m)}(t) \\
&- \sum_{i=\pi(m)+1}^{m} B_i(t)| > (\gamma_1 x + D_0\log(na))\log(na)) \\
\mathcal{P}^2_{N_0}(a) = \mathbf{P}(&\sup_{0\leq t\leq a} \sum_{l=j_0}^{N_0-1} |\sqrt{2^{l+1}}\alpha_{2^{l+1}}(t) - \sqrt{2^l}\alpha_{2^l}(t) \\
&- \sum_{i=2^l+1}^{2^{l+1}} B_i(t)| + |\sqrt{2^{j_0}}\alpha_{2^{j_0}}(t) \\
&- \sum_{i=1}^{2^{j_0}} B_i(t)| > (\gamma_2 x + (D_1/\ln 2)\log(na))\log(na))
\end{aligned}$$

where $0 < \gamma_1, \gamma_2 < 1$, $\gamma_1 + \gamma_2 = 1$ and $\sum_{l=N_0}^{N_0-1} = 0$.

Control of $\mathcal{P}^{j_0}(a)$. – We write $x + C_2\log(na) = y + C_2\log(2^{j_0}a)$, with $y = x + C_2\log(n2^{-j_0})$. We have $y + C_2\log(2^{j_0}a) > (2^{j_0}a)/12$, because $2^{j_0}a \leq 8$. We apply Proposition 3.4 if $a > 1/2$, and Proposition 3.5 if $a \leq 1/2$. We obtain:

$$\sup_{a\in[0,1]} \mathcal{P}^{j_0}(a) \leq \Lambda\exp(-\lambda x)$$

where Λ and λ are positive constants and

$$\mathcal{P}^{j_0}(1) \leq 10^{-99}\exp(-2.9x).$$

Control of $\mathcal{P}^1_{N_0}(a)$. – Let us define $Q_l(a)$ by

$$\begin{aligned}
Q_l(a) = \mathbf{P}(&\sup_{2^l<m\leq 2^{l+1}} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t) - \sqrt{2^l}\alpha_{2^l}(t) - \sum_{i=2^l+1}^{m} B_i(t)| \\
&> (\gamma_1 x + D_0\log(na))\log(na)).
\end{aligned}$$

We have

$$\mathcal{P}^1_{N_0}(a) \leq \sum_{l=j_0}^{N_0} Q_l(a).$$

For all l, we write $\gamma_1 x + D_0 \log(na) = y_l + D_0 \log(2^l a)$, with $y_l = \gamma_1 x + D_0 \ln(n2^{-l})$. If $y_l + C_0 \log(2^l a) > (2^l a)/12$, we use Proposition 3.4 for $a > 1/2$, Proposition 3.5 for $a \leq 1/2$ (see Remark 1), as well as the following stationarity property:

$$\mathcal{L}((\sqrt{m}\alpha_m - \sqrt{2^l}\alpha_{2^l}) \; ; \; 2^l < m \leq 2^{l+1}) = \mathcal{L}((\sqrt{m}\alpha_m) \; ; \; 1 \leq m \leq 2^l)$$
$$\mathcal{L}((\sum_{i=2^l+1}^{m} B_i) \; ; \; 2^l < m \leq 2^{l+1}) = \mathcal{L}((\sum_{i=1}^{m} B_i) \; ; \; 1 \leq m \leq 2^l).$$

We obtain:

$$\sup_{a\in[0,1]} Q_l(a) \leq \Lambda' \exp(-\lambda' y_l)$$

where Λ' and λ' are positive constants and

$$Q_l(1) \leq 4.7 \times 10^{-6} \exp(-0.18 y_l).$$

If $y_l + D_0 \log(2^l a) \leq (2^l a)/12$, we use Proposition 3.6. Finally we obtain:

$$\sup_{a\in[0,1]} \mathcal{P}^1_{N_0}(a) \leq \sum_{l=j_0}^{N_0} \Lambda'' \exp(-\lambda'' y_l) \leq \frac{\Lambda'' 2^{\lambda'' D_0}}{2^{\lambda'' D_0} - 1} \exp(-\lambda'' \gamma_1 x)$$

with $\Lambda'' = \max(\Lambda', \Lambda_0)$, $\lambda'' = \min(\lambda', \lambda_0)$ (where Λ_0, λ_0 are the constants of Proposition 3.6 *(i)*) and

$$\mathcal{P}^1_{N_0}(1) \leq 0.0017 \times \frac{4}{3} \exp(-\frac{\gamma_1 x}{21}).$$

Control of $\mathcal{P}^2_{N_0}(a)$. – Let us define $T_l(a)$, for $l \geq j_0$, and $T_{j_0-1}(a)$ by:

$$T_l(a) = \mathbf{P}\Big(\sup_{0\leq t\leq a} |\sqrt{2^{l+1}}\alpha_{2^{l+1}}(t) - \sqrt{2^l}\alpha_{2^l}(t) - \sum_{i=2^l+1}^{2^{l+1}} B_i(t)| > (\ln 2)\gamma_2 x + D_1 \log(na)\Big)$$

$$T_{j_0-1}(a) = \mathbf{P}\Big(\sup_{0\leq t\leq a} |\sqrt{2^{j_0}}\alpha_{2^{j_0}}(t) - \sum_{i=1}^{2^{j_0}} B_i(t)| > (\ln 2)\gamma_2 x + D_1 \ln(na)\Big).$$

We have

$$\mathcal{P}^2_{N_0}(a) \leq \sum_{l=j_0-1}^{N_0-1} T_l(a).$$

For $l \in \{j_0, \ldots, N_0-1\}$ we write $(\ln 2)\gamma_2 x + D_1 \log(na) = y_l + D_1 \ln(2^l a)$ with $y_l = (\ln 2)\gamma_2 x + D_1 \log(n2^{-l})$. We recall that the construction realized at the first step ($i = N$) is the same as the construction used by Komlós, Major and Tusnády (1975), Mason and Van Zwet (1987) and Bretagnolle and Massart (1989). Thus Theorem 2.1 gives:

$$\sup_{a\in[0,1]} T_l(a) \leq \Lambda_1 \exp(-\lambda_1 y_l)$$
$$\text{and } T_l(1) \leq 2\exp(-\frac{y_l}{6}).$$

The control of T_{j_0-1} is the same as the control of $Q_{j_0}(a)$. We obtain:

$$\sup_{a\in[0,1]} T_{j_0-1}(a) \leq \Lambda''' \exp(-\lambda''' x)$$
$$\text{and } T_{j_0-1}(1) \leq 10^{-99} \exp(-175(\ln 2)\gamma_2 x)$$

where Λ''' and λ''' are positive constants. Finally:

$$\sup_{a\in[0,1]} \mathcal{P}^2_{N_0}(a) \leq (\frac{\Lambda_1}{2^{\lambda_1 D_1} - 1} + \Lambda''') \exp(-\min(\lambda_1\gamma_2(\ln 2), \lambda''')x)$$

and

$$\mathcal{P}^2_{N_0}(1) \leq (\frac{2}{3} + 10^{-99}) \exp(-\frac{\lambda_1\gamma_2(\ln 2)x}{6}).$$

We choose γ_1 and γ_2 such that $\gamma_1/21 = \gamma_2(\ln 2)/6$, which completes the proof of Theorem 2.2.

3.3. Proof of Propositions 3.5 and 3.6

3.3.1. *Proof of Proposition 3.5*

Let us define $U_m(t)$ by $U_m(t) = \sqrt{m}\alpha_m(t)/(1-t)$. We have

$$\mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} |\sqrt{m}\alpha_m(t)| \geq x) \leq \inf_{r>0}(P_r + P'_r)$$

with

$$P_r = \mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} \exp(rU_m(t)) \geq \exp(rx))$$
$$P'_r = \mathbf{P}(\sup_{1\leq m\leq n} \sup_{0\leq t\leq a} \exp(-rU_m(t)) \geq \exp(rx)).$$

Let us set $V_m = \sup_{0\leq t\leq a} \exp(rU_m(t))$. Thus $(V_m, m \geq 1)$ is a submartingale and the Doob Inequality gives:

$$P_r \leq \exp(-rx)E(V_n) \leq \exp(-rx) \inf_{p>1} E(\sup_{0\leq t\leq a} (\exp(\frac{rU_n(t)}{p}))^p).$$

Since $(U_n(t), 0 \leq t \leq a)$ is a martingale, $(\exp((r/p)U_n(t)))^p$ is a submartingale for $p > 1$. Using the second Doob Inequality (1953) (see Shorack and Wellner (1986), page 871) we get:

$$P_r \leq \exp(-rx) \inf_{p>1} (\frac{p}{p-1})^p \mathbf{E}(\exp(rU_n(a))).$$

Similarly we get:

$$P'_r \leq \exp(-rx) \inf_{p>1} (\frac{p}{p-1})^p \mathbf{E}(\exp(-rU_n(a))).$$

Using $\inf_{p>1}(p/(p-1))^p = \exp(1)$ and the Bernstein Inequality (see Shorack and Wellner (1986), page 440, Inequality (4)) we obtain the first inequality of Proposition 3.5.

Using that $(\sum_{i=1}^m B_i(t)/(1-t),\ t \in [0,a])$ is a martingale the same technique gives the second inequality of Proposition 3.5.

3.3.2. *Proof of Proposition 3.6*

We prove parts *(i)* and *(ii)* together. Therefore we set $C_0 = 42$ for $a = 1$ and $C_0 \geq 42$ otherwise. Since $x + 42\log(2^N a) \leq (2^N a)/12$, we have $N \geq 13$. Let A be the integer such that $2^{A-N-1} < a \leq 2^{A-N}$ $(13 \leq A \leq N)$. Now it is enough to prove Proposition 3.6 for $a = 2^{A-N}$. The constants would only be modified by a multiplicative factor, except in the case $a = 1$.

We define $\Pi_{A,x}$ by:

$$\Pi_{A,x} = \mathbf{P}(\sup_{2^N+1\leq m\leq 2^{N+1}} \sup_{0\leq t\leq 2^{A-N}} |\Delta(m,t)| \geq \tau_{A,x})$$

$$\Delta(m,t) = (\sqrt{m}\alpha_m(t) - \sqrt{2^N}\alpha_{2^N}(t)) - (\sum_{i=1}^{m} B_i(t) - \sum_{i=1}^{2^N} B_i(t))$$

$$\tau_{A,x} = (x + C_0 \log(2^A))\log(2^A).$$

To obtain part *(i)* we have to prove the existence of positive constants Λ_0, λ_0 such that:

$$\sup_{13\leq A\leq N} \Pi_{A,x} \leq \Lambda_0 \exp(-\lambda_0 x)$$

and to obtain part *(ii)* we have to prove that

$$\Pi_{N,x} \leq 0.0017 \exp(-x/21).$$

In **I)** we bound $\Pi_{A,x}$ by a sum of probabilities:

$$\Pi_{A,x} \leq P(\Theta^{C}) + \Pi^1_{A,x} + \Pi^5_{A,x} + \mathbb{I}_{A\leq N-1} T^1_{A,x} + T^2_{A,x} + T^3_{A,x} + T^4_{A,x} + \mathbb{I}_{A\leq N-1} T^5_{A,x}.$$

In **II)** we control the terms of the right side defined in **I)**.

I) Upper bound of $\Pi_{A,x}$**.** – We choose a grid adapted to a (more precisely to A) and x. Let $L_0 := L_0(A,x)$ be the integer such that:

$$2^{L_0+A-N-1} \leq 0.8636(x + C_0 \log(2^A)) < 2^{L_0+A-N}.$$

Remark 2. – $N - A + 9 \leq L_0 \leq N - 3$.

For each $m \in \{2^N + 1, \ldots, 2^{N+1}\}$ we define $\pi_1(m)$ as the integer such that

$$(\pi_1(m) - 1)2^{L_0} < m \leq \pi_1(m) 2^{L_0}.$$

For each $t \in [0, 2^{A-N}]$ we define $\pi_2(t)$ as the integer such that

$$(\pi_2(t) - 1)2^{(L_0+A-N)-N} < t \leq \pi_2(t) 2^{(L_0+A-N)-N}.$$

We set $\mathcal{L} = [2^{N-L_0} + 1; 2^{N+1-L_0}]$, $\mathcal{K} = [1; 2^{N-L_0}]$, $u_L = L2^{L_0}$ and $s_K = K2^{(L_0+A-N)-N}$. We have

$$\Pi_{A,x} \leq \Pi^1_{A,x} + \Pi^2_{A,x}$$

with:

$$\Pi^1_{A,x} = P\Big(\sup_{2^N+1\leq m\leq 2^{N+1}} \sup_{0\leq t\leq 2^{A-N}} |\Delta(m,t) - \Delta(\pi_1(m), \pi_2(t))| \geq \alpha \tau_{a,x}\Big)$$

$$\Pi^2_{A,x} = P\Big(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\Delta(u_L, s_K)| \geq (1-\alpha)\tau_{A,x}\Big).$$

Study of $\Pi^2_{A,x}$**.** – We set $\bar{\Delta}(u_L, s_K) :=< U^N - W^N | \zeta_{]0,u_L-2^N]} \otimes (\zeta_{]0,s_K]} - s_K e_{N,0}) >$. By definition of $(B_i)_{i\geq 1}$ we get:

$$\Pi^2_{A,x} = P\Big(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\bar{\Delta}(u_L, s_K)| \geq (1-\alpha)\tau_{A,x}\Big).$$

The expansion of the vector $\zeta_{]0,u_L-2^N]}$ on the orthogonal basis of $\mathbb{R}^{2^N}$ $\tilde{\mathcal{B}}$ is:

$$\zeta_{]0,u_L-2^N]} = \sum_{i=L_0+1}^{N} \tilde{c}_i^L \tilde{e}_{f(i,L)} + \frac{u_L - 2^N}{2^N} e_{N,0}$$

with $f(i,L) := (i, f^*(i,L))$ where $f^*(i,L)$ is the integer defined by

$$f^*(i,L)2^i + 1 \leq (u_L - 2^N) \leq (f^*(i,L)+1)2^i$$

and where

$$\tilde{c}_i^L = \frac{< \zeta_{]0,u_L-2^N]} | \tilde{e}_{f(i,L)} >}{2^{i-2}}.$$

It is clear that $0 \leq \tilde{c}_i^L \leq 1$. More precisely:

$$\sum_{i=L_0+1}^{N} \tilde{c}_i^L \leq 2(N - L_0 + 0.5)/3.$$

Proof. – Let $d(a,b) := |b-a|$. We have $\tilde{c}_i^L = 2^{1-i}\mathrm{d}(L2^{L_0}, 2^i\ \mathbb{N})$.

Moreover $\mathrm{d}(L2^{i_0}, 2^i\ \mathbb{N}) + \mathrm{d}(L2^{L_0}, 2^{i-1}(2\ \mathbb{N}+1)) = 2^{i-1}$.

Therefore $\sum_{i=L_0+1}^{N} 2^{1-i}(\mathrm{d}(L2^{L_0}, 2^i\ \mathbb{N}) + \mathrm{d}(L2^{L_0}, 2^{i-1}(2\ \mathbb{N}+1)) = N - L_0$.

