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Dynamical Percolation

by

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ABSTRACT. – We study bond percolation evolving in time in such a way that the edges turn on and off independently according to a continuous time stationary 2-state Markov chain. Asking whether an infinite open cluster exists for a.e. t reduces (by Fubini's Theorem) to ordinary bond percolation. We ask whether "a.e. t" can be replaced by "every t" and show that for sub- and supercritical percolation the answer is yes (for any graph), while at criticality the answer is no for certain graphs. For instance, there exist graphs which do not percolate at criticality for a.e. t, but do percolate for some exceptional t. We show that for $\mathbb{Z}^d$, $d \geq 19$, there is a.s. no infinite open cluster for all t at criticality. We give a sharp criterion for a general tree to have an infinite open cluster for some t, in terms of the effective conductance of the tree (analogous to a criterion of R. Lyons for ordinary percolation on trees). Finally, we compute the Hausdorff dimension of the set of times for which an infinite open cluster exists on a spherically symmetric tree.

RÉSUMÉ. – Nous étudions un processus de percolation qui évolue dans le temps de telle manière que les arêtes changent d'état indépendamment les unes des autres selon les lois d'une chaîne de Markov continue et stationnaire à deux états. La question de l'existence t p.p. d'un amas infini et ouvert se réduit (par le Théorème de Fubini) au cas d'un processus de percolation ordinaire. Nous posons la question si "t p.p." peut être remplacé

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par "tout t" et montrons que c'est toujours le cas pour les processus de percolation souscritiques ou sûrcritiques, tandisque la résponse à cette question est non pour certains graphes dans le cas critique. Ainsi il y a des graphes qui ne percolent pas t p.p. dans le cas critique mais peuvent bien le faire pour certaines valeurs exceptionelles de t. Nous montrons aussi qu'il ne peut exister d'amas infini et ouvert dans $\mathbb{Z}^d$, $d \geq 19$ pour toutes les valeurs de t et ceci dans le cas critique. Nous donnons, dans le cas d'un arbre général, un critère tranchant de l'existence d'un amas infini et ouvert pour quelques valeurs de t inspiré par un critère de R. Lyons dans le cas de processus ordinaires de percolation sur des arbres. Finalement, nous déterminons la dimension de Hausdorff de l'ensemble de moments pour lesquels un amas infini et ouvert existe sur un arbre sphériquement symmétrique.

1. INTRODUCTION

Consider bond percolation on an infinite connected locally finite graph G, where for some $p \in [0,1]$ each edge (bond) of G is, independently of all others, open with probability p and closed with probability $1-p$. Write $P_{G,p}$ for this product measure, or simply P_p when no confusion arises. The main questions in percolation theory (see [10]) deal with the possible existence of infinite connected components (clusters) in the random subgraph of G consisting of all vertices and all open edges. Write $\mathcal{C}$ for the event that there exists such an infinite cluster. By Kolmogorov's 0-1 law, the probability of $\mathcal{C}$ is, for fixed G and p, either 0 or 1. Since $P_p(\mathcal{C})$ is nondecreasing in p, there exists a critical probability $p_c = p_c(G) \in [0,1]$ such that

$$P_p(\mathcal{C}) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c. \end{cases}$$

At $p = p_c$ we can have either $P_p(\mathcal{C}) = 0$ or $P_p(\mathcal{C}) = 1$, depending on G.

In this paper we consider a dynamical variant of percolation. Given $p \in (0,1)$, we want the set of open edges to evolve so that at any fixed time $t \geq 0$, the distribution of this set is P_p. The most natural way to accomplish this is to let the distribution at time 0 be given by P_p, and to let each edge change its status (open or closed) according to a continuous time, stationary 2-state Markov chain, independently of all other edges. For an edge e of G, write $\eta_t(\mathrm{e}) = 1$ if e is open at time t, and $\eta_t(\mathrm{e}) = 0$ otherwise. The entire configuration of open and closed edges at time t, denoted η_t, can then be regarded as an element of $X = \{0,1\}^E$ (where E is the edge set of G). The evolution of η_t is a Markov process, and can be viewed as the simplest type of particle system. Each edge flips (changes its value) at rate

$$\lambda(\eta_t, \mathrm{e}) = \begin{cases} p & \text{if } \eta_t(\mathrm{e}) = 0 \\ 1-p & \text{if } \eta_t(\mathrm{e}) = 1 \end{cases}$$

and the probability that two edges flip simultaneously is 0. Write $\boldsymbol{\Psi}_{G,p}$ (or $\boldsymbol{\Psi}_p$) for the underlying probability measure of this Markov process, and write $\mathcal{C}_t$ for the event that there is an infinite cluster of open edges in η_t. Since P_p is a stationary measure for this Markov process, Fubini's theorem implies that

$$\begin{cases} \boldsymbol{\Psi}_p(\ \mathcal{C}_t \text{ occurs for Lebesgue a.e. } t) = 1 & \text{if } P_p(\mathcal{C}) = 1 \\ \boldsymbol{\Psi}_p(\neg\mathcal{C}_t \text{ occurs for Lebesgue a.e. } t) = 1 & \text{if } P_p(\mathcal{C}) = 0 \end{cases}$$

where $\neg\mathcal{C}_t$ denotes the complement of $\mathcal{C}_t$. The main question studied in this paper is

For which graphs can the quantifier "for a.e. t" in the above statements be replaced by "for every t"?

To our knowledge, this question has not been studied before, although questions similar in spirit (when can "a.e." be replaced by "every"?) have been dealt with in a wide variety of contexts, e.g. random coverings [26], monotone couplings of percolation processes [1], and various path properties of Brownian motion [17]. Even closer in spirit is the study of quasi-everywhere properties of Brownian motion, *i.e.* properties that with probability 1 hold at all times for the canonical "Ornstein-Uhlenbeck" diffusion on Wiener space (see [9], [23] and the references therein).

The initial impetus for our work came from a question of Paul Malliavin concerning a probabilistic model of lightning. We now mention several other motivations.

1. In analogy with the work of Fukushima [9] where a capacity on Wiener space is introduced, dynamical percolation allows to distinguish different events of P_p-measure zero by introducing a Choquet capacity on analytic subsets of $X = \{0,1\}^E$:

$$\mathrm{Cap}(A) = \Psi_p[\exists t \in (0,1] \text{ such that } \eta_t \in A] \quad \text{for analytic } A \subset X\,.$$

One can then say that an event in X holds *quasi-everywhere* if its complement has zero capacity.

2. On the more applied side, there are close connections between percolation estimates on finite graphs and reliability theory (Compare

Chapter 2 in [10] with Chapter 7 in [3].) In [3] the failure probability of a network where individual component failures are independent, is interpreted as a disconnection probability of two vertices v, w in a random graph. In a time-dependent model it is natural to assume that component lifetimes and service times are exponentially distributed, whence the probability of a network failing throughout a time interval I is

$$\Psi_p[v \text{ and } w \text{ are in distinct components of } \eta_t\ \forall t \in I],$$

for an appropriate choice of time unit and parameter p.

3. The set of vertices in a tree that connect to the root in η_t for some $t \in I$, defines a *target percolation* in the terminology of [21], section 4. One of our main results, Theorem 1.5, establishes Conjecture 1 of [21], p. 124 for this class of target percolations.

4. Lyons' precise estimates in [20] for percolation probabilities on trees, have been applied in [24] to natural questions concerning intersections of ranges of stochastic processes in Euclidean space; we believe that a similar relation should exist between dynamical percolation on trees and certain path-valued processes in space.

We start by observing that for $p \neq p_c$, we can indeed replace "a.e. t" by "every t".

PROPOSITION 1.1. – *For any graph G we have*

$$\begin{cases} \Psi_p(\mathcal{C}_t \text{ occurs for every } t) = 1 & \text{if } p > p_c(G) \\ \Psi_p(\neg\mathcal{C}_t \text{ occurs for every } t) = 1 & \text{if } p < p_c(G)\,. \end{cases} \tag{1}$$

At the critical value $p_c(G)$ the situation is more delicate. We say that a graph G "exhibits **flickering** percolation" (in short, "G is flickering") if at $p = p_c(G)$ we have $P_{G,p}(\mathcal{C}) = 0$ but $\Psi_{G,p}\Big(\bigcup_{t>0} \mathcal{C}_t\Big) = 1$. (The latter probability is 0 or 1 for any graph.) In words, on a flickering graph at criticality, for almost all t there is no infinite cluster, but for some exceptional t the infinite cluster "flickers by". Call a graph G **tame** if there are a.s. no such exceptional times, *i.e.*, if $\Psi_{G,p_c}\Big(\bigcup_{t>0} \mathcal{C}_t\Big) = 0$.

THEOREM 1.2. – *There exists a flickering graph G_1. There also exists a graph G_2 such that for $p = p_c(G_2)$ we have $P_{G_2,p}(\mathcal{C}) = 1$, yet $\Psi_{G_2,p}(\bigcap_{t>0}\mathcal{C}_t) = 0$.*

The graphs for which percolation problems have been studied most extensively are the nearest-neighbor graphs on the cubical lattices $\mathbb{Z}^d$, and trees. On $\mathbb{Z}^2$, the critical value p_c is $1/2$ and $P_{p_c}(\mathcal{C}) = 0$ (see Kesten [14]); for $d > 2$ the precise value of $p_c(\mathbb{Z}^d)$ is not known. Hara and Slade [11] showed that $P_{p_c}(\mathcal{C}) = 0$ for $\mathbb{Z}^d$ if $d \geq 19$, and it is certainly believed that this holds for all d. On the regular tree $\mathbf{T}^k$, where each vertex has $k+1$ neighbors, a well-known branching process argument shows that $p_c = 1/k$ and $P_{p_c}(\mathcal{C}) = 0$.

THEOREM 1.3. – *Let G be either the integer lattice $\mathbb{Z}^d$ with $d \geq 19$ or a regular tree. Then G is tame*, i.e., $\mathbf{\Psi}_{G,p_c}(\neg\mathcal{C}_t$ *occurs for every* $t) = 1$.

For the homogeneous graphs considered in the theorem, it is natural to introduce the quantity $\theta(p)$ which is the probability under P_p that a given vertex percolates (*i.e.* has an open path to infinity). Clearly, $\theta(p) > 0$ if and only if $P_p(\mathcal{C}) = 1$. The $\mathbb{Z}^d$ part of Theorem 1.3 is based on the result of Hara and Slade [11] that in dimension $d \geq 19$,

$$\theta(p) = O(p - p_c)\,. \tag{2}$$

That this also holds for regular trees is a much easier fact. It is generally believed that (2) holds for $\mathbb{Z}^d$ for all $d \geq 6$, in which case we can replace $d \geq 19$ by $d \geq 6$ in Theorem 1.3. For $2 \leq d \leq 5$ it is believed (and for $d = 2$ known [16]) that (2) fails. We do not know what to conjecture as to whether the lattices $\mathbb{Z}^d$ for $2 \leq d \leq 5$ are flickering or tame.

