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Hitting times and spectral gap inequalities

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ABSTRACT. – The aim of this paper is to relate estimates on the hitting times of closed sets by a Markov process and a special class of inequalities involving the L_p ($p \leq 1$) norm of a function and its Dirichlet norm. These inequalities are weaker than the usual spectral gap inequality. In particular they hold for diffusion processes in $\mathbb{R}^n$ when the potential decreases polynomially. We derive uniform bounds for the moments of the hitting times. We also obtain estimates of the difference between the law of the hitting time of a "small" set and an exponential law.

RÉSUMÉ. – L'objet de cet article est d'étudier les liens entre, d'une part, des estimées des temps d'atteinte d'ensembles fermés par un processus de Markov et, d'autre part, une nouvelle famille d'inégalités liant la norme L_p ($p \leq 1$) d'une fonction et la forme de Dirichlet. Ces inégalités sont plus générales que l'inégalité de trou spectral. En particulier elles sont vérifiées pour une diffusion dans $\mathbb{R}^n$ dont le potentiel décroit polynomialement. Nous obtenons des majorations uniformes des petits moments des temps d'atteinte. Notre méthode permet également d'estimer la distance entre la loi du temps d'atteinte d'un "petit" ensemble et une loi exponentielle.

A.M.S. Classifications : 60 J 25, 60 J 60.

INTRODUCTION

Diffusions in $\mathbb{R}^n$

Let w be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}_+^*$. Also assume that $\int w(x)dx = 1$. Let X_t be the Markov process solution of the stochastic differential equation:

$$dX_t = d\beta_t + \frac{1}{2}\frac{\nabla w(X_t)}{w(X_t)}dt$$

The probability measure $d\mu(x) = w(x)dx$ is invariant and reversible for X.

For a closed set A, let $\tau_A = \inf\{t \geq 0 \ s.t. \ X_t \in A\}$ be the hitting time of A.

In this paper we shall use functional inequalities to study the links between, on one hand, estimates of τ_A and, on the other hand, the behaviour of w at infinity.

For $p > 0$, let

$$\Lambda(p) = \inf_{u \in C_0^\infty(\mathbb{R}^n), \int u d\mu = 0} \frac{\frac{1}{2}\int |\nabla u|^2 d\mu}{(\int |u|^p d\mu)^{2/p}}$$

and

$$\bar{\Lambda}(p) = \inf_{u \in \mathcal{H}} \frac{\frac{1}{2}\int |\nabla u|^2 d\mu}{(\int |u|^p d\mu)^{2/p}}$$

where $\mathcal{H} = \{u \in C_0^\infty \ s.t. \ u \geq 0 \ and \ \mu[u = 0] \geq \frac{1}{2}\}$.
Note that $\Lambda(p)$ is non-negative but might be 0. Also $\Lambda(p) \leq \Lambda(p')$ whenever $p \geq p'$. $\Lambda(2)$ is the first non vanishing eigenvalue of the generator of X in $L_2(\mu)$.
Our main results are the following:
For any closed set A,

$$\mu(A)^2 E_\mu[\tau_A] \leq 1/\Lambda(1)$$

(see part III)
Let $0 < p < 1$. If $\bar{\Lambda}(p) > 0$, then

$$\sup_A \mu(A)^{2/p} \|\tau_A\|_{*,p/(2-p),P_\mu} < +\infty$$

where $||F||_{*,p',P_\mu} = \sup_{t\geq 0} tP_\mu(|F| \geq t)^{1/p'}$ is the weak-L_{p',P_μ} norm of F. (See part III).

It is interesting to note that some converse to these inequalities is also true: if the first inequality is satisfied for some constant instead of $\Lambda(1)$, then $\Lambda(1)$ is non zero. Similarly, if the second inequality holds for some $p \in]0,1[$, then $\bar{\Lambda}(p')$ is non zero for all $p' < p$.

Let us consider the case $w(x) = |x|^{-\beta}$ $(\beta > n)$. It is easy to see that $\Lambda(2) = \bar{\Lambda}(2) = 0$ for any β. But we shall prove that, for $\beta > n$, there exists a $p \in]0,1]$ s.t. $\bar{\Lambda}(p) \neq 0$. In particular, for $\beta > 2+n$, then $\Lambda(1) \neq 0$. (See part VII). Therefore the quantities $\Lambda(p)$, $0 < p \leq 1$ seem to be quite well adapted to deal with slowly decreasing potentials.

These results do not depend on the fact that X is a diffusion in $I\!R^n$. Since it is more natural, we shall deal with general Markov processes. Note that we shall also consider non-symmetric processes.

General set-up

Let E be a locally compact separable Hausdorff space. Let μ be a Radon measure on E. We assume that μ is a probability.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form and its domain.

(By regular Dirichlet form we mean that the symmetric part of $\mathcal{E}$ is a symmetric Dirichlet form in the sense of Fukushima. The normal contractions operate on $\mathcal{E}$. We also assume that normal contractions operate on the adjoint form $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$. We assume that the following sector condition holds: $\mathcal{E}_1(u,v)^2 \leq M^2\mathcal{E}_1(u,u)\mathcal{E}_1(v,v)$, where, for $t > 0$, $(\mathcal{E}_t, \mathcal{F})$ is the bilinear form $\mathcal{E}_t(u,v) = \mathcal{E}(u,v) + t\int uv d\mu$. Since the symmetric part of $\mathcal{E}_1$ is closed, the set $\mathcal{F}$ endowed with the scalar product associated to the symmetric part of $\mathcal{E}_1$ is a Hilbert space. The sector condition implies that, for any $t > 0$, the bilinear form $(\mathcal{E}_t, \mathcal{F})$ is continuous on $\mathcal{F}$. By regularity, we mean that the set $\mathcal{F} \cap C_0$ is dense in $\mathcal{F}$ and in C_0, where C_0 is the space of continuous functions with compact support.)

Also assume that $1 \in \mathcal{F}$. (1 is the constant function whose value is 1) and that $\mathcal{E}(1,u) = \mathcal{E}(u,1) = 0$ for any $u \in \mathcal{F}$.

By the general theory of Dirichlet forms , there exists a Markov process, in fact a Hunt process, $(X_t, t \geq 0)(P_x, x \in E)$, associated to $(\mathcal{E}, \mathcal{F})$. Then $\mathcal{E}(u,v) = -\int v\mathcal{L}u d\mu$, where $\mathcal{L}$ is the generator of X and u, v are in the L_2 domain of $\mathcal{L}$. Also $\int (1/t)(P_t u - u)v d\mu \to -\mathcal{E}(u,v)$ when t tends to 0, where P_t is the L_2 semi-group of X and $u, v \in \mathcal{F}$.

Let E_x be the expectation w.r.t. P_x, the law of X when the initial law is a Dirac mass at point x. Also let $E_\mu = \int E_x d\mu$.

Since we have assumed that $\mathcal{E}(u,1)=0$, the process X is in fact strictly markovian. The probability measure μ is invariant under P_t *i.e.* when the law of X_0 is μ, then, for any $t \geq 0$, the law of X_t is also μ.

For any closed subset of E, A, and any $t > 0$, let h_t^A be the t-potential of A. By definition $h_t^A \in \mathcal{F}$ satisfies $\mathcal{E}_t(h_t^A, u) = 0$ for any quasi-continuous function $u \in \mathcal{F}$ s.t. $u = 0$ quasi-everywhere on A and $h_t^A = 1$ quasi-everywhere on A.

It is known that it is possible to identify h_t^A with the Laplace transform of the hitting time of A by X: let $\tau_A = \inf\{t \geq 0 \ s.t. \ X_t \in A\}$, then $h_t^A(x) = E_x[\exp(-t\tau_A)]$.

These results are part of the classical theory of Dirichlet forms, analytic potential theory and its probabilistic counterpart. They can be found in the book of Fukushima 1980 [5] for the symmetric case, and in the paper of S. Carrillo Menendez 1975 [4] for the general case.

The assumptions we have just described are sufficient to carry out our program but they are far from necessary. In particular our results still hold if E is not locally compact, provided it is a topological space and $(\mathcal{E}, \mathcal{F})$ satisfies some regularity condition (see Ma-Rockner, 1991 [6]). It is also possible to suppress the assumption of regularity for $(\mathcal{E}, \mathcal{F})$. In this latter case, one has to consider Borel sets instead of closed sets in parts III-IV.

Spectral gap inequalities and hitting times

For $p > 0$, let

$$\Lambda(p) = \inf_{u \in \mathcal{F}, \int u d\mu = 0} \frac{\mathcal{E}(u)}{||u||_p^2}$$

Also let

$$\bar{\Lambda}(p) = \inf \frac{\mathcal{E}(u)}{||u||_p^2}$$

where the inf is taken on functions $u \in \mathcal{F}$ s.t. $u \geq 0$ and $\mu(u = 0) \geq 1/2$.

Note that $\Lambda(p)$ is non-negative but might be 0. The inequality $\Lambda(p) > 0$ gets stronger when p increases. The inequality $\Lambda(2) > 0$ is the well known "spectral gap inequality". It is easy to prove that for any $t > 0$, any $u \in L_2$, $||P_t u - \int u d\mu||_2 \leq \exp(-\Lambda(2)t)||u||_2$.

Here we shall be interested in the implications of the weaker inequality $\Lambda(p) > 0$ for $0 < p \leq 1$.

