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Combining m-dependence with Markovness

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ABSTRACT. – Generally, no stationary sequence of random variables which is Markov of order n but not of order $n-1$ and m-dependent but not $(m-1)$-dependent exists if the state space of the sequence has small cardinality. We show that to ensure the existence for the Markov sequences of order $n = 1$ the number of attainable states must be at least $m + 2$ and that this bound is tight. Given a small state space such a sequence exists only for special n and m. On a two-element state space the smallest possible n and m are shown to be 3 and 2, respectively. This results from our parametric description of all binary m-dependent sequences, $m \geq 0$, that are Markov of order 3. © Elsevier, Paris

RÉSUMÉ. – Si l'espace d'états n'est pas suffisamment riche on ne peut pas construire, pour n et m quelconques, une suite aléatoire stationnaire de Markov d'ordre n et pas d'ordre $n-1$ qui est dans le même temps m-dépendante et pas $(m-1)$-dépendante. Nous montrons que pour les chaînes de Markov, $n = 1$, l'espace d'états doit avoir au moins $m + 1$ éléments et que ce nombre ne peut pas être amélioré. Pour les suites binaires les plus petits n et m admissibles sont 3 et 2, respectivement. C'est une conséquence

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de notre description paramétrique de toutes les suites binaires stationnaires m-dépendantes, $m \geq 0$, de Markov d'ordre 3. © Elsevier, Paris

1. INTRODUCTION

Let $\xi = (\xi_i;\ i \geq 1)$ be a strictly stationary sequence of random variables taking values in a finite state space S; speaking about a sequence we will always assume these properties. The sequence ξ is *Markov of order* $n \geq 0$ if $(\xi_i;\ 1 \leq i \leq k)$ is conditionally independent of $(\xi_i;\ i \geq k+n+1)$ given $(\xi_i;\ k+1 \leq i \leq k+n)$ for all $k \geq 1$. The sequence ξ is *dependent of order* $m \geq 0$ if $(\xi_i;\ 1 \leq i \leq k)$ is unconditionally independent of $(\xi_i;\ i \geq k+m+1)$, $k \geq 1$. For simplicity, we shorten the expressions "Markov of order n" and "dependent of order m" to n-Markov and m-dependent, correspondingly.

The aim of this note is to examine how these two properties interfere under restrictions on the cardinality of the state space. A more precise formulation will use the following notion of an index of a sequence. Let n_ξ be the smallest nonnegative integer n such that a sequence ξ is n-Markov, let m_ξ be the smallest $m \geq 0$ such that ξ is m-dependent and let d_ξ be the cardinality of the set of states which are attained with positive probabilities. Thus, we have $n_\xi \geq 0$, $m_\xi \geq 0$ and $d_\xi \geq 1$ with $n_\xi = 0$ if and only if $m_\xi = 0$. This expresses the sequence ξ is i.i.d. If ξ is Markov of no order $n \geq 0$ it is reasonable to write $n_\xi = \infty$ and similarly with the dependence and m_ξ; we shall, however, not deal with these cases at all. The triple $\langle n_\xi, m_\xi, d_\xi \rangle$ will be called *index* of ξ.

A natural question asks which triple can be equal to the index of a sequence ξ. In other words, given a triple of integers $\langle n, m, d \rangle$ does there exist a (stationary) sequence ξ such that $n_\xi = n$, $m_\xi = m$ and $d_\xi = d$, i.e. in the nontrivial case $n > 0$ and $m > 0$, such that it is n-Markov and not $(n-1)$-Markov, m-dependent and not $(m-1)$-dependent and takes exactly d states with positive probabilities?

We present answers only if $n = 1$ and partially if $d = 2$ here. In the second section devoted to the usual Markov chains (1-Markovness) we prove that a triple $\langle 1, m, d \rangle$ is the index of a sequence if and only if $1 \leq m \leq d-2$. Then we turn our attention entirely to the binary sequences, $S = \{0, 1\}$, and during a technical preparation in the third section we reveal that every $(n, m, 2)$-sequence, this is an abbreviation for n-Markov, m-dependent and two-element state space, is i.i.d. provided $n \leq 2$ or $m \leq 1$. For some years I conjectured this be valid for any n and m nonnegative. It is, however, not the case. We will see in Section 4 that all $(3, m, 2)$-sequences are 3-dependent and therefore the only two candidates for indices of 3-Markov binary sequences are $\langle 3, 2, 2\rangle$ and $\langle 3, 3, 2\rangle$. Both these triples are really indices and, moreover, we provide a complete characterization of the distributions of all $(3, 2, 2)$-sequences in the fifth section and all $(3, 3, 2)$-sequences in the sixth section, respectively.

Though Markov chains is an old topic, Markov chains with 1-dependence appeared for the first time in [1] and then in [2],[6] where the focus was on the structure of block-factors. Notes on binary sequences of this type are in [11] and [12]. Our question is akin to the problems around probabilistic conditional independence structures [8]; [4] and [5] settle an unconditional case for sequences of random variables. The lattest review of the field is in [7]. It is also worthwhile to mention the paper [9] where Markovness was combined with m-in dependence. That means any m variables of ξ are mutually independent.

2. MARKOV SEQUENCES OF FIRST ORDER

It is not unexpected that a solution of our problem for 1-Markov sequences will be based on an analysis of transition matrices. Let us remind that a sequence ξ with the state space $S = \{1, 2, \ldots, d\}$, $d = d_\xi$, is 1-Markov if and only if the probability of every event $\xi_1 = s_1 \cdots \xi_{k+1} = s_{k+1}$, denoted by $[s_1 \ldots s_{k+1}]$, is equal to $[s_1] p_{s_1 s_2} \cdots p_{s_k s_{k+1}}$, $k \geq 1$, where $[s] > 0$ is the probability of $\xi_1 = s$ and $p_{s,t}$ is the conditional probability of $\xi_2 = t$ given $\xi_1 = s$, $s, t \in S$. The (s, t)-entry of the k-th power of the transition matrix $\mathbf{P} = (p_{s,t};\ 1 \leq s,\ t \leq d)$ contains the conditional probability of $\xi_{k+1} = t$ given $\xi_1 = s$, $k \geq 1$.