Using $\mathrm{d}(L2^{L_0}, 2^{i-1}(2\ \mathbb{N}+1)) \geq \mathrm{d}(L2^{L_0}, 2^{i-1}\ \mathbb{N})$, we obtain :

$$\begin{aligned} N - L_0 &\geq (\sum_{i=L_0+1}^{N} (2^{1-i} + 2^{-i})\mathrm{d}(L2^{L_0}, 2^i\ \mathbb{N})) - 2^{-N}\mathrm{d}(L2^{L_0}, 2^N\ \mathbb{N}) \\ &\geq \frac{3}{2} \sum_{i=L_0+1}^{N} \tilde{c}_i^L - \frac{1}{2} \end{aligned}$$ □

By Remark 2 we get

$$\sum_{i=L_0+1}^{N} \tilde{c}_i^L \leq \frac{2A}{3}. \tag{3.11}$$

It follows that

$$\begin{aligned} |\bar{\Delta}(u_L, s_K)| \leq | &\sum_{i=L_0+1}^{N} \tilde{c}_i^L < U^N - W^N | \tilde{e}_{f(i,L)} \otimes (\zeta_{]0,s_K]} - s_K e_{N,0}) > | \\ &+ | < U^N - W^N | e_{N,0} \otimes (\zeta_{]0,s_K]} - s_K e_{N,0}) > |. \end{aligned}$$

The same expansion for $\zeta_{]0,s_K]}$ leads to:

$$\begin{aligned} |\bar{\Delta}(u_L, s_K)| \leq | &\sum_{i=L_0+1}^{N} \sum_{j=L_0+A-N+1}^{N} \tilde{c}_i^L \tilde{c}_j^K < U^N - W^N | \tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \\ &+ | < U^N - W^N | e_{N,0} \otimes (\zeta_{]0,s_K]} - s_K e_{N,0}) > | \end{aligned}$$

with $g(j,K) := (j, g^*(j,K))$ where $g^*(j,K)$ is the integer defined by

$$g^*(j,K)2^j + 1 \leq s_K \leq (g^*(j,K)+1)2^j$$

and where

$$\tilde{c}_j^K = \frac{< \zeta_{]0,s_K]} | \tilde{e}_{g(j,K)} >}{2^{j-2}}.$$

Now we denote by $\tilde{\Delta}^{f(i,L)}_{g(j,K)}$ the current term:

$$\tilde{\Delta}^{f(i,L)}_{g(j,K)} = < U^N - W^N | \tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > .$$

Since $s_K \leq 2^A$, we have $0 \leq \tilde{c}_j^K \leq \inf(1, 2^{A-j+1})$. This remark and Inequality (3.11) yield:

$$\Pi^2_{A,x} \leq \Pi^3_{A,x} + \mathbb{I}_{A \leq N-1} T^1_{A,x} + T^2_{A,x}$$

where

$$\begin{aligned}
\Pi^3_{A,x} &= P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} | \sum_{i=L_0+1}^{N} \sum_{j=L_0+A-N+1}^{A} \tilde{c}_i^L \tilde{c}_j^K \tilde{\Delta}^{f(i,L)}_{g(j,K)} | \\
&\geq (1-\alpha)\beta(1 - \frac{\mathbb{I}_{A\leq N-1}}{2})\tau_{A,x}) \\
T^1_{A,x} &= (2^{N-L_0})^2 (N-L_0) \sup_{L_0+1\leq i\leq N} \sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} P(\sum_{j=A+1}^{N} 2^{A-j+1} |\tilde{\Delta}^{f(i,L)}_{g(j,K)}| \\
&\geq \frac{3(1-\alpha)\beta}{4A} \tau_{A,x}) \\
T^2_{A,x} &= P(\sup_{K\in\mathcal{K}} | < U^N - W^N | e_{N,0} \otimes (\zeta_{]0,s_K]} - s_K e_{N,0}) > | \\
&\geq (1-\alpha)(1-\beta)\tau_{A,x}).
\end{aligned}$$

Therefore we have:

$$\Pi_{A,x} \leq \Pi^1_{A,x} + \Pi^3_{A,x} + \mathbb{I}_{A\leq N-1} T^1_{A,x} + T^2_{A,x}.$$

Study of $\Pi^3_{A,x}$. – The behaviour of $\tilde{\Delta}^{f(i,L)}_{g(j,K)}$ on $\Theta^{f(i,L)}_{g(j,K)}(0.35)$ is given by Lemma 3.3. For $2^{i+j-N} \geq 0.8636(x + C_0 \log(2^A))$, we can control $(\Theta^{f(i,L)}_{g(j,K)}(0.35))^{\mathrm{c}}$. Thus for each $i \geq L_0 + 1$ we define the integer $M(i)$ by:

$$2^{i+M(i)-N-1} \leq 0.8636(x + C_0 \log(2^A)) < 2^{i+M(i)-N}.$$

Remark 3. – $M(i) = L_0 + A - i$.

Moreover we define the event Θ by:

$$\Theta = \bigcap_{L=2^{N-L_0}+1}^{2^{N+1-L_0}} \bigcap_{K=1}^{2^{N+1-L_0}} \bigcap_{i=L_0+1}^{N} \bigcap_{j=M(i)+1}^{A} \Theta_{g(j,K)}^{f(i,L)}(0.35).$$

We have

$$\Pi_{A,x}^3 \leq P(\Theta^C) + \Pi_{A,x}^4 + \tilde{\Pi}_{A,x}^4$$

where

$$\begin{aligned}\Pi_{A,x}^4 &= P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} \sum_{i=L_0+1}^{N} \sum_{j=M(i)+1}^{A} \tilde{c}_i^L \tilde{c}_j^K |\tilde{\Delta}_{g(j,K)}^{f(i,L)}| \\ &\geq (1-\alpha)\beta\gamma(1-\frac{\mathbb{I}_{A\leq N-1}}{2})\tau_{A,x} \cap \Theta) \\ \tilde{\Pi}_{A,x}^4 &= P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\sum_{i=L_0+1}^{N} \sum_{j=L_0+A-N+1}^{M(i)} \tilde{c}_i^L \tilde{c}_j^K \tilde{\Delta}_{g(j,K)}^{f(i,L)}| \\ &\geq (1-\alpha)\beta(1-\gamma)(1-\frac{\mathbb{I}_{A\leq N-1}}{2})\tau_{A,x}).\end{aligned}$$

In the expression of $\tilde{\Pi}_{A,x}^4$ we can replace $\sum_{i=L_0+1}^{N}$ by $\sum_{i=L_0+1}^{N-1}$ because $j \leq M(N)$ is equivalent to $j < L_0 + A - N + 1$. We bound $\tilde{\Pi}_{A,x}^4$ using (3.10). Finally we have

$$\Pi_{A,x} \leq P(\Theta^C) + \Pi_{A,x}^1 + \Pi_{A,x}^4 + \mathbb{I}_{A\leq N-1} T_{A,x}^1 + T_{A,x}^2 + T_{A,x}^3 + T_{A,x}^4$$

with:

$$T_{A,x}^3 = P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\sum_{i=L_0+1}^{N-1} \sum_{j=L_0+A-N+1}^{M(i)} \tilde{c}_i^L \tilde{c}_j^K < W^N|\tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \geq$$
$$(1-\alpha)\beta(1-\gamma)\delta(1-\frac{\mathbb{I}_{A\leq N-1}}{2})\tau_{A,x})$$

$$T_{A,x}^4 = P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\sum_{i=L_0+1}^{N-1} \sum_{j=L_0+A-N+1}^{M(i)} \tilde{c}_i^L \tilde{c}_j^K < U^N|\tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \geq$$
$$(1-\alpha)\beta(1-\gamma)(1-\delta)(1-\frac{\mathbb{I}_{A\leq N-1}}{2})\tau_{A,x}).$$

Study of $\Pi_{A,x}^4$. – We set

$$\mathcal{T}^{L,K} := \sum_{i=L_0+1}^{N} \sum_{j=M(i)+1}^{A} \tilde{c}_i^L \tilde{c}_j^K |\tilde{\Delta}_{g(j,K)}^{f(i,L)}|. \tag{3.12}$$

We can apply lemma 3.3 (*1.(a)*), because by definition of $M(i)$: $j > M(i)$ yields $i + j - N \geq 10$. On the event Θ we have

$$|\tilde{\Delta}^{f(i,L)}_{g(j,K)}| \leq 3 + 0.6039(\xi^{f(i,L)}_{g(j,K)})^2 \tag{3.13}$$

$$+2^{-(i+j-N)}(1.6280(A^{f(i,L)}_{g(j,K)})^2 + 0.874210(B^{f(i,L)}_{g(j,K)})^2 + 0.032332(C^{f(i,L)}_{g(j,K)})^2)$$

with

$$\begin{aligned} A^{f(i,L)}_{g(j,K)} &=< U^N | \tilde{e}_{f(i,L)} \otimes e_{g(j,K)} > \\ B^{f(i,L)}_{g(j,K)} &=< U^N | e_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > \\ C^{f(i,L)}_{g(j,K)} &=< U^N | e_{f(i,L)} \otimes e_{g(j,K)} > -2^{i+j-N}. \end{aligned}$$

We prove below that

$$|C^{f(i,L)}_{g(j,K)}| \leq \sum_{u=j+1}^{A} 2^{i+j-u} |B^{f(i,L)}_{g(u,K)}| + 2^{j-A} |C^{f(i,L)}_{g(A,K)}|.$$

Proof. – We set

$$d_u := \begin{cases} 2^{j+1-u} \text{ if }]g^*(j,K)2^j; (g^*(j,K)+1)2^j] \subset]g^*(u,K)2^u; \\ (2g^*(u,K)+1)2^{u-1}] \\ -2^{j+1-u} \text{ if }]g^*(j,K)2^j; (g^*(j,K)+1)2^j] \subset](2g^*(u,K)+1)2^{u-1}; \\ (g^*(u,K)+1)2^u]. \end{cases}$$

It follows that: $e_{g(j,K)} = \sum_{u=j+1}^{N} d_u \tilde{e}_{g(u,K)} + 2^{j-N} e_{N,0} = \sum_{u=j+1}^{A} d_u \tilde{e}_{g(u,K)} + 2^{j-A} e_{g(A,K)}$. □

In the case $A = N$, we have $C^{f(i,L)}_{g(A,K)} = 0$. In order to obtain a good evaluation of the constants, we treat the case $A = N$ and the case $A \leq N-1$ separately. Using $(x+y)^2 \leq 2(x^2+y^2)$ and $\sum_{j=M(i)+1}^{A} 2^{j-A} \leq 2$ we get:

$$\sum_{j=M(i)+1}^{A} 2^{-(i+j-N)} (C^{f(i,L)}_{g(j,K)})^2 \leq (2 - \mathbb{I}_{A=N}) \sum_{j=M(i)+1}^{A} 2^{-(i+j-N)}$$
$$\times \left(\sum_{u=j+1}^{A} 2^{j+1-u} |B^{f(i,L)}_{g(u,K)}| \right)^2 + 4 \times \mathbb{I}_{A \leq N-1} \times 2^{-(i+A-N)} (C^{f(i,L)}_{g(A,K)})^2.$$

By the Cauchy-Schwarz Inequality and by permuting the variables u and

j we obtain:

$$\sum_{j=M(i)+1}^{A} 2^{-(i+j-N)}(C_{g(j,K)}^{f(i,L)})^2 \leq (2 - \mathbb{I}_{A=N})\frac{4}{(\sqrt{2}-1)^2}$$
$$\sum_{j=M(i)+1}^{A} 2^{-(i+j-N)}(B_{g(j,K)}^{f(i,L)})^2$$
$$+ 4 \times \mathbb{I}_{A\leq N-1} \times 2^{-(i+A-N)}(C_{g(A,K)}^{f(i,L)})^2. \tag{3.14}$$

Thus with (3.12), (3.13), (3.14) we get:

$$\mathcal{T}^{L,K}\mathbb{I}_\Theta \leq \mathcal{T}_1^{L,K}\mathbb{I}_\Theta + \mathbb{I}_{A\leq N-1}\mathcal{T}_2^{L,K}\mathbb{I}_\Theta$$

where

$$\begin{aligned}\mathcal{T}_1^{L,K} = \sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A} &(3 + 0.6039(\xi_{g(j,K)}^{f(i,L)})^2 \\ &+ (1.6280\mathbb{I}_{A=N} + 2.382\mathbb{I}_{A\leq N-1}) \\ &\times 2^{-(i+j-N)}(\tilde{c}_j^K(A_{g(j,K)}^{f(i,L)})^2 + \tilde{c}_i^L(B_{g(j,K)}^{f(i,L)})^2))\end{aligned}$$

$$\mathcal{T}_2^{L,K} = 4 \times 0.032332 \sum_{i=L_0+1}^{N} \tilde{c}_i^L 2^{-(i+A-N)}(C_{g(A,K)}^{f(i,L)})^2.$$

We have

$$\Pi_{A,x} \leq P(\Theta^{\mathrm{C}})+\Pi_{A,x}^1+\Pi_{A,x}^5+\mathbb{I}_{A\leq N-1}T_{A,x}^1+T_{A,x}^2+T_{A,x}^3+T_{A,x}^4+\mathbb{I}_{A\leq N-1}T_{A,x}^5$$

where

$$\Pi_{A,x}^5 = P(\sup_{L\in\mathcal{L}}\sup_{K\in\mathcal{K}} \mathcal{T}_1^{L,K}\mathbb{I}_\Theta \geq (1-\alpha)\beta\gamma(1 - \frac{3\mathbb{I}_{A\leq N-1}}{4})\tau_{A,x})$$
$$T_{A,x}^5 = P(\sup_{L\in\mathcal{L}}\sup_{K\in\mathcal{K}} \mathcal{T}_2^{L,K} \geq (1-\alpha)\beta\frac{\gamma}{4}\tau_{A,x}).$$

II) Control of the bound. – We choose $\alpha = 0.14026$, $(1-\alpha)(1-\beta) = 0.04509$, $(1-\alpha)(1-\beta)(1-\gamma)\delta = 0.03272$, $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 0.4374$, and thus $(1-\alpha)(1-\beta)\gamma = 0.34453$.