On the other hand, for $\mathbb{Z}^d$ in any dimension we can show that if $P_{p_c}(\mathcal{C}) = 0$, then $\mathbf{\Psi}_{p_c}$-a.s., for all t the union of all infinite open clusters at time t has zero density (see Corollary 4.2). We prove this via a simultaneous ergodic theorem which, we believe, is of independent interest. Let f be a bounded measurable function defined on $X = \{0,1\}^E$. The lattice $\mathbb{Z}^d$ acts on configurations by translation: for $v \in \mathbb{Z}^d$ and a configuration $\eta \in X$, let $S_v\eta \in X$ be η shifted by v. Let Λ_n denote the cube $\mathbb{Z}^d \cap [-n,n]^d$. For each fixed t, the usual pointwise ergodic theorem for $\mathbb{Z}^d$–actions yields

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} f(S_v\eta_t) = \mathbf{E}[f(\eta_t)] = \mathbf{E}[f(\eta_0)] \quad \mathbf{\Psi}_p\text{-a.s.} \tag{3}$$

where $|\cdot|$ denotes cardinality, and the rightmost equality in (3) is due to the stationarity of η in time. The next theorem shows that when f is continuous, the uncountable intersection of the events in (3) over all t, also has probability 1.

THEOREM 1.4. – *Let $f : X \to \mathbb{R}$ be a continuous function, where $X = \{0,1\}^E$ is equipped with the product topology. Then the event that*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} f(S_v\eta_t) = \mathbf{E}[f(\eta_0)] \text{ for all } t \geq 0$$

has $\mathbf{\Psi}_p$-probability 1.

Next, we consider dynamical percolation on general trees. Lyons [19] characterized p_c for any tree in terms of the Hausdorff dimension of the tree boundary. In [20] he obtained an exact criterion for $P_p(\mathcal{C}) > 0$ in terms of effective electrical resistance; effective resistance is easy to calculate on trees using the parallel and series laws (see Doyle and Snell [5]). Here we obtain such a criterion for dynamical percolation.

For an infinite tree Γ with a vertex ρ designated as the root, we write Γ_n for the set of vertices at distance exactly n from ρ, called the nth *level* of Γ. A tree is called **spherically symmetric** if all vertices on the same level have equally many children. (In §5, definitions for trees will be given more completely.)

THEOREM 1.5. – *Let $\{\eta_t\}$ be a dynamical percolation process with parameter $0 < p < 1$ on an infinite tree Γ. Assign each edge between levels $n-1$ and n of Γ the resistance p^{-n}/n. If in the resulting resistor network the effective resistance from the root to infinity is finite, then $\mathbf{\Psi}_{\Gamma,p}$-a.s. there exist times $t > 0$ such that Γ has an infinite open cluster, while if this resistance is infinite, then a.s. there are no such times. In particular, if Γ is spherically symmetric, then*

$$\mathbf{\Psi}_{\Gamma,p}(\cup_{t>0}\mathcal{C}_t) = 1 \text{ if and only if } \sum_{n=1}^{\infty} \frac{p^{-n}}{n|\Gamma_n|} < \infty \,. \tag{4}$$

Lyons [20] established a criterion for the percolation probability on a general tree Γ to be positive: Suppose that $0 < p < 1$ and assign each edge between levels $n-1$ and n resistance p^{-n}. Then $\mathbf{P}_{\Gamma,p}(\mathcal{C}) > 0$ iff the resulting effective resistance from the root to infinity is finite. Thus a spherically symmetric tree Γ with $p = p_c(\Gamma) \in (0,1)$, is flickering iff the series in (4) converges but $\sum_{n=1}^{\infty} \frac{p^{-n}}{|\Gamma_n|} = \infty$.

In the course of the proof of Theorem 1.5, we obtain bounds for the probability that there exists a time $t \in [0,1]$ for which there is an open path in η_t from the root to the nth level Γ_n. For example, on the regular tree $\mathbf{T}^k$ with $p = 1/k$, this probability is bounded between constant multiples of $1/\log n$. (The probability under $P_{1/k}$ that an open path exists from ρ to the nth level of $\mathbf{T}^k$, is bounded between constant multiples of $1/n$; this follows from Kolmogorov's theorem on critical branching processes, see [2].) For a general tree these bounds, given in Theorem 5.1, can be expressed in terms of the effective resistance from the root to Γ_n, and the ratio of the upper and lower bounds is an absolute constant.

For a flickering graph, the set of percolating times at criticality has zero Lebesgue measure, so it is natural to ask for its Hausdorff dimension. For spherically symmetric trees there is a complete answer.

THEOREM 1.6. – *Let* $p \in (0,1)$ *and let* Γ *be a spherically symmetric tree. If the set of times* $\{t \in [0,\infty) : \mathcal{C}_t$ *occurs*$\}$ *is a.s. nonempty, then* $\mathbf{\Psi}_p$*-a.s. this set has Hausdorff dimension*

$$\sup\left\{ \alpha \in [0,1] \ : \ \sum_{n=1}^{\infty} \frac{p^{-n} n^{\alpha-1}}{|\Gamma_n|} < \infty \right\}.$$

(*Note that this series converges for* $\alpha = 0$ *by* (4).)

Here are some interesting flickering trees:

Example 1.7. – Let Γ be the spherically symmetric tree where each vertex on level n has 4 children if $n = 1, 2, 4\ldots$ is a power of 2, and 2 children otherwise. Then it is easily seen that $n2^n \leq |\Gamma_n| \leq 2n2^n$ for all $n > 0$. Combining Theorem 1.6 with the result of Lyons quoted after Theorem 1.5, we see that $\mathbf{\Psi}_{1/2}$-a.s. the set of times for which percolation occurs on Γ has Hausdorff dimension 1 but Lebesgue measure 0.

Example 1.8. – Let $0 < p, \beta < 1$, and suppose that Γ is a spherically symmetric tree with $|\Gamma_n| = p^{-n} n^{\beta + o(1)}$ as $n \to \infty$. Then Theorem 1.6 implies that $\mathbf{\Psi}_p$-a.s. the set of times for which percolation occurs on Γ has Hausdorff dimension β.

We remark that although we restrict attention to bond percolation, most of our results and statements have immediate analogues for site percolation.

The rest of this paper is organized as follows. §2 contains the proofs of Proposition 1.1 and Theorem 1.2. In §3 we establish that (2) is a sufficient condition for a graph to be tame, and thus obtain Theorem 1.3. A key step is Lemma 3.4, which shows that on any graph, the set of times that a given vertex percolates in a fixed time interval is a.s. either empty or uncountable. In §4 we prove Theorem 1.4, and apply it to control the density of the infinite clusters on the lattice. §5 contains the proof of the electrical resistance criterion for general trees (Theorem 1.5) in a sharper form. The Hausdorff dimension result, Theorem 1.6, is established in §6; the lower bound on dimension is proved via a capacity estimate, and applies to general trees.

2. NONCRITICAL CASES, AND SOME GRAPHS WITH ATYPICAL TIMES

Consider a dynamical percolation process $\{\eta_t\}$ on a graph G. The definition of the process given in the introduction is equivalent to the following:

Choose η_0 according to P_p. To every edge e of G assign an independent Poisson process with rate 1. At each point t of this process replace $\eta_t(\mathrm{e})$ by an independent choice from the distribution $(1-p,p)$ on $\{0,1\}$.

Indeed suppose, for instance, that $\eta_0(\mathrm{e}) = 0$. Then the number N of points in the relevant Poisson process until the edge e opens is a geometric random variable with parameter p, and the sum of N mean 1 exponential variables (which are mutually independent and independent of N) has an exponential distribution with parameter p. The case $\eta_0(\mathrm{e}) = 1$ is similar, so the equivalence asserted above follows.

Note on measurability. – The space $D_X[0,\infty)$ of mappings from $[0,\infty) \to X$ which are right continuous and possess left limits, is a complete and separable metric space when endowed with the Skorohod metric (see [6]). The distribution of the process $\eta_{(\cdot)}$ is a Borel probability measure on this space. For any vertex v and any $L \geq 1$, the set of pairs $\{(\eta_{(\cdot)},t) : v \text{ is on an open path of length } L \text{ in } \eta_t\}$ is clearly a Borel set for the product topology on $D_X[0,\infty) \times [0,\infty)$. Intersecting these sets for all L shows that the set $\{(\eta_{(\cdot)},t) : v \text{ percolates in } \eta_t\}$ is also a Borel set. Other events considered in this paper, such as $\{\eta_{(\cdot)} : \exists t\ C_t \text{ occurs}\}$ can be obtained from these by countable unions and projections, and hence are in the completion of the Borel σ-field with respect to any prescribed Borel measure. (See [6], Appendix 11.)

Notation. – For $0 \leq a \leq b < \infty$ and any edge e of a graph G, we abbreviate

$$\inf_{[a,b]} \eta(\mathrm{e}) := \inf_{t\in[a,b]} \eta_t(\mathrm{e}).$$

and write $C_{[a,b]}^{\inf}$ for the event that there is an infinite cluster of edges with $\inf_{[a,b]} \eta(\mathrm{e}) = 1$.
Analogously, define $\sup_{[a,b]} \eta$, and let $C_{[a,b]}^{\sup}$ be the event that there is an infinite cluster of edges with $\sup_{[a,b]} \eta(\mathrm{e}) = 1$.

Proof of Proposition 1.1. – **(i)** Suppose $p > p_c$. Let $0 < \epsilon < p - p_c$ and observe that for every edge e,

$$\Psi_p\{\inf_{[0,\epsilon]} \eta(\mathrm{e}) = 1\} = p\exp(-(1-p)\epsilon) > p - \epsilon > p_c\,.$$

Since the events $\{\inf_{[0,\epsilon]} \eta(\mathrm{e}) = 1\}$ are mutually independent as e ranges over the edges of G, it follows from the definition of p_c that $\Psi_p[C_{[0,\epsilon]}^{\inf}] = 1$ and therefore

$$\Psi_p(C_t \text{ occurs for all } t \in [0,\epsilon]) = 1\,.$$

Repeating the argument for the intervals $[k\epsilon,(k+1)\epsilon]$ with integer k and using countable additivity, we obtain the supercritical part of the proposition.

(ii) A similar argument proves that for $p < p_c$ there is never an infinite open cluster. We take $\epsilon \in (0, p_c - p)$ and find that

$$\Psi_p\{\sup_{[0,\epsilon]} \eta(e) = 1\} = 1 - (1-p)\exp(-p\epsilon) < p + p\epsilon < p_c\,. \tag{5}$$

Therefore $\Psi_p\left(\mathcal{C}^{\sup}_{[0,\epsilon]}\right) = 0$, whence there is a.s. no infinite cluster for any $t \in [0, \epsilon]$. Countable additivity concludes the argument. □

We now turn to the proof of Theorem 1.2. To construct G_1 and G_2 we will start with graphs with prescribed p_c, and replace the edges of these graphs by certain aggregates of new edges. For $n \in \{1, 2 \ldots\}$, we define an n-**aggregate** between two vertices x and y to consist of 2^n disjoint paths (called *branches*) from x to y, each branch consisting of n edges. See Figure 1. Call an n-aggregate A_n open if it contains an open path from x to y, and call it closed otherwise.