It is not difficult to show that if $\Lambda(p) > 0$, for some $p \in]0,1]$, then P_t converges to equilibrium at an exponential speed in L_1 *i.e.* for any

$t > 0,\ u \in L_2,$

$$\left\| P_t u - \int u d\mu \right\|_1 \le e^{-c(p)\Lambda(p)t} \|u\|_2$$

where $c(p)$ is a strictly positive universal constant. You can replace Λ by $\bar{\Lambda}$ in this inequality. (See Theorem 2).

The aim of this paper is to express the inequality $\Lambda(p) > 0$ in terms of hitting times. Our main result deals with the case $p = 1$:

MAIN THEOREM

(i) For any closed set A,

$$\mu(A)^2 E_\mu[\tau_A] \le 1/\Lambda(1)$$

(ii) Assume that the Dirichlet form $\mathcal{E}$ is symmetric, or, more generally, that it satisfies the strong sector condition : there exists a constant M s.t. for any $u, v \in \mathcal{F}$, $|\mathcal{E}(u,v)| \le M\sqrt{\mathcal{E}(u)\mathcal{E}(v)}$, then there exists a constant C that only depends on M s.t

$$\sup_{A, \mu(A) \ge 1/2} E_\mu[\tau_A] \ge C/\Lambda(1)$$

The same estimates hold for $\bar{\Lambda}$ instead of Λ in (i) and (ii).

(See Proposition 4 for (i) and see the end of part III for (ii)).

The proof of this result is based on estimates of h_t^A: in Theorems 3 and 6, we shall prove that, for any $p \in]0,1]$, any $t > 0$ and any closed set A

$$1 - \int h_t^A d\mu \le C(p)\mu(A)^{-2/(2-p)}(t/\Lambda(p))^{p/(2-p)} \tag{0.1}$$

and

$$\left\| h_t^A - \int h_t^A d\mu \right\|_1 \le C(p)(t/\Lambda(p))^{p/2} \tag{0.2}$$

where $C(p)$ is a universal constant that only depends on p. Here also one can replace Λ by $\bar{\Lambda}$.

From the inequality (0.1) it is possible to deduce uniform bounds for the moments of τ_A: let $P_\mu[.] = \int P_x[.]d\mu$ be the law of X at equilibrium. Let F be a measurable function and $\|F\|_{*,p',P_\mu} = \sup_{t\ge 0} tP_\mu(|F| \ge t)^{1/p'}$ be the weak-L_{p',P_μ} norm of F. From (0.1) we deduce that, if $\Lambda(p) > 0$ or $\bar{\Lambda}(p) > 0$ for some $p \in]0,1]$, then

$$(0.3) \qquad \sup_A \mu(A)^{2/p} \|\tau_A\|_{*,p/(2-p),P_\mu} < +\infty$$

(See Proposition 3).

Some converse to inequality (0.1) is true: if we assume that $(\mathcal{E},\mathcal{F})$ is symmetric, *i.e.* that X is reversible, or, more generally, if we assume that $(\mathcal{E},\mathcal{F})$ satisfies the strong sector condition, and if there exists a constant c s.t. for any $t > 0$ and any closed subset A,

$$\int (1 - h_t^A) d\mu \leq \mu(A)^{-2/(2-p)} (t/c)^{p/(2-p)},$$

then $\bar{\Lambda}(p') > 0$ for any $p' < p$. (See Theorems 1 and 4). It is not clear wether you can replace $\bar{\Lambda}$ by Λ in this last statement.

The unpredictability property

Roughly speaking, we say the unpredictability property (U.P) holds if the law of the hitting time of a "small" subset of E by X_t is close to an exponential law. This terminology is justified by the loss of memory property of the exponential law. Proving the unpredictability property is the key step in the so-called "pathwise approach" of metastability (see Cassandro *et al.*, 1984 [3]).

We shall investigate the connections between the unpredictability property and generalized spectral gap inequalities.

Let A be a closed subset of E. If there exists a $t \geq 0$ s.t. $\int h_t^A d\mu = 1/2$, we denote by $T(A)$ the (unique) solution of the equation $\int h_{T(A)}^A d\mu = 1/2$. If not let $T(A) = \infty$. Note that if $\mu(A) < 1/2$ and $\tau_A < +\infty$ P_μ.a.s., then $T(A) < +\infty$. In a sense, $T(A)$ measures the size of A. Let us define the capacity of A by $cap(A) = \int h_1^A d\mu$. As a consequence of Hölder's inequality, we have $T(A) \log(1/cap(A)) \leq \log 2$, provided that $T(A) \leq 1$ or $cap(A) \leq 1/2$. Hence if $cap(A_n) \to 0$ then $T(A_n) \to 0$.

The aim of this section is to estimate the difference between the law of τ_A and an exponential law in terms of $T(A)$ and $\Lambda(p)$.

As a consequence of the inequality (0.2), we have the following estimate:

$$(0.4) \qquad \sup_t \left| \int h_{tT(A)}^A d\mu - \frac{1}{1+t} \right| \leq C(p)(T(A)/\Lambda(p))^{p/2}$$

for any $p \in]0,1]$. (See Proposition 6). (0.4) also holds for $\bar{\Lambda}$ instead of Λ.

In sections V and VI, we shall discuss the links between our generalized spectral gap inequalities, the usual spectral gap inequality $\Lambda(2) > 0$, and log-Sobolev inequalities (Section V) and the link between the spectral gap inequality and estimates of the hitting times (Section VI). The last section is devoted to the examples of diffusions in $\mathbb{R}^n$ we discussed at the beginning of this introduction. The paper is organized as follows:

Notations

In the paper, $C(\alpha)$, $C(\eta)$, $C(\alpha,\beta)$... are universal constants that only depend on the parameters α, η, α and β... They do not depend on the choice of $(\mathcal{E},\mathcal{F})$.

I. GENERALIZED SPECTRAL GAP INEQUALITIES

For $p \neq 1$, we cannot compare $\Lambda(p)$ and $\bar{\Lambda}(p)$. To avoid this difficulty we introduce two families of "spectral gap constants" and we prove that there exist two constants, $\mathcal{K}(\alpha,+\infty)$ and $\bar{\mathcal{K}}(\alpha,+\infty)$, s.t. $\Lambda(\alpha/(1+\alpha)) \leq \mathcal{K}(\alpha,+\infty)$ and $\bar{\Lambda}(\alpha/(1+\alpha)) \leq \bar{\mathcal{K}}(\alpha,+\infty)$ (Proposition 1) and besides the two constants $\mathcal{K}(\alpha,+\infty)$ and $\bar{\mathcal{K}}(\alpha,+\infty)$ only differ by an universal constant (Proposition 2). Once this is done, it will be sufficient for our purposes to prove estimates on the hitting times in terms of $\mathcal{K}(\alpha,+\infty)$.

In Theorem 1, we prove that, if $\bar{\mathcal{K}}(\alpha,+\infty) > 0$ then $\bar{\Lambda}(\beta/(1+\beta)) > 0$, for any $\beta < \alpha$. This result will play an important role when we want to estimate the spectral gap constants in terms of hitting times.

In the sequel α, p, q, η are parameters satisfying:

$\alpha \in]0,+\infty]$, $p \in]0,+\infty]$, $q \in]-\infty,0[\cup[1,+\infty]$, $\eta \in [-1,0[\cup]0,+\infty]$ and $1/p + 1/q = 1$, $\alpha = \eta q$. Note that $q \in]-\infty,0[$ when $p \in]0,1[$.

For $u \in L\log L$, let $E_1(u) = \int |u|\log|u|d\mu - \int |u|d\mu \log \int |u|d\mu$.

We define the following constants:

$$\mathcal{K}(\eta,p) = \inf_{u\in\mathcal{F},\int u d\mu=0} \frac{\mathcal{E}(u)}{\|u\|_1^2}\left(\frac{\|u\|_p}{\|u\|_1}\right)^{2/\eta}$$

$$\mathcal{K}(\alpha) = \inf_{u\in\mathcal{F},\int u d\mu=0} \frac{\mathcal{E}(u)}{\|u\|_1^2} \exp\left(\frac{2}{\alpha}\frac{E_1(u)}{\|u\|_1}\right)$$

Let $\mathcal{H}$ be the space of non-negative measurable functions u s.t. $\mu(u = 0) \geq 1/2$.

$$\bar{\mathcal{K}}(\eta,p) = \inf_{u\in\mathcal{F},u\in\mathcal{H}} \frac{\mathcal{E}(u)}{\|u\|_1^2}\left(\frac{\|u\|_p}{\|u\|_1}\right)^{2/\eta}$$

$$\bar{\mathcal{K}}(\alpha) = \inf_{u\in\mathcal{F},u\in\mathcal{H}} \frac{\mathcal{E}(u)}{\|u\|_1^2} \exp\left(\frac{2}{\alpha}\frac{E_1(u)}{\|u\|_1}\right)$$

Note that when $\eta = -1$, then $p = \alpha/(1+\alpha)$. Besides

$$\Lambda(p) = \mathcal{K}(-1,p)$$

and

$$\bar{\Lambda}(p) = \bar{\mathcal{K}}(-1,p)$$

Also

$$\Lambda(1) = \mathcal{K}(+\infty,p) = \mathcal{K}(\eta,1) = \mathcal{K}(+\infty)$$

and

$$\bar{\Lambda}(1) = \bar{\mathcal{K}}(+\infty,p) = \bar{\mathcal{K}}(\eta,1) = \bar{\mathcal{K}}(+\infty)$$

In the definitions of $\mathcal{K}$ and $\bar{\mathcal{K}}$, one can replace $\mathcal{F}$ by $\mathcal{F}\cap C_0$.