If the sequence ξ is, moreover, m-dependent, $m \geq 0$, then ξ_1 is independent of ξ_{m+2} and $\mathbf{P}^{m+1} = \mathbf{Q}$ where the matrix $\mathbf{Q}$ has constant columns, t-th one containing the probability $[t]$, $t \in S$. It is not difficult to see that, on contrary, this matrix equality implies that ξ is m-dependent. In fact, it implies ξ_k is independent of ξ_{k+m+1} what together with the conditional independence of ξ_k and $(\xi_i;\ i \geq k+m+2)$ given ξ_{k+m+1} yield ξ_k is independent of $(\xi_i;\ i \geq k+m+1)$. Repeating the same reasoning once again we obtain the desired m-dependence.

LEMMA 1. – *If a Markov sequence of first order with d-element state space, $d \geq 2$, is m-dependent, $m \geq 0$, then it is $(d-2)$-dependent.*

This assertion is nontrivial for $m > d-2$ as it provides reduction of the order of dependence due to "small" state space.

Proof. – Knowing that $\mathbf{P}^{m+1} = \mathbf{Q}$ we deduce that the spectra of both matrices are equal to $\{0, 1\}$. Since $\mathbf{P}$ is primitive (all entries of some of its powers are positive) the number 1 is an eigenvalue of $\mathbf{P}$ with algebraic multiplicity one, see [10]. We can write $\mathbf{P} = \mathbf{T}\mathbf{W}\mathbf{T}^{-1}$ where $\mathbf{T}$ is a regular matrix and $\mathbf{W}$ is the Jordan canonical form of $\mathbf{P}$, see [3]. The matrix $\mathbf{W}$ is block-diagonal. One of the blocks consists of the single eigenvalue 1 and the remaining blocks have zeros on their diagonals and ones on their superdiagonals. The k-th power of such a block of size $b \times b$, $b \geq 2$, is a zero matrix once $k \geq b$. Thus, we can conclude that each matrix $\mathbf{W}^k$, $k \geq d-1$, has only one nonzero entry; obviously it is the eigenvalue 1. Now, if $m+1 \geq d-1$ then $\mathbf{W}^{m+1} = \mathbf{W}^{d-1}$ and consequently $\mathbf{P}^{d-1} = \mathbf{P}^{m+1} = \mathbf{Q}$ what means that the examined sequence is $(d-2)$-dependent. ■

COROLLARY 1. – *Every (n, m, d)-sequence is, $d \geq 2$, dependent of order $(d^n - n - 1)$.*

Proof. – If ξ fulfils the assumptions, $n > 0$, we consider the sequence $\eta = (\eta_i;\, i \geq 1)$ of the random variables $\eta_i = (\xi_i, \ldots, \xi_{i+n-1})$. This sequence is obviously 1-Markov, $(n+m-1)$-dependent and $d_\eta \leq d_\xi^n \leq d^n$. By Lemma 1 it is $(d_\eta - 2)$-dependent what implies that ξ is dependent of order $(d^n - 2 - (n-1))$. ■

PROPOSITION 1. – *A triple of integers $\langle 1, m, d\rangle$ is the index of a sequence if and only if $1 \leq m \leq d-2$.*

Proof. – The necessity of the presented condition is a consequence of the previous lemma and the sufficiency will be approved below by a construction of the desired sequences.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_d$ be an arbitrary orthonormal base of the Euclidean space $\mathcal{R}^d$ such that $\mathbf{x}_1$ has all coordinates equal to $d^{-1/2}$. These vectors are taken as rows and their transpositions, columns, are obtained by using the superindex T. For example, $\mathbf{Q} = \mathbf{x}_1^T\mathbf{x}_1$ is a doubly stochastic matrix. If we set $\mathbf{U}_k = \sum_{j=2}^{k} \mathbf{x}_j^T \mathbf{x}_{j+1}$ for $1 \leq k \leq d-1$ then the powers of these matrices are $\mathbf{U}_k^\ell = \sum_{j=2}^{k+1-\ell} \mathbf{x}_j^T \mathbf{x}_{j+\ell}$, $1 \leq \ell \leq k$. This fact can be obtained by a simple induction argument. Note that $\mathbf{U}_k^k$, $1 \leq k \leq d-1$, are zero matrices and that for $\ell < k$ the matrix $\mathbf{U}_k^\ell$ is nonzero owing to $\mathbf{x}_2\mathbf{U}_k^\ell = \mathbf{x}_{2+\ell}$. In addition, $\mathbf{Q}\,\mathbf{U}_k^\ell = \mathbf{U}_k^\ell\,\mathbf{Q}$ are zero matrices for $1 \leq \ell \leq k \leq d-1$, too.

Now, let us have a triple $\langle 1, m, d\rangle$ and $1 \le m \le d-2$. The 1-Markov sequence ξ on a d-element state space with the transition matrix $\mathbf{P} = \mathbf{Q} + \varepsilon\, \mathbf{U}_{m+1}$, $\varepsilon \ne 0$ sufficiently small, and the uniform initial distribution is stationary because $\mathbf{x}_1 \mathbf{P} = \mathbf{x}_1$. In addition, $\mathbf{P}^k = \mathbf{Q} + \varepsilon^k\, \mathbf{U}^k_{m+1}$, $1 \le k \le m+1$, what enables to conclude that ξ is m-dependent but not $(m-1)$-dependent, i.e. $n_\xi = 1$, $m_\xi = m$ and $d_\xi = d$. ■

3. BINARY SEQUENCES: PRELIMINARIES

From now on we fix the state space as $S = \{0, 1\}$. States $s_k, \ldots, s_\ell$ from S, $k \le \ell$, will be concatenated into words and the word $s_k\, s_{k+1} \cdots s_\ell$ will be shortened to s_k^ℓ. The symbol S^k denotes the set of all words made of letters from S which have the length k, $k \ge 0$, e.g. $s_1^k \in S^k$ and $s_1^0 \in S^0$ is the empty word.