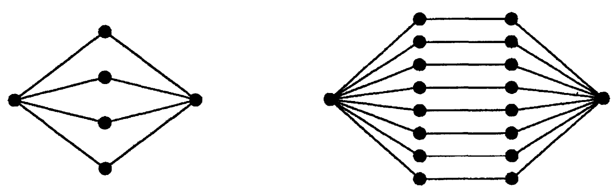

FIG. 1. – The n-aggregates A_2 and A_3.

LEMMA 2.1

$$\lim_{n\to\infty} P_p[A_n \textit{ is open}] = \begin{cases} 0 & \textit{if } p < \dfrac{1}{2} \\ 1 - e^{-1} & \textit{if } p = \dfrac{1}{2} \\ 1 & \textit{if } p > \dfrac{1}{2}. \end{cases}$$

Proof. – Each of the branches of A_n is open with probability p^n, so

$$P_p[A_n \text{ is closed }] = (1 - p^n)^{2^n}\,.$$

Letting $n \to \infty$ proves the lemma. □

Next, we consider the behavior of a single branch under the dynamics of our continuous time Markov chain. We let $\mathbf{E}_p$ denote expectation under Ψ_p.

LEMMA 2.2. – *Let B be a path of length n, and let $s > 0$. Denote by $\tau(B, s)$ the total amount of time in $[0, s]$ that B is open. Then*

(i) $\mathbf{E}_p[\tau(B, s)] = sp^n$.

(ii) $\mathbf{E}_p[\tau(B, s) \mid B \text{ is open at time } 0] \leq \frac{e^s}{nq}$, where $q = 1 - p$.

(iii) $\Psi_p\{\tau(B,s) > 0\} \geq se^{-s}nqp^n$.
(iv) $\Psi_p\{B$ is closed in η_0 and in $\eta_t\} = 1 - 2p^n + [p(p+qe^{-t})]^n$.

Proof. – **(i)** This is immediate from Fubini's theorem.

(ii) At each edge e of B, the chain $\eta_t(\mathrm{e})$ switches to the distribution (q,p) on $\{0,1\}$ at rate 1, and therefore for $t > 0$,

$$\Psi_p[B \text{ is open at time } t \mid B \text{ is open at time } 0] = (p+qe^{-t})^n . \tag{6}$$

Consequently, the conditional expectation $\mathbf{E}_p[\tau(B,s)\,|\,B$ is open at time 0] equals

$$\int_0^s (p+qe^{-t})^n\,dt \leq e^s \int_0^s (p+qe^{-t})^n e^{-t}\,dt \leq e^s \int_0^1 (p+qx)^n\,dx \leq \frac{e^s}{nq} .$$

(iii) By conditioning on the first time that B is open and using the strong Markov property, we infer from (ii) that

$$\mathbf{E}_p[\tau(B,s) \mid \tau(B,s) > 0] \leq \frac{e^s}{nq} .$$

Combining this with (i) yields

$$\Psi_p\{\tau(B,s) > 0\} = \frac{\mathbf{E}_p[\tau(B,s)]}{\mathbf{E}_p[\tau(B,s) \mid \tau(B,s) > 0]} \geq se^{-s}nqp^n .$$

(iv) This follows from the equation $\Psi_p\{B$ is open in η_0 and in $\eta_t\} = [p(p+qe^{-t})]^n$, which is a consequence of (6). □

The next lemma describes the behavior of large aggregates when the status of the edges evolves according to $\Psi_{\frac{1}{2}}$.

LEMMA 2.3. – *For any $s > 0$, we have*

$$\lim_{n\to\infty} \Psi_{\frac{1}{2}}(A_n \text{ is open for some } t \in [0,s]) = 1 \tag{7}$$

and

$$\lim_{n\to\infty} \Psi_{\frac{1}{2}}(A_n \text{ is closed for some } t \in [0,s]) = 1. \tag{8}$$

Proof. – Since the 2^n branches in A_n evolve independently, Lemma 2.2(iii) implies that

$$\Psi_{\frac{1}{2}}(A_n \text{ is closed for all } t \in [0,s]) \leq (1-se^{-s}n2^{-n-1})^{2^n} \to 0 \text{ as } n \to \infty ,$$

which establishes (7).

To prove (8), we apply Lemma 2.2(iv) and stationarity to obtain

$$\Psi_{\frac{1}{2}}\{B \text{ is closed in } \eta_r \text{ and in } \eta_t\} = 1 - 2^{1-n} + \left(\frac{1+e^{-|r-t|}}{4}\right)^n ,$$

for each branch B of A_n and all $r,t \in [0,s]$.

This yields a bound on the covariances of the indicators $I_t = \mathbf{1}_{\{A_n \text{ is closed in } \eta_t\}}$:

$$\mathbf{E}_{\frac{1}{2}}(I_r I_t) - \mathbf{E}_{\frac{1}{2}}(I_r)\mathbf{E}_{\frac{1}{2}}(I_t) = \left[1 - 2^{1-n} + \left(\frac{1+e^{-|r-t|}}{4}\right)^n\right]^{2^n} - [1 - 2^{1-n} + 4^{-n}]^{2^n} \leq 2^n \left(\frac{1+e^{-|r-t|}}{4}\right)^n, \tag{9}$$

where the inequality $x^{2^n} - y^{2^n} \leq 2^n|x-y|$ for $x, y \in (0,1)$ was used in the last step.

Let $Y_n = \int_0^s I_t \, dt$ denote the total amount of time in $[0, s]$ that A_n is closed. Lemma 2.1 implies that $\mathbf{E}_{\frac{1}{2}}(Y_n) \to se^{-1}$ as $n \to \infty$. By Fubini's Theorem and (9), we have

$$\mathbf{Var}_{\frac{1}{2}}(Y_n) = \int_0^s \int_0^s [\mathbf{E}_{\frac{1}{2}}(I_r I_t) - \mathbf{E}_{\frac{1}{2}}(I_r)\mathbf{E}_{\frac{1}{2}}(I_t)] \, dr \, dt \leq \int_0^s \int_0^s \left(\frac{1+e^{-|r-t|}}{2}\right)^n dr \, dt .$$

The right-hand side tends to zero by bounded convergence, so using Chebyshev's inequality (or Lemma 5.4), we deduce that $\mathbf{\Psi}_{\frac{1}{2}}[Y_n > 0] \to 1$ as $n \to \infty$. $\square$

Proof of Theorem 1.2. – We first construct G_2. Let G_2' be any graph for which
$0 < p_c(G_2') < 1 - e^{-1}$. (We may for definiteness take $G_2' = \mathbb{Z}^2$). Let $\{e_1, e_2, \ldots\}$ be an enumeration of the edges of G_2', and let $\{n_1, n_2, \ldots\}$ be a sequence of positive integers tending to ∞. We construct G_2 by replacing each edge e_i of G_2' by an n_i-aggregate. It follows from Lemma 2.1 that $p_c(G_2) = 1/2$, with percolation at criticality. We will show that if the sequence $\{n_i\}$ grows rapidly enough, then $\mathbf{\Psi}_{1/2}$-a.s., there are times when G_2 fails to percolate.

Start by using (8) to pick n_1 large enough so that

$$\mathbf{\Psi}_{\frac{1}{2}}\{A_{n_1} \text{ is closed for some } t \in [0,1]\} > 1 - 3^{-1} .$$

Then pick $\delta_1 > 0$ so small that

$$\mathbf{\Psi}_{\frac{1}{2}}\{A_{n_1} \text{ stays closed for a time interval of length} \geq \delta_1 \text{ in } [0,1]\} > 1 - 3^{-1}.$$

We continue inductively. Once n_i and δ_i are chosen, use (8) to pick n_{i+1} so large that

$$\Psi_{\frac{1}{2}}\{A_{n_{i+1}} \text{ is closed for some } t \in [0, \delta_i]\} > 1 - 3^{-i-1};$$

then pick $\delta_{i+1} > 0$ so small that

$$\Psi_{\frac{1}{2}}\{A_{n_{i+1}} \text{ stays closed for a time interval of length } \geq \delta_{i+1} \text{ in } [0, \delta_i]\} > 1 - 3^{-i-1}.$$

This defines the graph G_2. We now attempt to find a time $t \in [0, 1]$ when all the aggregates of G_2 are closed. First look for a closed time-interval $J_1 \subset [0, 1]$ of length δ_1, during which the aggregate A_{n_1} replacing e_1 stays closed. If such an interval J_1 exists, look for a closed interval $J_2 \subset J_1$ of length δ_2, during which the aggregate A_{n_2} replacing e_2 stays closed, and so on.

With probability greater than $\prod_{i=1}^{\infty}(1-3^{-i}) > 1/2$, we obtain an infinite nested sequence of closed intervals $[0, 1] \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ such that for each j, the aggregate A_{n_j} replacing e_j stays closed during J_j, and the length of J_j is δ_j. On that event, at the time $t \in \cap_{j=1}^{\infty} J_j$ all the aggregates A_{n_i} are closed. Hence

$$\Psi_{\frac{1}{2}}(\exists t \in [0, 1] \text{ such that } \mathcal{C}_t \text{ fails }) > \frac{1}{2}.$$

By Kolmogorov's 0-1 law this probability must be 1.

We now construct G_1. We only sketch this construction, as it is similar to the construction of G_2; a completely different example of a flickering graph is in Example 1.7.

Let G_1' be a graph for which $p_c(G_1') \in (1 - e^{-1}, 1)$. Replace the edges $\{e_i\}$ of G_1' by n_i-aggregates for a rapidly increasing sequence $\{n_i\}$. Using Lemma 2.1, we find that $p_c(G_1) = \frac{1}{2}$ with $P_{p_c}(\mathcal{C}) = 0$ on G_1. Applying (7) we get, as in the previous construction, that G_1 is flickering if the n_i are chosen to increase sufficiently fast. □

3. A SUFFICIENT CONDITION FOR TAMENESS

In this section we prove two lemmas concerning the times at which a single vertex percolates, and then apply them to show that regular trees and high-dimensional lattices are tame. For a graph G and a vertex v in G, write $\theta_v(p)$ for $P_p(v \text{ percolates})$. Let N_v denote the cardinality of the set $\{t \in [0,1] : v \text{ percolates in } \eta_t\}$. Recall that $\mathbf{E}_p$ denotes expectation under $\mathbf{\Psi}_p$.

LEMMA 3.1. – *Let G be a graph, let v be a vertex of G and denote by p_c the critical value for percolation on G. Suppose there exists a constant C such that*

$$\theta_v(p) \le C(p - p_c)$$

for all $p > p_c$. Then $\mathbf{E}_{p_c}[N_v] < \infty$.