Now let p', q', η' be chosen as p, q, η and s.t. $p' \geq p$. As an immediate corollary of Hölder's inequality, we get the

PROPOSITION 1. - *Assume that $p' \geq p$ and $\alpha = \eta' q'$. Then*

(i) $\mathcal{K}(\eta,p) \leq \mathcal{K}(\eta',p')$
(ii) if $p < 1 < p'$, $\mathcal{K}(\eta,p) \leq \mathcal{K}(\alpha) \leq \mathcal{K}(\eta',p')$
(iii) $\bar{\mathcal{K}}(\eta,p) \leq \bar{\mathcal{K}}(\eta',p')$
(iv) if $p < 1 < p'$, $\bar{\mathcal{K}}(\eta,p) \leq \bar{\mathcal{K}}(\alpha) \leq \bar{\mathcal{K}}(\eta',p')$
(v) $\Lambda(\alpha/(1+\alpha)) \leq \mathcal{K}(\frac{\alpha}{2},2) \leq \mathcal{K}(\alpha,+\infty)$
(vi) $\bar{\Lambda}(\alpha/(1+\alpha)) \leq \bar{\mathcal{K}}(\frac{\alpha}{2},2) \leq \bar{\mathcal{K}}(\alpha,+\infty)$

Proof. – The Proposition is a corollary of Hölder's inequality. With the notations of the Proposition, we have

$$q \log \frac{\|u\|_p}{\|u\|_1} \le q' \log \frac{\|u\|_{p'}}{\|u\|_1} \tag{1.1}$$

And if $p < 1 < p'$,

$$q \log \frac{\|u\|_p}{\|u\|_1} \le \frac{E_1(u)}{\|u\|_1} \le q' \log \frac{\|u\|_{p'}}{\|u\|_1} \tag{1.2}$$

(Let p or p' tend to 1 in (1.1)).

(v) is a consequence of (i), and (vi) is a consequence of (iii).

The link between $\mathcal{K}$ and $\bar{\mathcal{K}}$ is described in the following Proposition.

PROPOSITION 2. – *Assume that* $1 < p \le +\infty$ *and* $0 < \eta \le +\infty$. *Then*

(i) $\mathcal{K}(\eta, p) \ge C(\eta)\bar{\mathcal{K}}(\eta, p)$

(ii) $\bar{\mathcal{K}}(\eta, p) \ge C(\eta)\mathcal{K}(\eta, p)$

(iii) $\mathcal{K}(\alpha) \ge C(\alpha)\bar{\mathcal{K}}(2\alpha)$

(iv) $4\Lambda(1) \ge \bar{\Lambda}(1) \ge \frac{1}{4}\Lambda(1)$

Proof. – Note that, since $p > 1$ and $\alpha > 0$, then $\eta > 0$.

To prove (i), let $u \in \mathcal{F} \cap L_p$ be s.t. $\int u d\mu = 0$. Then either $u^+ \in \mathcal{H} \cap \mathcal{F}$ or $u^- \in \mathcal{H} \cap \mathcal{F}$. Assume that $u^+ \in \mathcal{H} \cap \mathcal{F}$. Then $\mathcal{E}(u^+) \le \mathcal{E}(u)$, since u^+ is a normal contraction of u. Obviously, $\|u^+\|_p \le \|u\|_p$. Besides $\|u\|_1 = 2\int u^+ d\mu$.

Thus,

$$\frac{\mathcal{E}(u)}{\|u\|_1^2}\left(\frac{\|u\|_p}{\|u\|_1}\right)^{2/\eta} \ge C(\eta)\frac{\mathcal{E}(u^+)}{\|u^+\|_1^2}\left(\frac{\|u^+\|_p}{\|u^+\|_1}\right)^{2/\eta}$$

To prove (ii), let $u \in \mathcal{F} \cap \mathcal{H}$, $A = \{x \ s.t. \ u(x) = 0\}$. By definition of $\mathcal{H}$, $\mu(A) \ge 1/2$.

Let $\hat{u} = u - \int u d\mu$. Then $\int \hat{u} d\mu = 0$ and $\mathcal{E}(\hat{u}) = \mathcal{E}(u)$, $\|\hat{u}\|_1 \ge \mu(A)\|u\|_1 \ge 1/2\|u\|_1$. Besides $\|\hat{u}\|_p \le \|u\|_p + \|u\|_1 \le 2\|u\|_p$.

Thus

$$\frac{\mathcal{E}(\hat{u})}{\|\hat{u}\|_1^2}\left(\frac{\|\hat{u}\|_p}{\|\hat{u}\|_1}\right)^{2/\eta} \le C(\eta)\frac{\mathcal{E}(u)}{\|u\|_1^2}\left(\frac{\|u\|_p}{\|u\|_1}\right)^{2/\eta}$$

and (ii) is proved.

Let $u \in \mathcal{F}$ and $\int u d\mu = 0$. Assume that $u^+ \in \mathcal{H}$. In order to establish (iii), it is clearly sufficient to prove that

$$E_1(u) \ge E_1(u^+) - \log(4)\int u^+ d\mu \tag{1.3}$$

But, for any $p > 1$, $\int |u|^p d\mu = \int |u^+|^p d\mu + \int |u^-|^p d\mu \geq \int |u^+|^p d\mu + (\int |u^-| d\mu)^p = \int |u^+|^p d\mu + (\int |u^+| d\mu)^p$. Therefore, $\|u\|_p^p - \|u\|_1 \geq \|u^+\|_p^p - \|u^+\|_1 + \|u^+\|_1^p - \|u^+\|_1$. Dividing this last inequality by p, and letting p tend to 1, we get (1.3).

(iv) is a consequence of (i) and (ii) for $\eta = +\infty$.

Proposition 1 (iii) implies that $\bar{\mathcal{K}}(-1, \frac{\alpha}{1+\alpha}) \leq \bar{\mathcal{K}}(\alpha, +\infty)$. We shall now describe a converse to that inequality.

THEOREM 1. – *Let $\alpha \in]0, +\infty[$. For any $\beta < \alpha/(1+\alpha)$, there exists a constant, $C(\alpha, \beta)$, s.t.*

$$\bar{\mathcal{K}}(-1, \beta) \geq C(\alpha, \beta)\bar{\mathcal{K}}(\alpha, +\infty)$$

Proof. – The proof is based on the ideas of Bakry *et al*, 1995 [2].

We first have to introduce the scale of Lorentz norms: for a measurable function u, and $a, b \in]0, +\infty[$, let

$$\|u\|_{a,b} = \left(b \int_0^\infty t^b \mu(|u| \geq t)^{b/a} \frac{dt}{t} \right)^{1/b}$$

be the Lorentz norm of u.

Observe that $\|u\|_{b,b} = \|u\|_b$ and that

(1.4)

$$(1-2^{-b})\Sigma_{k\in\mathbb{Z}} 2^{bk} \mu(|u| \geq 2^k)^{b/a} \leq \|u\|_{a,b}^b \leq (2^b - 1)\Sigma_{k\in\mathbb{Z}} 2^{bk} \mu(|u| \geq 2^k)^{b/a}$$

Let $u \in \mathcal{F} \cap \mathcal{H}$. For $k \in \mathbb{Z}$, define $u_k(x) = (u(x) - 2^k)^+ \wedge 2^k$. Then $u_k \in \mathcal{H} \cap \mathcal{F}$.

By definition of $\bar{\mathcal{K}}$,

$$\frac{\mathcal{E}(u_k)}{\|u_k\|_1^2} (\frac{\|u_k\|_\infty}{\|u_k\|_1})^{2/\alpha} \geq \bar{\mathcal{K}}(\alpha, +\infty)$$

$\|u_k\|_1 \geq 2^k \mu(u \geq 2^{k+1})$ and $\|u_k\|_\infty \leq 2^k$. Therefore

$$\mathcal{E}(u_k) \geq \bar{\mathcal{K}}(\alpha, +\infty) 2^{2k} \mu(u \geq 2^{k+1})^{2(1+\alpha)/\alpha}$$

Corollary 2.3. of Bakry *et al.*, 1995 [2] implies that $\Sigma_{k\in\mathbb{Z}} \mathcal{E}(u_k) \leq 6\mathcal{E}(u)$. Hence

$$\mathcal{E}(u) \geq C\bar{\mathcal{K}}(\alpha, +\infty) \sum_{k\in\mathbb{Z}} 2^{2k} \mu(u \geq 2^k)^{2(1+\alpha)/\alpha}$$

Using (1.4), we get that

$$\mathcal{E}(u) \geq C\bar{\mathcal{K}}(\alpha, +\infty)\|u\|^2_{\frac{\alpha}{1+\alpha},2} \tag{1.5}$$

The end of the proof of the Theorem is a direct consequence of the following Lemma:

LEMMA 1. – *For any $\beta \in]0,1[$, $b < \beta$, there exists a constant, $C(b,\beta)$, s.t. for any function u*

$$\|u\|_{\beta,2} \geq C(b,\beta)\|u\|_b$$

Proof. – Let $\phi(t) = \mu(|u| \geq t)$. ϕ is decreasing and $\phi(0) = 1$.

$$\begin{aligned}
& b\int_0^\infty t^{b-1}\phi(t)dt \\
=& b\int_0^T t^{b-1}\phi(t)dt + b\int_T^\infty t^{b-1}\phi(t)dt \\
\leq& T^b + b\int_T^\infty t^{b-1}\frac{1}{t}\left(\int_0^t \phi(s)ds\right)dt \text{ since } \phi \text{ is decreasing} \\
\leq& T^b + C(b,\beta)\int_T^\infty t^{b-2}\left(\int_0^t s\phi(s)^{2/\beta}ds\right)^{\beta/2} t^{1-\beta}dt \text{ by Hölder's inequality} \\
\leq& T^b + C(b,\beta)T^{b-\beta}\left(\int s\phi(s)^{2/\beta}ds\right)^{\beta/2}
\end{aligned}$$

Choose $T = (\int s\phi(s)^{2/\beta}ds)^{1/2}$ to get the result.

