ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 33, n° 4 (1997), p. 407-436
<http://www.numdam.org/item?id=AIHPB_1997__33_4_407_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 4, 1997, p. 407-436. *Probabilités et Statistiques*

Quadratic variations and estimation of the local Hölder index of a Gaussian process

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ABSTRACT. – We study the convergence of a generalisation of the quadratic variations of a Gaussian process. We build a convergent estimator of the local Hölder index of the sample paths and prove a central limit theorem.

RÉSUMÉ. – Nous étudions la convergence d'une généralisation des variations quadratiques d'un processus gaussien. Ces variations sont ensuite utilisées pour construire un estimateur consistant de l'exposant de Hölder des trajectoires. Nous calculons la vitesse de convergence de l'estimateur et démontrons un théorème de la limite centrale.

Classification A.M.S. : 60 G 12, 60 G 35.

1. INTRODUCTION

We propose a method for the almost surely identification of the local Hölder index of a Gaussian process, using a discrete observation of one sample path. The Hölder index of a Gaussian process gives precise information on the sample path (e.g. Adler, 1990, Th. 5.1 for extrema distributions, Ibragimov and Rozanov, 1978, p. 22 for smoothness of the sample paths), on the rate of convergence of non-parametric estimate of the covariance function (Istas and Laredo, 1993) and on the asymptotic behaviour of the wavelet decomposition coefficients of X (Istas, 1992). We obtain a convergence theorem and a central limit theorem for the estimation of this index. This estimation uses a generalisation of the quadratic variation. We show the convergence of the generalised quadratic variations and a central limit theorem.

Let $X(t)$ be a Gaussian process on [0,1] with mean value $m(t) = E(X(t))$ and covariance function $R(t,t') =' E(X(t) - m(t))(X(t') - m(t'))$. Consider the quadratic variation of process X on $[0,1]$ at scale $1/n$:

$$V_n = \sum_{k=1}^{n} (X(k/n) - X((k-1)/n))^2.$$

A preliminary result on quadratic variations of Gaussian non-differentiable process is Baxter's Theorem (e.g. Grenander (1981) ch. 5) giving assumptions on the smoothness of the mean value $m(t)$ and of the covariance function $R(t,t')$ outside the diagonal (*i.e.* for t, t') that ensure the (a.s.) convergence of the quadratic variation V_n as n tends to infinity.

Guyon and Leon (1989) introduced an important generalisation of these variations for a Gaussian stationary non-differentiable process with covariance function $c(t)$. Let H be a real function. The H-variation of process X on $[0,1]$ is defined by:

$$V_{H,n} = \sum_{k=1}^{n} H\left(\frac{X(k/n) - X((k-1)/n)}{(2(c(0) - c(1/n)))^{1/2}}\right). \tag{1}$$

Guyon and Leon (1989) studied the convergence in distribution of the H-variations for non-differentiable Gaussian processes. Their covariance function $c(t)$ satisfies:

$$c(t) = 1 - t^{\beta} L(t), \tag{2}$$

where β is a real parameter, $0 < \beta < 2$, and L is a slowly varying function at zero.

Applying the results concerning the H-variations to the quadratic variations ($H(x) = x^2$), they distinguish two cases:

– if $0 < \beta < 3/2$, $\sqrt{n}\left(\frac{V_{H,n}}{n} - 1\right)$ converges to a centred normal distribution;

– if $3/2 < \beta < 2$, $n^{2-\beta}\left(\frac{V_{H,n}}{n} - 1\right)$ converges to a centred non-normal distribution.

The generalisation of these variations for Gaussian fields is studied in Guyon (1987) and Leon and Ortega (1989). Another generalisation for generalised Gaussian processes and quadratic variations along curves is done in Adler and Pyke (1993). Quadratic variations (smoothed by convenient kernels) are applied to the estimation of the diffusion coefficient of a diffusion process in Genon-Catalot *et al.* (1992), Brugière (1991) and Soulier (1991).

We define general quadratic variations, substituting a general discrete difference operator to the simple difference $X(k\Delta) - X((k-1)\Delta)$. Process X is centred and has stationary increments. We consider observations of the process at times $j\Delta$ for $j = 1$ to n, with mesh $\Delta(n)$ tending to zero as n tends to infinity.