Proof. – Let $m > 1$. For every edge e of G and every $i \in \{1, 2, \ldots, m\}$ we have

$$\begin{aligned}\mathbf{\Psi}_{p_c}&\left\{\exists t \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \text{ such that } \eta_t(\mathrm{e}) = 1\right\}\\ &= 1 - (1 - p_c)\exp(-p_c/m) \le p_c + p_c/m\,.\end{aligned}$$

Therefore the probability that v percolates in $\{\mathrm{e} : \exists t \in [(i-1)/m, i/m] \text{ with } \eta_t(\mathrm{e}) = 1\}$ is at most $\theta_v(p_c + p_c/m)$. Let $N_{v,m}$ be the number of $i \in \{1, 2, \ldots m\}$ such that v is in an infinite cluster of $\{\mathrm{e} : \exists t \in [(i-1)/m, i/m] \text{ with } \eta_t(\mathrm{e}) = 1\}$. We have

$$\mathbf{E}_{p_c}[N_{v,m}] \le m\theta_v(p_c + \frac{p_c}{m}) \le Cp_c \quad \text{for all } m > 1.$$

Since $N_v \le \liminf_{m\to\infty} N_{v,m}$, Fatou's lemma completes the proof. □

LEMMA 3.2. – *Let $0 < p < 1$ and let G be any graph where $P_{G,p}(\mathcal{C}) = 0$. Consider the process $\{\bar{\eta}_t\}$ obtained from $\{\eta_t\}$ by setting, for every edge e, the set $\{t : \bar{\eta}_t(e) = 1\}$ to be the closure of the set $\{t : \eta_t(e) = 1\}$. Then $\mathbf{\Psi}_p$-a.s., for every vertex v we have*

$$\{t \in [0,\infty) : v \text{ percolates in } \bar{\eta}_t\} = \{t \in [0,\infty) : v \text{ percolates in } \eta_t\}\,.$$

In particular, a.s. this set of times is closed.

Proof. – For each edge e, the process $\{\eta_t(\mathrm{e})\}$ changes its value only at a countable (random) set of times $\{t_j(\mathrm{e})\}_{j\ge1}$. At these times, η and $\bar{\eta}$ agree a.s. on all other edges. The evolution of different edges is independent and the status of a single edge does not affect the existence of infinite clusters, so the assumption on $P_{G,p}(\mathcal{C})$ implies that a.s., at *all* times $\{t_j(\mathrm{e})\}_{j\ge1}$, all open clusters in $\bar{\eta}$ are finite. Since the set of edges is countable, we are done. □

Next, we recall a basic correlation inequality. A probability measure on a partially ordered topological space is said to have **positive correlations** if any two increasing continuous functions have a nonnegative covariance with respect to this measure – see §II.2 in Liggett [18] for background.

LEMMA 3.3 (Harris). – *Let G be a graph with edge set E, and denote $X = \{0,1\}^E$. Let N be a positive integer, and equip X^N with the product topology and the usual (coordinatewise) partial order. Then for any $0 < p < 1$ and any $0 \le t_1 < \ldots < t_N$, the $\mathbf{\Psi}_{G,p}$ distribution of $\eta_{t_1}, \ldots, \eta_{t_N}$, which is a measure on X^N, has positive correlations.*

It is enough to verify this lemma for finite graphs, where it follows from a more general inequality due to Harris [12], asserting that all attractive spin systems with transitions only between comparable states, have positive correlations; see Corollary II.2.21 in [18].

LEMMA 3.4. – *Let G be a graph and let v be a vertex of G. Suppose that $p \in (0,1)$ satisfies $\theta_v(p) = 0$. Then $\mathbf{\Psi}_p$-a.s., the number N_v of times in $[0,1]$ that v percolates is either 0 or ∞.*

Proof. – We proceed similarly to the proof of Lemma 4.2 in [21]. Suppose, for contradiction, that $\mathbf{\Psi}_p(N_v = k) > 0$ for some $0 < k < \infty$. Let $\Lambda_1^* \subset \Lambda_2^* \subset \Lambda_3^* \subset \cdots$ be finite edge sets whose union contains all edges of G, and denote by $\mathcal{F}_n$ the σ-field generated by $\{\eta_t(\mathrm{e}) : \mathrm{e} \in \Lambda_n^*,\ t \in [0,1]\}$. By the martingale convergence theorem,

$$\lim_{n\to\infty} \mathbf{\Psi}_p(N_v = k|\mathcal{F}_n) = \mathbf{1}_{\{N_v=k\}} \qquad \mathbf{\Psi}_p\text{-a.s.}$$

Therefore, using the obvious regular conditional distribution given $\mathcal{F}_n$, we can find n and a realization ω of $\{\eta_t(\mathrm{e}) : \mathrm{e} \in \Lambda_n^*,\ t \in [0,1]\}$ such that $\mathbf{\Psi}_p[N_v = k|\mathcal{F}_n](\omega) \ge 0.9$. Let $\widetilde{\mathbf{\Psi}}_p$ be the conditional distribution of $\mathbf{\Psi}_p$ given ω, so $\widetilde{\mathbf{\Psi}}_p(N_v = k) \ge 0.9$. Explicitly, $\widetilde{\mathbf{\Psi}}_p$ is obtained by setting η_t to ω in Λ_n^*, and independently running dynamical percolation outside Λ_n^*. For $s \ge 0$, denote by A_s the event $\{v \text{ percolates for at least one } t \in [0,s)\}$. Clearly $\widetilde{\mathbf{\Psi}}_p(A_s)$ is a left-continuous function of s. For any prescribed s, we have $\widetilde{\mathbf{\Psi}}_p[v \text{ percolates at time } s] = 0$, so by Lemma 3.2 $\widetilde{\mathbf{\Psi}}_p(A_s)$ is also right-continuous in s. Hence there exists $\gamma \in (0,1)$ such that $\widetilde{\mathbf{\Psi}}_p(A_\gamma) = 0.5$. Consider the event $B_\gamma :=$ $\{v \text{ percolates for at least } k \text{ different times in } [\gamma, 1]\}$. Clearly, $\widetilde{\mathbf{\Psi}}_p(B_\gamma) \ge$ 0.4, because $\{N_v = k\} \subseteq A_\gamma \cup B_\gamma$. The events A_γ and B_γ are increasing, and therefore the conditional probabilities $\widetilde{\mathbf{\Psi}}_p[A_\gamma \mid \eta_\gamma]$ and $\widetilde{\mathbf{\Psi}}_p[B_\gamma \mid \eta_\gamma]$ are increasing functions of η_γ, since the dynamical percolation process is attractive (see [18]) and reversible.

Thus by the Markov property and Lemma 3.3,

$$\widetilde{\mathbf{\Psi}}_p[A_\gamma \cap B_\gamma] = \int \widetilde{\mathbf{\Psi}}_p[A_\gamma \mid \eta_\gamma] \cdot \widetilde{\mathbf{\Psi}}_p[B_\gamma \mid \eta_\gamma] \, d\widetilde{\mathbf{\Psi}}_p$$
$$\geq \widetilde{\mathbf{\Psi}}_p[A_\gamma] \cdot \widetilde{\mathbf{\Psi}}_p(B_\gamma) \geq 0.2 .$$

Since the events $\{N_v = k\}$ and $A_\gamma \cap B_\gamma$ are disjoint, we conclude that $\widetilde{\mathbf{\Psi}}_p(N_v = k) \leq 1 - 0.2 = 0.8$, which is the desired contradiction. □

Remark. – **(i)** Lemma 3.2 states that for any vertex v, $\{t \in [0,1] : v \text{ percolates in } \eta_t\}$ is a.s. a closed set. By the previous lemma applied to all rational time intervals instead of $[0,1]$, this set a.s. has no isolated points, and therefore by elementary topology it is either empty or uncountable.

(ii) Steve Evans has pointed out that the lemma above can also be derived from the general theory of Markov processes.
The proof of Theorem 1.3 is now very short.

Proof of Theorem 1.3. – The graphs considered in the theorem satisfy the assumption of Lemma 3.1 for every vertex v (see [15] for $\mathbf{T}^k$ and [11] for $\mathbb{Z}^d$), whence $\mathbf{E}_{p_c}[N_v] < \infty$ for all v. It follows that every vertex v satisfies $\mathbf{\Psi}_{p_c}\{N_v < \infty\} = 1$, and hence, by Lemma 3.4 we conclude that $\mathbf{\Psi}_{p_c}\{N_v = 0\} = 1$. □

Remark. – For the planar lattice $\mathbb{Z}^2$, the results of [16] imply that, in the terminology of the proof of Lemma 3.1, $\lim_{m\to\infty} \mathbf{E}[N_{v,m}] = \infty$.

4. A SIMULTANEOUS ERGODIC THEOREM AND DENSITY OF CLUSTERS

Denote by E the set of edges in $\mathbb{Z}^d$, and write $\mathrm{e} \in \Lambda_n^*$ if $\mathrm{e} \in E$ has at least one endpoint in the cube Λ_n. If ξ and ξ' are in $X = \{0,1\}^E$, we define

$$\bar{d}(\xi, \xi') := \limsup_{n\to\infty} \frac{1}{|\Lambda_n^*|} \sum_{\mathrm{e}\in\Lambda_n^*} \mathbf{1}_{\{\xi(\mathrm{e})\neq\xi'(\mathrm{e})\}}.$$

LEMMA 4.1. – *Let $f : X \to \mathbf{R}$ be continuous. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\bar{d}(\xi, \xi') \leq \delta$ then*

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} |f(S_v\xi) - f(S_v\xi')| \leq \varepsilon. \tag{10}$$

Proof. – Let $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \ldots$ be an enumeration of the edges of $\mathbb{Z}^d$. Let $\varepsilon > 0$. Since f is continuous, there exists k such that if $\xi^{(1)}, \xi^{(2)} \in X$

satisfy $\xi^{(1)}(\mathrm{e}_i) = \xi^{(2)}(\mathrm{e}_i)$ for all $i \leq k$, then $|f(\xi^{(1)}) - f(\xi^{(2)})| \leq \varepsilon/2$. Now suppose that $\bar{d}(\xi, \xi') < \delta$. Then for any edge e we have

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} \mathbf{1}_{\{S_v\xi(\mathrm{e})\neq S_v\xi'(\mathrm{e})\}} \leq d\delta\,, \tag{11}$$

since the ratio $|\Lambda_n^*|/|\Lambda_n|$ tends to d as $n \to \infty$. For each vertex v such that $S_v\xi(\mathrm{e}_i) = S_v\xi'(\mathrm{e}_i)$ for all $i \leq k$, the corresponding summand in (10) is at most $\varepsilon/2$ by the choice of k. The other summands are controlled by (11), so that altogether

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} |f(S_v\xi) - f(S_v\xi')| \leq \varepsilon/2 + 2kd\delta \cdot \max_X |f|\,. \tag{12}$$

Finally, choose δ such that the right-hand side is smaller than ε. □

Proof of Theorem 1.4. – By countable additivity, it suffices to prove the Theorem with t restricted to $[0, 1]$. Given $\varepsilon > 0$, choose δ from Lemma 4.1. Partition the unit interval into subintervals $J_1, \ldots J_N$ of length $< \delta$. Letting $0 = t_0, t_1, \ldots t_{N-1}, t_N = 1$ denote the endpoints of these intervals, the ergodic theorem yields

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} f(S_v\eta_t) = \mathbf{E}[f(\eta_0)] \text{ for } t = t_1, t_2, \ldots, t_N \quad \mathbf{\Psi}_p\text{-a.s.} \tag{13}$$

For any edge e and any $i \in \{1, \ldots, N\}$, let $F(\mathrm{e}, i)$ be the event that $\eta_t(\mathrm{e})$ changes its value at least once during the time interval J_i. The probability that an exponential random variable with mean ≥ 1 takes a value $\leq \delta$ is less than δ, so the strong law of large numbers gives

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n^*|} \sum_{e\in\Lambda_n^*} \mathbf{1}_{F(\mathrm{e},i)} \leq \delta \text{ for } i = 1, \ldots, N \quad \mathbf{\Psi}_p\text{-a.s.} \tag{14}$$

Any $t \in [0, 1]$ is in some subinterval J_i; on the event in (14) we have $\bar{d}(\eta_t, \eta_{t_i}) \leq \delta$, so Lemma 4.1 implies that

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} |f(S_v\eta_t) - f(S_v\eta_{t_i})| \leq \varepsilon \quad \text{for all } t \in J_i \quad \mathbf{\Psi}_p\text{-a.s.} \tag{15}$$

In conjunction with (13) and the triangle inequality, this completes the proof. □

We note that an analogue of Theorem 1.4 holds if X is replaced by $\{0, 1\}^{\mathbb{Z}^d}$, and furthermore that it extends to more general settings. All we need in the proof is that the flip rates are bounded, whence we have the simultaneous ergodic theorem for a wide variety of interacting particle systems (including e.g. the contact process, the voter model, and Glauber dynamics for the Ising model; see [18]) if these are started with ergodic translation invariant stationary distributions.