II. ESTIMATES OF THE SEMI-GROUP

Let $(P_t, t \geq 0)$ be the L_2 semi-group associated to $(\mathcal{E}, \mathcal{F})$.

Our aim is to prove that if one of the constants $\mathcal{K}$ or $\bar{\mathcal{K}}$ is non zero, then the process X converges exponentially quickly to equilibrium. Owing to Propositions 1, 2 and Theorem 1, it is sufficient to consider the case $\mathcal{K}(\eta, 2) \neq 0$.

THEOREM 2. – *For any $\eta \in]0, +\infty]$, there exists a constant, $c(\eta) > 0$ s.t. for any $t \geq 0$ and any function $u \in L_2$,*

$$\|P_t u - \int u d\mu\|_1 \leq e^{-c(\eta)\mathcal{K}(\eta,2)\, t}\|u\|_2$$

This inequality also holds for $\Lambda(2\eta/(1+2\eta))$ or $\bar{\Lambda}(2\eta/(1+2\eta))$ instead of $\mathcal{K}(\eta,2)$.

We break the proof into two Lemmas:

LEMMA 2. – *Let $\eta \in]0,+\infty[$. For any function $u \in \mathcal{F}$ s.t. $\int u d\mu = 0$*

$$\|u\|_1^2 \leq \frac{1}{\mathcal{K}(\eta,2)}\mathcal{E}(u) + \frac{\eta^\eta}{(1+\eta)^{1+\eta}}\|u\|_2^2$$

Proof. – Note that for any $x, y > 0$, $x^{\frac{\eta}{1+\eta}} y^{\frac{1}{1+\eta}} = (\frac{1+\eta}{\eta}x)^{\frac{\eta}{1+\eta}}((\frac{\eta}{1+\eta})^\eta y)^{\frac{1}{1+\eta}} \leq x + \frac{\eta^\eta}{(1+\eta)^{1+\eta}}y$. Applying this inequality to $x = \mathcal{E}(u)/\mathcal{K}(\eta,2)$ and $y = \|u\|_2^2$, since

$$\left(\frac{\mathcal{E}(u)}{\mathcal{K}(\eta,2)}\right)^{\frac{\eta}{1+\eta}} \|u\|_2^{\frac{2}{1+\eta}} \geq \|u\|_1^2$$

we get the Lemma.

LEMMA 3. – *Assume that there exist $\varepsilon \in [0,1[$ and $\lambda > 0$ s.t. for any function $u \in \mathcal{F}$ s.t. $\int u d\mu = 0$,*

$$\|u\|_1^2 \leq \frac{1}{\lambda}\mathcal{E}(u) + \varepsilon\|u\|_2^2$$

Then, for any function $u \in L_2$,

$$\left\|P_t u - \int u d\mu\right\|_1 \leq e^{-\lambda(1-\varepsilon)t}\|u\|_2$$

Proof. – Let $u \in \mathcal{F}$ s.t. $\int u d\mu = 0$.

Let $u(t) = \int |P_t u|^2 d\mu$ and $v(t) = (\int |P_t u| d\mu)^2$.

Then

$$\begin{aligned}\frac{d}{dt}(e^{-2\lambda\varepsilon t}u(t)) &= -e^{-2\lambda\varepsilon t}2\lambda\left(\varepsilon u(t) + \frac{1}{\lambda}\mathcal{E}(P_t u)\right)\\ &\leq -2\lambda e^{-2\lambda\varepsilon t}v(t)\end{aligned}$$

So

$$e^{-2\lambda\varepsilon t}u(t) \leq u(0) - 2\lambda\int_0^t e^{-2\lambda\varepsilon s}v(s)ds$$

Besides $v(t) \leq u(t)$. Therefore

$$e^{-2\lambda\varepsilon t}v(t) \leq u(0) - 2\lambda\int_0^t e^{-2\lambda\varepsilon s}v(s)ds$$

Applying Gronwall's Lemma, we get that

$$v(t) \leq u(0)e^{-2\lambda(1-\varepsilon)t}$$

Proof of Theorem 2. – If $\eta < +\infty$, then $(\frac{\eta}{1+\eta})^{\frac{\eta}{1+\eta}}(\frac{1}{1+\eta})^{\frac{1}{1+\eta}} < 1$ and the conclusion of the Theorem follows at once from Lemmas 2 and 3. If $\eta = +\infty$, then the inequality of Lemma 3 is satisfied for $\varepsilon = 0$ and $\lambda = \mathcal{K}(+\infty)$.

By Propositions 1 and 2, one can replace $\mathcal{K}(\eta, 2)$ by $\Lambda(2\eta/(1+2\eta))$ or $\bar{\Lambda}(2\eta/(1+2\eta))$.

III. ESTIMATES OF THE MOMENTS OF THE HITTING TIMES

Estimates of the potentials

In the next Theorem, we shall prove that the inequality $\mathcal{K}(\alpha, +\infty) > 0$ is equivalent to uniform estimates of the t-potentials of sets of μ-measure bigger than $1/2$.

Let $\varepsilon \in]0, 2]$. Let $\bar{\mathcal{I}}(\varepsilon)$ be the best constant in the inequality: for any $t > 0$, for any closed set A s.t. $\mu(A) \geq 1/2$,

$$\int (1 - h_t^A) d\mu \leq (t/\bar{\mathcal{I}}(\varepsilon))^{\varepsilon/2}$$

In other words $\bar{\mathcal{I}}(\varepsilon) = \inf t^{-1}(\int(1 - h_t^A)d\mu)^{2/\varepsilon}$, where the inf is taken on $t > 0$ and closed sets A s.t. $\mu(A) \geq 1/2$. Note that $\bar{\mathcal{I}}(\varepsilon)$ might be 0.

More generally, for $a \in]0, 1]$, let $\bar{\mathcal{I}}_a(\varepsilon)$ be the best constant in the inequality: for any $t > 0$, for any closed set A s.t. $\mu(A) \geq a$,

$$\int (1 - h_t^A) d\mu \leq (ta^{-\frac{2+\varepsilon}{\varepsilon}}/\bar{\mathcal{I}}_a(\varepsilon))^{\varepsilon/2}$$

THEOREM 3. – *For any $\alpha \in]0, +\infty]$ and $a \in]0, 1]$,*

$$\bar{\mathcal{I}}_a\left(\frac{2\alpha}{\alpha+2}\right) \geq C(\alpha)\mathcal{K}(\alpha, +\infty)$$

This inequality also holds for $\Lambda(\alpha/(1+\alpha))$ instead of $\mathcal{K}(\alpha, +\infty)$.

Proof. – We first consider the case $a = 1/2$.

By Proposition 2, it is equivalent to prove the Theorem for $\mathcal{K}(\alpha, +\infty)$ or $\bar{\mathcal{K}}(\alpha, +\infty)$.

Let A be a closed set s.t. $\mu(A) \geq 1/2$.

Remember that $\mathcal{E}_t(h_t^A, u) = 0$ for any quasi-continuous function $u \in \mathcal{F}$ s.t. $u = 0$ quasi everywhere on A.

We apply this equality to $u = 1 - h_t^A$ to get

$$\mathcal{E}(h_t^A) = t \int h_t^A(1 - h_t^A) d\mu \tag{3.1}$$

Therefore

$$\mathcal{E}(h_t^A) \leq t \int (1 - h_t^A) d\mu \tag{3.2}$$

Since $\mu(A) \geq 1/2$, $1 - h_t^A \in \mathcal{H}$. Besides $\|1 - h_t^A\|_\infty \leq 1$. Hence

$$\begin{aligned}\int (1 - h_t^A) d\mu &\leq (\bar{\mathcal{K}}(\alpha, +\infty)^{-1} \mathcal{E}(h_t^A))^{\alpha/(2(1+\alpha))} \|1 - h_t^A\|_\infty^{\frac{1}{1+\alpha}} \\ &\leq (\bar{\mathcal{K}}(\alpha, +\infty)^{-1} t \int (1 - h_t^A) d\mu)^{\alpha/(2(1+\alpha))}\end{aligned}$$

Therefore

$$\int (1 - h_t^A) d\mu \leq (t/\bar{\mathcal{K}}(\alpha, +\infty))^{\alpha/(\alpha+2)}$$

i.e.

$$\bar{\mathcal{I}}\left(\frac{2\alpha}{\alpha+2}\right) \geq C(\alpha)\mathcal{K}(\alpha, +\infty)$$

Let us now consider the general case $a \in]0, 1]$.

Let $\mathcal{H}_a$ be the space of non-negative measurable functions u s.t. $\mu(u = 0) \geq a$.