In Theorem 1, we give conditions on the discrete difference and on the covariance of X that ensure the almost surely convergence of the general quadratic variations. In Theorem 2, we give stronger conditions that ensure the asymptotic normality of the general quadratic variations. Then we use the quadratic variations to estimate the Hölder index of process X from discrete observations at times $j\Delta$ for $j = 1$ to n. In Theorem 3, we define estimators of this index and give conditions that ensure their almost surely convergence and asymptotic normality. The preceding results are easily extended to non-centred processes under weak smoothness assumptions on the expectation function, in Corollaries 1, 2 and 3.

Other methods for the estimation of the local Hölder index have been proposed. A short survey is given in Lang (1994). Hall and Wood (1993) showed that the box counting estimation has a very large asymptotical bias. Hall *et al.* (1994) studied an estimation using the level crossings.

Hydrology is an application field for the previous study. The estimation of the smoothness is used for the characterisation of rainfall time series or the spatial variability of rainfields (Hubert and Carbonnel, 1988). Estimation of the local Hölder index is also used in pattern recognition to define zones of homogenous roughness (Pentland, 1984).

2. GENERAL ASSUMPTIONS

2.1. Assumptions on the process

X is a centred Gaussian process with stationary increments on $[0, +\infty[$, observed at times $j\Delta$ for $j = 1$ to n and $\Delta(n) > 0$. The mesh $\Delta(n)$ tends to 0 as n tends to infinity. For notational simplicity, we denote it by Δ. We define T by $T = \sup_n(n\Delta(n))$. It is the bound (possibly infinite) of the observation interval.

Define the variogram function $v(t)$ as half the variance of the increments of X:

$$v(t) = \frac{1}{2}E((X(s+t) - X(s))^2).$$

Note that for a stationary process X with covariance function $c(t) = \operatorname{cov}(X(0), X(t)), v(t) = c(0) - c(t)$.

We use assumptions and parameters defining the behaviour of $v(t)$ at zero:

Assumption (A1)

Denote D the greatest integer such that v is $2D$ times differentiable.

We assume that there exists a real s such that $0 < s < 2$ and a real $C > 0$ such that:

$$v^{(2D)}(t) = v^{(2D)}(0) + C(-1)^D|t|^s + r(t) \text{ and } r(t) = o(|t|^s) \text{ at zero.} \quad (3)$$

Notes:

– Integer D is the order of differentiability in quadratic mean of the process (Cramer and Leadbetter 1967). By definition, X has a quadratic mean derivative X' if and only if:

$$\frac{X(t+h) - X(t)}{h} \xrightarrow{q.m} X'(t) \text{ as } h \text{ tends to } 0.$$

– For a process X satisfying (A1), denote $h = D + s/2$. Then parameter h is the local Hölder index of the process (Ibragimov and Rozanov (1978), p. 22). The definition of the local Hölder index of the process is the following: for an integer p, we say that a function f is p-Hölderian if its p^{th} derivative exists and is continuous. For $0 < r < 1$ and n integer, we say that f is $(p + r)$-Hölderian if $f^{(p)}$ is r-Lipschitz, that is:

There exist $C > 0$ and $\varepsilon > 0$ such that if $|x - y| < \varepsilon$ then $|f^p(x) - f^p(y)| \leq C|x - y|^r$.

The local Hölder index of a process is equal to the sup of the r such that the sample paths of the process are (a.s.) r-Hölderian.

2.2. Examples of processes satisfying assumption (A1)

Assumption (A1) holds true for the classic examples of non-differentiable processes with a known local Hölder index. Processes X with covariance $c_0(t) = \exp(-|t|^s)$ for $0 < s < 2$ satisfy (A1), with $D = 0$. The same holds true for processes X with covariance $c_1(t) = (1-|t|^s)^+$ for $0 < s < 1$, with $D = 0$. The standard Brownian motion satisfies (A1) with $D = 0$ and $s = 1$.

Consider the function $c_2(t) = (1 + |t|)\exp(-|t|)$. It is a covariance function because its Fourier transform is positive:

$$f(\lambda) = \frac{2}{\pi(1+\lambda^2)^2} \geq 0.$$

Process X with covariance function c_2 is differentiable and satisfies (A1), with $D = 1$ and $s = 1$. Fractional Brownian motions defined by Mandelbrot and Van Ness (1968) satisfy clearly assumption (A1).