A sequence ξ is n-Markov, $n \ge 0$, if and only if for all $k \ge 0$ and $s_1^{n+k} \in S^{n+k}$ the probability $[s_1^{n+k}]$ of the event $\xi_1 = s_1 \cdots \xi_{n+k} = s_{n+k}$ can be factorized as follows

$$[s_1^{n+k}] = [s_1^n] \textstyle\prod_{j=1}^k (s_j^{j+n}).$$

In this formula the numbers (s_j^{j+n}), conditional probabilities, are defined by the equalities $[s_1^{n+1}] = [s_1^n]\,(s_1^{1+n})$. If $[s_1^n] = 0$ the choice of (s_1^{1+n}) is arbitrary and will not affect our next computations.

An n-Markov sequence is m-dependent, $m \ge 0$, if and only if for all s_1^n, $s_{n+m+1}^{n+m+n} \in S^n$ the following equality

$$\begin{aligned} 0 = \Box_{n,m}(s_1^n,\, s_{n+m+1}^{n+m+n}) = \\ = [s_1^n] \left([s_{n+m+1}^{n+m+n}] - \textstyle\sum_{s_{n+1}^{n+m} \in S^m} \prod_{j=1}^{n+m} (s_j^{j+n}) \right) \end{aligned}$$

takes place. That means $(\xi_{k+i};\, 1 \le i \le n)$ is independent of $(\xi_{k+n+m+i};\, 1 \le i \le n)$, $k \ge 0$, and this fact implies m-dependence in a similar way as was done above with $n = 1$. We shall also need the symbol $\Box^*_{n,m}(s_1^n, s_{n+m+1}^{n+m+n})$ denoting the difference in parentheses. Sometimes an argument in $\Box_{n,m}(s_1^n, \cdot)$ is omitted to work with a function on S^n.

An n-Markov sequence, $n \ge 1$, is $(n-1)$-Markov if and only if

$$0 = \triangle_{n-1}(s_2^n) = [0\, s_2^n\, 0]\,[1\, s_2^n\, 1] - [0\, s_2^n\, 1]\,[1\, s_2^n\, 0]$$

for all $s_2^n \in S^{n-1}$. The equalities express that the variable ξ_k is independent of ξ_{k+n} given $(\xi_{k+i};\, 1 \le i \le n-1)$. We shall also need the symbol

$\triangle^*_{n-1}(s_2^n)$ denoting the above difference with the brackets replaced by parentheses. Thus, $\triangle_{n-1}(s_2^n) = [0\, s_2^n]\,[1\, s_2^n]\,\triangle^*_{n-1}(s_2^n)$.

Beside the foregoing basic observations we want to summarize and label some other useful facts concerning $(n, m, 2)$-sequences, n, $m \geq 1$, $(k \geq 0)$

1. $\square_{n,m-1}(s_2^n\, 0, \cdot) + \square_{n,m-1}(s_2^n\, 1, \cdot) = 0$
2. $[0\, s_2^n]\,\square_{n,k}(1\, s_2^n, \cdot) = [1\, s_2^n]\,\square_{n,k}(0\, s_2^n, \cdot)$ if $\triangle_{n-1}(s_2^n) = 0$
3. $\square^*_{n,m-1}(s_2^n\, 0, \cdot) = \square^*_{n,m-1}(s_2^n\, 1, \cdot) = 0$ if $\triangle_{n-1}(s_2^n) \neq 0$
4. $\square_{n,m-1}(\cdot, 0\, s_2^n) = \square_{n,m-1}(\cdot, 1\, s_2^n) = 0$ if $\triangle^*_{n-1}(s_2^n) \neq 0$

Some comments are in order. The expression in 1. equals the sum of $\square_{n,m}(0\, s_2^n, \cdot)$ and $\square_{n,m}(1\, s_2^n, \cdot)$. The validity of 2. is clear if $[0\, s_2^n]$ or $[1\, s_2^n]$ is zero; if they are both positive we use $(0\, s_2^{n+1}) = (1\, s_2^{n+1})$. To see 3. we write

$$0 = \square_{n,m}(s_1^n, \cdot) = \textstyle\sum_{t \in S} [s_1^n\, t]\,\square^*_{n,m-1}(s_2^n\, t, \cdot)$$

and match these equalities into pairs corresponding to $0\, s_2^n$ and $1\, s_2^n$; the assumption $\triangle_{n-1}(s_2^n) \neq 0$ means that the determinant of two equations in a pair is nonzero. Finally, the validity of 4. is obtained similarly from

$$0 = \square_{n,m}(\cdot, s_{n+m+1}^{n+m+n}) = \textstyle\sum_{t \in S} (s_{n+m}^{2n+m})\,\square_{n,m-1}(\cdot, t\, s_{n+m+1}^{2n+m-1}).$$

LEMMA 2. – *If ξ is a $(n, m, 2)$-sequence where $n \leq 2$ or $m \leq 1$ then ξ is i.i.d.*

Proof. – Using Corollary 1 we can restrict ourselves to $m = 1$ and $n \geq 2$. We shall demonstrate by contradiction that $\triangle_{n-1}$ is identically zero and then apply the induction argument.

Let $\triangle_{n-1}(t_2^n) \neq 0$ for some $t_2^n \in S^{n-1}$. By fact 3. we have

$$[s_{n+2}^{2n+1}] = \textstyle\prod_{j=2}^{n+1} (s_j^{j+n})$$

as soon as the word s_2^{2n+1} begins with t_2^n. We multiply both sides by $[s_{n+1}^{2n}]$ and sum over s_{2n+1} what gives

$$[s_{n+1}^{2n}]\,[s_{n+2}^{2n}] = [s_{n+1}^{2n}] \textstyle\prod_{j=2}^{n} (s_j^{j+n}).$$

The conclusion is $[s_{n+1}^{2n+1}]\,[s_{n+2}^{2n}] = [s_{n+1}^{2n}]\,[s_{n+2}^{2n+1}]$ for all $s_{n+1}^{2n+1} \in S^{n+1}$ contradicting the assumption $\triangle_{n-1}(t_2^n) \neq 0$. ∎

4. BINARY 3-MARKOV SEQUENCES

From Corollary 1 we know that every $(3, m, 2)$-sequence is 4-dependent. We will see in a moment that it is even 3-dependent. By Lemma 2 the only nontrivial m's to be examined are then 2 and 3. The aim of this section is to prove some auxiliary results about these two cases.