We will now apply Theorem 1.4 to control the supremum over time of the density of the infinite clusters.

COROLLARY 4.2. – *Let $G = \mathbb{Z}^d$ and suppose that $P_{p_c}(\mathcal{C}) = 0$. Let $\mathcal{C}_t(v)$ be the event that v percolates in η_t, and denote by $\mathbf{1}_{\mathcal{C}_t(v)}$ the indicator of this event. Then*

$$\boldsymbol{\Psi}_{p_c}\left[\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}\mathbf{1}_{\mathcal{C}_t(v)} = 0 \text{ for all } t\right] = 1.$$

Proof. – Denote by E the set of edges in $\mathbb{Z}^d$. For any configuration $\xi \in \{0,1\}^E$ define $f_k(\xi) := 1$ if there is an open path in ξ from the origin to $\mathbb{Z}^d \setminus \Lambda_k$, and $f_k(\xi) := 0$ otherwise. Clearly,

$$\limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n}\mathbf{1}_{\mathcal{C}_t(v)} \le \limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n} f_k(S_v\eta_t)$$

for all t and any k. Theorem 1.4 and countable additivity imply that $\boldsymbol{\Psi}_{p_c}$-a.s.,

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n} f_k(S_v\eta_t) = \mathbf{E}_{p_c}[f_k(\eta_0)] \quad \text{for all real } t\ge 0 \text{ and integer } k\ge 1\,.$$

The assumption $P_{p_c}(\mathcal{C}) = 0$ implies that $\mathbf{E}_{p_c}[f_k(\eta_0)] \to 0$ as $k \to \infty$, so the corollary is proved. $\square$

5. CAPACITY CRITERIA FOR FLICKERING TREES

Let Γ be a finite or infinite tree with root ρ. (All trees herein are assumed to be locally finite.) Let $|v|$ denote the distance from the vertex v to the root ρ, *i.e.* the number of edges on the unique path connecting v to ρ. Let $\Gamma_n = \{v : |v| = n\}$ denote the nth level of Γ. A vertex $v \neq \rho$ which has degree one is called a **leaf** of Γ. If Γ is finite, then let its boundary $\partial\Gamma$ be the set of leaves of Γ; if Γ is infinite, we let $\partial\Gamma$ denote the set of infinite non-self-intersecting paths (called **rays**) from ρ. If v and w are vertices of Γ, write $v \le w$ if v is on the path connecting ρ and w. Let $v \wedge w$ denote the greatest lower bound for v and w; pictorially, this is where the paths from

ρ to v and w diverge. For an infinite tree, if $x, y \in \partial\Gamma$, extend this notation by letting $x \wedge y$ be the greatest vertex in both x and y. This completes the basic notation for trees, and we turn to the notation for capacities.

Let $K : \partial\Gamma \times \partial\Gamma \to \mathbb{R}$ be a nonnegative symmetric function. If μ is a signed measure on $\partial\Gamma$, define the **energy** of μ with respect to the kernel K to be

$$\mathcal{E}_K(\mu) = \iint K(x,y)\, d\mu(x)\, d\mu(y).$$

The capacity of $\partial\Gamma$ with respect to the kernel K is defined by

$$\mathrm{Cap}_K(\partial\Gamma) = [\inf\{\mathcal{E}_K(\mu) \,:\, \mu \text{ is a probability measure on } \partial\Gamma\}]^{-1}.$$

When $K(x,y) = h(|x \wedge y|)$ for all $x, y \in \partial\Gamma$, where h is a nonnegative, nondecreasing function, we write $\mathcal{E}_h$ instead of $\mathcal{E}_K$ and Cap_h instead of Cap_K.

We now state the main result of this section. Since we introduce an auxiliary random killing time τ, we denote the underlying probability measure $\mathbf{P}$ rather than Ψ_p. The event that there is an open path from the root to $\partial\Gamma$ in η_t is denoted $\{\rho \overset{t}{\leftrightarrow} \partial\Gamma\}$.

THEOREM 5.1. – *Consider dynamical percolation $\{\eta_t\}$ with parameter $0 < p < 1$ on a tree Γ which is either finite or infinite with $P_{\Gamma,p}(\mathcal{C}) = 0$. Let τ be a random variable with an exponential distribution of mean 1, which is independent of the process $\{\eta_t\}$. Let*

$$h(n) = \frac{p^{-n}}{n+1} \cdot \frac{1-p^{n+1}}{1-p} \quad \text{for } n \geq 0. \tag{16}$$

Then the event $A = \{\exists t \in [0,\tau] \,:\, \rho \overset{t}{\leftrightarrow} \partial\Gamma\}$ satisfies

$$\frac{1}{2} Cap_h(\partial\Gamma) \leq \mathbf{P}(A) \leq 2\widetilde{C} Cap_h(\partial\Gamma), \tag{17}$$

where we can take $\widetilde{C} = 18.8871$. Furthermore, if Γ is infinite, let Υ_∞ denote the random set of positive times at which the root ρ percolates. If $Cap_h(\partial\Gamma) > 0$ then Υ_∞ is a.s. uncountable, while if $Cap_h(\partial\Gamma) = 0$ then Υ_∞ is a.s. empty.

Remarks

(i) It is easy to verify that h is increasing and $h(n) \leq p^{-n}$ for all n. These properties also follow from the interpretation of h given in Lemma 5.2(iii) below. In the sequel, we will sometimes write $h(v)$ instead of $h(|v|)$ when v is a vertex.

(ii) The event A is easier to work with than the perhaps more natural event $B = \{\exists t \in [0,1] : \rho \overset{t}{\leftrightarrow} \partial\Gamma\}$. Noting that $\mathbf{P}(B) \leq \mathbf{P}(A|\tau > 1) \leq \mathbf{P}(A)/e^{-1}$ and $\mathbf{P}(A) \leq \sum_{k=0}^{\infty} e^{-k}\mathbf{P}(B) = \mathbf{P}(B)/(1-e^{-1})$, we obtain

$$\frac{1-e^{-1}}{2}\mathrm{Cap}_h(\partial\Gamma) \leq \mathbf{P}(B) \leq 2e\widetilde{C}\mathrm{Cap}_h(\partial\Gamma).$$

(iii) The easiest way to compute capacity on a tree is via its connection to effective conductance. In describing this connection we follow Lyons ([19], [20]) where more details can be found. Given a probability measure μ on $\partial\Gamma$, and a vertex v of Γ, define $\mu(v)$ to be the μ-measure of the set of leaves that are separated from the root by v, provided Γ is finite; if Γ is infinite, consider instead the set of rays going through v. This defines a one-to-one correspondence between probability measures on $\partial\Gamma$ and so-called **unit flows** on Γ from ρ to $\partial\Gamma$. For nondecreasing h with $h(-1) = 0$, a simple summation by parts (detailed in [19] and in [22]) shows that

$$\mathcal{E}_h(\mu) = \sum_{v\in\Gamma}(h(|v|) - h(|v|-1))\mu(v)^2. \tag{18}$$

Except for the summand $h(0)$ which corresponds to $v = \rho$, this sum is exactly the **energy dissipation** (in the terminology of Doyle and Snell [5]) of the flow μ provided that the edge leading to each vertex v from its parent is assigned resistance $\nabla h(|v|) := h(|v|) - h(|v|-1)$. By Thomson's principle (see [5]) the minimum of the energy dissipation over all unit flows is the effective resistance $\mathcal{R}_{\text{eff}}(\Gamma, \nabla h)$ between ρ and $\partial\Gamma$. Using the definition of capacity and (18) we get

$$\mathrm{Cap}_h(\partial\Gamma) = \frac{1}{h(0) + \mathcal{R}_{\text{eff}}(\Gamma, \nabla h)}.$$

For the kernel h in Theorem 5.1, the ratio of $\nabla h(n)$ and p^{-n}/n is bounded above and below by positive finite constants that only depend on p. Therefore

Theorem 5.1 implies Theorem 1.5.

To get the last assertion of Theorem 1.5, take Γ to be spherically symmetric, and note that $\mathcal{R}_{\text{eff}}(\Gamma, \nabla h)$ is (by symmetry) not altered if we short-circuit all vertices on a fixed level of Γ to each other. Doing this for all levels, the resistance between level $n-1$ and level n becomes $\frac{\nabla h(n)}{|\Gamma_n|}$ so that

$$\mathcal{R}_{\text{eff}}(\Gamma, \nabla h) = \sum_{n=1}^{\infty} \frac{\nabla h(n)}{|\Gamma_n|}$$

which converges iff the sum in (4) converges.

Before proving Theorem 5.1, we prove a lemma concerning the behavior of a pair of paths.

Notation. – Denote by $\{v \overset{t}{\leftrightarrow} w\}$ the event that there is an open path in η_t between the vertices v and w. Similarly, when x is a ray of the tree, $\{\rho \overset{t}{\leftrightarrow} x\}$ means that x is open at time t. Thus $\{\rho \overset{t}{\leftrightarrow} \partial\Gamma\} = \bigcup_{x\in\partial\Gamma} \{\rho \overset{t}{\leftrightarrow} x\}$.

For $s > 0$ let $T_v(s) := \int_0^s \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}}\, dt$ be the amount of time in $[0, s]$ when the path from the root to v is open.