Control of $P(\Theta^{\mathrm{c}})$. – We remark that

$$\bigcap_{L=2^{N-L_0}+1}^{2^{N+1-L_0}} \bigcap_{j=M(i)+1}^{A} \bigcap_{K=1}^{2^{N+1-L_0}} \Theta_{g(j,K)}^{f(i,L)}(0.35)$$
$$= \bigcap_{l=0}^{2^{N-i}-1} \bigcap_{k=0}^{2^{A-(M(i)+1)}-1} \Theta_{M(i)+1,k}^{i,l}(0.35).$$

Thus, by Lemma 3.3 and Remarks 2 and 3, we have for $C_0 \geq 42$:

$$P(\Theta^{\mathrm{c}}) \leq \sum_{i=L_0+1}^{N} 2^{N-L_0} \times 4\exp(-2^{A+L_0-N}h(0.35))$$
$$\leq 2^{-18}\exp(-x/21).$$

Control of $\Pi_{A,x}^1$. We write:

$$\Delta(m,t) - \Delta(\pi_1(m),\pi_2(t)) = \Delta(m,t) - \Delta(\pi_1(m),t) + \Delta(\pi_1(m),t) - \Delta(\pi_1(m),\pi_2(t)).$$

Using:

$$\mathcal{L}(\sqrt{m}\alpha_m - \sqrt{m'}\alpha'_m;\ m' < m \leq m'') = \mathcal{L}(\sqrt{m}\alpha_m;\ 1 \leq m \leq m'' - m')$$
$$\mathcal{L}(\sum_{i=m'+1}^{m} B_i\ ;\ 1 \leq m \leq m'' - m') = \mathcal{L}(\sum_{i=1}^{m} B_i\ ;\ 1 \leq m \leq m'' - m')$$

and the fact that, for each m,

$$\mathcal{L}(\sqrt{m}\alpha_m(t_0+s) - \sqrt{m}\alpha_m(t_0);\ s \in]0,b])$$
$$\mathcal{L}(\sum_{i=1}^{m} B_i(t_0+s) - \sum_{i=1}^{m} B_i(t_0);\ s \in]0,b])$$

do not depend on t_0, and using moreover the bound (3.10), we obtain:

$$\Pi_{A,x}^1 \leq T_{A,x}^6 + T_{A,x}^7 + T_{A,x}^8 + T_{A,x}^9$$

where

$$T^6_{A,x} = \mathrm{Card}(\mathcal{L})P(\sup_{1\le m\le 2^{L_0}} \sup_{0\le t\le 2^{A-N}} |\sum_{i=1}^{m} B_i(t)| \ge (1-\alpha)\frac{\delta}{2}\tau_{A,x})$$

$$T^7_{A,x} = \mathrm{Card}(\mathcal{L})P(\sup_{1\le m\le 2^{L_0}} \sup_{0\le t\le 2^{A-N}} |\sqrt{m}\alpha_m(t)| \ge (1-\alpha)\frac{\delta}{2}\tau_{A,x})$$

$$T^8_{A,x} = \mathrm{Card}(\mathcal{L})\mathrm{Card}(\mathcal{K}) \sup_{1\le L\le 2^{N-L_0}} P(\sup_{0\le t\le 2^{(A+L_0-N)-N}} |\sum_{i=1}^{u_L} B_i(t)| \ge (1-\alpha)(1-\delta)(1-\epsilon)\tau_{A,x})$$

$$T^9_{A,x} = \mathrm{Card}(\mathcal{L})\mathrm{Card}(\mathcal{K}) \sup_{1\le L\le 2^{N-L_0}} P(\sup_{0\le t\le 2^{(A+L_0-N)-N}} |\sqrt{u_L}\alpha_{u_L}(t)| \ge (1-\alpha)(1-\delta)\epsilon\tau_{A,x}).$$

By Remark 2 we have:

$$\mathrm{Card}(\mathcal{L}),\ \mathrm{Card}(\mathcal{K}) \le 2^{A-9}. \tag{3.15}$$

Proposition 3.5 provides the control of $T^6_{A,x}$ and $T^7_{A,x}$ in the case $A \le N-1$: for C_0 large enough, there exists positive constants D_1, λ_1 such that:

$$T^6_{A,x} + T^7_{A,x} \le D_1 \exp(-\lambda_1 x).$$

To control $T^6_{N,x}$ and $T^7_{N,x}$ we apply Proposition 3.4 and obtain:

$$T^6_{N,x} + T^7_{N,x} \le 2 \times 2^{-12} \exp(-x/21).$$

The control of $T^8_{A,x}$ and $T^9_{A,x}$ follows from the maximal inequalities proved by Shorack and Wellner (1986) and Csáki (1974) (see also Lemmas 2 and 3 of Bretagnolle and Massart (1989)):

PROPOSITION 3.7. – *For all real $b \in]0,1[$ we have:*

$$\mathbf{P}(n \sup_{t\in[0,b]} |\hat{F}_n(t) - t| \ge x) \le 2\exp(-nb(1-b)h(\frac{x}{nb}))$$

where $h(t) = (1+t)\ln(1+t) - t$.

PROPOSITION 3.8. – *Let B be a Brownian bridge. For all real $b \in]0,1/2]$ we have:*

$$\mathbf{P}(\sup_{t\in[0,b]} \sqrt{n}|B(t)| \ge x) \le 2\exp(-\frac{x^2}{2nb(1-b)}).$$

Remark 4. – The function h satisfies $h(t) \geq 3t^2/(6+2t)$ (this is Bernstein Inequality, see also Shorack and Wellner (1986), page 441.)

We apply Propositions 3.7 and 3.8 with $b = 2^{(L_0+A-N)-N}$. By Remark 2 we have $b \leq 1/8$. We obtain

$$T^8_{A,x} + T^9_{A,x} \leq 2 \times 2^{-17} \exp(-x/21).$$

Control of $\Pi^5_{A,x}$. – For simplification's sake, we detail only the case $A = N$. In the case $A \leq N-1$ the method is the same but the constants are modified. We denote with a star the constants which are different in this case.

By Lemma 3.3 *(1.b.)* (recall that $j > M(i)$ yields $i + j - N \geq 10$) and using the relation $(x+y)^2 \leq (111/10)x^2 + (111/101)y^2$ we get:

$$\begin{aligned}\mathcal{T}_1^{L,K}\mathbb{I}_\Theta \leq & \sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A}(3 + 180^*.708\, 2^{-(i+j-N)}) + 0.6039 \\ & \times \sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A}(\xi^{f(i,L)}_{g(j,K)})^2 \\ & + 0^*.6039 \times \sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A}(\tilde{c}^K_j(\xi^{**}{}^{f(i,L)}_{g(j,K)})^2 + \tilde{c}^L_i(\xi^{*}{}^{f(i,L)}_{g(j,K)})^2).\end{aligned}$$

Using Inequality (3.11), Remark 3 and $\sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A} 2^{-(i+j-N)} \leq (N-L_0)/(0.8636(x + C_0\log(2^A)))$ we get that:

$$\begin{aligned}\mathcal{T}_1^{L,K}\mathbb{I}_\Theta \leq & \frac{3(N-L_0)(N-L_0+1)}{2} + 0^*.553(N-L_0) + 0.6039 \\ & \times \sum_{i=L_0+1}^{N}\sum_{j=M(i)+1}^{A}(\xi^{f(i,L)}_{g(j,K)})^2 \\ & + \frac{2(N-L_0+1)0^*.6039}{3}\Big(\sup_{A+L_0-N+1\leq j\leq A}\sum_{i=A+L_0-j+1}^{N}(\xi^{**}{}^{f(i,L)}_{g(j,K)})^2 \\ & + \sup_{L_0+1\leq i\leq N}\sum_{j=A+L_0-i+1}^{A}(\xi^{*}{}^{f(i,L)}_{g(j,K)})^2\Big).\end{aligned}$$

We have to control $\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} \mathcal{T}_1^{L,K} \mathbb{1}_\Theta$. Notice that:

$$\{g(j,K); K \in \mathcal{K}; A+L_0-N+1 \le j \le A\}$$
$$= \{(j,k); A+L_0-N+1 \le j \le A; 0 \le k \le 2^{A-j}-1\}$$
$$\{f(i,L); L \in \mathcal{L}; L_0+1 \le i \le N\}$$
$$= \{(i,l); L_0+1 \le i \le N; 0 \le l \le 2^{N-i}-1\} \quad (3.16).$$

Thus using Remarks 2 and 3 and (3.15), (3.16) we get:

$$\Pi^5_{A,x} \le (2^{A-9})^2 P(Z_1 \ge z_1) + 2^{A-9} \times \Big(\sum_{j=A+L_0-N+1}^{A} 2^{A-j} + \sum_{j=L_0+1}^{N} 2^{N-i} \Big) P(Z_2 \ge z_2) \tag{3.17}$$

where Z_1 and Z_2 are variables with chi-2 distribution with respectively $(A-9)(A-8)/2$ and $A-9$ degrees of freedom and where

$$z_1 = \frac{1}{0.6039}\bigg((1-\alpha)\beta\gamma\epsilon\Big(1 - \frac{3\mathbb{1}_{A\le N-1}}{4}\Big)\tau_{A,x} - (A-9)\Big(\frac{0^*.553}{3} + \frac{A-8}{2}\Big)\bigg)$$

$$z_2 = \frac{3}{0^*.6039 \times 2(A-8)}\bigg((1-\alpha)\beta\gamma\Big(\frac{1-\epsilon}{2}\Big)\Big(1 - \frac{3\mathbb{1}_{A\le N-1}}{4}\Big)\tau_{A,x} - (A-9)\Big(\frac{0^*.553}{3} + \frac{A-8}{2}\Big)\bigg).$$

We choose $(1-\alpha)\beta\gamma\epsilon = 0.068$ and we control the right side of (3.17) with Cramer-Chernov result for a variable $Z(d)$ with chi-2 distribution with d degrees of freedom:

$$\mathbf{P}(Z(d) \ge z) \le \exp\Big(-\frac{2z}{5} + \frac{d\ln(5)}{2}\Big).$$

After some calculations we obtain:

$$\Pi^5_{N,x} \le 3 \times 10^{-11} \exp(-x/21).$$

Control of $T^1_{A,x}$. – Using $\sum_{j=A+1}^{\infty} (j-A) 2^{A-j+1} = 4$ and Remark 2 we get:

$$T^1_{A,x} \le \frac{A-9}{0.8636 \times C_0 \log(2^A)} \frac{(2^A)^2}{2^9} \sup_{L_0+1\le i\le N} \sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} \sum_{j=A+1}^{N} P(|\tilde{\Delta}^{f(i,L)}_{g(j,K)}| \ge z(j-A))$$

where $z = (3/16)(1-\alpha)\beta(\ln 2)(x + C_0 \log(2^A))$. Thus Lemma 3.3 *(2)* gives the result for C_0 large enough.

Control of $T^2_{A,x}$. – The first part of the construction is the same as the construction of Komlós, Major and Tusnády (1975-Theorem 3), Mason and Van Zwet (1987) or Bretagnolle and Massart (1989). Thus we apply Theorem 2.1. Let μ be defined by $\mu = (1-\alpha)(1-\beta) \times 13 \ln 2$. For $C_0 \geq C_1/\mu$ we obtain:

$$T^2_{A,x} \leq \Lambda_1 \exp(-\lambda_1 x)$$

where $C_1, \Lambda_1, \lambda_1$ are the constants of Theorem 2.1 *(i)*. For $C_0 \geq 42$ we obtain by Theorem 2.1 *(ii)*:

$$T^2_{N,x} \leq 10^{-3} \exp(-x/21).$$

Control of $T^3_{A,x}$. – Let us recall that $< W|\tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} >$ are independent Gaussian variables with expectation 0 and variance $2^{i+j-N}/16$. The term

$$\sum_{i=L_0+1}^{N-1} \sum_{j=L_0+A-N+1}^{M(i)} \tilde{c}_i^L \tilde{c}_j^K < W|\tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} >$$

is a Gaussian variable with expectation 0 and variance:

$$\sum_{i=L_0+1}^{N-1} \sum_{j=L_0+A-N+1}^{M(i)} (\tilde{c}_i^L \tilde{c}_j^K)^2 \frac{2^{i+j-N}}{16}.$$

This variance is bounded by $0.8636(A-10)(x + C_0 \log(2^A))/4$ (Remark 2). Using (3.15) it follows that:

$$\begin{aligned} T^3_{A,x} &\leq 2^{2A-17} \exp(-(0.03272)^2 (1 - \frac{\mathbb{I}_{A \leq N-1}}{2})^2 \times \frac{(A \ln 2)^2}{A-10} \\ &\quad \times \frac{2}{0.8636}(x + C_0 \log(2^A))) \\ &\leq \begin{cases} 2^{-17} \exp(-x/84) \text{ if } A \leq N-1 \text{ and } C_0 \geq 168 \\ 2^{-17} \exp(-x/21) \text{ if } A = N \text{ and } C_0 = 42. \end{cases} \end{aligned}$$

Control of $T^4_{A,x}$. – Let r be equal to $(1-\alpha)\beta(1-\gamma)(1-\delta)(1 - (\mathbb{I}_{A \leq N-1}/2)) \times (3/2)(\ln 2)(x + C_0 \log(2^A))$. We set $J_0 = L_0 + A - N$. Using (3.11), (3.15) and (3.16) we have:

$$\begin{aligned} T^4_{A,x} &\leq 2^{A-9} (\sum_{i=L_0+1}^{N} 2^{N-i}) \sup_{K \in \mathcal{K}} \sup_{L_0+1 \leq i \leq N} \sup_{0 \leq l \leq 2^{N-i}-1} \\ &P(|\sum_{j=J_0+1}^{M(i)} \tilde{c}_j^K < U_N|\tilde{e}_{i,l} \otimes \tilde{e}_{g(j,K)} > | \geq r). \end{aligned}$$

Let us denote by B, resp. C, the intervals $]g^*(M(i),K)2^{M(i)};K2^{J_0}]$ and $]K2^{J_0};(g^*(M(i),K)+1)2^{M(i)}]$. We have $\zeta_B+\zeta_C = e_{g(M(i),K)}$ (ζ is defined in Section 3.1). We prove below that:

$$\sum_{j=J_0+1}^{M(i)} \tilde{c}_j^K \tilde{e}_{g(j,K)} = \zeta_B - \frac{|B|}{2^N} e_{N,0} - \frac{|B|}{|B|+|C|}(\zeta_B+\zeta_C - \frac{|B|+|C|}{2^N} e_{N,0}).$$

Proof. – The expansion of ζ_B on $\tilde{\mathcal{B}}$ is:

$$\zeta_B = \sum_{j=J_0+1}^{M(i)} \tilde{c}_j^K \tilde{e}_{g(j,K)} + \sum_{j=M(i)+1}^{N} \frac{|B|}{2^{j-1}} \epsilon_j^K \tilde{e}_{g(j,K)} + \frac{|B|}{2^N} e_{N,0}$$

where

$$\epsilon_j^K = \begin{cases} 1 & \text{if }]g^*(M(i),K)2^{M(i)},(g^*(M(i),K)+1)2^{M(i)}] \\ & \subset]g^*(j,K)2^j;(2g^*(j,K)+1)2^{j-1}] \\ -1 & \text{if }]g^*(M(i),K)2^{M(i)},(g^*(M(i),K)+1)2^{M(i)}] \\ & \subset](2g^*(j,K)+1)2^{j-1};(g^*(j,K)+1)2^j]. \end{cases}$$

We conclude by expanding $e_{g(M(i),K)}$ on $\tilde{\mathcal{B}}$:

$$e_{g(M(i),K)} = \sum_{j=M(i)+1}^{N} \epsilon_j^K \frac{2^{M(i)}}{2^{j-1}} \tilde{e}_{g(j,K)} + \frac{2^{M(i)}}{2^N} e_{N,0}. \square$$

We set $I_1 = \vee(B,C)$ and $I_2 = \wedge(B,C)$. (We denote by $\vee$ (resp. $\wedge$) the interval with the largest (resp. smallest) length.) We have:

$$\begin{aligned} &|\sum_{j=J_0+1}^{M(i)} \tilde{c}_j^K < U_N|\tilde{e}_{i,l} \otimes \tilde{e}_{g(j,K)} > | \\ &\quad \leq 0.5 \times | < U_N|\tilde{e}_{i,l} \otimes (\zeta_{I_1} - |I_1|e_{N,0}) > | \\ &\quad\quad + | < U_N|\tilde{e}_{i,l} \otimes (\zeta_{I_2} - |I_2|e_{N,0}) > |. \end{aligned}$$

Using the definition of $\tilde{e}_{i,l}$, we have:

$$T_{A,x}^4 \leq 2 \times (2^{A-9})^2 \sup_{K\in\mathcal{K}} \sup_{L_0+1\leq i\leq N}$$
$$\left(P(|\tilde{\mathcal{B}}(2^{i-1}, \frac{|I_1|}{2^N})| \geq \frac{2r}{3}) + P(|\tilde{\mathcal{B}}(2^{i-1}, \frac{|I_2|}{2^N})| \geq \frac{2r}{3})\right)$$

where $\tilde{\mathcal{B}}(n,p)$ is the centered binomial $\mathcal{B}(n,p)-np$. We apply the following result due to Bennett (1962) and Wellner (1978):

PROPOSITION 3.9. – *Let Z be a binomial with mean m. Then, for any positive x and any sign ϵ, we have:*

$$\mathbf{P}(\epsilon(Z-m) \geq x) \leq \exp(-mh(\frac{x}{m}))$$

where the function h is the same as in Proposition 3.7.