LEMMA 3. – *For every* $(3, m, 2)$*-sequence* $\triangle_2(01) = 0$ *or* $\triangle_2(10) = 0$.

Proof. – Let us suppose that both $\triangle_2(01)$ and $\triangle_2(10)$ are nonzero. By fact 3. we deduce

$$0 = \square^*_{3,m-1}(01t, \cdot) = \square^*_{3,m-1}(10t, \cdot), \quad t \in S.$$

If $\triangle_2(00) = 0$ then by fact 2.

$$0 = [000]\,\square_{3,m-1}(100, \cdot) = [100]\,\square_{3,m-1}(000, \cdot)$$

and since $[100] \neq 0$ (otherwise $\triangle_2(10) = 0$) we employ 1. to obtain

$$0 = \square_{3,m-1}(00t, \cdot), \quad t \in S.$$

If $\triangle_2(00) \neq 0$ we have this equality immediately by 3. The same reasoning applies symmetrically to $\triangle_2(11)$. Thus, we see that the sequence is $(m-1)$-dependent. By induction, it is i.i.d., a contradiction. ■

LEMMA 4. – *A* $(3, m, 2)$*-sequence is i.i.d. if and only if both numbers* $\triangle_2(01)$ *and* $\triangle_2(10)$ *are equal to zero.*

Proof. – One implication is trivial. If $\triangle_2(01) = \triangle_2(10) = 0$ and both $\triangle_2(00)$ and $\triangle_2(11)$ are nonzero we recall 3., 2. and 1. and, similarly as in the proof above, keep lowering of the order of dependence. Hence, by symmetry, let $\triangle_2(00) = 0$. Then $\triangle_2(11) = 0$ would imply the sequence is 2-Markov and by Lemma 2 also i.i.d.

From $\triangle_2(11) \neq 0$ we deduce $\square_{3,m-1}(s1t, \cdot) = 0$ for $s, t \in S$ using 3., 2. and 1. as usually. Further, 1. and 2. enable to write the following four linear equations ($s \in S$)

$$\square_{3,m-1}(s00, \cdot) + \square_{3,m-1}(s01, \cdot) = 0,$$
$$[00s]\,\square_{3,m-1}(10s, \cdot) - [10s]\,\square_{3,m-1}(00s, \cdot) = 0.$$

If the determinant $\triangle_1(0)$ of the system of equations is nonzero then the order of dependence of the sequence decreases. Analogically as soon as

$[00] = 0$. Let $\triangle_1(0) = 0$ and $[00] \neq 0$. Since $[110] \neq 0$ we know that $[s0t] = [s0][0t]/[0] > 0$ for $s, t \in S$ and thus

$$(110s)\,(10st) = (10s)\,(0st) = (0s)\,(00st) = (000s)\,(00st).$$

This equality implies $0 = \square^*_{3,m-1}(110, \cdot) = \square^*_{3,m-1}(000, \cdot)$ and then $0 = \square_{3,m-1}(s0t, \cdot)$ for all $s, t \in S$, having decrease of the order of dependence, too. ■

LEMMA 5. – *In every* $(3, m, 2)$*-sequence* $\triangle_2(00) = 0$ *or* $\triangle_2(11) = 0$.

Proof. – By Lemma 3, Lemma 4 and symmetry we can assume $\triangle_2(10) \neq 0$ and $\triangle_2(01) = 0$. We start from the opposite $\triangle_2(00)\triangle_2(11) \neq 0$ aiming at a contradiction. Note that $[s_1^3]$ is positive for $s_1^3 \in S$. Since $\square_{3,4}(\cdot, \cdot) = 0$ by Corollary 1, one has $\square_{3,3}(s0t, \cdot) = 0$ and $\square_{3,2}(00s, \cdot) = 0$ by means of 3., s, $t \in S$. Owing to 4. $\square_{3,1}(00s, t1u) = 0$ and $\square_{3,0}(00s, t11) = 0$ for any s, t, $u \in S$. The choice $st = 00$ in the latter equality gives $[01] = (0000)(0001)$. Then we substitute $st = 01$ and $st = 11$ and find $(0001) = (1111)$. On the other hand, from $\square_{3,2}(00s, \cdot) = 0$ and $\triangle_2(00) \neq 0$ we have also $\square_{3,1}(00s, t00) = 0$ again by 4., s, $t \in S$. This provides for $st = 01$

$$[100] = (0000)(0001)(010)(0100) + (0001)(011)(0110)(1100)$$

where the left product equals $[0100]$. Thus, $[1100]$ is equal to the right product and then

$$[110] = (0001)(011)(0110).$$

Let us multiply both sides by (0000) whence $(0000)[011] = [0110]$. The contradiction sounds $(0110) = (0000) = (1110)$. ■

COROLLARY 2. – $\langle 3, 4, 2\rangle$ *is not an index.*

Proof. – Let a sequence ξ has the index $\langle 3, 4, 2\rangle$. Then index of the sequence of triples $\eta = ((\xi_i, \xi_{i+1}, \xi_{i+2});\ i \geq 1)$ is $\langle 1, 6, d\rangle$ whence $d = 8$ by Proposition 1. From the proof of Lemma 1 we know that the transition matrix of η has the rank 7. But, due to Lemma 3 and Lemma 5 this matrix has at least two pairs of equal rows, a contradiction. ■

LEMMA 6. – *Let* $\triangle_2(10) \neq 0$ *and* $\triangle_2(01) = 0$ *in a* $(3, m, 2)$*-sequence* ξ. *Then this sequence is* 2*-dependent if and only if both numbers* $\triangle_2(00)$ *and* $\triangle_2(11)$ *are equal to zero.*

Proof. – Let us observe that $[s_1^3] > 0$ for $[s_1^3] \in S - \{000, 111\}$ and then the 1-Markov sequence η constructed by grouping triples of consequent variables as above has the index $\langle 1, m, d\rangle$ where $6 \leq d \leq 8$. The transition matrix of η has the two rows indexed 001 and 101 identical. If $\triangle_2(ss) = 0$ and $[sss] > 0$ then also the two rows indexed by $0ss$ and $1ss$ coincide, $s \in S$. Hence, the matrix has rank at most 5. Then η is 4-dependent and ξ must be 2-dependent.