2.3. Assumptions on the variations

Variations

We consider a finite sequence of reals $a_0, ..., a_p$ with zero sum and we associate a finite difference with it:

$$\Delta_a X_j = \sum_{i=0}^{p} a_i X((i+j)\Delta) \qquad \text{for} \quad 1 \leq j \leq n-p.$$

Note that $\Delta_a X_j$ is a stationary process. Then we define the quadratic variation corresponding to a:

$$V(a,n,\Delta) = \frac{1}{n}\sum_{j=1}^{n-p}\left(\frac{\Delta_a X_j^2}{\sigma_{a,\Delta}^2} - 1\right),$$

where $\sigma_{a,\Delta}^2$ denotes the variance of $\Delta_a X_j$.

We simplify the expression "quadratic variation depending on sequence a" and call it quadratic a-variation (this notation does not correspond to the H-variations of Guyon and Leon, 1989). We study the convergence of $V(a,n,\Delta)$ as n tends to infinity.

Integer $M(a)$ denotes the order of the first non-zero moment of the sequence a. It is defined by:

$$\sum_{i=0}^{p} a_i i^k = 0 \text{ for } 0 \leq k < M(a) \text{ and } \sum_{i=0}^{p} a_i i^{M(a)} \neq 0.$$

We know that $\sum_{k=0}^{p} \sum_{l=0}^{p} a_k a_l |k-l|^{2q}$ equals zero for $q = 0, 1, ..., M(a)-1$.

Assumption (A2)

We assume that for any real s such that $0 < s < 2M(a)$ and s is not an even integer:

$$\sum_{k=0}^{p} \sum_{l=0}^{p} a_k a_l |k-l|^{s} \neq 0.$$

Example. – The finite second order difference operator defined by $a_0 = 1$, $a_1 = -2$, $a_2 = 1$ satisfies the previous conditions with $M(a) = 2$.

3. RESULTS ON THE QUADRATIC a-VARIATIONS

Four parameters essentially drive the smoothness of the variogram function v. Parameter D has been defined in (A1). Parameter s gives the behaviour of v at the origin. We introduce parameter q, related to the smoothness of function v in a fixed neighbourhood of 0 and parameter d, related to the smoothness of function v outside this neighbourhood. We distinguish two cases. First, when q and d can be chosen large enough with respect to s, using a convenient sequence a with $2M(a)$ more than q and d, we do not need additional conditions on $\Delta(n)$ to prove a.s. convergence. But when v is such that we cannot choose q and d large enough, we need a stronger condition on mesh $\Delta(n)$ and Hölder index h. The same distinction occurs for the central limit theorem and, actually, the rates of convergence are different.

3.1. General case

In the case of simple quadratic variations ($a_0 = 1$, $a_1 = -1$), the operator Δ_a is a discrete differentiation operator of order 1. In section 3.1, we use sequences a with first non-zero moment $M(a) \geq 1$. The corresponding operator Δ_a is a discrete differentiation operator of order $M(a)$. We apply these operators to D-differentiable processes, using sequences a with $M(a) > D$. In additional corollaries, we generalise the results to non-centred processes, under conditions on the smoothness of the expectation.

Convergence of the quadratic a-variations

We give conditions for the expansion of the variogram function v and for the choice of the sequence a and the mesh $\Delta(n)$ that ensure the convergence of the quadratic a-variations.

THEOREM 1. – *Let X be a centred process with stationary increments satisfying* (A1), i.e. *their variogram function v satisfies :*

$$v^{(2D)}(t) = v^{(2D)}(0) + C(-1)^D|t|^s + r(t) \text{ with } r(t) = o(|t|^s) \text{ at zero.}$$

(i) *Assume that there exist three reals $\delta > 0$, $G > 0$, $\gamma > s$ and an integer $q > \gamma + 1/2$ such that the remainder $r(t)$ is q times differentiable on $]0, \delta]$ and $|r^{(q)}(t)| \leq G|t|^{\gamma-q}$.*

If $\delta < T$, we assume that for some integer $d \geq s + 1/2$, v is $2D + d$ times differentiable on $]\delta, T]$ and that

$$\int_\delta^T |v^{(2D+d)}(t)|dt < \infty.$$

We choose a sequence a satisfying (A2) *with $2M(a) \geq \max\{2D + q, 2D + d\}$.*

Then $V(a, n, \Delta)$ tends (a.s.) to zero as n tends to infinity.