Since the function $t^{-1}h(t)$ is increasing in t it follows that, according to remark 4:

$$T^4_{A,x} \leq 8 \times (2^{A-9})^2 \sup_{L_0+1 \leq i \leq N} \exp(-\frac{2r}{(9m/r)+2})$$

with $m = 2^{i-N-1}2^{M(i)}$. Remark 3 implies that $9m/r \leq 6 \times (2 - \mathbb{I}_{A=N}) \times 0.8636/((1-\alpha)\beta(1-\gamma)(1-\delta)(\ln 2))$. Thus we get:

$$T^4_{A,x} \leq \begin{cases} 2^{-15}\exp(-x/84) \text{ if } A \leq N-1 \text{ and } C_0 \geq 168 \\ 2^{-15}\exp(-x/21) \text{ if } A = N \text{ and } C_0 = 42. \end{cases}$$

Control of $T^5_{A,x}$. – Let us remark that $C^{f(i,L)}_{g(A,K)} \sim \mathcal{B}(2^i, 2^{A-N}) - 2^{i+A-N}$. Thus by Remark 2, Inequality (3.11) and Proposition 3.9 we get:

$$T^5_{A,x} \leq \frac{2(A-9)}{0.8636 \times C_0 \log(2^A)} \frac{(2^A)^2}{2^9} \sup_{L_0+1 \leq i \leq N} \exp(-2^{i+A-N}h((\frac{z}{2^{i+A-N}})^{1/2}))$$

where $z = (1-\alpha)\beta\gamma \times (3/2)(\ln 2)(x + C_0 \log(2^A))/(16 \times 0.032332)$. Using $h(t) \geq (3/8)\inf(t,t^2)$ (Remark 4) and $2^{i+A-N} > 2 \times 0.8636(x + C_0 \log(2^A))$ we obtain the result for C_0 large enough.

3.4. Proof of Lemma 3.3

Proposition 3.9 directly provides the control of $(\Theta^{i,l}_{j,k}(\epsilon))^{\mathrm{C}}$.

*Proof of part (*1.a.*).* – By Proposition 3.1, given $(U^{i,l}_{j,k}U^{i,l}_{j-1,2k}, U^{i-1,2l}_{j,k})$, the law of $U^{i-1,2l}_{j-1,2k}$ is the law $\mathcal{H}(U^{i,l}_{j,k}, U^{i,l}_{j-1,2k}, U^{i-1,2l}_{j,k})$. The conditional expectation is:

$$m^{i,l}_{j,k} = \frac{U^{i,l}_{j-1,2k}U^{i-1,2l}_{j,k}}{U^{i,l}_{j,k}}$$

and the approximation of Lemma 2.5 of the conditional variance is:

$$(\sigma^{i,l}_{j,k})^2 = U^{i,l}_{j,k} \frac{U^{i,l}_{j-1,2k}}{U^{i,l}_{j,k}} \frac{U^{i,l}_{j-1,2k+1}}{U^{i,l}_{j,k}} \times \frac{U^{i-1,2l}_{j,k}}{U^{i,l}_{j,k}} \frac{U^{i-1,2l+1}_{j,k}}{U^{i,l}_{j,k}}.$$

We set

$$\delta_{j,k}^{i,l} = \frac{U_{j,k}^{i-1,2l} - U_{j,k}^{i-1,2l+1}}{U_{j,k}^{i,l}}$$
$$\delta_{j,k}'^{i,l} = \frac{U_{j-1,2k}^{i,2l} - U_{j-1,2k+1}^{i,l}}{U_{j,k}^{i,l}}.$$

It is easy to verify that for $\epsilon \leq 0.35$:

$$\sup_{\Theta_{j,k}^{i,l}(\epsilon)} |\delta_{j,k}^{i,l}\delta_{j,k}'^{i,l}| \leq (0.35)^2 \leq 1/8.$$

Moreover for $\epsilon \leq 0.35$ and $i+j-N \geq 8$ we get:

$$\inf_{\Theta_{j,k}^{i,l}(\epsilon)} (\sigma_{j,k}^{i,l})^2 = \inf_{\Theta_{j,k}^{i,l}(\epsilon)} \frac{U_{j,k}^{i,l}}{16}(1-(\delta_{j,k}^{i,l})^2)(1-(\delta_{j,k}'^{i,l})^2) \geq 4.5. \tag{3.18}$$

Thus we can apply the hypergeometric lemma (special case of Lemma 2.5). On the event $\Theta_{j,k}^{i,l}(\epsilon)$, $\epsilon \leq 0.35$, we have:

$$|U_{j-1,2k}^{i-1,2l} - m_{j,k}^{i,l} - \sigma_{j,k}^{i,l}\xi_{j,k}^{i,l}| \leq 3 + 0.41(\xi_{j,k}^{i,l})^2.$$

By definition of $U_{j,k}^{i,l}$ and of $\xi_{j,k}^{i,l}$ we have:

$$< U^N|\tilde{e}_{i,l}\otimes\tilde{e}_{j,k} > = \frac{1}{4}(U_{j-1,2k}^{i-1,2l} - U_{j-1,2k+1}^{i-1,2l} - U_{j-1,2k}^{i,2l+1} + U_{j-1,2k+1}^{i-1,2l+1})$$
$$< W^N|\tilde{e}_{i,l}\otimes\tilde{e}_{j,k} > = \xi_{j,k}^{i,l}(\frac{2^{i+j-N}}{16})^{1/2}.$$

We set:

$$A = < U^N|\tilde{e}_{i,l}\otimes e_{j,k} >$$
$$B = < U^N|e_{i,l}\otimes\tilde{e}_{j,k} >$$
$$C = U_{j,k}^{i,l} - 2^{i+j-N}.$$

We have:

$$< U^N - W^N|\tilde{e}_{i,l}\otimes\tilde{e}_{j,k} >= U_{j-1,2k}^{i-1,2l} - m_{j,k}^{i,l} - \sigma_{j,k}^{i,l}\xi_{j,k}^{i,l} + \frac{AB}{U_{j,k}^{i,l}} + E$$

where E is the following error term:

$$E = \xi_{j,k}^{i,l}(2^{(i+j-N-4)/2} - \sigma_{j,k}^{i,l}).$$

Since $U^{i,l}_{j,k} \geq (1-\epsilon)2^{i+j-N}$ we get:

$$| < U^N - W^N | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > | \leq 3 + 0.41(\xi^{i,l}_{j,k})^2 + \frac{2^{-(i+j-N)}}{2(1-\epsilon)}(A^2+B^2) + |E|.$$

Control of E – Using relation (3.18) we get:

$$\begin{aligned} |E| &\leq \frac{4}{\tau} 2^{-(i+j-N)/2} |\xi^{i,l}_{j,k}| \, |(\sigma^{i,l}_{j,k})^2 - \frac{2^{i+j-N}}{16}| \\ &\leq \frac{2^{-(i+j-N)/2}}{4\tau} |\xi^{i,l}_{j,k}| \, (2\epsilon(|A|+|B|) + |C|) \end{aligned}$$

with $\tau = 1 + (1+\epsilon)(1-\epsilon)^{3/2}$. Using the inequality $|xy| \leq 0.5((\lambda x)^2 + (y/\lambda)^2)$ we obtain finally:

$$|E| \leq \frac{a^{-2} + b^{-2} + c^{-2}}{2} |\xi^{i,l}_{j,k}|^2 + \frac{2^{-(i+j-N)}}{8\tau^2}(\epsilon^2 a^2 A^2 + \epsilon^2 b^2 B^2 + \frac{c^2}{4} C^2).$$

We choose $a^2 = 163.5, b^2 = 19.9874, c^2 = 3.0163$ and the proof of part (*1.a.*) is complete.

Proof of part (1.b.). – Conditionally to $\sigma\{U^{i,l}_{J,k};\ J \geq j;\ 0 \leq k \leq 2^{N-J}-1\}$, the variables $U^{i,l}_{j-1,2k};\ k = 0, \ldots, 2^{N-j}-1$ are independent with distribution $\mathcal{B}(U^{i,l}_{j,k}, 1/2)$. Tusnady's Lemma (Lemma 2.4) and Skorohod's Theorem (1976) provide the existence of a sequence of independent standard normal variables $(\xi^{*i,l}_{j,k})$, $j = 1, \ldots, N$, $k = 0, \ldots, 2^{N-j}-1$ such that:

$$| < U^N | e_{i,l} \otimes \tilde{e}_{j,k} > | \leq 1 + \sqrt{\frac{U^{i,l}_{j,k}}{4}} \, |\xi^{*i,l}_{j,k}|.$$

Since $U^{i,l}_{j,k} \leq (1+\epsilon)2^{i+j-N}$ the first inequality of part (*1.b.*) holds. Conditionally to $\sigma\{U^{I,l}_{j,k};\ I \geq i;\ 0 \leq l \leq 2^{N-I}-1\}$, the variables $U^{i-1,2l}_{j,k}$ are independent with distribution $\mathcal{H}(2^i, U^{i,l}_{j,k}, 2^{i-1})$. The hypergometric lemma, and more precisely the special case *1.* of Lemma 2.5, and Skorohod's Theorem (1976) provide the existence of a sequence of independent standard normal variables $(\xi^{**i,l}_{j,k})$, $j = 1, \ldots, N$, $k = 0, \ldots, 2^{N-j}-1$ such that for $2^{i+j-N}(1-\epsilon)^2/8 \geq 4.5$ we get:

$$| < U^N | \tilde{e}_{i,l} \otimes e_{j,k} > | \leq 3 + \sqrt{s^{i,l}_{j,k}} |\xi^{**i,l}_{j,k}|,$$

where $s^{i,l}_{j,k}$ is the approximation of Lemma 2.5 of the conditional variance:

$$s^{i,l}_{j,k} = \frac{2^i}{4} \frac{U^{i,l}_{j,k}}{2^i} \frac{2^i - U^{i,l}_{j,k}}{2^i}.$$

Indeed if $j = N$ then $< U^N|\tilde{e}_{i,l} \otimes e_{j,k} >= 0$ and if $j \leq N-1$ and $2^{i+j-N}(1-\epsilon)^2/8 \geq 4.5$ then $s_{j,k}^{i,l} \geq U_{j,k}^{i,l}(1-\epsilon)/8 \geq 4.5$.

As $U_{j,k}^{i,l} \leq (1+\epsilon)2^{i+j-N}$ the second inequality of part (*1.b.*) holds.

Proof of part (2). – Let $P_{ij} := \mathbf{P}(| < U^N - W^N|\tilde{e}_{i,l} \otimes \tilde{e}_{j,k} > | \geq t)$. Suppose $t \geq t_0$ (otherwise take $(D,\mu) = (\exp(t_0), 1)$). If $t \leq 2^{i+j-N}$, we use part (*1.*). We obtain:

$$P_{ij} \leq \mathbf{P}((\Theta_{j,k}^{i,l}(0.35))^{\mathrm{C}}) + \mathbf{P}(|\xi| \geq \alpha\sqrt{t})$$
$$+6\mathbf{P}(|Z - 2^{i+j-N-1}| \geq \beta 2^{(i+j-N)/2}\sqrt{t})$$

where α and β are positive constants, $\xi \sim \mathcal{N}(0,1)$ and where Z is a binomial with mean $2^{i+j-N-1}$. Using Cramer-Chernov's result for normal and binomial distribution (Proposition 3.9) and Remark 4 we obtain the result. If $t > 2^{i+j-N}$, we use the bound (3.10) and relations (3.19). It follows that:

$$P_{ij} \leq \mathbf{P}(|\xi| \geq \alpha' t 2^{-(i+j-N-4)/2}) + 4\mathbf{P}(|Z - 2^{i+j-N-2}| \geq \beta' t)$$

where α' and β' are positive constants, $\xi \sim \mathcal{N}(0,1)$ and where Z is a binomial with mean $2^{i+j-N-2}$. We conclude as previously.

4. PROOF OF THEOREM 2.3

The proof of this theorem is analogous to the proof of Theorem 2.2. Thus we only give the general scheme.

4.1. Identification of $D^{(n)}$

We suppose $2^N + 1 \leq n \leq 2^{N+1}$. The Gaussian process used in the construction is now a bidimensional Wiener process $\{W(s,t); 0 \leq s \leq 1, 0 \leq t \leq 1\}$. (Recall that W is a continuous Gaussian process such that $E(W(s,t)) = 0$ and $E(W(s,t)W(s',t')) = (s \wedge s')(t \wedge t')$). For all functions $f(s,t)$, we set $f(]a,b]\times]c,d]) = f(a,c) - f(a,d) - f(b,c) + f(b,d)$. We define the $\mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ vector $W_N^{(n)}$ by:

$$W_N^{(n)} = ((\sqrt{n}W(]\frac{l}{2^N}, \frac{l+1}{2^N}]\times]\frac{k}{2^N}, \frac{k+1}{2^N}]); \ 1 \leq k \leq 2^N); \ 1 \leq l \leq 2^N).$$