On the other hand, let ξ be 2-dependent. If $\triangle_2(00) \neq 0$ then by the fact 3. we have $\square_{3,1}(s0t, \cdot) = 0$ for $s, t \in S$ and then by 3. again $\square^*_{3,0}(00t, \cdot) = 0$ for $t \in S$. We can argue as in the proof of Lemma 2 to arrive at a contradiction and thus necessarily $\triangle_2(00) = 0$. If $\triangle_2(11) \neq 0$ then by 3. we get $\square^*_{3,1}(11t, \cdot) = 0$, $t \in S$, and by the fact 4.

$$\square^*_{3,0}(11t,\ s_5 1 s_7) = 0, \quad t, s_5, s_7 \in S.$$

The choice $ts_5 = 01$ leads to

$$[11s] = (1101)\,(1011)\,(011s), \quad s \in S,$$

and then (add the above equations) to $[11s] = [11]\,(011s)$. Hence $\triangle_2(11)$ equals zero, a contradiction. ■

Under the assumptions of Lemma 6, the sequence ξ is 3-dependent and not 2-dependent if and only if exactly one of the numbers $\triangle_2(00)$ and $\triangle_2(11)$ is equal to zero. This follows from Corollary 1, Corollary 2, Lemma 5 and Lemma 6.

5. $(3, 2, 2)$-SEQUENCES

In this section we will describe parametrically all binary 2-dependent sequences that are Markov of order 3.

PROPOSITION 2. – *Let ξ be a binary 3-Markov sequence such that*

$$\triangle_2(10) \neq 0 = \triangle_2(00) = \triangle_2(01) = \triangle_2(11) = 0.$$

Then ξ is 2-dependent if and only if $[0st1] = [0s][t1]$, $s, t \in S$.

Proof. – If ξ is a $(3, 2, 2)$-sequence with $\triangle_2$'s as above then by 3. and 4. we derive

$$\square^*_{3,1}(10s, \cdot) = \square^*_{3,0}(10s,\ t10) = 0, \quad s, t \in S.$$

Since $[s_1^3]$ are positive if s_1^3 is different from 000 and 111 we obtain $[t10] = (0st)\,(0st1)\,(st10)$ and for $st \neq 11$ immediately the desired equalities. But,

$$\textstyle\sum_{s,\,t\in S}[0st1] - [0s][t1] = 0$$

by 2-dependence and we have also $[0111] = [01][11]$.

In the opposite direction, we deduce first from $[0ss1] = [0s][s1]$ that the probabilities $[000]$ and $[111]$ are positive, too. Then

$$[s_5 1 s_7 s_8] = (s_2 0 s_4 s_5)\,(0 s_4 s_5 1)\,(s_4 s_5 1 s_7)\,(s_5 1 s_7 s_8)$$

because $(s_2 0 s_4 s_5) = (0 s_4 s_5)$, $(s_5 1 s_7 s_8) = (1 s_7 s_8)$ and $[0 s_4 s_5 1] = [0 s_4][s_5 1]$. We multiply the above equation by $(s_1 s_2 0 s_4)$, sum over s_4 and s_5 and arrive at $\square^*_{3,2}(s_1^2 0,\, 1 s_7^8) = 0$.

Further, we are going to verify the equality $\square^*_{3,2}(s_1^2 1,\, 1 s_7^8) = 0$ for $s_1^2,\, s_7^8 \in S^2$, which is equivalent to

$$[1 s_7^8] - \sum_{s_5\in S} \frac{[s_5 1 s_7 s_8]}{[s_2 1]\,[s_5 1]} \sum_{s_4\in S} [s_2 1 s_4 s_5]\,(1 s_4 s_5 1) = 0.$$

This will be clear if we show that

$$\nabla(s_2, s_5) = [s_2 1]\,[s_5 1] - \textstyle\sum_{s_4\in S}[s_2 1 s_4 s_5]\,(1 s_4 s_5 1) = 0$$

for every s_2 and s_5 from S. But,

$$[01]^{-1}\nabla(0,1) = [11] - [01]\,(011) - [11]\,(111) = 0$$

is straightforward and

$$\nabla(1,0) = [11]\,[01] - [1100]\,(001) - (111)\,[1101] = 0$$

owing to $(001) = (111)$ which is a consequence of

$$[000][11] - [011][00] = [001]^{-1}\big([0001][11][00] - [0011][00][01]\big) = 0.$$

The fact

$$\textstyle\sum_{s_2\in S}\nabla(s_2, s_5) = [1][s_5 1] - \sum_{s_4\in S}[s_4 s_5 1] - [0 s_4 s_5 1] = 0$$

implies that ∇ is identically zero, indeed.

At this moment we know $\Box^{*}_{3,2}(\cdot, 1s_7^8) = 0$ for $s_7^8 \in S^2$. The multiplication by (s_0^3) and the summation over $s_3 \in S$ will give $\Box^{*}_{3,3}(\cdot, 1s_8^9) = 0$. Now, we sum over s_9 and compare the result with $\Box^{*}_{3,2}(\cdot, 11s_8)$. We have $\Box^{*}_{3,2}(\cdot, 01s_8) = 0$. Repeating the same trick we arrive at $\Box^{*}_{3,2}(\cdot, 001) = 0$. But, the sum of $\Box^{*}_{3,2}(\cdot, s_6^8)$ over $s_6^8 \in S^3$ is zero and thus $\Box^{*}_{3,2}(\cdot, 000) = 0$. Since $\Box_{3,2}$ is identically zero the sequence ξ is 2-dependent. ■

THEOREM 1. – *Let α, $\beta \in \mathcal{R}$ satisfy the two inequalities*

$$\pm(4\alpha - 3\beta - \beta^3) \leq 1 - \beta^2.$$

The binary 3-Markov sequence $\zeta^{\alpha,\beta}$, which has its distribution of first four variables proportional to the function given by the following table is 2-dependent.