Let $u \in \mathcal{H}_a$ and let $\hat{u} = u - \int u d\mu$. Then it is easy to see that $\mathcal{E}(\hat{u}) = \mathcal{E}(u)$, $\|\hat{u}\|_\infty \leq \|u\|_\infty$ and $\|\hat{u}\|_1 \geq a\|u\|_1$. Therefore

$$\inf_{u \in \mathcal{F} \cap \mathcal{H}_a} \frac{\mathcal{E}(u)}{\|u\|_1^2} \left(\frac{\|u\|_\infty}{\|u\|_1}\right)^{2/\alpha} \geq \mathcal{K}(\alpha, +\infty) a^{2\frac{\alpha+1}{\alpha}}$$

Let A be a closed set s.t. $\mu(A) \geq a$. Then $1 - h_t^A \in \mathcal{H}_a$. We can now repeat the same argument as for $a = 1/2$ to prove that

$$\bar{\mathcal{I}}_a\left(\frac{2\alpha}{\alpha+2}\right) \geq \mathcal{K}(\alpha, +\infty)$$

The fact that one can replace $\mathcal{K}(\alpha, +\infty)$ by $\Lambda(\alpha/(1+\alpha))$ is a consequence of Proposition 1.

In order to prove a converse to Theorem 3, we need a further assumption on the Dirichlet form $(\mathcal{E}, \mathcal{F})$: we shall say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the **strong sector condition** with constant $M \in [1, +\infty[$ if for any $u, v \in \mathcal{F}$,

$$|\mathcal{E}(u,v)| \leq M\sqrt{\mathcal{E}(u)\mathcal{E}(v)}$$

Note that the strong sector condition is fulfilled by symmetric Dirichlet forms with $M = 1$.

THEOREM 4. – *Assume that the strong sector condition holds with constant* $M \in]0, +\infty[$. *Then, for any* $\alpha \in]0, +\infty]$,

$$\bar{\mathcal{I}}\left(\frac{2\alpha}{\alpha+2}\right) \leq C(\alpha, M)\mathcal{K}(\alpha, +\infty)$$

Remark. – By Theorem 1, one can replace $\mathcal{K}(\alpha, +\infty)$ by $\bar{\Lambda}(p')$ in the inequality of the Theorem, provided that $p' < \alpha/(1+\alpha)$.

Proof. Let $\varepsilon = 2\alpha/(2+\alpha)$ and $I = \bar{\mathcal{I}}(\varepsilon)$.

Let $u \in \mathcal{H} \cap C_0$. Let $A = \{x \ s.t. \ u(x) = 0\}$. Then A is closed and $\mu(A) \geq 1/2$. Therefore $1 - h_t^A \in \mathcal{H}$.

From the strong sector condition, we get that

$$\mathcal{E}(u) \geq M^{-2}\frac{\mathcal{E}(h_t^A, u)^2}{\mathcal{E}(h_t^A)} \tag{3.3}$$

It follows from (3.2) and the definition of I that

$$\mathcal{E}(h_t^A) \leq t(t/I)^{\varepsilon/2} \tag{3.4}$$

On the other hand, since $u = 0$ on A, $\mathcal{E}_t(h_t^A, u) = 0$ *i.e.* $\mathcal{E}(h_t^A, u) = -t\int uh_t^A d\mu$. Besides

$$\begin{aligned}\int uh_t^A d\mu &\geq \int ud\mu - \|u\|_\infty \int (1-h_t^A)d\mu \\ &\geq \int ud\mu - \|u\|_\infty (t/I)^{\varepsilon/2}\end{aligned}$$

Thus

$$|\mathcal{E}(h_t^A, u)| \geq t\left(\int ud\mu - \|u\|_\infty (t/I)^{\varepsilon/2}\right)$$

Using this last inequality and (3.4) in (3.3), we obtain that for $t \leq I(\int u d\mu / \|u\|_\infty)^{2/\varepsilon}$,

$$\mathcal{E}(u) \geq M^{-2} t^{(2-\varepsilon)/2} I^{\varepsilon/2} \left(\int u d\mu - \|u\|_\infty (t/I)^{\varepsilon/2} \right)^2$$

We choose $t = I(\int u d\mu / (2\|u\|_\infty))^{2/\varepsilon}$ to get

$$\mathcal{E}(u) \geq C(M, \varepsilon) I \left(\int u d\mu \right)^{(\varepsilon+2)/\varepsilon} \|u\|_\infty{}^{1-2/\varepsilon}$$

i.e. we have proved that

$$\bar{\mathcal{K}}\left(\frac{2\varepsilon}{2-\varepsilon}, +\infty \right) \geq C(M, \varepsilon) I$$

But $2\varepsilon/(2-\varepsilon) = \alpha$ and, by Proposition 2, $\bar{\mathcal{K}}(\alpha, +\infty) \leq C(\alpha)\mathcal{K}(\alpha, +\infty)$. Thus the proof of the Theorem is complete.

Estimates of the moments of the hitting times

From the inequalities of Theorem 4 it is easy to derive an upper bound on the moments of the hitting times under P_μ. (P_μ is the law of the process X when the initial law is μ.)

Instead of L_p spaces, it is more convenient to work in weak L_p spaces: for $0 < p < +\infty$ let

$$\|F\|_{*,p,P_\mu} = \sup_{t \geq 0} t P_\mu[|F| \geq t]^{1/p}$$

be the weak-L_p norm of F.

Remember that for $p' < p$, there exists a constant, $C(p, p')$, s.t. $\|F\|_{p',P_\mu} \leq C(p, p') \|F\|_{*,p,P_\mu}$, where $\|F\|_{p',P_\mu}$ is the $L_{p'}(P_\mu)$ norm of F.

As a consequence of Theorem 4, we have the

PROPOSITION 3. – *For any $\alpha \in]0, +\infty]$, and any closed subset A,*

$$\|\tau_A\|_{*,\alpha/(2+\alpha),P_\mu} \leq C(\alpha) \mathcal{K}(\alpha, +\infty)^{-1} \mu(A)^{-2(\alpha+1)/\alpha}$$

This inequality also holds for $\Lambda(\alpha/(1+\alpha))$ instead of $\mathcal{K}(\alpha, +\infty)$.

Proof.

$$\begin{aligned}\int (1 - h_t^A) d\mu &= E_\mu[1 - e^{-t\tau_A}] \\ &\geq E_\mu[(1 - e^{-t\tau_A}) 1_{t\tau_A \geq 1}] \geq (1 - 1/e) P_\mu[t\tau_A \geq 1]\end{aligned}$$

Hence, using Theorem 3, we have

$$P_\mu[\tau_A \geq t] \leq C(\alpha)(\mathcal{K}(\alpha,+\infty)t\mu(A)^{(2\alpha+2)/\alpha})^{-\alpha/(2+\alpha)}$$

Therefore

$$tP_\mu[\tau_A \geq t]^{(2+\alpha)/\alpha} \leq C(\alpha)\mathcal{K}(\alpha,+\infty)^{-1}\mu(A)^{-2(\alpha+1)/\alpha}$$

Case $\alpha = +\infty$, proof of the main theorem

When $\alpha = +\infty$, it is possible to prove a slightly stronger result than Proposition 3: remember that $\Lambda(1) = \inf_{u\in\mathcal{F},\int ud\mu=0} \mathcal{E}(u)/\|u\|_1^2$. Applying this definition to the function $u - \int ud\mu$, for $u \in \mathcal{H}_a$, we get

$$\inf_{u\in\mathcal{F}\cap\mathcal{H}_a} \frac{\mathcal{E}(u)}{\|u\|_1^2} \geq a^2\Lambda(1)$$

Let A be a closed set. Applying the last inequality to the function $1 - h_t^A \in \mathcal{H}_{\mu(A)}$, we get

$$\mu(A)^2\Lambda(1)\Big(\int(1-h_t^A)d\mu\Big)^2 \leq \mathcal{E}(h_t^A) = t\int h_t^A(1-h_t^A)d\mu$$

And since $(\int h_t^A d\mu)^2 \leq \int (h_t^A)^2 d\mu$, we have

$$\mu(A)^2\Lambda(1)\Big(\int(1-h_t^A)d\mu\Big)^2 \leq t\Big(\int h_t^A d\mu - \Big(\int h_t^A d\mu\Big)^2\Big)$$

i.e.

$$\int h_t^A d\mu \geq \frac{\mu(A)^2\Lambda(1)}{\mu(A)^2\Lambda(1)+t}$$

Therefore

$$\frac{1}{t}(1 - E_\mu[e^{-t\tau_A}]) \leq \frac{1}{\mu(A)^2\Lambda(1)+t}$$

Letting t tend to 0, we get

$$E_\mu[\tau_A] \leq \frac{1}{\mu(A)^2\Lambda(1)}$$

Thus we have proved the

PROPOSITION 4. – *For any closed set A,*

$$E_\mu[\tau_A] \le \frac{1}{\mu(A)^2\Lambda(1)}$$

We can now complete the proof of the main theorem (ii):
Assume that $\mathcal{E}$ satisfies the strong sector condition with constant M.
Let $L = \sup_{A,\mu(A)\ge 1/2} E_\mu[\tau_A]$. Since, for any $t > 0$, $1-\int h_t^A d\mu \le tE_\mu[\tau_A]$, we have $\bar{\mathcal{I}}(2) \ge 1/L$.

From Theorem 4 it then follows that $\mathcal{K}(+\infty, +\infty) \ge C(M)/L$, where $C(M)$ is a constant that only depends on M.

But $\mathcal{K}(+\infty, +\infty) = \Lambda(1)$. Thus the proof of part (ii) of the main theorem is complete.