Using $W_N^{(n)}$, we can construct a vector $U_N^{(n)} \in \mathbb{R}^{2^N} \otimes \mathbb{R}^{2^N}$ with the same distribution as $A_N^{(n)}$ where $A_N^{(n)}$ is defined by:

$$A_N^{(n)} = ((n\hat{G}_n(]\frac{l}{2^N}, \frac{l+1}{2^N}]\times]\frac{k}{2^N}, \frac{k+1}{2^N}]); \ 1 \leq k \leq 2^N); \ 1 \leq l \leq 2^N)$$

i.e. with the multinomial distribution $\mathcal{M}_{2^N\times 2^N}(n, 2^{-N}\times 2^{-N}, \ldots, 2^{-N}\times 2^{-N})$. We set $U^{i,l}_{j,k} := < U^{(n)}_N | e_{i,l}\otimes e_{j,k} >$. The variables used for the construction are $< W^{(n)}_N|\tilde{e}_{i,l}\otimes e_{N,0} >, < W^{(n)}_N|e_{N,0}\otimes\tilde{e}_{j,k} >, < W^{(n)}_N|\tilde{e}_{i,l}\otimes\tilde{e}_{j,k} >$ which are Gaussian variables with mean 0 and respective variance $\lambda 2^{i-2}, \lambda 2^{j-2}$ and $\lambda 2^{i+j-N-4}$, where $\lambda = n2^{-N}$. As $\tilde{\mathcal{B}}\otimes\tilde{\mathcal{B}}$ is an orthogonal basis of $\mathbb{R}^{2^N}\otimes\mathbb{R}^{2^N}$, all these variables ($1\le i,j\le N,\ 0\le l\le 2^{N-i}-1,\ 0\le k\le 2^{N-j}-1$) are independent. The construction of the vector $U^{(n)}_N$ is not fully the same as the construction of the vector U^N in Section 3.1 ($A^{(n)}_N$ and A^N do not have the same distribution). However, the description of the first step $i = N$ (first line of the array (3.9)) remains the same:

$$\begin{cases} U^{N,0}_{N,0} = n \\ U^{N,0}_{j-1,2k} = \Psi^{-1}_{U^{N,0}_{j,k}}\circ\Phi((\lambda 2^{j-2})^{-1/2} < W^{(n)}_N|e_{N,0}\otimes\tilde{e}_{j,k} >) \\ U^{N,0}_{j-1,2k+1} = U^{N,0}_{j,k} - U^{N,0}_{j-1,2k} \\ 1\le j\le N \quad 0\le k\le 2^{N-j}-1 \end{cases}$$

but for $i-1\in\{N-1, N-2, \ldots, 0\}$ the description of the steps $j = N$ (first column of the array (3.9)) becomes:

$$\begin{cases} U^{i-1,2l}_{N,0} = \Psi^{-1}_{U^{i,l}_{N,0}}\circ\Phi((\lambda 2^{i-2})^{-1/2} < W^{(n)}_N|\tilde{e}_{i,l}\otimes e_{N,0} >) \\ U^{i-1,2l+1}_{N,0} = U^{i,l}_{N,0} - U^{i-1,2l}_{N,0} \\ 0\le l\le 2^{N-i}-1 \end{cases}$$

and remains the same for $j-1\in\{N-1, N-2, \ldots, 0\}$ (array (3.9) without the first line and the first column):

$$\begin{cases} U^{i-1,2l}_{j-1,2k} = \Phi^{-1}_{U^{i,l}_{j,k}, U^{i,l}_{j-1,2k}, U^{i-1,2l}_{j,k}}\circ\Phi((\lambda 2^{i+j-N-4})^{-1/2} < W^{(n)}_N|\tilde{e}_{i,l}\otimes\tilde{e}_{j,k} >) \\ U^{i-1,2l}_{j-1,2k+1} = U^{i-1,2l}_{j,k} - U^{i-1,2l}_{j-1,2k} \\ U^{i-1,2l+1}_{j-1,2k} = U^{i,l}_{j-1,2k} > -U^{i-1,2l}_{j-1,2k} \\ U^{i-1,2l+1}_{j-1,2k+1} = U^{i,l}_{j-1,2k+1} - U^{i-1,2l+1}_{j-1,2k+1} \\ 0\le l\le 2^{N-i}-1 \quad 0\le k\le 2^{N-j}-1. \end{cases}$$

The vector $U_N^{(n)} = ((U_{0,k}^{0,l};\ 0 \le k \le 2^N - 1);\ 0 \le l \le 2^N - 1)$ has the same distribution as the vector $A_N^{(n)}$ (see Proposition 3.2). The construction of $U_{j,k}^{i,l}$ for $j \neq N$ remains the same as in Section 3.1. Thus, Inequalities *(1.a.)* and *(2.)* of Lemma 3.3 (about the behaviour of $< U_N^{(n)} - W_N^{(n)} | \tilde{e}_{i,l} \otimes \tilde{e}_{j,k} >$) still hold. The first inequality of part *(1.b.)* remains the same and the second inequality of part *(1.b.)* becomes of the same type as the first inequality. Indeed, conditionally to $\sigma\{U_{j,k}^{I,l},\ I \ge i,\ 0 \le l \le 2^{N-I} - 1\}$, the variables $U_{j,k}^{i-1,2l}$ are independent with distribution $\mathcal{B}(U_{j,k}^{i,l}, 1/2)$.

By Skohorod's Theorem (1976) there exists a bidimensional Wiener process $\{\tilde{W}^{(n)}(s,t),\ 0 \le s \le 1,\ 0 \le t \le 1\}$ such that $(A_N^{(n)}, \tilde{W}^{(n)})$ has the same distribution as $(U_N^{(n)}, W)$. We set $D^{(n)}(s,t) = \tilde{W}^{(n)}(s,t) - st\tilde{W}^{(n)}(1,1)$.

4.2. Proof of Inequality (2.5)

To obtain Inequality (2.5) of Theorem 2.3 we have to prove the existence of positive constants Λ_3, λ_3 such that

$$\sup_{(a,b)\in[0,1]^2} \mathcal{P}_n(a,b) \le \Lambda_3 \exp(-\lambda_3 x)$$

where $\mathcal{P}_n(a,b)$ is the following probability:

$$\begin{aligned}\mathcal{P}_n(a,b) =& \mathbf{P}(\sup_{0\le s\le b}\ \sup_{0\le t\le a} |n(\hat{G}_n(s,t) - st) - \sqrt{n}D^{(n)}(s,t)| \\ & \ge (x + C_3 \log(nab)) \log(nab)).\end{aligned}$$

If $x + C_3 \log(nab) > (nab)/12$, then the bound (3.10) and Propositions 4.1, 4.2 below give Inequality (2.5) for C_3 large enough.

PROPOSITION 4.1. – *a) For all $a, b \in [0,1]$ such that $0 \le ab \le 1/2$ we have:*

$$\mathbf{P}(\sup_{(s,t)\in[0,b]\times[0,a]} |n(\hat{G}_n(s,t) - st)| \ge x) \le 2e \exp(-nab(1-ab)h(\frac{x}{nab}))$$

where the function h is defined in Proposition 3.7.

b) There exists an universal positive constant C such that:

$$\mathbf{P}(\sup_{(s,t)\in[0,1]\times[0,1]} |\sqrt{n}(\hat{G}_n(s,t) - st)| \ge x) \le Cx^2 \exp(-2x^2).$$

PROPOSITION 4.2. – *Let $D(s,t) = W(s,t) - stW(1,1)$ where W is a bidimensional Wiener process.*

a) For all $a, b \in [0,1]$ *such that* $0 \le ab \le 1/2$ *we have:*

$$\mathbf{P}(\sup_{(s,t)\in[0,b]\times[0,a]} |D(s,t)| \ge x) \le 2e\exp(-\frac{x^2(1-ab)}{2ab}).$$

b) There exists an universal positive constant C *such that:*

$$\mathbf{P}(\sup_{(s,t)\in[0,1]\times[0,1]} |D(s,t)| \ge x) \le Cx^2\exp(-2x^2)$$

The proof of the part a) of these propositions is the same as the proof of Proposition 3.5. The part b) of Proposition 4.1 is due to Talagrand (1994) and the part b) of Proposition 4.2 is due to Adler and Brown (1986).

If $x + C_3\log(nab) \le (nab)/12$, we impose $C_3 \ge 42$ and we have $nab \ge 4206$. There exists $A, B \in \mathbb{N}$, $12 \le A, B \le N$ such that $2^{A-N-1} < a \le 2^{A-N}$ and $2^{B-N-1} < b \le 2^{B-N}$. Remark that $A + B - N \ge 12$. So it is enough to prove the existence of positive constants D, μ such that

$$\sup_{12\le A,B\le N} \Pi_{A,B,x} \le D\exp(-\mu x)$$

where

$$\Pi_{A,B,x} = \mathbf{P}(\sup_{(s,t)\in[0,2^{B-N}]\times[0,2^{A-N}]} |n(\hat{G}_n(s,t) - st) - \sqrt{n}D^{(n)}(s,t)| \ge \tau_{A,B,x})$$

$$\tau_{A,B,x} = \frac{25}{36}(x + C_3\log(2^{A+B-N}))\log(2^{A+B-N}).$$

We set:

$$\Delta(s,t) = n(\hat{G}_n(s,t) - st) - \sqrt{n}D^{(n)}(s,t).$$

Let $L_1 := L_1(A,B,x)$ and $L_2 := L_2(A,B,x)$ be the integers such that:

$$2^{L_1+A-N} \le 0.8636(x + C_3\log(2^{A+B-N})) < 2^{L_1+A-N+1}$$
$$2^{L_2+B-N} \le 0.8636(x + C_3\log(2^{A+B-N})) < 2^{L_2+B-N+1}.$$

Remark 5. – $N - A + 8 \le L_1 \le B - 3$ *and* $N - B + 8 \le L_2 \le A - 3$

We set $\mathcal{L} = [1, 2^{B-L_1}]$ and $\mathcal{K} = [1, 2^{A-L_2}]$. Let us denote by $\pi_1(s)$ the left projection of s on $\{L2^{L_1-N}; L \in \mathcal{L}\}$ and $\pi_2(t)$ the left projection of t on $\{K2^{L_2-N}; K \in \mathcal{K}\}$. We write:

$$\begin{aligned}\Delta(s,t) =&\Delta(s,t) - \Delta(\pi_1(s),t) + \Delta(\pi_1(s),t)\\ &- \Delta(\pi_1(s),\pi_2(t)) + \Delta(\pi_1(s),\pi_2(t)).\end{aligned}$$

Using the stationarity of the increments of the unidimensional processes $n(\hat{G}_n(.,t) - .t)$, $n(\hat{G}_n(s,.) - s.)$, $\sqrt{n}D(.,t)$, $\sqrt{n}D(s,.)$ and part $a)$ of Propositions 4.1 and 4.2 (see also Section 3.3.2 **II**: Control of $\Pi^1_{A,x}$), we obtain for C_3 large enough:

$$\Pi_{A,B,x} \leq D_1 \exp(-\mu_1 x) + \Pi^2_{A,B,x}$$

where

$$\Pi^2_{A,B,x} = P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\Delta(L2^{L_1-N}, K2^{L_2-N})| \geq \frac{\tau_{A,B,x}}{2}).$$

By definition of $D^{(n)}$ we get:

$$\Pi^2_{A,B,x} = P(\sup_{L\in\mathcal{L}} \sup_{K\in\mathcal{K}} |\bar{\Delta}(L2^{L_1-N}, K2^{L_2-N})| \geq \frac{\tau_{A,B,x}}{2})$$

where

$$\begin{aligned}\bar{\Delta}(L2^{L_1-N}, K2^{L_2-N}) = &< U_N^{(n)} - W_N^{(n)} | \zeta_{]0,L2^{L_1}]} \otimes \zeta_{]0,K2^{L_2}]} > \\ &- L2^{L_1-N} K2^{L_2-N} < U_N^{(n)} - W_N^{(n)} | e_{N,0} \otimes e_{N,0} > .\end{aligned}$$

The term $\bar{\Delta}(L2^{L_1-N}, K2^{L_2-N})$ may be written:

$$\begin{aligned}&< U_N^{(n)} - W_N^{(n)} | \zeta_{]0,L2^{L_1}]} \otimes (\zeta_{]0,K2^{L_2}]} - K2^{L_2-N} e_{N,0}) > \\ &+ K2^{L_2-N} < U_N^{(n)} - W_N^{(n)} | (\zeta_{]0,L2^{L_1}]} - L2^{L_1-N} e_{N,0}) \otimes e_{N,0} > .\end{aligned}$$

The expansion of $\zeta_{]0,L2^{L_1}]}$ and $\zeta_{]0,K2^{L_2}]}$ on $\tilde{\mathcal{B}}$ gives:

$$\Pi^2_{A,B,x} \leq Q_{A,B,x} + \tilde{T}^2_{A,B,x} + \tilde{\tilde{T}}^2_{A,B,x}$$

where

$$\begin{aligned}Q_{A,B,x} = &P(\sup_{L\in\mathcal{L}, K\in\mathcal{K}} | \sum_{i=L_1+1}^{N} \sum_{j=L_2+1}^{N} \tilde{c}_i^L \tilde{c}_j^K \\ &< U_N^{(n)} - W_N^{(n)} | \tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \geq \frac{\tau_{A,B,x}}{6}) \\ \tilde{T}^2_{A,B,x} = &P(\sup_{K\in\mathcal{K}} | < U_N^{(n)} - W_N^{(n)} | e_{N,0} \otimes (\zeta_{]0,K2^{L_2}]} - K2^{L_2-N} e_{N,0}) > | \\ &\geq \frac{\tau_{A,B,x}}{6} \times 2^{N-B}) \\ \tilde{\tilde{T}}^2_{A,B,x} = &P(\sup_{L\in\mathcal{L}} | < U_N^{(n)} - W_N^{(n)} | (\zeta_{]0,L2^{L_1}]} - L2^{L_1-N} e_{N,0}) \otimes | e_{N,0} > | \\ &\geq \frac{\tau_{A,B,x}}{6} \times 2^{N-A}).\end{aligned}$$

The terms $\tilde{T}^2_{A,B,x}$ and $\tilde{\tilde{T}}^2_{A,B,x}$ are analogous to the term $T^2_{A,x}$ in the proof of Proposition 3.6. By construction, the processes $n(\hat{G}_n(1,t) - t)$ and $n(\hat{G}_n(s,1) - s)$ are such that Theorem 2.1 holds. Using the inequality $2^{N-B}\log(2^{A+B-N}) \geq (\log(n2^{A-N}))/2$ and the analogous for $\tilde{\tilde{T}}^2_{A,B,x}$ we obtain:

$$\Pi^2_{A,B,x} \leq D_2 \exp(-\mu_2 x) + Q_{A,B,x}.$$

Moreover

$$\begin{aligned} Q_{A,B,x} &\leq R^{(I,J)}_{A,B,x} + \mathbb{I}_{B\leq N-1} R^{(I^{\complement},J)}_{A,B,x} \\ &+ \mathbb{I}_{A\leq N-1} R^{(I,J^{\complement})}_{A,B,x} + \mathbb{I}_{A\leq N-1}\mathbb{I}_{B\leq N-1} R^{(I^{\complement},J^{\complement})}_{A,B,x} \end{aligned}$$

where $I = \{L_1+1,\ldots,B\}$, $J = \{L_2+1,\ldots,A\}$, where

$$\begin{aligned} R^{(I,J)}_{A,B,x} = P(\sup_{L\in\mathcal{L},K\in\mathcal{K}} |\sum_{i\in I}\sum_{j\in J} \tilde{c}^L_i \tilde{c}^K_j \\ < U^{(n)}_N - W^{(n)}_N | \tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \geq \frac{\tau_{A,B,x}}{24}) \end{aligned}$$

and where the others terms are defined in the same way. We treat the term $R^{(I,J)}_{A,B,x}$ in the same way as the term $\Pi^3_{A,x}$ in the proof of Proposition 3.6. The terms $R^{(I^{\complement},J)}_{A,B,x}$ and $R^{(I,J^{\complement})}_{A,B,x}$ are analogous to the term $T^1_{A,x}$ in the proof of Proposition 3.6. The term $R^{(I^{\complement},J^{\complement})}_{A,B,x}$ is very similar: using $\sum_{i=B+1}^{\infty}(i-B)2^{B-i+1} = 4$ (and the same for j) we obtain:

$$\begin{aligned} R^{(I^{\complement},J^{\complement})}_{A,B,x} \leq 2^{B-L_1+A-L_2} \sup_{L\in\mathcal{L},K\in\mathcal{K}} \sum_{i=B+1}^{N}\sum_{j=A+1}^{N} \\ P(| < U^{(n)}_N - W^{(n)}_N | \tilde{e}_{f(i,L)} \otimes \tilde{e}_{g(j,K)} > | \geq \frac{(i-A)(i-B)\tau_{A,B,x}}{24\times 16}). \end{aligned}$$

We conclude using Lemma 3.3 *(2.)* for C_3 large enough.