(ii) *Assume that $s > 3/2$. Assume that there exists $\delta > 0$, such that r is exactly 2 times differentiable on $]0, \delta]$ and $|r''(t)| = O(|t|^{s-2})$.*

If $\delta < T$, assume that v is exactly $2D + 2$ times differentiable on $]\delta, T]$ and $\int_\delta^T |v^{(2D+d)}(t)|dt < \infty$.

We choose a sequence a satisfying (A2) *with $M(a) \geq D + 1$.*

We choose $\Delta(n)$ such that $\sum_{n\geq 1} \frac{1}{n^2\Delta(n)^{4s-6}} < \infty$.

Then $V(a, n, \Delta)$ tends (a.s.) to zero as n tends to infinity.

Note:

The conditions in (i) imply that v is smoother outside zero than at zero. Sequence a has a first non-zero moment $M(a)$ such that $2M(a)$ is greater than the parameters describing the smoothness of v.

In (ii), the previous conditions are not satisfied because parameter s defining the smoothness at zero and q and d defining the smoothness outside zero are close. In this case, we need an additional relation between n and Δ.

COROLLARY 1 (Case of a non-centred process). – *Let X be a non-centred process and h its Hölder index. Let a be a finite sequence. Denote by $V(a, n, \Delta)$ the quadratic a-variations of X. Denote by m the expectation of X and $Y = X - m$. Assume that Y satisfies assumption* (A1).

(i) *Assume that the remainder* $r(t)$ *corresponding to* Y, *the sequence* a *and the sequence* $\Delta(n)$ *satisfy the conditions of Theorem* 1 (i). *Assume that* m *is* l*-Hölderian with* $l > h$. *Then* $V(a,n,\Delta)$ *tends (a.s.) to zero as* n *tends to infinity.*

(ii) *Assume that the remainder* $r(t)$ *corresponding to* Y, *the sequence* a *and the sequence* $\Delta(n)$ *satisfy the conditions of Theorem* 1 (ii). *Assume that* m *is* l*-Hölderian with* $l > h$. *Then* $V(a,n,\Delta)$ *tends (a.s.) to zero as* n *tends to infinity.*

Central limit theorem

THEOREM 2. – *Let* X *be a centred process with stationary increments satisfying* (A1), i.e. *their variogram function* v *satisfies:*

$$v^{(2D)}(t) = v^{(2D)}(0) + C(-1)^D|t|^s + r(t) \text{ with } r(t) = o(|t|^s) \text{ at zero.}$$

(i) *Assume that there exist three reals* $\delta > 0$, $G > 0$, $\gamma > s$ *and an integer* $q \geq 2$, *greater than* $\gamma + 1/2$, *such that the remainder* $r(t)$ *is* q *times differentiable on* $]0,\delta]$ *and* $|r^{(q)}(t)| \leq G|t|^{\gamma-q}$.

If $\delta < T$, *we assume that for some integer* $d \geq 2$, *greater than* $s + 1/2$, v *is* $2D + d$ *times differentiable on* $]\delta, T]$ *and that*

$$\int_\delta^T |v^{(2D+d)}(t)|dt < \infty.$$

We choose a sequence a *satisfying* (A2) *with* $2M(a) \geq \max\{2D + q, 2D + d\}$.

If $s > 1$, *we choose* $\Delta(n)$ *such that* $n\Delta(n)^{2(s-1)} \to \infty$.

Then $\sqrt{n}V(a,n,\Delta)$ *tends in distribution to a centred normal variable, as n tends to infinity.*

(ii) *Assume that* $s > 3/2$. *Assume that there exists* $\delta > 0$, *such that* r *is exactly* 2 *times differentiable on* $]0,d]$ *and* $|r''(t)| = O(|t|^{s-2})$.

If $\delta < T$, *assume that* v *is exactly* $2D + 2$ *times differentiable on* $]\delta, T]$ *and* $\displaystyle\int_\delta^T |v^{(2D+d)}(t)|dt < \infty$.

We choose a sequence a *satisfying* (A2) *with* $M(a) = D + 1$.

We choose $\Delta(n)$ *such that* $n\Delta(n) \to \infty$.