Extensions

We do not assume anymore that the measure μ is invariant for the process X. We also only suppose that $(\mathcal{E}, \mathcal{F})$ is sub-markovian, *i.e.* $P_t 1 \le 1$ a.s. We shall now describe how it is possible to extend our results to this more general situation. For the sake of simplicity, we only consider the case $\alpha = +\infty$.

For a closed set A, let $h_0^A(x) = P_x[\tau_A < +\infty]$.

Then $h_0^A \in \mathcal{F}$ and equation (3.1) now reads: for any $t \ge 0$,

$$\mathcal{E}(h_0^A - h_t^A) = t \int h_t^A (h_0^A - h_t^A) d\mu \tag{3.5}$$

The statement of Theorem 3 thus becomes:

for any closed set A s.t $\mu(A) \ge 1/2$, for any $t \ge 0$,

$$\int (h_0^A - h_t^A) d\mu \le t/\bar{\Lambda}(1) \tag{3.6}$$

As in Proposition 4, it is easy to deduce from (3.6) that, for any closed set A s.t. $\mu(A) \ge 1/2$,

$$E_\mu[1_{\tau_A < +\infty} \tau_A] \le 1/\bar{\Lambda}(1) \tag{3.7}$$

IV. ESTIMATES OF THE LAW OF THE HITTING TIMES OF SMALL SETS: THE UNPREDICTABILITY PROPERTY

Estimates of the potentials

As in part III, for $\varepsilon \in]0,2]$, let $\mathcal{I}(\varepsilon)$ be the best constant in the inequality: for any $t > 0$, for any closed set A,

$$\|h_t^A - \int h_t^A d\mu\|_1 \leq (t/\mathcal{I}(\varepsilon))^{\varepsilon/2}$$

THEOREM 5. – *For any* $\alpha \in]0,+\infty]$,

$$\mathcal{I}\left(\frac{\alpha}{\alpha+1}\right) \geq C(\alpha)\mathcal{K}(\alpha,+\infty)$$

This inequality also holds for $\Lambda(\alpha/(1+\alpha))$ instead of $\mathcal{K}(\alpha,+\infty)$.

Proof. – We proceed as for Theorem 3: by (3.2), $\mathcal{E}(h_t^A) \leq t$. Since $\int(h_t^A - \int h_t^A d\mu)d\mu = 0$, we have

$$\begin{aligned}\left\|h_t^A - \int h_t^A d\mu\right\|_1 &\leq (\mathcal{K}(\alpha,+\infty)^{-1}\mathcal{E}(h_t^A))^{\alpha/(2(1+\alpha))}\left\|\int h_t^A d\mu - h_t^A\right\|_\infty^{\frac{1}{1+\alpha}}\\ &\leq (\mathcal{K}(\alpha,+\infty)^{-1}t)^{\alpha/(2(\alpha+1))}\end{aligned}$$

To replace $\mathcal{K}(\alpha,+\infty)$ by $\Lambda(\alpha/(1+\alpha))$, use Proposition 1.

Estimates of the law of the hitting times of small sets

Our aim is to show that if any of the constants $\mathcal{K}(\alpha,+\infty)$ is non zero, then the unpredictability property holds. This will be an easy consequence of the next Theorem but we first need a preliminary result: Remember that the quantity $T(A)$ has been defined in the introduction.

PROPOSITION 5. – *Let A be a closed set s.t. $T(A) < +\infty$, then, for any $t > 0$,*

$$\left|\int h_{tT(A)}^A d\mu - \frac{1}{1+t}\right| \leq 2\left\|h_{T(A)}^A - \frac{1}{2}\right\|_1 \tag{4.1}$$

Proof. – Let $\hat{X}$ be the dual process of X w.r.t μ. By definition, the Dirichlet form of $\hat{X}$ is $\hat{\mathcal{E}}(f,g) = \mathcal{E}(g,f)$. We shall use a $\hat{}$ to denote any quantity associated to $\hat{X}$. Thus $\hat{h}_t^A$ is the Laplace transform of the hitting time of A by $\hat{X}$.

Let A be a closed subset of E. As in the proof of Theorem 3, for any $t, s > 0$, and any quasi-continuous function $u \in \mathcal{F}$ s.t. $u = 0$ quasi everywhere on A, we have $\mathcal{E}_t(h_t^A, u) = \mathcal{E}_s(u, \hat{h}_s^A) = 0$. We apply this equality to $u = 1 - h_t^A$ and $u = 1 - \hat{h}_s^A$ to get

$$\int \hat{h}_t^A d\mu = \int h_t^A d\mu$$

and

$$(t-s)\int h_t^A \hat{h}_s^A d\mu = t\int h_t^A d\mu - s\int h_s^A d\mu \tag{4.2}$$

Let $\varepsilon = \int |h_{T(A)}^A - \frac{1}{2}| d\mu$

Since $h_{tT(A)}^A$ is bounded by 1,

$$\left| \int \hat{h}_{tT(A)}^A h_{T(A)}^A d\mu - (1/2)\int h_{tT(A)}^A d\mu \right| \leq \varepsilon$$

So, by 4.2,

$$\left| \frac{t\int h_{tT(A)}^A d\mu - (1/2)}{t-1} - \frac{1}{2}\int h_{tT(A)}^A d\mu \right| \leq \varepsilon$$

i.e.

$$\left| \int h_{tT(A)}^A d\mu - \frac{1}{1+t} \right| \leq \varepsilon \frac{2|t-1|}{t+1}$$

Hence

$$\left| \int h_{tT(A)}^A d\mu - \frac{1}{1+t} \right| \leq 2\left\| h_{T(A)}^A - \frac{1}{2} \right\|_1$$

Let A be a closed subset of E. Applying Theorem 5 to A and $T(A)$, we get

$$\left\| h_{T(A)}^A - \frac{1}{2} \right\|_1 \leq C(\alpha)(T(A)/\mathcal{K}(\alpha, +\infty))^{\alpha/(2(1+\alpha))}$$

Therefore, using 4.1,

$$\sup_t \left| \int h_{tT(A)}^A d\mu - \frac{1}{1+t} \right| \leq C(\alpha)(T(A)/\mathcal{K}(\alpha, +\infty))^{\alpha/(2(\alpha+1))} \tag{4.3}$$

Thus we have proved the

PROPOSITION 6. – *Let A be a closed set s.t. $T(A) < +\infty$. Then*

$$\sup_t \left| \int h_{tT(A)}^A d\mu - \frac{1}{1+t} \right| \leq C(\alpha)(T(A)/\mathcal{K}(\alpha, +\infty))^{\alpha/(2(\alpha+1))}$$

Remark. – By Proposition 1, one can replace $\mathcal{K}(\alpha, +\infty)$ by $\Lambda(\alpha/(1+\alpha))$ in the statement of the Proposition and (4.3).

V. GENERALIZED SPECTRAL GAP AND LOG-SOBOLEV INEQUALITIES

Let

$$\Lambda(2) = \inf_{u \in \mathcal{F}, \int u d\mu = 0} \frac{\mathcal{E}(u)}{\int u^2 d\mu}$$

The inequality $\mathcal{K}(-1, \frac{\alpha}{\alpha+1}) > 0$ is strictly weaker than the usual spectral gap inequality $\Lambda(2) > 0$. It is interesting to wonder "how much weaker ?".

For $u \in \mathcal{F} \cap L^2 \log L$, let

$$E_2(u) = \int u^2 \log u^2 d\mu - \int u^2 d\mu \log \int u^2 d\mu$$

We say that $(\mathcal{E}, \mathcal{F})$ satisfies a **log-Sobolev** inequality with constants c and m if, for any function $u \in \mathcal{F}$, then $u \in L^2 \log L$ and

$$E_2(u) \leq c(\mathcal{E}(u) + m \int u^2 d\mu) \tag{5.1}$$

It is known (see Bakry, 1992 [1]) that log-Sobolev inequalities are equivalent to contraction properties of the semi-group. For instance, if $(\mathcal{E}, \mathcal{F})$ satisfies (5.1) then for any $t > 0$ and any function u, $\|P_t u\|_q \leq e^{m'} \|u\|_2$, where $q = 1 + \exp(4t/c)$ and $m' = mc(1/2 - 1/q)$. Note that $m' = 0$ if $m = 0$, and then P_t is in fact a contraction from L_2 to L_q.

Following Bakry, we say that the log-Sobolev inequality (5.1) is **tight** if $m = 0$.

It is known that $(\mathcal{E}, \mathcal{F})$ satisfies a tight log-Sobolev inequality if and only if it satisfies a log-Sobolev inequality and $\Lambda(2) > 0$. We shall prove a slightly more general result:

PROPOSITION 7. – *The following five assertions are equivalent:*

(i) *$(\mathcal{E}, \mathcal{F})$ satisfies a tight log-Sobolev inequality;*

(ii) *there exists $\alpha \in]0, +\infty]$ s.t. $(\mathcal{E}, \mathcal{F})$ satisfies a log-Sobolev inequality and $\Lambda(\alpha/(1+\alpha)) > 0$;*

(iii) *there exists $\alpha \in]0, +\infty]$ s.t. $(\mathcal{E}, \mathcal{F})$ satisfies a log-Sobolev inequality and $\mathcal{K}(\alpha, +\infty) > 0$;*

(iv) *for any $\alpha \in]0, +\infty]$, $(\mathcal{E}, \mathcal{F})$ satisfies a log-Sobolev inequality and $\Lambda(\alpha/(1+\alpha)) > 0$;*

(v) *for any $\alpha \in]0, +\infty]$, $(\mathcal{E}, \mathcal{F})$ satisfies a log-Sobolev inequality and $\mathcal{K}(\alpha, +\infty) > 0$.*

Besides the following inequality holds:

$$1 \leq \left(\frac{\Lambda(2)}{\mathcal{K}(\eta,2)}\right)^{\eta/(\eta+1)} e^{c(\Lambda(2)+m)} \tag{5.2}$$

for any η.