Appendix : proof of Lemma 2.5

A. PRELIMINARIES

We recall the definition and some properties of hypergeometric distributions. Let $\mathcal{E}$ be a set with cardinality n and let $\mathcal{A}$ and $\mathcal{B}$ be two independent subsets of $\mathcal{E}$, with respectively the uniform distribution on the subsets with cardinality n_1 and n_2. The hypergeometric variable $X \sim \mathcal{H}(n, n_1, n_2)$ is the number of elements of $\mathcal{E}$ which belongs to

$\mathcal{A} \cap \mathcal{B}$. We define the variables U, V, W as the number of elements of $\mathcal{A} \cap \mathcal{B}^c$, $\mathcal{A}^c \cap \mathcal{B}$, $\mathcal{A}^c \cap \mathcal{B}^c$:

U	W	$n - n_2$
X	V	n_2
n_1	$n - n_1$	n

$$\mathbf{P}(X = x) = \frac{n_1!\,(n - n_1)!n_2!\,(n - n_2)!}{n!x!\,u!\,v!\,w!}. \tag{A.1}$$

The conditions $0 \le x \le n_1 \wedge n_2$, $0 \le u \le n_1 \wedge (n - n_2)$, $0 \le v \le (n - n_1) \wedge n_2$, $0 \le w \le (n - n_1) \wedge (n - n_2)$ give for x:

$$0 \vee (n_1 + n_2 - n) \le x \le n_1 \wedge n_2.$$

We recall the notations of Lemma 2.5: $p = n_1/n$, $p' = n_2/n$, $p + q = 1$, $p' + q' = 1$, $\delta = p - q$, and $\delta' = p' - q'$. We suppose $|\delta\delta'| \le 1 - \eta$, where $\eta > 0$. We have $E(X) = npp'$, $V(X) = (n/(n-1))\sigma^2$ where $\sigma^2 = npp'qq'$. The hypergeometric variable of Lemma 2.5 is defined, using a standard normal variable Y, by

$$X = \Phi^{-1}_{n,np,np'} \circ \Phi(Y)$$

where $\Phi_{n,np,np'}$ and Φ are the cumulative distribution functions of the hypergeometric distribution $\mathcal{H}(n, np, np')$ and of the standard normal distribution. We study the centered variable $X^0 = X - npp'$ which takes its values on $[(-npp') \vee (-nqq'); np'q \wedge npq'] \cap \mathbb{N} - npp'$. We have to prove the two inequalities below

$$|X^0| \le a + \sigma|Y| + b|\delta\delta'|Y^2 \tag{A.2}$$
$$|X^0 - \sigma Y| \le c + dY^2 \tag{A.3}$$

and need to obtain explicit constants when $|\delta\delta'| \le 1/8$. First we remark that $np'q \wedge npq', npp' \wedge nqq' \le 4\sigma^2/(1 - |\delta\delta'|)$. Then, if $\sigma^2 \le \sigma_0^2$, Inequalities (A.2) and (A.3) hold with $a = 4\sigma_0^2/\eta$, $b = 0$, $c = (4\sigma_0^2/\eta) + 0.5$, and $d = 0.5$. Then, to prove Lemma 2.5, it is enough to establish (A.2) and (A.3) for $\sigma^2 \ge \sigma_0^2$ and obtain explicit constants a, b, c, d when $\sigma_0^2 = 4.5$ and

$|\delta\delta'| \leq 1/8$. Properties of X (see (A.1)) involve $X^0 + U^0 = 0$. Since $U = \Phi^{-1}_{n,np,nq'} \circ \Phi(-Y)$, we have to prove only that

$$X^0 \leq a + \sigma Y + b|\delta\delta'|Y^2 \tag{A.4}$$

$$\sigma Y - c - dY^2 \leq X^0 \leq \sigma Y + c + dY^2 \tag{A.5}$$

for $X^0 \geq 0$. It is clear that the second part of (A.5) follows from (A.4) with $(c, d) = (a, b)$. We do not prove (A.4) directly. First, we choose a constant α such that:

$$X^0 \leq \begin{cases} \sigma Y + a & \text{if } \delta\delta' \leq 0 \\ \max(\sigma Y + a, \sigma Y + \alpha + \frac{\delta\delta'}{2}(\frac{X^0}{\sigma})^2) & \text{if } \delta\delta' > 0 \end{cases} \tag{A.6}$$

with $\alpha = 1.835$ when $|\delta\delta'| \leq 1/8$ and $\sigma_0^2 = 4.5$. Below, we prove that (A.4) can be deduced from (A.6).

Proof. – When $\sigma Y + a < \sigma Y + \alpha + (\delta\delta'/2)(X^0/\sigma)^2$ and $\delta\delta' > 0$ we obtain:

$$\sigma Y + \alpha \geq X^0(1 - \frac{\delta\delta'}{2}\frac{X^0}{\sigma^2}) \text{ from (A.6) .}$$

In the case $0 < \delta\delta' \leq 1/8$ the relation $X^0 \leq 4\sigma^2/(1 - \delta\delta')$ gives $\sigma Y + \alpha \geq 5X^0/7$ and (A.6) becomes:

$$X^0 \leq \alpha + \delta\delta'(\frac{7\alpha}{5\sigma})^2 + \sigma Y + (\frac{7}{5})^2\delta\delta' Y^2.$$

Notice that $a = 3$, $\alpha = 1.835$ and $\sigma_0^2 = 4.5$ give the good constants.

In the case $\delta\delta' > 1/8$ we apply the same method when $X^0 \leq \sigma^2/2$ (we impose σ_0^2 large enough). For $X^0 > \sigma^2/2$ we remark that Y is an increasing function of X^0. Then (A.6) gives $Y \geq \lambda\sigma$ where λ is a constant. Using $X^0 \leq 4\sigma^2/(1 - \delta\delta')$, we have $(X^0/\sigma)^2 \leq 4\lambda^2 Y^2/\eta^2$. □

We still have to prove that

$$\sigma Y - c - dY^2 \leq X^0 \leq \begin{cases} \sigma Y + a & \text{if } \delta\delta' \leq 0 \\ \max(\sigma Y + a, \sigma Y + \alpha + \frac{\delta\delta'}{2}(\frac{X^0}{\sigma})^2) & \text{if } \delta\delta' > 0 \end{cases}$$

for $X^0 \geq 0$. We refer to the quantile transformation method (Csörgö and Revesz (1981), pages 133-134). With this method we obtain the inequalities we need from the probability inequalities. Then it is enough to show the following lemma:

LEMMA A.1. – *We suppose* $|\delta\delta'| \le 1 - \eta$ *for some positive* η. *Let* ξ *be in* $\{\xi = k - npp'; k \in \mathbb{N}; 0 \le \xi \le \min(np'q, npq')\}$. *There exists a value* $\sigma_0^2(\eta) > 0$ *and some positive constants* a, α, c, d *which depend only on* η *such that for* $\sigma^2 \ge \sigma_0^2(\eta)$ *we have:*

$$\mathbf{P}(\sigma Y - c - dY^2 \ge \xi - 1) \le \mathbf{P}(X^0 \ge \xi) \tag{A.7}$$

$$\mathbf{P}(X^0 \ge \xi) \le \begin{cases} \mathbf{P}(\sigma Y \ge \xi - a) & \text{if } \delta\delta' \le 0 \\ \max(\mathbf{P}(\sigma Y \ge \xi - a), \mathbf{P}(\sigma Y \ge \xi - \alpha - \frac{\delta\delta'}{2}(\frac{\xi}{\sigma})^2)) & \text{if } \delta\delta' > 0. \end{cases} \tag{A.8}$$

Moreover, one can take $a = c = 3, d = 0.41, \alpha = 1.835$ *when* $|\delta\delta'| \le 1/8$ *and* $\sigma^2 \ge 4.5$.

Remark. – Inequality (A.3) is sufficient to prove Theorems 2.2 and 2.3. The proof of Inequality (A.2) may be simplier in the case $\delta\delta' = 0$. But the proof of Inequality (A.3) (more precisely the "the right side" of (A.5)) needs "the right side" (A.4) of Inequality (A.2). Thus Lemma 2.5 cannot be deduced from Lemma 2.4.

B. SOME TECHNICAL LEMMAS AND THEIR COROLLARIES

LEMMA B.1. – *(Bounds for* $\ln \mathbf{P}(X^0 = \xi)$.*) Let* $f(\xi) := \ln(\sqrt{2\pi}\sigma) + \ln \mathbf{P}(X^0 = \xi)$. *For* $0 \le \xi \le \lambda\sigma^2$, $0 < \lambda < 1$ *and* $\sigma^2 > 0$, *we have:*

$$f(\xi) = -\xi^2/2\sigma^2 - \xi\delta\delta'/2\sigma^2 + \xi^3\delta\delta'/6\sigma^4 + \rho(\xi)$$

with

$$\begin{cases} \rho(\xi) \ge -1/6(1-\lambda)\sigma^2 + \xi^2 S_2/4(1+\lambda)^2\sigma^4 - \xi^4 S_3/12(1-\lambda)^3\sigma^6 \\ \rho(\xi) \le \xi^2 S_2/4(1-\lambda)^2\sigma^4 - \xi^4 S_3/12(1+\lambda)^3\sigma^6 \end{cases}$$

where $S_k = (p^k + q^k)(p'^k + q'^k)$, $\delta = p - q$ *and* $\delta' = p' - q'$. *Moreover we have* $1/4 \le S_2 \le (1 + (\delta\delta')^2)/2$ *and* $1/16 \le S_3 \le (1 + 3(\delta\delta')^2)/4$.

Proof. – Set $\alpha_1 = p'q$, $\alpha_2 = pp'$, $\alpha_3 = pq'$, $\alpha_4 = qq'$ and, for $k \in \mathbf{N}^*$,

$$\gamma_k = \frac{k!}{(k/e)^k\sqrt{2\pi k}}$$

Our conditions $0 \leq \xi \leq \lambda\sigma^2$, $\lambda < 1$, and $\sigma^2 > 0$ yield that $n\alpha_i + (-1)^i\xi \in \mathbb{N}^*$ for $i = 1, \ldots, 4$. Then, we use Stirling's formula, and we obtain:

$$\mathbf{P}(X^0 = \xi) = CS(\xi, n, p, p') \times \frac{1}{\sqrt{2\pi}\sigma} \exp(-\alpha(\xi) - \beta(\xi))$$

where

$$CS(\xi, n, p, p') = \gamma_{np}\gamma_{nq}\gamma_{np'}\gamma_{nq'} / (\gamma_n \prod_{i=1}^{4} \gamma_{n\alpha_i + (-1)^i\xi})$$

$$\alpha(\xi) = \sum_{i=1}^{4} (n\alpha_i + (-1)^i\xi) \ \ln(1 \ + \ (-1)^i\xi/(n\alpha_i)\ )$$

$$\beta(\xi) = 1/2 \sum_{i=1}^{4} \ln(1 \ + \ (-1)^i\xi/(n\alpha_i)\ ).$$

Using the fact that γ_k is decreasing in k, that $1 \leq \gamma_k \leq 1 + (2/12k)$ and using inequalities: $np \leq n$, $npp' + \xi \leq np'$, $nqq' + \xi \leq nq'$, $np'q - \xi \leq nq$, and $npq' - \xi \leq np$, we get:

$$\prod_{i=1}^{4} \frac{1}{1 + \frac{2}{12(n\alpha_i + (-1)^i\xi)}} \leq CS(\xi, n, p, p') \leq 1.$$

Using inequalities $\ln(1+x) \leq x$ for all $x \geq 0$, $n\alpha_i + (-1)^i\xi \geq n\alpha_i - \lambda\sigma^2 \geq n\alpha_i(1 - \lambda)$ and $\sum_i 1/n\alpha_i = 1/\sigma^2$, we get:

$$-\frac{1}{6(1-\lambda)\sigma^2} \leq \ln CS(\xi, n, p, p') \leq 0.$$

We now expand $\alpha(\xi)$ and $\beta(\xi)$ using Taylor's formula of the fourth order and of the second order respectively:

$$\alpha(\xi) = \sum_{i=1}^{4} \frac{\xi^2}{2n\alpha_i} - \frac{(-1)^i\xi^3}{6(n\alpha_i)^2} + \frac{\xi^4}{12(n\alpha_i)^3} \times \frac{1}{(1 + \frac{(-1)^i\theta\xi}{n\alpha_i})^3}$$

$$\beta(\xi) = \sum_{i=1}^{4} \frac{(-1)^i\xi}{2n\alpha_i} - \frac{\xi^2}{4(n\alpha_i)^2} \times \frac{1}{(1 + \frac{(-1)^i\theta\xi}{n\alpha_i})^2}$$

with $0 < \theta < 1$. We obtain the result by using the following formulas:

$$\sum_{i=1}^{4} \frac{1}{n\alpha_i} = \frac{1}{\sigma^2} \ ; \ \sum_{i=1}^{4} \frac{(-1)^i}{n\alpha_i} = \frac{\delta\delta'}{\sigma^2} \ ; \ \sum_{i=1}^{4} \frac{(-1)^i}{(n\alpha_i)^2} = \frac{\delta\delta'}{\sigma^4}$$

and that, for $0 < \theta < 1$, $0 \leq \xi \leq \lambda\sigma^2$,

$$1 - \lambda \leq 1 + \frac{(-1)^i \theta \xi}{n\alpha_i} \leq 1 + \lambda.$$

LEMMA B.2. – *(Upper bound of* $r(\xi) = \mathbf{P}(X^0 = \xi + 1)/\mathbf{P}(X^0 = \xi)$.*) For all* $\xi \geq 0$ *we have:*

$$r(\xi) \leq \begin{cases} \exp(-\xi/\sigma^2) & \text{if } -1 \leq \delta\delta' \leq 0 \\ \exp(-\xi(1-\delta\delta')/2\sigma^2) & \\ \exp(-\xi/\sigma^2 \ + \ \xi^2\delta\delta'/2\sigma^4). & \end{cases}$$

Proof. –

$$r(\xi) = \frac{(np'q - \xi)(npq' - \xi)}{(npp' + \xi + 1)(nqq' + \xi + 1)} \leq \exp(\Delta)$$

where $\Delta = \ln(1 - \ \xi/np'q) + \ln(1 - \ \xi/npq') - \ln(1 + \ \xi/npp') - \ln(1 + \xi/nqq')$. Using inequalities $\ln(1-x) \leq -x - x^2/2$ and $-\ln(1+x) \leq -x + x^2/2$ for all $x \geq 0$ we obtain the first and the third inequalities of Lemma B.2. Moreover $\Delta \leq \ln(1 - (\xi/np'q)) + \ln(1 - (\xi/npq'))$. Since $p'q + pq' = (1 - \delta\delta')/2$, we obtain the second inequality of Lemma B.2.

Corollaries of Lemmas B.1 and B.2.