0000	$(1-\beta)^2\,(1+\beta^2-2\alpha)$
1000	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
0100	$(1-\beta)^3\,(1+\beta)+4(1-\beta)\,(\beta-\alpha)$
1100	$(1-\beta^2)^2+8\beta(\beta-\alpha)$
0010	$(1+\beta^2-2\alpha)\,(1-\beta^2+2\beta-2\alpha)$
1010	$(1-\beta^2)^2-4(\alpha-\beta)^2$
0110	$(1-\beta)^2\,(1+\beta^2+2\alpha)$
1110	$(1-\beta^2)\,(1+\beta^2+2\alpha)$
0001	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
1001	$(1+\beta)^2\,(1+\beta^2-2\alpha)$
0101	$(1-\beta^2)^2$
1101	$(1+\beta)^3\,(1-\beta)+4(1+\beta)\,(\alpha-\beta)$
0011	$(1+\beta^2)^2-4\alpha^2$
1011	$(1+\beta^2+2\alpha)\,(1-\beta^2+2\alpha-2\beta)$
0111	$(1-\beta^2)\,(1+\beta^2+2\alpha)$
1111	$(1+\beta)^2\,(1+\beta^2+2\alpha)$

Every $(3,2,2)$-sequence ξ is equal in distribution to some $\zeta^{\alpha,\beta}$ or to the sequence obtained from some $\zeta^{\alpha,\beta}$ by interchanging zeros and ones.

Proof. – The proportionality factor is $1/16$. The conditions imposed on α and β restrict the parameters to be between -1 and 1, see Figure 1, and

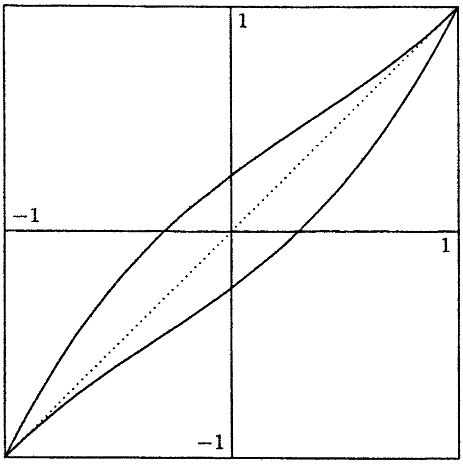

Fig. 1. – Parameters (α, β) are from the region bordered by two cubic curves.

guarantee that all entries of the table are nonnegative; notably the critical inequalities are $[0100] \geq 0$ and $[1101] \geq 0$.

It is easy to verify that $\sum_{t \in S}[t\, s_1^3] - [s_1^3 t] = 0$, $s_1^3 \in S_3$, i.e. that every sequence $\zeta^{\alpha,\beta}$ is strictly stationary. To this end we remark that

$$\begin{array}{ll} 8\,[000] = (1-\beta)(1+\beta^2-2\alpha) & 8\,[010] = (1-\beta)(1-\beta^2+2\beta-2\alpha) \\ 8\,[111] = (1+\beta)(1+\beta^2+2\alpha) & 8[101] = (1+\beta)(1-\beta^2+2\alpha-2\beta) \\ 8\,[100] = (1+\beta)(1+\beta^2-2\alpha) & 8[110] = (1-\beta)(1+\beta^2+2\alpha) \end{array}$$

and $[100] = [001]$, $[110] = [011]$. It is also not difficult to see that $\zeta^{\alpha,\beta}$, $\alpha \neq \beta$, fulfils the assumptions of Proposition 2, especially

$$(0001) = (1001) = \frac{1+\beta}{2} = (0111) = (1111)$$
$$(0011) = (1011) = \frac{1+\beta^2+2\alpha}{2(1+\beta)}$$

and $\triangle_2(10) \neq 0$. Hence, all sequences $\zeta^{\alpha,\beta}$ are 2-dependent. Note that $\zeta^{\alpha,\alpha}$ are i.i.d., $|\alpha| \leq 1$ (the dotted segment in Figure 1).

In the opposite direction, let ξ be a (3,2,2)-sequence. By Lemmas 3 and 6 we know that up to switching between zeros and ones $\triangle_2(st) = 0$ for $st \neq 10$. If also $\triangle_2(10) = 0$ then ξ is i.i.d. by Lemma 4 and thus equal in distribution to some $\zeta^{\alpha,\alpha}$. If $\triangle_2(10) \neq 0$ then $[0st1] = [0s]\,[t1]$ for $s,\ t \in S$ by Proposition 2. Thus, with $st = 11$ here, the quadratic equation in $[111]$

$$[111]^2 - [11]\,[111] + [11]^2\,[01] = 0$$

has nonnegative discriminant equal to $[11]^2\,(1-4[01])$. We set $\beta^2 = 1-4[01]$ and then we have $2[111] = [11]\,(1+\beta)$. If we take $\alpha = 2[1]-1$ we can compute $4[00] = 1+\beta^2-2\alpha$ and $4[11] = 1+\beta^2+2\alpha$. Using the listed properties of ξ and the stationarity it is easy, but a bit laborious, to compute first the probabilities $[s_1 s_2 s_3]$ and then to construct the whole table. ■

COROLLARY 3. – *The triple* $\langle 3,2,2\rangle$ *is the index of* $\zeta^{0.1,0}$.

Remark 1. – Let us mention that the sequence $\zeta^{-\alpha,\alpha}$, $\alpha \neq 0$ small, has the same index as $\zeta^{0.1,0}$. In addition, every its two consequent variables are independent. But, no of the sequences $\zeta^{\alpha,\beta}$, $\alpha \neq \beta$, is 2-independent.

2. From the topological point of view the class of $(3,2,2)$-sequences with the week topology is homeomorphic to two closed circles (disks) pasted together along its diameters; the common diameter corresponds to the i.i.d. sequences and switching between circles to switching between 0 and 1.

3. Reversing time in a $(3,2,2)$-sequence indexed by integer numbers one obtains again a $(3,2,2)$-sequence. In our parametrization this corresponds to the transition $0 \leftrightarrow 1$ and $(\alpha,\beta) \leftrightarrow (-\alpha,-\beta)$ simultaneously.