Then $\sqrt{n}\Delta(n)^{s-3/2}V(a,n,\Delta)$ *tends in distribution to a centred normal variable as* n *tends to infinity.*

COROLLARY 2 (Case of a non-centred process). – *Let* X *be a non-centred process and* h *its Hölder index. Let* a *be a finite sequence. Denote by*

$V(a, n, \Delta)$ the quadratic a-variations of X. Denote by m the expectation of X and $Y = X - m$. Assume that Y satisfies assumption (A1).

(i) *Assume that the remainder $r(t)$ corresponding to Y, the sequence a and the sequence $\Delta(n)$ satisfy the conditions of Theorem* 2 (i). *Assume that m is l-Hölderian with $l > h$ such that $\sqrt{n}\Delta(n)^{l-h}$ tends to zero. Then $\sqrt{n}V(a, n, \Delta)$ tends in distribution to a centred normal variable, as n tends to infinity.*

(ii) *Assume that the remainder $r(t)$ corresponding to Y, the sequence a and the sequence $\Delta(n)$ satisfy the conditions of Theorem* 2 (ii). *Assume that m is l-Hölderian with $l > h$ such that $\sqrt{n}\Delta(n)^{s-3/2+l-h}$ tends to zero. Then $\sqrt{n}\Delta(n)^{s-3/2}\ V(a, n, \Delta)$ tends in distribution to a centred normal variable, as n tends to infinity.*

Proofs are given in section 5.

3.2. Non-differentiable processes

In this paragraph, we specify the case of non differentiable processes ($D = 0$), and compare to the results of Guyon and Leon (1989).

Convergence of the quadratic a-variations

Assume that process X satisfies conditions of Theorem 1 (i). If $s < 3/2$, we can choose a sequence a such that $M(a) = 1$, namely the sequence ($a_0 = 1$, $a_1 = -1$) defining the simple quadratic variations and we find the same result of convergence than Guyon and Leon. If $s \geq 3/2$, we use a sequence a with $M(a) > 1$, that is, we apply to the process a discrete differentiation operator of a larger order. Assume now that process X satisfies conditions of Theorem 1 (ii). Then it is no use to choose a sequence a with $M(a) > 1$. We need a additional relation between n and $\Delta(n)$. If $s \geq 7/4$, this relation implies that the observation interval is no longer bounded, as it was in Guyon and Leon (1989).

Central limit theorem

For a non-differentiable process observed on a fixed interval ($\Delta(n) = 1/n$), Guyon and Leon (1989) showed that if $s < 3/2$, the classic quadratic variation converges to a normal distribution at rate $\sqrt{n}$. In the case $s > 3/2$, the limit distribution is the second Wiener chaos with rate n^{2-s}. In (i), we claim that the limit distribution is normal with a rate $\sqrt{n}$ for the quadratic a-variation, if $n\Delta(n)^{2(s-1)} \rightarrow \infty$. This condition is satisfied for $\Delta(n) = 1/n$ only if $s < 3/2$. If $s \geq 3/2$, we have to choose a sequence $\Delta(n)$ such that $n\Delta(n) \rightarrow \infty$ (the observation interval is no more bounded). For a process X satisfying (ii), we also have to choose a sequence $\Delta(n)$ such that $n\Delta(n)\to\infty$; the limit distribution is normal, but now the rate of convergence is lower than $\sqrt{n}$.

Examples. – Processes X with covariance $c_0(t) = \exp(-|t|^s)$ satisfy assumption (A1). Function $v(t) = |t|^s + r(t)$ with $r(t)$ of order $|t|^{2s}$ at zero and $r^{(q)}(t)$ of order $|t|^{2s-q}$ at zero. We can find convenient parameters to ensure conditions (i): if $s < 1/2$, we choose $q = 1$, $d = 1$, $M(a) \geq 1$; if $1/2 \leq s < 3/2$, we choose $q = 2$, $d = 1$, $M(a) \geq 1$; if $3/2 \leq s < 2$, we choose $q = 3$, $d = 1$, $M(a) \geq 2$.

Define now processes X with covariance $c_0(t) = exp(-|t|^s + w(t))$, where $s > 3/2$ and $w(t)$ is a concave exactly two times differentiable function. Then process X does not satisfy conditions (i). It satisfies conditions (ii), but we have to ensure extra conditions on $\Delta(n)$.