LEMMA 5.2. – *Let u and w be vertices of Γ. With the notation of Theorem 5.1 in force,*

(i) $\mathbf{E}[T_w(\tau)] = p^{|w|}$

(ii) $\mathbf{E}[T_w(\tau) \mid T_w(\tau) > 0] = \mathbf{E}[T_w(\tau) \mid \rho \overset{0}{\leftrightarrow} w] = h(w)p^{|w|}$

(iii) $\mathbf{P}[T_w(\tau) > 0] = h(w)^{-1}$

(iv) $\mathbf{E}[T_u(\tau)T_w(\tau)] = 2h(u \wedge w)p^{|u|+|w|}$

(v) If $u \le w$ *then* $\mathbf{E}[T_u(\tau)T_w(\tau) \mid \rho \overset{0}{\leftrightarrow} w] \le 2h(u)h(w)p^{|u|+|w|}$.

Proof. – Let $q = 1 - p$.

(i) This is immediate from Fubini's Theorem.

(ii) The first equality follows from the lack of memory of the exponential distribution. Verifying the second equality requires a calculation like the one in Lemma 2.2:

$$\begin{aligned}\mathbf{E}[T_w(\tau) \mid \rho \overset{0}{\leftrightarrow} w] &= \int_0^\infty \mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{0}{\leftrightarrow} w]\mathbf{P}(\tau > t)\, dt \\ &= \int_0^\infty (p + qe^{-t})^{|w|} e^{-t}\, dt = \left.\frac{-(p+qe^{-t})^{|w|+1}}{(|w|+1)q}\right|_{t=0}^{\infty}.\end{aligned}$$

(iii) The required probability is the ratio of the expectations in (i) and (ii).

(iv) Since the process $\{\eta_t\}$ is reversible,

$$\begin{aligned}\mathbf{E}[T_u(\tau)T_w(\tau)] &= \mathbf{E}\int_0^\tau\int_0^\tau \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} u\}} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} w\}}\, dt\, ds \\ &= 2\int_0^\infty\int_s^\infty \mathbf{P}[\rho \overset{s}{\leftrightarrow} u]\mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{s}{\leftrightarrow} u] e^{-t}\, dt\, ds\,. \qquad (19)\end{aligned}$$

Observe that for $t > s$,

$$\mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{s}{\leftrightarrow} u] = p^{|w|-|u\wedge w|}\mathbf{P}[\rho \overset{t}{\leftrightarrow} u \wedge w \mid \rho \overset{s}{\leftrightarrow} u \wedge w] .$$

Change variables $\tilde{t} = t - s$ in (19) to get

$$\begin{aligned}
&\mathbf{E}[T_u(\tau)T_w(\tau)] \\
&= 2p^{|w|-|u\wedge w|} \int_0^\infty \int_0^\infty \mathbf{P}[\rho \overset{s}{\leftrightarrow} u]e^{-s-\tilde{t}}\mathbf{P}[\rho \overset{\tilde{t}}{\leftrightarrow} u \wedge w \mid \rho \overset{0}{\leftrightarrow} u \wedge w]\, d\tilde{t}\, ds \\
&= 2p^{|w|-|u\wedge w|}\mathbf{E}[T_u(\tau)] \cdot \mathbf{E}[T_{u\wedge w}(\tau) \mid \rho \overset{0}{\leftrightarrow} u \wedge w] .
\end{aligned}$$

Substituting parts (i) and (ii) of the lemma into the last equation proves (iv).

(v) Applying the strong Markov property at the stopping time $\min\{t : \rho \overset{t}{\leftrightarrow} w\}$, we get

$$\mathbf{E}[T_u(\tau)T_w(\tau)] \geq \mathbf{P}[T_w(\tau) > 0] \cdot \mathbf{E}[T_u(\tau)T_w(\tau) \mid \rho \overset{0}{\leftrightarrow} w] .$$

Invoking (iii) and (iv) concludes the proof.

□

We now recall a basic fact from Potential Theory.

LEMMA 5.3. – *Suppose that Γ is a finite tree Let $K : \partial\Gamma \times \partial\Gamma \to \mathbb{R}$ be a nonnegative symmetric kernel. Then there exists a probability measure μ_0 on $\partial\Gamma$ which minimizes K-energy, and such a measure satisfies $\int_{\partial\Gamma} K(x,y)\, d\mu_0(y) \geq \mathcal{E}_K(\mu_0)$ for all $x \in \partial\Gamma$.*

Proof. – Since Γ is a finite tree, $\mathcal{E}_K(\mu)$ depends continuously on μ, so the existence of a minimizing measure follows from compactness of the set of probability measures on $\partial\Gamma$.

Now let $x \in \partial\Gamma$ and $0 < \epsilon < 1$. The choice of μ_0 implies that

$$\mathcal{E}_K(\mu_0 + \epsilon(\boldsymbol{\delta}_x - \mu_0)) \geq \mathcal{E}_K(\mu_0) ,$$

where $\boldsymbol{\delta}_x$ is the Dirac measure at x. Expanding, we find that

$$2\epsilon\left(\int_{\partial\Gamma} K(x,y)\, d\mu_0(y) - \mathcal{E}_K(\mu_0)\right) + \epsilon^2\mathcal{E}_K(\boldsymbol{\delta}_x - \mu_0) \geq 0 .$$

Dividing by ϵ and letting $\epsilon \to 0$ shows that

$$\int_{\partial\Gamma} K(x,y)\, d\mu_0(y) \geq \mathcal{E}_K(\mu_0) \text{ for all } x \in \partial\Gamma .$$

□

The following lemma will be used several times in the sequel.

LEMMA 5.4 (One-sided Chebyshev inequality). – *Let Y be a random variable such that* $\mathbf{E}[Y] > 0$ *and* $\mathbf{Var}[Y] < \infty$. *Then for any* $0 \le a \le 1$ *we have*

$$\mathbf{P}(Y > (1-a)\mathbf{E}[Y]) \ge \frac{a^2}{a^2 + \mathbf{Var}[Y]/\mathbf{E}[Y]^2}\,. \tag{20}$$

Proof. – The random variable $X := 1 - Y/\mathbf{E}[Y]$ has mean zero, so inequality (7.5) in Feller [7], §V, gives that $\mathbf{P}(X < a) \ge \frac{a^2}{a^2+\mathbf{Var}[X]}$. This is equivalent to (20). □

Proof of Theorem 5.1. – We first prove the theorem when Γ is a **finite tree**. The lower bound on $\mathbf{P}(A)$ is proved via the second moment method. Let μ be a probability measure on $\partial\Gamma$, and consider the random variable

$$Z := \sum_{v\in\partial\Gamma} T_v(\tau)p^{-|v|}\mu(v)\,. \tag{21}$$

Lemma 5.2(i) implies that $\mathbf{E}(Z) = 1$. Part (iv) of the same lemma gives

$$\mathbf{E}[Z^2] = \sum_{v\in\partial\Gamma}\sum_{w\in\partial\Gamma} \mathbf{E}[T_v(\tau)T_w(\tau)]p^{-|v|-|w|}\mu(v)\mu(w) = 2\mathcal{E}_h(\mu)\,. \tag{22}$$

Using the Cauchy-Schwarz inequality (or Lemma 5.4 with $a = 1$) we find that

$$\mathbf{P}[A] \ge \mathbf{P}[Z > 0] \ge \frac{\mathbf{E}[Z]^2}{\mathbf{E}[Z^2]} = \frac{1}{2\mathcal{E}_h(\mu)}\,.$$

Taking the supremum of the right-hand side over all probability measures μ on $\partial\Gamma$ proves the lower bound on $\mathbf{P}[A]$ in (17).

Proving the **upper bound** $\mathbf{P}(A) \le 2\widetilde{C}\mathrm{Cap}_h(\partial\Gamma)$ is harder. Let μ_0 be a probability measure of minimal h-energy on $\partial\Gamma$. Embed the tree Γ in the upper half-plane, with the root at the origin. If $x, y \in \partial\Gamma$ and x is to the left of y or $x = y$, we write $x \preceq y$ (or equivalently, $y \succeq x$). Lemma 5.3 implies that the following two sets form a cover of $\partial\Gamma$:

$$\partial\Gamma_L := \left\{x \in \partial\Gamma\ :\ \int_{y\succeq x} h(x\wedge y)\,d\mu_0(y) \ge \frac{1}{2}\mathcal{E}_h(\mu_0)\right\} \tag{23}$$

$$\partial\Gamma_R := \left\{x \in \partial\Gamma\ :\ \int_{y\preceq x} h(x\wedge y)\,d\mu_0(y) \ge \frac{1}{2}\mathcal{E}_h(\mu_0)\right\}$$

Let $A_L = \{\exists t \in [0,\tau]\ :\ \rho \overset{t}{\leftrightarrow} \partial\Gamma_L\}$ and define A_R analogously.

For any $x \in \partial\Gamma_L$ we define $\tau_x := \inf\{t > 0 : \rho \overset{t}{\leftrightarrow} x\}$, and for any vertex v, let

$$T_v(x,\tau) := \int_{\tau_x \wedge \tau}^{\tau} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}}\, dt$$

denote the amount of time in the interval $[\tau_x \wedge \tau, \tau]$ that v is connected to the root. We now come to the **key step.** For every $x \in \partial\Gamma_L$ write

$$S_x := \sum_{y \succeq x} p^{-|x \wedge y|} T_{x\wedge y}(x,\tau)\mu_0(y)\,.$$

Given $a \in (0,1)$, define a random subset $W = W(a)$ of $\partial\Gamma_L$ by:

$$W := \left\{ x \in \partial\Gamma_L \ : \ T_x(\tau) > 0 \text{ and } S_x \geq \frac{1-a}{2}\mathcal{E}_h(\mu_0) \right\}$$

(recall that $T_x(\tau) = \displaystyle\int_0^\tau \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} x\}}\, dt$). In words, W consists of leaves in $\partial\Gamma_L$ that have an open path to the root before time τ, and the vertices along this path remain connected to the root for a "substantial" amount of time. Our first task is to bound the conditional probability $\mathbf{P}[W \neq \emptyset \mid A_L]$ from below. Let $x^* \in \partial\Gamma_L$ be the first leaf in $\partial\Gamma_L$ to be connected to the root, *i.e.* a leaf for which τ_{x^*} is minimal. (If there is a tie, let x^* be the leftmost of the minimizers.) Observe that $A_L = \{\tau_{x^*} < \tau\}$.