6. $(3,3,2)$-SEQUENCES

In this section we will describe parametrically all binary 3-dependent sequences that are Markov of order 3. Since all of them which are 2-dependent were investigated in the previous section we concentrate on the non-2-dependent ones. This will close the description of all $(3,m,2)$-sequences.

PROPOSITION 3. – *Let* ξ *be a binary* 3*-Markov sequence satisfying* $\triangle_2(1s) \neq 0 = \triangle_2(0s)$ *for* $s \in S$. *Then* ξ *is* 3*-dependent if and only if* $[111] = (001)^3$, $[0s01] = [0s][01]$, $s \in S$, *and* $(11st) = (00t)$, $s, t \in S$.

Proof. – Obviously, $[s_1^3] > 0$ for $s_1^3 \neq 000$. If ξ is 3-dependent then by 3. $\square_{3,2}(1st,\cdot) = 0$ and by 4. $\square^*_{3,1}(1st,u1v) = 0$ and $\square^*_{3,0}(1st,u11) = 0$, $s,t,u,v \in S$. For $su = 00$ the last equation gives $[000] > 0$ and $[011] = (0t0)(0t01)(011)$, i.e. $[0t][01] = [0t01]$; as a useful consequence we have $[01] = (000)(001)$. The choice $su = 01$ provides $[01] = (11t0)(1t01)$ what reads as $(000) = (1100)$ if $t = 0$ and as $(000) = (1110)$ if $t = 1$. And finaly, $stu = 111$ leads to $[111] = (1111)^3 = (001)^3$.

For the reverse implication we will first demonstrate $\square^*_{3,0}(1st,u11) = 0$. For $u = 0$ this is equivalent to $[01] = (1st0)(st01)$ which certainly holds if $s = 0$; if $s = 1$ we have $[01] = (000)(001)$. For $u = 1$ we want to see that $[111] = (1st1)(st11)(t111)$ or, rewritten, $[t11] = (1st1)(st11)(111t)$. If $t = 0$ this means $[011] = (001)(011)(000)$. If $t = 1$ this amounts $[s11] = (1s11)(111s)(1111)$ which is fulfilled for $s = 1$ as $(1111) = (001)$ and which holds also for $s = 0$ having $[011] = (011)(000)(001)$.

The next step is to show $\square^*_{3,1}(1st, u1v) = 0$. For $u = 1$ this can be obtained from $\square^*_{3,0}(1st, u'11) = 0$ by multiplication with $(u'11v)$ and summation over u'. For $u = 0$ we want to verify

$$[01] = \textstyle\sum_{v\in S}(1stv)(stv0)(tv01)$$

which is clear for $t = 0$: the product equals $(1s0v)[01]$. If $t = 1$ then we rewrite it into

$$(000) = \textstyle\sum_{v\in S}(1s1v)(s1v0)$$

being satisfied for $s = 1$. For $s = 0$ we have

$$\begin{aligned}(000)[01] &= [0100] + [0110] = [01] - [0101] - [0111]\\ &= [01] - [01]^2 - (1110)[111],\end{aligned}$$

which can be casted into $(001)[01] = [01]^2 + (000)(001)^3$ and $(001) = [01] + (001)^2$.

Knowing that $\square^*_{3,1}(1st, u1v) = 0$ we can obtain $\square^*_{3,2}(1st, 1vu) = 0$ and $\square^*_{3,2}(s1t, u1v) = 0$. These two equalities imply $\sum_{t\in S}\square^*_{3,2}(11s, 00t) = 0$ where both summands must equal zero due to $(001)\square^*_{3,2}(\cdot, 000) = (000)\square^*_{3,2}(\cdot, 001)$ (cf. the fact 2.). Hence, $\square^*_{3,2}(11s, \cdot) = 0$ and then $\square^*_{3,3}(11s, \cdot) = 0$ and $\square^*_{3,3}(s11, \cdot) = 0$ arguing as usually. But owing to the equality $(110t)(10tu) = (s00t)(00tu)$ we have $\square^*_{3,2}(110, \cdot) = \square^*_{3,2}(s00, \cdot)$ whence $\square^*_{3,3}(s00, \cdot) = 0$ and $\square^*_{3,4}(s00, \cdot) = 0$. We can write now

$$0 = \square_{3,4}(000, \cdot) + \square_{3,4}(100, \cdot) = \square_{3,3}(000, \cdot) + \square_{3,3}(001, \cdot)$$

and deduce $\square^*_{3,3}(s01, \cdot) = 0$. Thence $\square_{3,3}(\cdot, \cdot) = 0$ and ξ is 3-dependent. ■

THEOREM 2. – *Let α, $\beta \in \mathcal{R}$ satisfy the two inequalities*

$$-(1-\beta^2)^2 \leq 8\alpha - 8\beta \leq (1-\beta)^3\,(1+\beta).$$

The binary 3-Markov sequence $\theta^{\alpha,\beta}$, which has its distribution of first four variables proportional to the function given by the following table is 3-dependent.

0000	$(1-\beta)^2(1+\beta^2-2\alpha)$
1000	$(1-\beta^2)(1+\beta^2-2\alpha)$
0100	$(1-\beta)^3(1+\beta)+8(\beta-\alpha)$
1100	$(1-\beta^2)^2+4(1-\beta)(\alpha-\beta)$
0010	$(1+\beta^2-2\alpha)(1+3\beta-\beta^2+\beta^3-4\alpha)/(1-\beta)$
1010	$(1-\beta^2+2\alpha-2\beta)(1+3\beta-\beta^2+\beta^3-4\alpha)/(1-\beta)$
0110	$(1-\beta^2)^2+8(\alpha-\beta)$
1110	$(1-\beta^2)(1+\beta)^2$
0001	$(1-\beta^2)(1+\beta^2-2\alpha)$
1001	$(1+\beta)^2(1+\beta^2-2\alpha)$
0101	$(1-\beta^2)^2$
1101	$(1+\beta)^3(1-\beta)+4(1+\beta)(\alpha-\beta)$
0011	$(1+\beta^2-2\alpha)(1-3\beta-\beta^2-\beta^3+4\alpha)/(1-\beta)$
1011	$(1-\beta^2+2\alpha-2\beta)(1-3\beta-\beta^2-\beta^3+4\alpha)/(1-\beta)$
0111	$(1-\beta^2)(1+\beta)^2$
1111	$(1+\beta)^4$