Consider now process X with covariance $c_0(t) = \exp(-|t|^{1.7})$ and choose a sequence a such that $M(a) = 1$, then sequence a does not fulfill conditions (i). In this case, the conclusions are the same than in case (ii), even if function r is not only two times differentiable: if we ensure the conditions on $\Delta(n)$, the rate of convergence is $\sqrt{n}\Delta(n)^{0.2}$. This process is simulated and called P2 in section 6.

4. ESTIMATION OF THE LOCAL HÖLDER INDEX

We use the previous results of convergence to build an estimator of h. We consider now the empirical quadratic variation:

$$U(a,n,\Delta) = \frac{1}{n}\sum_{j=1}^{n-p}\left(\sum_{i=0}^{p} a_i X((i+j)\Delta)\right)^2.$$

Definition of the estimators

Let a sequence a be chosen, the quadratic variation $U(a,n,\Delta)$ is asymptotically equivalent to $\sigma^2_{a,\Delta}$, and we will show in the preliminary calculations of section 5.1 that:

$$\sigma^2_{a,\Delta} \underset{n\to\infty}{\approx} -C(-1)^D \sum_{k=0}^{p}\sum_{l=0}^{p} a_k a_l |k-l|^{2h}\Delta^{2h}.$$

Considering several sequences $(a^i)_{i=1\ldots I}$, we dispose of a system of I combinations of the reals $(C(j\Delta)^{2h})_{j=1\ldots p}$ for the constant C defined in A1-(3). Using a linear regression, we can estimate the values of $(C(j\Delta)^{2h})_{j=1\dots p}$ from the observations of the $U(a^i, n, \Delta)$.

We denote p^i the length of the sequence a^i. We define a matrix A with size $I \times p$, where $p = \max((p^i)_{i=1\dots I})$, by

$$A_{i,j} = 2 \sum_{k=0}^{p^j - j} a_k^i a_{k+j}^i \text{ for } j = 1, \dots, p^i \text{ and } A_{i,j} = 0 \text{ otherwise.} \quad (4)$$

Let U be the vector of the $U(a^i, n, \Delta)$ and Z the vector of the $(C(-1)^D (j\Delta)^{2h})_{j=1\dots p}$. Then U tends to the product AZ.

We assume that the a^i are such that A is full rank. We estimate Z by $\hat{Z}$ result of the regression: $\hat{Z} = (A'A)^{-1} A'U$.

We compute now a linear regression on the logarithms of the co-ordinates of $|\hat{Z}|$. We define the estimators of h and C by:

$$\hat{h} = \frac{\sum_{j=1\dots p} \log|\hat{Z}_j| \log(j\Delta) - \frac{1}{p} \sum_{j=1\dots p} \log|\hat{Z}_j| \sum_{j=1\dots p} \log(j\Delta)}{2\left(\sum_{j=1\dots p} ((\log(j\Delta))^2 - \frac{1}{p}\left(\sum_{j=1\dots p} \log(j\Delta)\right)^2\right)},$$

$$\log(\hat{C}) = \frac{1}{p}\Big(\sum_{j=1\dots p} \log|\hat{Z}_j| - 2\hat{h} \sum_{j=1\dots p} \log(j\Delta)\Big).$$

THEOREM 3. – *Let X be a centred process with stationary increments satisfying* (A1).

Let $\alpha > 0$ and define the observation mesh Δ by $\Delta(n) = n^{-\alpha}$.

Let a^i be finite sequences. Assume that the a^i are such that A defined in (4) *is full rank.*

(i) *Assume that the remainder $r(t)$ in* (A1) *and sequences a^i satisfy the conditions of Theorem* 1 (i) *then:*

$$(\hat{h}, \hat{C}) \xrightarrow[\text{a.s.}]{} (h, C), \text{ as } n \text{ tends to infinity.}$$

(ii) *Assume that the remainder $r(t)$ in* (A1) *and sequences a^i satisfy the conditions of Theorem* 1 (ii).

Assume that $\alpha < 1/(4s - 6)$ then:

$$(\hat{h}, \hat{C}) \xrightarrow[\text{a.s.}]{} (h, C), \text{ as } n \text{ tends to infinity.}$$

(iii) *Assume that the remainder $r(t)$ in* (A1) *and sequences a^i satisfy the conditions of Theorem* 2 (i).