Proof. – It follows from the remarks at the beginning of this section, Proposition 1 and Theorem 1 that we only have to prove that if $(\mathcal{E},\mathcal{F})$ satisfies (5.1) and if $\mathcal{K}(\alpha,+\infty) > 0$ for some α, then $\Lambda(2) > 0$. Because of Theorem 1, we can in fact assume that $\mathcal{K}(\eta,2) > 0$ for some η. So that the proof of the Theorem will be complete if we prove (5.2).

As for 1.2, Hölder's inequality implies that

$$\frac{E_2(u)}{||u||_2^2} \geq 2\log\frac{||u||_2}{||u||_1}$$

Using this inequality in (5.1) yields

$$2\log\frac{||u||_2}{||u||_1} \leq c\left(\frac{\mathcal{E}(u)}{||u||_2^2} + m\right)$$

But if $\int u d\mu = 0$, then

$$\frac{||u||_1}{||u||_2} \leq \left(\frac{\mathcal{E}(u)}{\mathcal{K}(\eta,2)||u||_2^2}\right)^{\eta/(2(1+\eta))}$$

Thus

$$0 \leq \frac{\eta}{1+\eta}\log\left(\frac{\mathcal{E}(u)}{\mathcal{K}(\eta,2)||u||_2^2}\right) + c\left(\frac{\mathcal{E}(u)}{||u||_2^2} + m\right)$$

Taking the inf over functions u s.t. $\int u d\mu = 0$, we get (5.2).

VI. THE SPECTRAL GAP INEQUALITY AND HITTING TIMES

In this section we shall discuss the extension of our results to the case $p = 2$.

As in Proposition 2, one easily proves that

$$\Lambda(2) \leq 2\bar{\Lambda}(2)$$

PROPOSITION 8.- (i) *For any closed set A s.t. $\mu(A) \geq 1/2$,*

$$E_\mu[\tau_A] \leq (1 - \mu(A))/\bar{\Lambda}(2) \tag{6.1}$$

(ii) *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies the strong sector condition with constant M. There exists a universal constant, C, that only depends on M, s.t*

$$\sup_{A,\mu(A)\geq 1/2} (1 - \mu(A))^{-1} E_\mu[\tau_A] \geq C/\bar{\Lambda}(2) \tag{6.2}$$

Proof. – Let $\Omega = E - A$.

Since $\mu(A) \geq 1/2$, $1 - h_t^A \in \mathcal{H}$, and, by definition of $\bar{\Lambda}(2)$,

$$\bar{\Lambda}(2) \|1 - h_t^A\|_2^2 \leq \mathcal{E}(1 - h_t^A) \tag{6.3}$$

But $\mathcal{E}(1 - h_t^A) = t \int h_t^A (1 - h_t^A) d\mu \leq t \int (1 - h_t^A) d\mu$, and $\int (1 - h_t^A) d\mu \leq \|1 - h_t^A\|_2 \sqrt{\mu(\Omega)}$. Thus (6.3) implies that

$$\bar{\Lambda}(2) \int (1 - h_t^A) d\mu \leq t\mu(\Omega)$$

Letting t converge to 0, we get (i).

Now let $L = \sup_{A,\mu(A)\geq 1/2} (1 - \mu(A))^{-1} E_\mu[\tau_A]$.

Since the function $t \rightarrow \int h_t^A d\mu$ is convex, we have, for any $t \geq 0$,

$$1 - \int h_t^A d\mu \leq Lt\mu(\Omega)$$

Let $u \in \mathcal{H}$. Let $A = \{x \text{ s.t. } u(x) = 0\}$. Using the same argument as in the proof of Theorem 4, we get

$$\begin{aligned} &\mathcal{E}(u) \\ &\geq \frac{1}{M^2} \frac{(\mathcal{E}(h_t^A, u))^2}{\mathcal{E}(h_t^A)} \\ &\geq \frac{1}{M^2} \frac{t^2 (\int u h_t^A d\mu)^2}{t \int (1 - h_t^A) d\mu} \\ &\geq \frac{1}{M^2} \frac{(\int u h_t^A d\mu)^2}{L\mu(\Omega)} \end{aligned}$$

Letting t tend to 0, we obtain that

$$\mathcal{E}(u) \geq \frac{(\int u d\mu)^2}{M^2 L\mu(\Omega)} \tag{6.4}$$

Following the proof of Theorem 1, let us define, for $k \in \mathbb{Z}$, $u_k(x) = (u(x) - 2^k)^+ \wedge 2^k$. We apply (6.4) to the function u_k to get

$$2^{2k}\mu(u \geq 2^{k+1})^2 \leq M^2 L\mu(u \geq 2^k)\mathcal{E}(u_k)$$

since $\int u_k d\mu \geq 2^k\mu(u \geq 2^{k+1})$ and $\mu(u_k \neq 0) \leq \mu(u \geq 2^k)$.

Therefore

$$2^{2k}\mu(u \geq 2^{k+1}) \leq M\sqrt{L}\sqrt{\mathcal{E}(u_k)}\sqrt{2^{2k}\mu(u \geq 2^k)}$$

Summing over $k \in \mathbb{Z}$ and using Hölder's inequality, we get

$$\sum_k 2^{2k}\mu(u \geq 2^{k+1}) \leq M\sqrt{L}\sqrt{\sum_k \mathcal{E}(u_k)\sum_k 2^{2k}\mu(u \geq 2^k)}$$

Using (1.4) and the fact that $\sum_k \mathcal{E}(u_k) \leq 6\mathcal{E}(u)$, we obtain that

$$||u||_2^2 \leq 16\sqrt{6}M\sqrt{L}\sqrt{\mathcal{E}(u)}||u||_2^2$$

i.e. $\bar{\Lambda}(2) \geq 3.2^9 M^2/L$.

VII DIFFUSIONS IN $\mathbb{R}^n$

As in the introduction, let w be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}_+^*$. Also assume that $\int w(x)dx = 1$. Let X_t be the Markov process solution of the stochastic differential equation:

$$dX_t = d\beta_t + \frac{1}{2}\frac{\nabla w(X_t)}{w(X_t)}dt$$

The probability measure $d\mu(x) = w(x)dx$ is invariant and reversible for X.

The Dirichlet form of X is given by: $\mathcal{F}$ is the set of functions u s.t. $u \in L_2(\mu)$ and $\nabla u \in L_2(\mu)$, and $\mathcal{E}(u) = \frac{1}{2}\int |\nabla u|^2 d\mu$.

Our aim is to prove that, if w decreases quickly enough at infinity, then $\bar{\Lambda}(p) \neq 0$ for some p. We shall be mainly concerned with the special case $w(x) = |x|^{-\beta}$ for $|x| \geq 1$ $(\beta > n)$.

In the following computation, we shall use polar coordinates: $r = |x|$ is the radial part, and $\tau = x/|x|$ is the angular part of x. Let us denote by S the unit sphere in $\mathbb{R}^n$ and by $d\lambda$ the Haar measure on S.

Let u be a smooth function on $I\!R^n$. Then, for $r > 1$,

$$\Big|u(r,\tau) - u(1,\tau)| = \Big|\int_1^r \nabla u(t,\tau).\tau dt\Big| \leq \int_1^r |\nabla u(t,\tau)|dt$$

Let $p \in]0,1[$ and $\varepsilon \in]0,1]$.

$$\Big(\int_{|x|\geq 1} |u(x) - u(x/|x|)|^p d\mu(x)\Big)^{1/p}$$
$$\leq \Big(\int_{|x|\geq 1} w(x)dx(\int_1^{|x|} |\nabla u(t,\tau)|dt)^p\Big)^{1/p}$$
$$= \Big(\int_{|x|\geq 1} w(x)^\varepsilon dx\Big(w(x)^{(1-\varepsilon)/p}\int_1^{|x|} |\nabla u(t,\tau)|dt\Big)^p\Big)^{1/p}$$
$$\leq \Big(\int_{|x|\geq 1} w(x)^{\varepsilon/(1-p)}dx\Big)^{(1-p)/p} \int_{|x|\geq 1} w(x)^{(1-\varepsilon)/p}dx \int_1^{|x|} |\nabla u(t,\tau)|dt$$

by Hölder's inequality.