Since τ_{x^*} is a stopping time for the dynamical percolation process, the strong Markov property and Lemma 5.2(ii) imply that for each $x \in \partial\Gamma_L$ and each $v \leq x$ we have

$$\mathbf{E}[T_v(x,\tau) \mid A_L, x^* = x] = \mathbf{E}[T_v(\tau) \mid \rho \overset{0}{\leftrightarrow} v] = h(v)p^{|v|}\,. \tag{24}$$

Similarly, Lemma 5.2(v) implies that for $u, w \leq x$,

$$\begin{aligned}
&\mathbf{E}[T_u(x,\tau)T_w(x,\tau) \mid A_L, x^* = x] \\
&\leq 2h(u)h(w)p^{|u|+|w|} \\
&= 2\mathbf{E}[T_u(x,\tau) \mid A_L, x^* = x] \cdot \mathbf{E}[T_w(x,\tau) \mid A_L, x^* = x]\,.
\end{aligned}$$

Therefore the definitions of S_x and $\partial\Gamma_L$ imply that

$$\mathbf{E}[S_x \mid A_L, x^* = x] = \sum_{y \succeq x} h(x \wedge y)\mu_0(y) \geq \frac{1}{2}\mathcal{E}_h(\mu_0) \tag{25}$$

and

$$\mathbf{E}[S_x^2 \mid A_L, x^* = x] \leq 2\mathbf{E}[S_x \mid A_L, x^* = x]^2\,. \tag{26}$$

We can now apply the one-sided Chebyshev inequality (Lemma 5.4). Using (25) and (26) we obtain

$$\mathbf{P}[x \in W | \, A_L, x^* = x\,] = \mathbf{P}\left[S_x \geq \frac{1-a}{2} \mathcal{E}_h(\mu_0) \middle| \, A_L, x^* = x \right] \geq \frac{a^2}{a^2+1}\,.$$

Therefore

$$\mathbf{P}[W \neq \emptyset \mid A_L] \geq \frac{a^2}{a^2+1}\,. \tag{27}$$

Let L_W denote the leftmost vertex in the random set W. (So that L_W is only defined when W is nonempty.) We now bound $\mathcal{E}_h(\mu_0)$, by using a "first entrance decomposition" for $\mathbf{E}[Z]$. The crucial observation is that for any two leaves $x \preceq y$, given the indicator of the event $\{\rho \overset{t}{\leftrightarrow} x \wedge y\}$, the event $\{\rho \overset{t}{\leftrightarrow} y\}$ is conditionally independent of the whole evolution on edges on the path from the root to x and to the left of it. We also have that

$$\mathbf{P}[\{\rho \overset{t}{\leftrightarrow} y\} \, \big| \, \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} x \wedge y\}}] = p^{|y|-|x\wedge y|}\mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} x \wedge y\}}\,.$$

Integrating over $t \in [\boldsymbol{\tau}_x \wedge \tau, \, \tau]$, we infer that for $x \preceq y$ in $\partial\Gamma$,

$$\mathbf{E}[T_y(x,\tau) \mid L_W = x, \, T_{x\wedge y}(x,\tau)] = p^{|y|-|x\wedge y|} T_{x\wedge y}(x,\tau)$$

whence

$$\begin{aligned} \mathbf{E}[T_y(\tau) \mid L_W = x] &\geq \mathbf{E}[T_y(x,\tau) \mid L_W = x] \\ &= p^{|y|-|x\wedge y|}\mathbf{E}[T_{x\wedge y}(x,\tau) \mid L_W = x]\,. \end{aligned} \tag{28}$$

By Lemma 5.2(i)

$$\begin{aligned} 1 &= \sum_{y\in\partial\Gamma} \mathbf{E}[T_y(\tau)] p^{-|y|}\mu_0(y) \\ &\geq \sum_{y\in\partial\Gamma} \sum_{x\preceq y} \mathbf{E}[T_y(\tau) \mid L_W = x] \cdot \mathbf{P}[L_W = x] p^{-|y|}\mu_0(y) \\ &\geq \sum_{x\in\partial\Gamma} \sum_{y\succeq x} p^{-|x\wedge y|}\mathbf{E}[T_{x\wedge y}(x,\tau) \mid L_W = x]\mu_0(y)\mathbf{P}[L_W = x] \\ &= \sum_{x\in\partial\Gamma} \mathbf{E}[S_x \mid L_W = x] \cdot \mathbf{P}[L_W = x]\,. \end{aligned}$$

Recalling the definition of W, we get

$$1 \geq \frac{1-a}{2} \sum_{x\in\partial\Gamma} \mathcal{E}_h(\mu_0) \cdot \mathbf{P}[L_W = x] = \frac{1-a}{2}\mathcal{E}_h(\mu_0) \cdot \mathbf{P}[W \neq \emptyset]\,.$$

Thus the inequality (27) yields

$$\mathrm{Cap}_h(\partial\Gamma) = \mathcal{E}_h(\mu_0)^{-1} \geq \frac{1-a}{2}\mathbf{P}[W \neq \emptyset] \geq \frac{(1-a)a^2}{2(a^2+1)}\mathbf{P}(A_L).$$

Consequently $\mathbf{P}(A_L) \leq \widetilde{C}\mathrm{Cap}_h(\partial\Gamma)$ where

$$\widetilde{C} = \inf_{0<a<1}\left(\frac{2(a^2+1)}{(1-a)a^2}\right) \in [18.8870, 18.8871].$$

Translating the above argument into Hebrew (*i.e.*, interchanging left and right) yields $\mathbf{P}(A_R) \leq \widetilde{C}\mathrm{Cap}_h(\partial\Gamma)$; since $A = A_L \cup A_R$ (see (23)), the inequality $\mathbf{P}(A) \leq 2\widetilde{C}\mathrm{Cap}_h(\partial\Gamma)$ is proved for finite trees.

We now extend the result to **infinite trees without leaves** by a simple limiting argument. Given such an infinite tree Γ, let $\Gamma^{(n)}$ denote the finite tree consisting of levels 0 to n of Γ; the boundary of this tree coincides with Γ_n. Recall the process $\{\bar{\eta}_t\}$ introduced in Lemma 3.2, and denote $\bar{\Upsilon}_\tau^n := \{t \in [0,\tau] \, : \, \rho\leftrightarrow\Gamma_n \text{ in } \bar{\eta}_t\}$.

Compactness of $\partial\Gamma$ in the standard topology (see [20]) implies that $\cap_n \bar{\Upsilon}_\tau^n = \{t \in [0,\tau] \, : \, \rho\leftrightarrow\partial\Gamma \text{ in } \bar{\eta}_t\}$. Lemma 3.2 states that $\mathbf{P}$-a.s. this set is identical to

$$\Upsilon_\tau := \{t \in [0,\tau] \, : \, \rho \overset{t}{\leftrightarrow} \partial\Gamma\}.$$

Thus by compactness of the sets $\bar{\Upsilon}_\tau^n$, we have $\mathbf{P}[A] = \lim_{n\to\infty}\mathbf{P}[A^{(n)}]$, where A is the event that $\Upsilon_\tau \neq \emptyset$ and $A^{(n)}$ is the event that $\bar{\Upsilon}_\tau^n \neq \emptyset$. Using the easily verified fact that $\lim_{n\to\infty}\mathrm{Cap}_h(\partial\Gamma^{(n)}) = \mathrm{Cap}_h(\partial\Gamma)$, the inequalities (17) already proved for finite trees, yield that (17) also holds for infinite trees without leaves.

Finally, if Γ is an **infinite tree with leaves**, consider the leafless subtree Γ' which is the union of all rays in Γ. Since $\partial\Gamma = \partial\Gamma'$, the estimate (17) for Γ is equivalent to the same estimate for Γ'.

It only remains to prove the zero-one law for infinite trees stated at the end of the theorem. Denote

$$\Upsilon_{[s,\infty)} := \{t \in [s,\infty) \, : \, \rho \overset{t}{\leftrightarrow} \partial\Gamma\}.$$

If $\mathrm{Cap}_h(\partial\Gamma) = 0$ then $\Upsilon_{[0,\infty)} = \emptyset$ a.s. by countable additivity and Remark (ii) after Theorem 5.1.

Conversely, suppose that $\mathrm{Cap}_h(\partial\Gamma) > 0$. Then

$$\begin{aligned}&\mathbf{P}\{\Upsilon_{[n,\infty)} \neq \emptyset \text{ for all integers } n > 0\}\\ &\quad = \lim_{n\to\infty}\mathbf{P}\{\Upsilon_{[n,\infty)} \neq \emptyset\} \geq \frac{1}{2}\mathrm{Cap}_h(\partial\Gamma),\end{aligned} \tag{29}$$

by (17) and stationarity. Since $\{\eta_t\}$ is an ergodic process under time evolution (see [25]), the probability in (29) must be 1. It remains to establish the last assertion of the theorem. If $\theta_\rho(p) > 0$ (*i.e.* the root percolates at a fixed time with positive probability) then $\Upsilon_{[0,\infty)}$ a.s. has infinite Lebesgue measure. On the other hand, if $\theta_\rho(p) = 0$, then applying Lemmas 3.2 and 3.4 to all rational time intervals instead of $[0, 1]$ shows that a.s. $\Upsilon_{[0,\infty)}$ is a closed set with no isolated points; therefore it is a.s. uncountable. □

6. HAUSDORFF DIMENSION OF PERCOLATING TIMES

We start by formulating an extension of the lower bound in Theorem 1.6. Let $\alpha \in (0,1)$, and let $D \subset \mathbb{R}$ be a closed set. The α-energy of a finite Borel measure σ on D is

$$\mathcal{E}_\alpha(\sigma) = \int_D \int_D |x-y|^{-\alpha}\, d\sigma(x)\, d\sigma(y).$$

The α-**capacity** of D is

$$\mathrm{Cap}_\alpha(D) = \left[\inf_\sigma \mathcal{E}_\alpha(\sigma)\right]^{-1}$$

where the infimum is over probability measures σ on D and by convention, $\infty^{-1} = 0$.

The following classical theorem relates capacity to dimension.

THEOREM 6.1 (Frostman [8]). – *For any closed set $D \subset \mathbb{R}$, the critical parameter* $\inf\{\beta > 0 \mid Cap_\beta(D) = 0\}$ *is exactly the Hausdorff dimension of D.*

For the definition of Hausdorff dimension and the proof of Theorem 6.1, see Kahane [13].

For $\alpha \in (0,1)$, we will relate the α-capacity of the set of times at which the root percolates, to the capacity of the tree boundary in the kernel

$$h_\alpha(n) := \frac{p^{-n}}{1-\alpha} \cdot \left(\frac{1-p^{n+1}}{(n+1)(1-p)}\right)^{1-\alpha}. \tag{30}$$

We use freely the notation established in the beginning of the previous section.

THEOREM 6.2. – *Consider a dynamical percolation process with parameter p on a tree Γ, and let τ be an independent exponential random variable* *of mean 1. Suppose that for some $\alpha \in (0,1)$, the boundary $\partial\Gamma$ supports a probability measure μ with $\mathcal{E}_{h_\alpha}(\mu) < \infty$. Then the random set*

$$\Upsilon_\tau := \{t \in [0,\tau] \,:\, \rho \overset{t}{\leftrightarrow} \partial\Gamma\}$$

supports a (random) measure σ of (random) total mass $\|\sigma\|$ such that

(i) $\mathbf{E}[\|\sigma\|] = 1$

(ii) $\mathbf{P}[\|\sigma\| \geq 1 - a] \geq \dfrac{a^2}{a^2 + 2\mathcal{E}_h(\mu) - 1}$ *for any $a \in (0,1)$*

(iii) $\mathbf{E}[\mathcal{E}_\alpha(\sigma)] \leq 2\mathcal{E}_{h_\alpha}(\mu)$.

Consequently (letting $a \to 1$ in (ii)), $\mathbf{P}[Cap_\alpha(\Upsilon_\tau) > 0] \geq \frac{1}{2}\mathcal{E}_h(\mu)^{-1}$.

(Note that necessarily $\mathbf{P}[\|\sigma\| = 0] \geq \mathbf{P}[\Upsilon_\tau = \emptyset] > 0$.)