- Corollary 1: For all $\sigma^2 \geq 4$ and $0 \leq \xi \leq 1$, we have:

$$\ln(\sqrt{2\pi}\sigma\mathbf{P}(X^0 = \xi)) \leq \frac{1}{2\sigma^2}.$$

- Corollary 2: For all $\xi > 1$, we have:

$$\mathbf{P}(X^0 \geq \xi) \leq \frac{\exp(-\omega(\xi-1)/\sigma^2)}{1 - \exp(-\omega(\xi-1)/\sigma^2)} \mathbf{P}(X^0 = \xi - 1),$$

$$\text{with } \omega = \begin{cases} 1 & \text{if } -1 \leq \delta\delta' \leq 0 \\ (1-\delta\delta')/2 & \text{if } 0 < \delta\delta' \leq 1 \end{cases}.$$

- Corollary 3: Let $\phi(y) = \exp(-y^2/2)$. For all $\xi \geq 0$ and for all strictly positive real Λ, we define $\Lambda(\xi)$ by:

$$\Lambda(\xi) = \Lambda + \frac{\delta\delta'}{2}\left(\frac{\xi+1}{\sigma}\right)^2 \mathbb{I}_{\delta\delta'>0}.$$

Then, we have:

$$\frac{\mathbf{P}(X^0 = \xi + 1)}{\mathbf{P}(X^0 = \xi)} \leq \exp\left(\frac{-2\Lambda + 1}{2\sigma^2}\right) \frac{\phi((\xi + 1 - \Lambda(\xi))/\sigma)}{\phi((\xi - \Lambda(\xi))/\sigma)}.$$

Proof. – For Corollary 1 we use Lemma B.1 with $\lambda = 1/4$:

$$2\sigma^2 \ln(\sqrt{2\pi}\sigma \mathbf{P}(X^0 = \xi)) \leq f(\xi) := -\xi^2 - \delta\delta'\xi + \frac{\delta\delta'\xi^3}{3\sigma^2} + \frac{8\xi^2}{9\sigma^2}.$$

Since $f''(\xi) \leq 0$ for $\sigma^2 \geq 4$ and $0 \leq \xi \leq 1$, the function g defined by $g(\xi) = f(\xi) + \delta\delta'\xi$ is concave. Moreover $g(0) = g'(0) = 0$, thus $g(\xi) \leq 0$. Using the first and second inequalities of Lemma B.2, we obtain Corollary 2. Using the first inequality of Lemma B.2, in the case where $\delta\delta' \leq 0$, and the third inequality of Lemma B.2, in the case where $\delta\delta' > 0$, we obtain Corollary 3.

LEMMA B.3. – *(Lower bound for $r(\xi) = \mathbf{P}(X^0 = \xi + 1)/\mathbf{P}(X^0 = \xi)$.) If $|\delta\delta'| \leq 1 - \eta = \theta$ and $\xi + 1 \leq \lambda\sigma^2$ with $\lambda < 1$, then we have:*

$$\ln(r(\xi)) \geq -\lambda - \frac{(1+\theta)^2}{4}(-\ln(1-\lambda) - \lambda).$$

Proof. –

$$r(\xi) = \frac{(np'q - \xi)(npq' - \xi)}{(np'p + \xi + 1)(nqq' + \xi + 1)} \geq \frac{(np'q - (\xi + 1))(npq' - (\xi + 1))}{(np'p + \xi + 1)(nqq' + \xi + 1)}.$$

We set $\xi' = \xi + 1$, $u = \xi'/(np'q)$, $v = \xi'/(npq')$, $w = \xi'/(npp')$, $s = \xi'/(nqq')$ and we get: $\ln(r(\xi) \geq \ln(1-u)(1-v) - \ln(1+w)(1+s)$. Using $\ln(1+u) \leq u$ and using the fact that for all u satisfying $0 \leq u \leq \tilde{u} < 1$, we have $\ln(1-u) \geq -u - \alpha u^2$ with $\alpha = (-\ln(1-\tilde{u}) - \tilde{u})/\tilde{u}^2$ we obtain: $\ln(r(\xi)) \geq -(\xi'/\sigma^2) - \alpha(u^2 + v^2)$ with $\alpha = (-\ln(1-\lambda) - \lambda)/\lambda^2$ and $u^2 + v^2 = (\xi'^2/\sigma^4)((p'q)^2 + (pq')^2)$. Since $|\delta\delta'| \leq 1 - \eta = \theta$, we have $(p'q)^2 + (pq')^2 \leq (1+\theta)^2/4$ and we get the result.

C. PROOF OF INEQUALITY (A.8) OF LEMMA A.1

We define $\Delta(\xi)$ by:

$$\Delta(\xi) = \begin{cases} 2 & \text{if } \delta\delta' \leq 0 \\ 1.835 + (\delta\delta'\xi^2)/(2\sigma^2) & \text{if } 0 < \delta\delta' \leq 1/8 \\ 2.9 + (\delta\delta'\xi^2)/(2\sigma^2) & \text{if } \delta\delta' > 1/8. \end{cases}$$

Let $\xi^{(1)}$ be a value to be defined later, and let $\xi^{(2)} = \wedge\{k - npp'; k - npp' > (\sigma^2/2) - 1\}$. Then $(\sigma^2/2) - 1 < \xi^{(2)} \leq (\sigma^2/2)$. Since $\sigma^2 \geq 4.5$, the function $\xi - \Delta(\xi)$ is increasing on $[0; \xi^{(2)} + 1]$. Remark that $\xi - \Delta(\xi)$ is increasing for all ξ if $\delta\delta' \leq 1/8$ (we recall that $\xi \leq 4\sigma^2/(1 - |\delta\delta'|)$).

Scheme of the proof

1) For $0 \leq \xi \leq \xi^{(1)}$ we prove that

$$\mathbf{P}(\xi \leq X^0 \leq \xi^{(1)}) \leq \begin{cases} \mathbf{P}(\sigma Y \in [\xi - 3, \xi^{(1)} - 1] & \text{if } \delta\delta' \leq 1/8 \\ \mathbf{P}(\sigma Y \in [\xi - 4, \xi^{(1)} - 2] & \text{if } \delta\delta' > 1/8. \end{cases} \tag{C.9}$$

2) For $\xi^{(1)} < \xi \leq \xi^{(2)}$ we prove that

$$\mathbf{P}(X^0 = \xi) \leq \mathbf{P}(\sigma Y \in [\xi - \Delta(\xi); \xi + 1 - \Delta(\xi + 1)]), \tag{C.10}$$

and verify that

$$\xi^{(1)} + 1 - \Delta(\xi^{(1)} + 1) \geq \begin{cases} \xi^{(1)} - 1 & \text{if } \delta\delta' \leq 1/8 \\ \xi^{(1)} - 2 & \text{if } \delta\delta' > 1/8. \end{cases} \tag{C.11}$$

Let $\phi(y) = \exp(-y^2/2)$. Using $\mathbf{P}(Y \in [a, b])) \geq (b - a)(\sqrt{2\pi}\sigma)^{-1}\phi(y)$ we see that (C.10) is fulfilled as soon as $D(\xi) \geq 0$, with:

$$D(\xi) := g(\xi) + \ln(1 - \Delta(\xi + 1) + \Delta(\xi))$$

where

$$g(\xi) := \ln \phi((\xi + 1 - \Delta(\xi + 1))/\sigma) - f(\xi)$$

and f defined in Lemma B.1.

3) For $\xi > \xi^{(2)}$ we prove that $D(\xi) \geq 0$ when $\delta\delta' \leq 1/8$. When $\delta\delta' > 1/8$ we prove that

$$\mathbf{P}(X^0 \geq \xi) \leq \mathbf{P}(\sigma Y \geq \xi - \Delta(\xi)) \qquad \text{(C.12).}$$

The function $\xi - \Delta(\xi)$ is concave and larger than 3σ at $\xi^{(2)}$ for $\sigma^2 \geq 50$. Let $\xi^{(3)} = \vee\{k - npp'; k - npp' - \Delta(k - npp')) \geq 3\sigma\}$. Since we have $P(X^0 \geq \xi) \leq P(X^0 \geq \xi^{(3)}) \leq P(\sigma Y \geq \xi^{(3)} - \Delta(\xi^{(3)})) \leq P(\sigma Y \geq \xi - \Delta(\xi))$ for $\xi > \xi^{(3)}$, it is enough to prove (C.12) for $\xi^{(2)} < \xi \leq \xi^{(3)}$. The Gaussian inequality:

$$\mathbf{P}(Y \geq y) \geq \frac{y^2 - 1}{y^3} \frac{\phi(y)}{\sqrt{2\pi}}$$

implies, with $\xi - \Delta(\xi) \geq 3\sigma$, that

$$\mathbf{P}(\sigma Y \geq \xi - \Delta(\xi)) \geq \frac{8\sigma^2}{9(\xi - 1)} \frac{\phi((\xi - \Delta(\xi))/\sigma)}{\sqrt{2\pi}\sigma}.$$

Corollary 2, with $\omega = (1 - \delta\delta')/2$, yields:

$$\mathbf{P}(X^0 \geq \xi) \leq (\exp(\omega(\xi - 1)/\sigma^2) - 1)^{-1}\mathbf{P}(X^0 = \xi - 1).$$

Moreover, the function Q defined by $Q(t) = 8(\exp(\omega t) - 1)/(9t)$ is increasing. Thus Inequality (C.12) is verified as soon as $D'(\xi) := g(\xi - 1) + \ln(Q(\frac{\xi^{(2)}}{\sigma^2})) \geq 0$ for $\xi^{(2)} < \xi \leq \xi^{(3)}$.

End of the proof.

1) Case $0 \leq \xi \leq \xi^{(1)}$. First, we define $\xi^{(1)}$:

$$\xi^{(1)} = \begin{cases} \wedge\{k - npp'; k - npp' \geq 1\} \text{ if } \delta\delta' \leq 1/8 \\ \wedge\{k - npp'; k - npp' \geq 2\} \text{ if } \delta\delta' > 1/8. \end{cases}$$

Inequality (C.9) follows from:

$$\mathbf{P}(\xi \leq X^0 \leq \xi^{(1)}) \leq \mathbf{P}(\sigma Y \in [0, \xi^{(1)} - \xi + 2]). \tag{C.13}$$

From Corollary 1, we get:

$$\mathbf{P}(\xi \leq X^0 \leq \xi^{(1)}) \leq \frac{(\xi^{(1)} - \xi + 1)\exp(1/2\sigma^2)}{\sqrt{2\pi}\sigma}.$$

Since $\sigma^2 \geq \sigma_0^2$, (C.13) is verified as soon as:

$$(\xi^{(1)} - \xi + 1)\exp(1/2\sigma_0^2) \leq \int_0^{\xi^{(1)} - \xi + 2} \exp(-t^2/2\sigma_0^2)\, dt.$$

If $\delta\delta' \leq 1/8$, then $\xi^{(1)} - \xi + 1 \in \{1, 2\}$ and this inequality is true for $\sigma_0^2 = 4.5$. If $\delta\delta' > 1/8$, then $\xi^{(1)} - \xi + 1 \in \{1, 2, 3\}$ and this inequality is true for $\sigma_0^2 = 11$.

2) Case $\xi^{(1)} < \xi \leq \xi^{(2)}$. Inequalities (C.11) are satisfied with $\sigma_0^2 = 4.5$ if $\delta\delta' \in]0, 1/8]$, and $\sigma_0^2 = 80$ if $\delta\delta' > 1/8$. To prove that $D(\xi) \geq 0$, we use the upper bound of $\mathbf{P}(X^0 = \xi)$, given by Lemma B.1.

If $\delta\delta' \leq 0$ we get:

$$\begin{aligned} D(\xi) = g(\xi) &\geq -\frac{\xi^2}{2\sigma^2} + \frac{\xi}{\sigma^2} - \frac{1}{2\sigma^2} + \frac{\xi^2}{2\sigma^2} + \frac{\delta\delta'\xi}{2\sigma^2} - \frac{S_2\xi^2}{\sigma^4} \\ &\geq (1 + \frac{\delta\delta'}{2})\frac{\xi}{\sigma^2} - \frac{1 + (\delta\delta')^2}{2}\left(\frac{\xi}{\sigma^2}\right)^2 - \frac{1}{2\sigma^2}. \end{aligned}$$

This function is concave in ξ/σ^2, positive at $2/\sigma^2$ and at $1/2$ as soon as $\sigma^2 \geq 4/(3 + \delta\delta'(2 - \delta\delta'))$ (in particular, one can take $\sigma^2 \geq 4.5$ for $|\delta\delta'| \leq 1/8$).

If $\delta\delta' > 0$ we get:

$$g(\xi) \geq \xi(\Delta(\xi+1)-1)/\sigma^2 - (\Delta(\xi+1)-1)^2/(2\sigma^2) + \delta\delta'\xi/(2\sigma^2) \\ - \delta\delta'\xi^3/(6\sigma^4) - S_2\xi^2/\sigma^4.$$

Moreover, with $\sigma^2 \geq 4.5$ (if $\delta\delta' \leq 1/8$), or with $\sigma^2 \geq 80$ (if $\delta\delta' > 1/8$), we have $\xi - \Delta(\xi) \geq 0$ for $\xi^{(1)} < \xi \leq \xi^{(2)}$. Thus:

$$\xi(\Delta(\xi+1)-1)/\sigma^2 - (\Delta(\xi+1)-1)^2/(2\sigma^2) \geq \xi(\Delta(\xi+1)-1)/(2\sigma^2).$$

It follows:

$$g(\xi) \geq \frac{\xi}{2\sigma^2}(\alpha + \delta\delta' + \frac{\delta\delta'\xi^2}{6\sigma^2} - \frac{\xi}{\sigma^2}) \tag{C.14}$$

with $\alpha = 0.835$ if $\delta\delta' \leq 1/8$, and with $\alpha = 1.9$ if $\delta\delta' > 1/8$. Recall that $D(\xi) = g(\xi) + \ln(1 - \Delta(\xi+1) + \Delta(\xi))$. We have:

$$\ln(1 - \Delta(\xi+1) + \Delta(\xi)) \geq \ln(1 - \gamma\delta\delta'\xi/2\sigma^2))$$

with $\gamma = 5/2$ if $\delta\delta' \leq 1/8$ and with $\gamma = 7/3$ if $\delta\delta' > 1/8$. Let h denote the function $h(x) = 1 + x^{-1}\ln(1-\gamma x)$. We get:

$$D(\xi) \geq \frac{\xi}{2\sigma^2}(\alpha + \frac{\delta\delta'\xi^2}{6\sigma^2} - \frac{\xi}{\sigma^2} + \delta\delta' h(\frac{\delta\delta'\xi}{2\sigma^2})).$$

The function h is decreasing. If $\delta\delta' \leq 1/8$, then $(\delta\delta'\xi)/(2\sigma^2) \leq 1/32$ and we obtain $D(\xi) \geq 0$. If $\delta\delta' > 1/8$, we have to distinguish two cases. For $\xi^2/(6\sigma^2) \geq 1.126$, we get:

$$D(\xi) \geq (\xi/(2\sigma^2))(1.9 - 0.5 + 1.126 + h(0.25)) \geq 0.$$

For $\xi^2/(6\sigma^2) < 1.126$, then $\xi/\sigma^2 \leq 0.214$ as soon as $\sigma^2 \geq 150$. Hence:

$$D(\xi) \geq (\xi/(2\sigma^2))(1.9 - 0.214 + h(0.107)) \geq 0.$$

3) Case $\xi > \xi^{(2)}$. From Corollary 3, we have:

$$g(\xi) \geq \frac{2\Lambda - 1}{2\sigma^2} + \ln(\phi(\xi - \Delta(\xi+1))/\sigma)) - f(\xi - 1)$$

with $\Lambda = 1$ if $\delta\delta' \leq 0$, $\Lambda = 0.835$ if $0 < \delta\delta' \leq 1/8$, and $\Lambda = 1.9$ if $\delta\delta' > 1/8$. We have for all $\xi > \xi^{(2)}$ if $\delta\delta' \leq 1/8$, and for $\xi^{(2)} < \xi \leq \xi^{(3)}$ if $\delta\delta' > 1/8$: $0 \leq \xi - \Delta(\xi+1) \leq \xi - \Delta(\xi)$. Thus $g(\xi) \geq (2\Lambda - 1)/(2\sigma^2) + g(\xi - 1)$.