If $s \leq 1$, *assume that for some* $\tau > 0$, $r(t) = o(|t|^{s+t})$ *at zero and choose* $\alpha > 1/2\tau$.

If $s > 1$, *assume that for some* $\tau > s - 1$, $r(t) = o(|t|^{s+t})$ *at zero and choose* α *such that* $1/2\tau < \alpha < 1/(2s - 2)$.

Then $\sqrt{n}(\hat{h} - h)$ *converges in distribution to a centred Gaussian variable as* n *tends to infinity.*

$\frac{\sqrt{n}}{\log(n)}(\hat{C} - C)$ *converges in distribution to a centred Gaussian variable as* n *tends to infinity.*

(iv) *Assume that the remainder* $r(t)$ *in* (A1) *and sequences* a^i *satisfy the conditions of Theorem* 2 (ii).

Assume that for some $\tau > 0$, $r(t) = o(|t|^{s+t})$ *at zero and choose* $1/(2\tau + 2s - 3) < \alpha < 1$.

Then $n^{1/2-\alpha(s-3/2)}(\hat{h} - h)$ *converges in distribution to a centred Gaussian variable as* n *tends to infinity.*

$\frac{n^{1/2-\alpha(s-3/2)}}{\log(n)}(\hat{C} - C)$ *converges in distribution to a centred Gaussian variable as* n *tends to infinity.*

COROLLARY 3 (Case of a non-centred process). – *Let* X *be a non-centred process and* a^i *be finite sequences. Denote by* $(V(a^i, n, \Delta))$ *the quadratic* a^i*-variations of* X. *Denote by* m *the expectation of* X *and* $Y = X - m$. *Assume that* Y *satisfies assumption* (A1). *Let* $\alpha > 0$ *and define the observation mesh* Δ *by* $\Delta(n) = n^{-\alpha}$.

(i) *Assume that the remainder* $r(t)$ *corresponding to* Y *and the sequences* a^i *satisfy the conditions of Theorem* 3 (i). *Assume that* m *is* l*-Hölderian with* $l > h$. *Then* $(\hat{h}, \hat{C}) \underset{\text{a.s.}}{\longrightarrow} (h, C)$, *as* n *tends to infinity.*

(ii) *Assume that the remainder* m *corresponding to* Y, *the sequences* a^i *and parameter* α *satisfy the conditions of Theorem* 3 (ii). *Assume that* m *is* l*-Hölderian with* $l > h$. *Then* $(\hat{h}, \hat{C}) \underset{\text{a.s.}}{\longrightarrow} (h, C)$, *as* n *tends to infinity.*

(iii) *Assume that the remainder* $r(t)$ *corresponding to* Y, *the sequences* a^i *and parameter* α *and* τ *satisfy the conditions of Theorem* 3 (iii). *Assume that* m *is* l*-Hölderian with* $l > h$ *such that* $\sqrt{n}\Delta^{l-h}$ *tends to zero. Then* $\sqrt{n}(\hat{h} - h)$ *and* $\frac{\sqrt{n}}{\log(n)}(\hat{C} - C)$ *converge as in Theorem* 3.

(iv) *Assume that the remainder* $r(t)$ *corresponding to* Y, *the sequences* a^i *and parameters* α *and* τ *satisfy the conditions of Theorem* 3 (iv).

Assume that m *is* l*-Hölderian with* $l > h$ *such that* $n^{1/2-\alpha(s-3/2)}\Delta^{l-h}$ *tends to zero. Then* $n^{1/2-\alpha(s-3/2)}(\hat{h} - h)$ *and* $\frac{n^{1/2-\alpha(s-3/2)}}{\log(n)}(\hat{C} - C)$ *converge as in Theorem* 3.