Every $(3,3,2)$*-sequence* ξ *which is not* 2*-dependent equals in distribution to some* $\theta^{\alpha,\beta}$, $\alpha \neq \beta$, *up to the switching of zeros and ones or up to the time reversal.*

Proof. – The proportionality factor is $1/16$. The conditions imposed on α and β restrict the parameters to be strictly between -1 and 1, see Figure 2, and guarantee that all entries of the table are nonnegative (the critical inequalities are $[0100] \geq 0$ and $[0110] \geq 0$). By continuity, $\theta^{1,1}$ will be a constant sequence. The dashed curves in Figure 2 remind the restrictions from Theorem 1 which can be here interpreted as $[010] \geq 0$ and $[011] \geq 0$.

It is easy to verify that the sequences $\theta^{\alpha,\beta}$ are strictly stationary. For this purpose we write down

$8\,[000] = (1-\beta)(1+\beta^2-2\alpha)$ $\quad 8\,[111] = (1+\beta)^3$

$8\,[100] = (1+\beta)(1+\beta^2-2\alpha)$ $\quad 8\,[010] = (1+3\beta-\beta^2+\beta^3-4\alpha)$

$8\,[101] = (1+\beta)(1-\beta^2+2\alpha-2\beta)$ $\quad 8\,[110] = (1-3\beta-\beta^2-\beta^3+4\alpha)$.

From

$$(0001) = (1001) = \frac{1+\beta}{2} = (1101) = (1111)$$

$$(0011) = (1011) = \frac{1-3\beta-\beta^2-\beta^3+4\alpha}{2(1-\beta^2)}$$

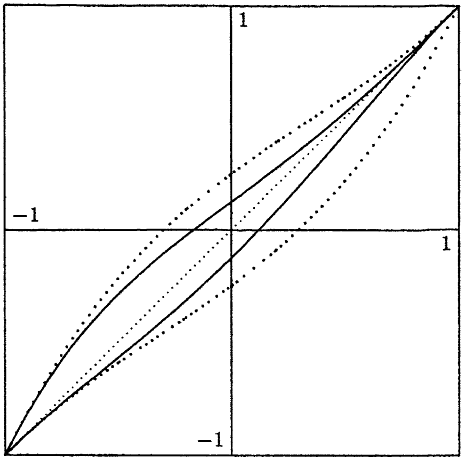

Fig. 2. – Parameters (α, β) are from the region bordered by two biquadratic curves.

we see that the sequences $\theta^{\alpha,\beta}$ satisfy the assumptions of Proposition 3 and are thus 3-dependent.

Let ξ be a (3,3,2)-sequence which is not 2-dependent. Switching zeros and ones and reversing time, if necessary, we know that $\triangle_2(0s) = 0$ and $\triangle_2(1s) \neq 0$, $s \in S$; see the lemmas of Section 4. By Proposition 3, $[000][001] = [0001][00] = [00]^2[01]$, which can be casted into a quadratic equation in [000] similarly as in the proof of Theorem 1. The equation has nonnegative discriminant equal to $[11]^2\,(1 - 4[01])$. We set $\beta^2 = 1 - 4[01]$, $\alpha = 2[1] - 1$ and then we can easily compute [000], [001] and [101]. Using $[111] = (011)^3$ we obtain [111], [110] and [010]. From $(11st) = (00t)$ and $(s0tu) = (0tu)$ the whole table can be computed with a little effort. ■

COROLLARY 4. – *The triple* $\langle 3, 3, 2\rangle$ *is the index of* $\theta^{0,0.1}$.

Remark. – From the topological point of view the union of the classes of $(3, m, 2)$-sequences over $m \geq 0$ finite, with the week topology, is homeomorphic to six closed disks pasted together along its diameters. The common diameter corresponds to the i.i.d. sequences, two disks to the $(3, 2, 2)$-sequences and the remaining four disks without their common diameter to the $(3, 3, 2)$-sequences that are not 2-dependent.

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REFERENCES

[1] AARONSON J., GILAT D. and KEANE M. S., On the structure of 1-dependent Markov shifts, *J. Theor. Probab.* Vol. **5**, 1992, pp. 545-561.

[2] BURTON R. M., GOULET M. and MEESTER R. W. J., On 1-dependent processes and k-block factors, *The Annals of Probability* Vol. **21**, 1993, pp. 2157-2168.

[3] HORN R. A. and JOHNSON C. R., *Matrix Analysis*, Cambridge University Press, Cambridge.

[4] MATÚŠ F., Independence and Radon projections on compact groups, PhD. theses (in Slovak), Institute of Information Theory and Automation, Prague, 1988.

[5] MATÚŠ F., Stochastic independence, algebraic independence and abstract connectedness, *Theor. Comp. Science* Vol. **134**, 1994, pp. 455-471.

[6] MATÚŠ F., On two-block-factor sequences and one-dependence, *Proc. Amer. Math. Soc.* Vol. **14**, 1996, pp. 1237-1242.

[7] MATÚŠ F., Conditional independence structures examined via minors, *Annals of Mathematics and AI* Vol. **21**, 1997, pp. 99-128.

[8] PEARL J., *Probabilistic Reasoning in Intelligent Systems*, Morgan Kaufman, San Meteo, California, 1988.

[9] ROSENBLATT M. and SLEPIAN D., Nth order Markov chains with every N variables independent, *J. Soc. Indust. Appl. Math.* Vol. **10**, 1962, pp. 537-549.

[10] SENETA E., *Non-Negative Matrices*, G. Allen & Unwin Ltd., London, 1973.

[11] VALK V. de, One-dependent processes, PhD. theses, Delft Univ. Technology, Delft Univ. Press, 1988.

[12] VALK V. de, One-dependent processes: two-block-factors and non-two-block-factors, CWI Tract 85, Amsterdam, 1994.

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