The following lemma will be useful in the proof.

LEMMA 6.3. – *With the assumptions of Theorem 6.2, let $w \leq x$ be vertices of Γ. Then*

$$\int_0^\infty t^{-\alpha}\mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{0}{\leftrightarrow} x]e^{-t}\,dt \leq p^{|w|}h_\alpha(w)\,, \tag{31}$$

where $h_\alpha(w)$ is an abbreviation for $h_\alpha(|w|)$.

Proof. – For $s \geq 0$ define $F_w(s) := \int_0^s \mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{0}{\leftrightarrow} w]e^{-t}\,dt$. Clearly $F_w(s) \leq s$, and Lemma 5.2(ii) implies that $F_w(s) \leq h(w)p^{|w|}$ for all $s \geq 0$. Since $\mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{0}{\leftrightarrow} x] = \mathbf{P}[\rho \overset{t}{\leftrightarrow} w \mid \rho \overset{0}{\leftrightarrow} w]$, the left-hand side of (31) can be written, using integration by parts, as

$$\begin{aligned}
\int_0^\infty t^{-\alpha}\,dF_w(t) &= t^{-\alpha}F_w(t)\Big|_{t=0}^{\infty} + \int_0^\infty F_w(t)\alpha t^{-\alpha-1}\,dt \\
&\leq 0 + \int_0^{h(w)p^{|w|}} t \cdot \alpha t^{-\alpha-1}\,dt \\
&\quad + \int_{h(w)p^{|w|}}^\infty h(w)p^{|w|}\alpha t^{-\alpha-1}\,dt \\
&= \left(\frac{\alpha}{1-\alpha} + 1\right)(h(w)p^{|w|})^{1-\alpha}
\end{aligned} \tag{32}$$

By comparing the definitions of h and h_α, we see that the right-hand sides of (31) and (32) are equal. □

Proof of Theorem 6.2. – Observe that the last statement of the theorem follows immediately from the preceding assertions (i)-(iii). To prove these assertions we first assume that Γ is a **finite tree**, and define the random measure σ on $[0,\infty)$ by

$$\sigma(D) := \sum_{x\in\partial\Gamma} p^{-|x|}\mu(x)\int_{D\cap[0,\tau]} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} x\}}\, dt \qquad \text{for Borel sets } D\subset[0,\infty)\,.$$

Clearly, σ is supported on Υ_τ. The total mass $||\sigma||$ is exactly the random variable Z introduced in (21), so that as already remarked, (i) follows easily from Fubini's Theorem. The inequality (ii) follows from Lemma 5.5 and (22). It remains to verify (iii), and this is analogous to Lemma 5.2(iv):

$$\begin{aligned}\mathbf{E}[\mathcal{E}_\alpha(\sigma)] &= \sum_{x\in\partial\Gamma} p^{-|x|}\mu(x)\sum_{y\in\partial\Gamma} p^{-|y|}\mu(y)\\ &\quad\times \mathbf{E}\int_0^\tau\!\!\int_0^\tau |t-s|^{-\alpha}\mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} x\}}\mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} y\}}\, dt\, ds\\ &= 2\sum_{x,y\in\partial\Gamma}\mu(x)\mu(y)\int_0^\infty\!\!\int_s^\infty |t-s|^{-\alpha}p^{-|x|}\\ &\quad\times \mathbf{P}[\rho \overset{s}{\leftrightarrow} x]p^{-|y|}\mathbf{P}[\rho \overset{t}{\leftrightarrow} y \mid \rho \overset{s}{\leftrightarrow} x]e^{-t}\, dt\, ds\,. \end{aligned} \tag{33}$$

Observe that $p^{-|x|}\mathbf{P}[\rho \overset{s}{\leftrightarrow} x] = 1$ and

$$\mathbf{P}[\rho \overset{t}{\leftrightarrow} y \mid \rho \overset{s}{\leftrightarrow} x] = p^{|y|-|x\wedge y|}\mathbf{P}[\rho \overset{t}{\leftrightarrow} x\wedge y \mid \rho \overset{s}{\leftrightarrow} x]\,.$$

Change variables $\tilde{t} = t-s$ and integrate ds to get that for any two leaves x, y, the double integral in (33) can be written as

$$p^{-|x\wedge y|}\int_0^\infty \tilde{t}^{-\alpha}\mathbf{P}[\rho \overset{\tilde{t}}{\leftrightarrow} x\wedge y \mid \rho \overset{0}{\leftrightarrow} x]e^{-\tilde{t}}\, d\tilde{t} \le h_\alpha(x\wedge y)\,,$$

where the last inequality follows from Lemma 6.3. Therefore

$$\mathbf{E}[\mathcal{E}_\alpha(\sigma)] \le 2\sum_{x,y\in\partial\Gamma}\mu(x)\mu(y)h_\alpha(x\wedge y)$$

as asserted in (iii).

Next, if Γ is an **infinite tree without leaves**, we define for each n the measure σ_n corresponding to the truncation of Γ to the first n levels. For a vertex u of Γ, we use, as in Remark (iii) after Theorem 5.1, the notation $\mu(u) = \mu(\{x\in\partial\Gamma : u\in x\})$. If φ is a compactly supported continuous function on $[0,\infty)$, then the sequence of integrals

$$\int_0^\infty \varphi(t)\, d\sigma_n(t) = \sum_{u\in\Gamma_n} p^{-|u|}\mu(u)\int_0^\tau \varphi(t)\mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} u\}}\, dt \tag{34}$$

forms a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$, where $\mathcal{F}_n$ is generated by the process $\{\eta_t(\mathrm{e})\}$ restricted to edges e in the first n levels of Γ. This martingale is bounded in L^2 since $\mathbf{E}[\|\sigma_n\|^2] \leq 2\mathcal{E}_h(\mu)$ by (22).

By separability of the compactly supported continuous functions and the convergence theorem for L^2 martingales, we see that with probability 1 the sequence of random measures σ_n converges weakly to a random measure σ on the closed set Υ_τ (whence also $\sigma_n \times \sigma_n$ converges weakly to $\sigma \times \sigma$.) Clearly this random measure σ satisfies (i) and (ii).

For every finite $M > 0$, with probability one the integrals

$$\int_0^\infty \int_0^\infty \left(|t-s|^{-\alpha} \wedge M\right) d\sigma_n(s)\, d\sigma_n(t)$$

converge to the corresponding integral with σ replacing σ_n. Thus by Fatou's lemma

$$\mathbf{E} \int_0^\infty \int_0^\infty \left(|t-s|^{-\alpha} \wedge M\right) d\sigma(s)\, d\sigma(t) \leq 2\mathcal{E}_{h_\alpha}(\mu)\,.$$

Letting $M \to \infty$ and using Lebesgue's monotone convergence theorem proves (iii) in this case.

Finally, the case where Γ is an **infinite tree with leaves** reduces to the previous case as in Theorem 5.1. □

To prove the upper bound on dimension in Theorem 1.6 we will need Lyons' estimate for ordinary percolation.

LEMMA 6.4 (specialized from Theorem 2.1 of Lyons [20]). – *Let Γ be a (finite or infinite) tree and let $p \in (0,1)$. Denote $g(k) = p^{-k}$. Then*

$$Cap_g(\partial\Gamma) \leq P_p[\rho \leftrightarrow \partial\Gamma] \leq 2Cap_g(\partial\Gamma)\,.$$

In particular, if Γ is spherically symmetric, then

$$P_p[\rho \leftrightarrow \Gamma_n] \leq 2\left[(1-p)\sum_{k=1}^{n} \frac{p^{-k}}{|\Gamma_k|}\right]^{-1}. \tag{35}$$

An alternative proof of this lemma, in which the percolation probability is interpreted as a hitting probability for a certain Markov chain, is in Benjamini, Pemantle and Peres [4].

Proof of Theorem 1.6. – By timewise ergodicity of $\{\eta_t\}$, the Hausdorff dimension in the statement is an a.s. constant. As the lower bound on dimension follows directly by combining Theorem 6.2 and Theorem 6.1, it only remains to establish the upper bound. Suppose that $0 < \alpha, p < 1$, and Γ is a spherically symmetric tree such that

$$\sum_{k=1}^{\infty} \frac{p^{-k} k^{\alpha-1}}{|\Gamma_k|} = \infty\,. \tag{36}$$

We will show this implies that the random set $\Upsilon_1 := \{t \in [0,1] : \rho \overset{t}{\leftrightarrow} \partial\Gamma\}$ has Hausdorff dimension at most α almost surely. A routine countable-additivity argument (as in the proof of Theorem 5.1) will then complete the proof.

For any time interval I of length $1/n$ and any edge e, we have $\boldsymbol{\Psi}_p[\exists t \in I : \eta_t(\mathrm{e}) = 1] \le p(1 + 1/n)$ by (5). Therefore Lemma 6.4 implies that there exists $C = C(p)$ such that

$$\begin{aligned} \boldsymbol{\Psi}_p[\exists t \in I : \rho \overset{t}{\leftrightarrow} \Gamma_n] &\le C\left[\sum_{k=1}^{n} \frac{(p+p/n)^{-k}}{|\Gamma_k|}\right]^{-1} \\ &\le Ce\left[\sum_{k=1}^{n} \frac{p^{-k}}{|\Gamma_k|}\right]^{-1}. \end{aligned} \tag{37}$$

Let

$$N_n := \sum_{j=1}^{n} \mathbf{1}_{\{\exists t \in [(j-1)/n, j/n] : \rho \overset{t}{\leftrightarrow} \Gamma_n\}}\,.$$

Clearly, $\mathbf{E}[N_n]$ is obtained by multiplying the left-hand side of (37) by n. Thus

$$\sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{\mathbf{E}[N_n]} \ge \frac{1}{Ce} \sum_{n=1}^{\infty} n^{\alpha-2} \sum_{k=1}^{n} \frac{p^{-k}}{|\Gamma_k|} = \frac{1}{Ce} \sum_{k=1}^{\infty} \frac{p^{-k}}{|\Gamma_k|} \sum_{n=k}^{\infty} n^{\alpha-2}\,. \tag{38}$$

The right-hand sum diverges by the hypothesis (36). Therefore we have, for any $\epsilon > 0$, that

$$0 = \liminf_{n\to\infty} \frac{\mathbf{E}[N_n]}{n^{\alpha+\epsilon}} \ge \mathbf{E}\left[\liminf_{n\to\infty} \frac{N_n}{n^{\alpha+\epsilon}}\right],$$

by Fatou's Lemma. Thus

$$\liminf_{n\to\infty} \frac{N_n}{n^{\alpha+\epsilon}} = 0 \quad a.s.$$

Since this is true for all $\epsilon > 0$, and Υ_1 is covered by N_n intervals of length $1/n$, the definition of Hausdorff dimension implies that the dimension of Υ_1 is at most α. □

We conclude with an unsolved problem.

Question. – Given $0 < p < 1$, which infinite trees Γ satisfy

$$\boldsymbol{\Psi}_{\Gamma,p}[\forall t\ \exists \text{ an infinite cluster in } \eta_t] = 1\,?$$

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