The second factor in the rhs can be written as

$$\int_1^\infty dt \int_S d\lambda(\tau)|\nabla u(t,\tau)| \int_{r\geq t} r^{n-1}drw(r,\tau)^{(1-\varepsilon)/p}$$
$$= \int_1^\infty dt \int_S d\lambda(\tau)t^{n-1}w(t,\tau)|\nabla u(t,\tau)|t^{1-n}w(t,\tau)^{-1}$$
$$\times \int_t^\infty r^{n-1}drw(r,\tau)^{(1-\varepsilon)/p}$$
$$\leq \sqrt{\int_1^\infty t^{1-n}dt \int_S d\lambda(\tau)w(t,\tau)^{-1}\Big(\int_t^\infty r^{n-1}drw(r,\tau)^{(1-\varepsilon)/p}\Big)^2}\sqrt{2\mathcal{E}(u)}$$

Therefore

$$\Big(\int_{|x|\geq 1} |u(x) - u(x/|x|)|^p d\mu(x)\Big)^{2/p} \leq C(w,p,\varepsilon)\mathcal{E}(u) \tag{7.1}$$

Where

$$C(w,p,\varepsilon) = 2\Big(\int_{|x|\geq 1} w(x)^{\varepsilon/(1-p)}dx\Big)^{(2-2p)/p}$$
$$\int_1^\infty t^{1-n}dt \int_S d\lambda(\tau)w(t,\tau)^{-1}(\int_t^\infty r^{n-1}drw(r,\tau)^{(1-\varepsilon)/p})^2$$

Simple scaling arguments imply that, for any $\alpha > 0$, one has

$$(7.2)\qquad \Big(\int_{|x|\geq\alpha} |u(x) - u(\alpha x/|x|)|^p d\mu(x)\Big)^{2/p} \leq C_\alpha(w,p,\varepsilon)\mathcal{E}(u)$$

Where

$$C_\alpha(w,p,\varepsilon) =$$

$$2\alpha^{n(1-2/p)}\Big(\int_{|x|\geq\alpha} w(x)^{\varepsilon/(1-p)}dx\Big)^{(2-2p)/p}\dots$$

$$\dots\int_\alpha^\infty t^{1-n}dt\int_S d\lambda(\tau)w(t,\tau)^{-1}(\int_t^\infty r^{n-1}drw(r,\tau)^{(1-\varepsilon)/p})^2$$

Note that $C_\alpha(w,p,\varepsilon) \leq C(w,p,\varepsilon)$ for $\alpha \geq 1$.

Let $R > 1$. We integrate inequality (7.2) w.r.t. $1_{1\leq\alpha\leq R}\alpha^{n-1}d\alpha$, to get that

(7.3)

$$\int_1^R \alpha^{n-1}d\alpha\Big(\int_{|x|\geq r} |u(x) - u(\alpha x/|x|)|^p d\mu(x)\Big)^{2/p} \leq C(w,p,\varepsilon)\frac{R^n}{n}\mathcal{E}(u)$$

Let $R > 0$. The Laplace operator satisfies a Poincaré inequality on the ball of radius R *i.e.* there exists a constant C s.t. for any smooth function u

$$\int_{|x|\leq R} dx(u(x) - \int_{|y|\leq R} u(y)dy)^2 \leq C\int_{|x|\leq R} |\nabla u|^2(x)dx$$

Since w is bounded away from infinity and 0 on compact sets, we also have (for a different value of the constant C),

$$\int_{|x|\leq R} d\mu(x)(u(x) - \int_{|y|\leq R} u(y)d\mu(y))^2 \leq C\mathcal{E}(u)$$

Therefore

$$(7.4)\qquad \Big(\int_{|x|\leq R} d\mu(x)|u(x) - \int_{|y|\leq R} u(y)d\mu(y)|\Big)^2 \leq C\mathcal{E}(u)$$

Let now $a > 0$. As in the proofs of Proposition 2 or Theorem 3, we deduce from (7.4) that, for non-negative smooth functions u s.t. $\mu[\{x\ s.t.\ |x| \leq R\} \cap \{x\ s.t.\ u(x) = 0\}] \geq a$, we have

$$\Big(\int_{|x|\leq R} d\mu(x)|u(x)|\Big)^2 \leq C\mathcal{E}(u)$$

Therefore

$$(7.5)\qquad \left(\int_{|x|\leq R} d\mu(x)|u(x)|^p\right)^{2/p} \leq C\mathcal{E}(u)$$

The constant C in (7.5) depends on R, w and a.

Let $u \in \mathcal{H}$. Let $a \in]0, \frac{1}{2}[$. Let R be big enough so that for any set A s.t $\mu[A] \geq \frac{1}{2}$, one has $\mu[A \cap \{x \ s.t. \ |x| \leq R\}] \geq a$. Therefore $\mu[\{x \ s.t. \ |x| \leq R\} \cap \{x \ s.t. \ u(x) = 0\}] \geq a$. Let $r \in [1, R]$.

$$\begin{aligned}\int |u(x)|^p d\mu(x) = &\int_{|x|\geq r} |u(x)|^p d\mu(x) + \int_{|x|\leq r} |u(x)|^p d\mu(x)\\ \leq &\Bigg(\int_{|x|\geq r} |u(x) - u(rx/|x|)|^p d\mu(x)\\ &+ \int_{|x|\geq r} |u(rx/|x|)|^p d\mu(x) + \int_{|x|\leq r} |u(x)|^p d\mu(x)\Bigg)\end{aligned}$$

If we integrate this last inequality w.r.t. $1_{1\leq r\leq R} r^{n-1}dr$ we shall get three terms. The first one is

$$\begin{aligned}&\int_1^R r^{n-1}dr \int_{|x|\geq r} |u(x) - u(rx/|x|)|^p d\mu(x)\\ &\quad \leq C\left(\int_1^R r^{n-1}dr\left(\int_{|x|\geq r} |u(x) - u(rx/|x|)|^p d\mu(x)\right)^{2/p}\right)^{p/2}\end{aligned}$$

Here C depends on R. By (7.3) this quantity is bounded by $\left(C(w,p,\varepsilon)\frac{R^n}{n}\mathcal{E}(u)\right)^{p/2}$.

The second term is

$$\begin{aligned}&\int_1^R r^{n-1}dr \int_{|x|\geq r} |u(rx/|x|)|^p d\mu(x)\\ &= \int_1^R r^{n-1}dr \int_S d\lambda(\tau)|u(r,\tau)|^p w(r,\tau) w^{-1}(r,\tau) \int_{\rho\geq r} \rho^{n-1}d\rho w(\rho,\tau)\Big)\\ &\leq C\int_1^R r^{n-1}dr \int_S d\lambda(\tau)|u(r,\tau)|^p w(r,\tau)\end{aligned}$$

provided that

$$\sup_{\tau} \int_{\rho \geq 1} \rho^{n-1} d\rho w(\rho, \tau) < \infty$$

By (7.5),

$$\int_1^R r^{n-1} dr \int_S d\lambda(\tau) |u(r,\tau)|^p w(r,\tau) \leq C(\mathcal{E}(u))^{p/2}$$

The third term is

$$\begin{aligned} \int_1^R r^{n-1} dr \int_{|x| \leq r} & |u(x)|^p d\mu(x) \\ & \leq C \int_{|x| \leq R} |u(x)|^p d\mu(x) \\ & \leq C(\mathcal{E}(u))^{p/2} \end{aligned}$$

by (7.5).

Gathering these calculus, we get that

$$\left(\int |u(x)|^p d\mu(x) \right)^{2/p} \leq C\mathcal{E}(u) \tag{7.6}$$

if $C(w,p,\varepsilon) < \infty$ and $\sup_\tau \int_{\rho \geq 1} \rho^{n-1} d\rho w(\rho,\tau) < \infty$.

Let us now consider the case $p = 1$. Using the same arguments as in the proof of (7.2), it is possible to show that, for $r \geq 1$,

$$\left(\int_{|x| \geq r} |u(x) - u(rx/|x|)| d\mu(x) \right)^2 \leq C(w)\mathcal{E}(u) \tag{7.7}$$

where

$$C(w) = 2 \int_1^\infty t^{1-n} dt \int_S d\lambda(\tau) w(t,\tau)^{-1} \left(\int_t^\infty r^{n-1} dr w(r,\tau) \right)^2$$

From (7.7) and (7.5) one deduces that for any non-negative smooth function u s.t. $\mu[\{x \ s.t. \ u(x) = 0\}] \geq 1/2$, we have

$$\left(\int |u(x)| d\mu(x) \right)^2 \leq C\mathcal{E}(u) \tag{7.8}$$

provided that $C(w) < \infty$ and $\sup_\tau \int_{\rho \geq 1} \rho^{n-1} d\rho w(\rho,\tau) < \infty$.

(7.8) means that $\bar{\Lambda}(1) \geq 1/C$. Proposition 2 (iv) then implies that $\Lambda(1) \geq 1/4C$. Therefore we have proved the

PROPOSITION 9. – *Assume that* $\sup_\tau \int_1^\infty r^{n-1} dr w(r,\tau) < \infty$.

(i) *If*

$$\int_1^\infty t^{1-n} dt \int_S d\lambda(\tau) w(t,\tau)^{-1} \left(\int_t^\infty r^{n-1} dr w(r,\tau) \right)^2 < \infty$$

then $\Lambda(1) \neq 0$.

(ii) *Let* $p \in]0,1[$. *If there exists* $\varepsilon \in]0,1]$ *s.t.*

$$\int_{|x|\geq 1} w(x)^{\varepsilon/(1-p)} dx < \infty$$

and

$$\int_1^\infty t^{1-n} dt \int_S d\lambda(\tau) w(t,\tau)^{-1} \left(\int_t^\infty r^{n-1} dr w(r,\tau)^{(1-\varepsilon)/p} \right)^2 < \infty$$

then $\bar{\Lambda}(p) \neq 0$.

Example. – We choose $w(x) = |x|^{-\beta}$, $\beta > n$.

Let $p \in]0,1]$. A direct application of Proposition 9 yields that $\bar{\Lambda}(p) \neq 0$ if $\beta > n + 2p/(2-p)$. In particular we see that for any $\beta > n$, there exists a $p \in]0,1]$ s.t. $\bar{\Lambda}(p) \neq 0$.

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