In the case $\delta\delta' \leq 1/8$ we obtain:

$$D(\xi) \geq \ln(1 - \Delta(\xi+1) + \Delta(\xi)) + \frac{2\Lambda - 1}{2\sigma^2}(\xi - \xi^{(2)}) + g(\xi^{(2)}).$$

Since $\ln(1 - \Delta(\xi+1) + \Delta(\xi)) + (2\Lambda - 1)(\xi - \xi^{(2)})/(2\sigma^2)$ is increasing in ξ (because $\sigma^2 \geq 4.5$) we get $D(\xi) \geq D(\xi^{(2)}) \geq 0$.

In the case $\delta\delta' > 1/8$ we obtain:

$$D'(\xi) \geq \ln(Q(\frac{\xi^{(2)}}{\sigma^2})) + g(\xi^{(2)}).$$

We have supposed $\sigma^2 \geq 150$. Thus $0.49 < \xi^{(2)}/\sigma^2 \leq 0.5$. Using the lower bound of $g(\xi^{(2)})$ given by (C.14) we obtain:

$$D'(\xi) \geq \ln(\frac{8 \times ((\exp(\omega t)) - 1)}{9 \times 0.49}) + \delta\delta'(0.245 + \frac{\sigma^2}{102}) + 0.343.$$

The right side is an increasing function of $\delta\delta'$ if $\sigma^2 \geq 102/(1 - \delta\delta')$ and is positive at $\delta\delta' = 1/8$ if $\sigma^2 \geq 379$.

D. PROOF OF INEQUALITY (A.7) OF LEMMA A.1

Let $\theta = 1 - \eta$. On the set $X^0 \geq (\sigma^2/(4d)) - c'$, where $c' = c - 1$, it is clear that Inequality (A.7) holds because

$$-c' - dY^2 + \sigma Y \leq \frac{\sigma^2}{4d} - c' \leq X^0.$$

We suppose now $0 \leq X^0 \leq (\sigma^2/(4d)) - c$. Since $P(\sigma Y - dY^2 - c \geq \xi - 1) \leq P(\sigma Y \geq (\sigma^2/2d)(1 - \sqrt{1-x}))$, where $x := (4d(\xi + c'))/\sigma^2$, it suffices to prove that for all $\xi \in [0, (\sigma^2/(4d)) - c]$:

$$\mathbf{P}(X^0 \geq \xi) \geq \mathbf{P}(\sigma Y \geq \frac{\sigma^2}{2d}(1 - \sqrt{1-x})) \qquad \text{(D.15)}.$$

If x is large, we will prove Inequality (D.15), and if x is small we will prove the following inequality step by step:

$$\mathbf{P}(X^0 = \xi) \geq \mathbf{P}(\sigma Y \in [\frac{\sigma^2}{2d}(1 - \sqrt{1-x})\ ;\ \frac{\sigma^2}{2d}(1 - \sqrt{1-x'})\ ]\ ), \quad \text{(D.16)}$$

where x and x' are defined by: $x = 4d(\xi + c')/\sigma^2$ and $x' = 4d(\xi + c)/\sigma^2$.

Relation between $P(X^0 \geq \xi)$ and $P(X^0 = \xi)$. – With the notation of Lemma B.3: $r(\xi) = P(X^0 = \xi + 1)/P(X^0 = \xi)$, we have:

$$P(X^0 \geq \xi) = P(X^0 = \xi)(1 + \sum_i \prod_{j=0}^{i} r(\xi + j)).$$

Since $\xi + c \leq \sigma^2/(4d)$, we can apply Lemma B.3 when $\lambda = 1/(4d) < 1$:

$$\ln(P(X^0 \geq \xi)) \geq \ln(P(X^0 = \xi)) + \ln(\frac{1 - \alpha^{c+1}}{1 - \alpha})$$

with

$$\alpha = \exp(-\frac{1}{4d} + \frac{(1+\theta)^2}{4}(\frac{1}{4d} + \ln(1 - \frac{1}{4d})).$$

Inequality (D.15) is replaced by:

$$\ln(P(X^0 = \xi)) + \ln(\frac{1 - \alpha^{c+1}}{1 - \alpha}) \geq \mathbf{P}(\sigma Y \geq \frac{\sigma^2}{2d}(1 - \sqrt{1 - x})). \quad \text{(D.17)}$$

Uppers bounds for the Gaussian probabilities. – Using $\mathbf{P}(Y \geq y) \leq (y\sqrt{2\pi})^{-1} \exp(-y^2/2)$ we have:

$$\ln(\mathbf{P}(\sigma Y \geq \frac{\sigma^2}{2d}(1 - \sqrt{1 - x}))) \leq -\ln(\sigma\sqrt{2\pi}) - \ln\left(\frac{1 - \sqrt{1 - x}}{2d}\right)$$
$$- \frac{\sigma^2}{8d^2}(1 - \sqrt{1 - x})^2. \quad \text{(D.18)}$$

Moreover,

$$\mathbf{P}(\sigma Y \in [\frac{\sigma^2}{2d}(1 - \sqrt{1 - x})\ ;\ \frac{\sigma^2}{2d}(1 - \sqrt{1 - x'})])$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\sigma}{2d}(1-\sqrt{1-x})}^{\frac{\sigma}{2d}(1-\sqrt{1-x'})} \exp(-y^2/2)\, dy = \frac{\sigma}{4d\sqrt{2\pi}} \int_x^{x'} h(u)\, du,$$

where $h(u) = (1-u)^{-1/2} \exp(-(\sigma^2/(8d^2))(1-\sqrt{1-u})^2)$. The study of h shows that $\int_x^{x'} h(u)\, du \leq (x - x')h(x)$ as soon as $\sigma^2 \geq 16d^2$, $c \geq (d+1)^2$ and $x \leq x_0$, where:

$$x_0 = (1/2) + (4d(d-1)/\sigma^2) + (1/2)\sqrt{1 - (16d^2)/(\sigma^2)}.$$

With these conditions we have:

$$\ln(\mathbf{P}(\sigma Y \in [\frac{\sigma^2}{2d}(1 - \sqrt{1 - x})\ ;\ \frac{\sigma^2}{2d}(1 - \sqrt{1 - x'})]))$$
$$\leq -\ln(\sigma\sqrt{2\pi}) - \frac{1}{2}\ln(1 - x) - \frac{\sigma^2}{8d^2}(1 - \sqrt{1 - x})^2. \quad \text{(D.19)}$$

The conditions on the constants are:

$$\begin{cases} \sigma_0^2 \geq 16d^2 \\ c \geq (d+1)^2 \\ 4d > 1. \end{cases} \qquad (\mathcal{C})$$

End of the proof. Let us denote by g the function which appears in both Inequalities (D.19) and (D.18):

$$g(x) = \ln(\sigma\sqrt{2\pi}) + \frac{\sigma^2}{8d^2}(1-\sqrt{1-x})^2.$$

Using (D.17) and (D.18), we see that Inequality (D.15) is satisfied as soon as:

$$\ln(P(X^0 = \xi)) + g(x) \geq -\ln(\frac{1-\alpha^{c+1}}{1-\alpha}) - \ln(\frac{1-\sqrt{1-x}}{2d}).$$

Using (D.19), we see that Inequality (D.16) is satisfied as soon as $x \leq x_0$ and:

$$\ln(P(X^0 = \xi)) + g(x) \geq -\frac{1}{2}\ln(1-x).$$

Thus we need a lower bound for $\ln(P(X^0 = \xi)) + g(x)$. We have $(1-\sqrt{1-x})^2 \geq (x^2/4) + (x^3/8) + (5x^4/64)$ with $x = 4d(\xi + c')/\sigma^2$. Using Lemma B.1, we get:

$$\ln(P(X^0 = \xi)) + g(x) \geq \sum_{i=0}^{4} a_i \xi^i$$

with:

$$\begin{aligned} a_0 &= \frac{1}{2\sigma^2}(c'^2 - \frac{1}{3(1-\lambda)}) + \frac{dc'^3}{\sigma^4} + \frac{5d^2c'^4}{2\sigma^6} \\ a_1 &= \frac{1}{2\sigma^2}(2c' - \theta) + \frac{3dc'^2}{\sigma^4} + \frac{10d^2c'^3}{\sigma^6} \\ a_2 &= \frac{1}{\sigma^4}(3c'd + \frac{1}{16(1+\lambda)^2}) + \frac{15d^2c'^2}{\sigma^6} \\ a_3 &= \frac{1}{\sigma^4}(d - \frac{\theta}{6}) + \frac{10d^2c'}{\sigma^6} \\ a_4 &= \frac{1}{\sigma^6}(\frac{5d^2}{2} - \frac{(1+3\theta^2)}{48(1-\lambda)^3}). \end{aligned}$$

Since $\lambda \leq 1/4d$, a_4 is positive when $d = 0.51$ (if $\theta = 1$) or $d = 0.41$ (if $\theta = 1/8$). Then, with $\sigma^2 \geq 4.5$, $c = 4$ in the general case $\theta = 1$ or $c = 3$ in the case $\theta = 1/8$, the three conditions of $(\mathcal{C})$ are satisfied. We now detail the case $\theta = 1/8$, $\sigma_0^2 = 4.5$, $c = 3$, and $d = 0.41$ (the general case $\theta = 1$ is analogous). We replace the constants by their value, but keep $\lambda = 1/4d$ in the coefficient a_0. We will later give a lower bound for this coefficient. We get $\alpha \leq 0.4895$, $x_0 \geq 0.602$ and:

$$\ln(P(X^0 = \xi)) + g(x) \geq \sum_{i=0}^{3} b_i \xi^i$$

with:

$$\begin{aligned} b_0 &= \frac{1}{\sigma^2}(2 - \frac{1}{6(1-\lambda)}) + \frac{3.28}{\sigma^4} + \frac{6.724}{\sigma^6} \\ b_1 &= \frac{1.9375}{\sigma^2} + \frac{4.92}{\sigma^4} + \frac{13.448}{\sigma^6} \\ b_2 &= \frac{2.484}{\sigma^4} + \frac{10.086}{\sigma^6} \\ b_3 &= \frac{0.389}{\sigma^4} + \frac{3.362}{\sigma^6}. \end{aligned}$$

Case $x \leq 0.602$. In this case $(\ln(1-x))/2 \geq -(x/2) - (x^2/4) - 0.317x^3$. We replace x by $4d(\xi + c')/\sigma^2$ and obtain:

$$\ln(P(X^0 = \xi)) + g(x) + \frac{1}{2}\ln(1-x) \geq \sum_{i=0}^{3} c_i \xi^i$$

where $c_2, c_3 \geq 0$ and:

$$\begin{aligned} c_0 &= \frac{1}{\sigma^2}(0.36 - \frac{1}{6(1-\lambda)}) + \frac{0.5904}{\sigma^4} - \frac{4.47}{\sigma^6} \\ c_1 &= \frac{1.1175}{\sigma^2} + \frac{2.2304}{\sigma^4} - \frac{3.34}{\sigma^6}. \end{aligned}$$

The right side is not really polynomial: c_0 is a function of λ where λ is such that $\xi/\sigma^2 \leq \lambda$ (Lemma B.1). If $\xi \leq 0.3\sigma^2$, then $\lambda \leq 0.3$, and all the coefficients c_i, $i = 0, 1, 2, 3$ are positive. If $0.3\sigma^2 \leq \xi \leq 0.61\sigma^2$ (since $\xi/\sigma^2 \leq 1/(4d)$), then $\lambda \leq 0.61$ and we get $c_0 + c_1\xi \geq (\sigma^4(1.1175\xi - 0.0674) + 4.87)/\sigma^6$. Using $\xi \geq 0.3\sigma^2 \geq 1.35$, we obtain the positivity.

Case $x \geq 0.602$. In this case we have:

$$\ln(P(X^0 = \xi)) + g(x) + \ln(\frac{1-\alpha^{c+1}}{1-\alpha}) + \ln(\frac{1-\sqrt{1-x}}{2d}) \geq b_0 + b_1\xi - 0.1852.$$

Now $\lambda \leq 0.61$ (since $\xi/\sigma^2 \leq 1/(4d)$). If $\sigma^2 \leq 10.5$, we obtain $b_0 \geq 0.1852$. If $\sigma^2 \geq 10.5$, we use $\xi/\sigma^2 = x/(4d) - c'/\sigma^2 \geq 0.17$, and obtain $b_1\xi \geq 0.32$.

E. BOUND OF $|X^0|$ CAN NOT BE IMPROVED

We show that the term $|\delta\delta'|Y^2$ of Inequality (A.2) cannot be avoided. Using properties of $\mathcal{H}(n, np, np')$, it is sufficient to prove this result for $X^0 \geq 0$ and $\delta\delta' \geq \rho > 0$. In Section C we have obtained the existence of constants α and b such that:

$$\mathbf{P}(X^0 \geq \xi) \leq \mathbf{P}\left(\sigma Y \geq \xi - \alpha - b\delta\delta'\frac{\xi^2}{\sigma^2}\right).$$

Let us suppose ξ of order $\sigma^{4/3}\ln\sigma$, more precisely $\sigma^{4/3}\ln\sigma - 1 < \xi \leq \sigma^{4/3}\ln\sigma$ (we recall that $\xi = k - npp'; k \in \mathbb{N}$). Using the lower bound of $\mathbf{P}(X^0 = \xi)$ (Lemma B.1) and inequality $\mathbf{P}(Y \geq y) \leq (y\sqrt{2\pi})^{-1}\exp(-y^2/2)$, we obtain:

$$\frac{\xi^2}{2\sigma^2} - \frac{\xi^3\delta\delta'}{6\sigma^4} + \ln\sigma + \epsilon(\sigma) > \frac{y^2}{2} + \ln y,$$

$$\text{where} \quad y = \frac{\xi}{\sigma} - \frac{\alpha}{\sigma} - b\delta\delta'\frac{\xi^2}{\sigma^3} \quad \text{and} \quad \lim_{\sigma\to\infty}\epsilon(\sigma) = 0.$$

Using the definition of ξ, we get:

$$\frac{\sigma^{2/3}(\ln\sigma)^2}{2} + \delta\delta'(b - \frac{1}{6})(\ln\sigma)^3 + \epsilon'(\sigma) + \ln\sigma \geq \frac{\sigma^{2/3}(\ln\sigma)^2}{2} + \frac{\ln\sigma}{3} + \ln(\ln\sigma),$$

where $\lim_{\sigma\to\infty}\epsilon'(\sigma) = 0$. This leads to $b \geq 1/6$ as soon as $\delta\delta' \geq \rho > 0$.

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