Note on the computation of the variances of estimators $\hat{h}$ and $\hat{C}$

For the quadratic variations corresponding to two sequences a and b, denote by $N(n,\Delta)$ the normalisation factor of both $V(a,n,\Delta)$ and $V(b,n,\Delta)$. The limit of the covariance $N^2(n,\Delta)\text{cov}(V(a,n,\Delta),V(b,n,\Delta))$ is expressed as follows :

$$\frac{2\sum_{|j|\le p}(\sum_{k,l} a_k b_l |j+k-l|^{2h})^2 + 2\sum_{|j|\le p}\left(\sum_{k,l} a_k b_l |k-l|^{2D}\int_0^1 \frac{(1-\eta)^{2D-1}}{(2D-1)!}(j+\eta(k-l))^s d\eta\right)^2}{\sum_{k,l} a_k a_l |k-l|^{2h}\sum_{k,l} b_k b_l |k-l|^{2h}}.$$

If D equals zero, this formula is:

$$\frac{2\sum_{j\in Z}(\sum_{k,l} a_k b_l |j+k-l|^{2h})^2}{(\sum_{k,l} a_k a_l |k-l|^{2h}\sum_{k,l} b_k b_l |k-l|^{2h})}.$$

These formulas allow the computation of the variances of estimators $\hat{h}$ and $\hat{C}$.

5. PROOFS

We represent the quadratic a-variations as a sum of squares of independent Gaussian variables.

5.1. Preliminary calculations

$[x]$ denotes the integer part of x. A' denotes the transpose of matrix A. (Const) denotes an indefinite positive constant that may change from one occurrence to another.

Recall that $\sigma^2_{a,\Delta}$ denotes the variance of $\Delta_a X_j$. Denote $\rho_\Delta(j)$ the coefficient of correlation of $\Delta_a X_0$ and $\Delta_a X_j$ and ρ_Δ the $(n-p)$ square matrix, whose components are the $\rho_\Delta(j-i)$. We denote the eigenvalues of this matrix by $(\lambda_{j,n})_{j=1..n-p}$ and P is the orthonormal matrix such that $\text{Diag}(\lambda_{j,n}) = P'\rho_\Delta P$. Then the Gaussian variables defined by:

$$\xi_{i,n} = (\lambda_{i,n})^{-1/2}\sum_{j=1}^{n-p} P_{ji}\frac{\Delta_a X_j}{\sigma^2_{a,\Delta}},$$

are normalised independent centred Gaussian variables and

$$\sum_{j=1}^{n-p} \frac{\Delta_a X_j^2}{\sigma_{a,\Delta}^2} = \sum_{j=1}^{n-p} \lambda_{j,n} \xi_{j,n}^2,$$

so that

$$V(a, n, \Delta) = \frac{1}{n} \sum_{j=1}^{n-p} \lambda_{j,n} (\xi_{j,n}^2 - 1). \tag{5}$$

We denote $s_n^2 = \mathrm{Var}\left(\sum_{j=1}^{n-p} \frac{\Delta_a X_j^2}{\sigma_{a,\Delta}^2}\right)$.

We have $s_n^2 = 2\sum_{j=1}^{n-p} \lambda_{j,n}^2 = 2\sum_{j=1}^{n-p}\sum_{i=1}^{n-p} \rho_\Delta^2(j-i) = 2\sum_{|j|<n-p} (n-p-|j|)\rho_\Delta^2(j)$.

First we compute an equivalent to the coefficients of correlation $\rho_\Delta(j-i)$. As we often use the Taylor expansion about $j\Delta$, we give a notation to its integral remainder. For a sequence a, a mesh Δ, an order r and a function f, we denote:

$$R(j, \Delta, r, f) = \sum_{k=0}^{p}\sum_{l=0}^{p} a_k a_l (k-l)^r \int_0^1 \frac{(1-\eta)^{r-1}}{(r-1)!} f((j+(k-l)\eta)\Delta)d\eta. \tag{6}$$

LEMMA 1. – *Let a be a finite sequence. If f is continuously differentiable up to order $q \leq 2M(a)$, then:*

$$\sum_{k=0}^{p}\sum_{l=0}^{p} a_k a_l f((j+k-l)\Delta) = \Delta^q R(j, \Delta, r, f^{(q)}).$$

Proof of Lemma 1. – We use the Taylor expansion of function f at $j\Delta$ with integral remainder:

$$\begin{aligned} f((j+k-l)\Delta) &= f(j\Delta) + \cdots + \frac{((k-l)\Delta)^{q-1}}{(q-1)!} f^{(q-1)}(j\Delta) \\ &\quad + ((k-l)\Delta)^q \int_0^1 \frac{(1-\eta)^{q-1}}{(r-1)!} f^{(q)}((j+(k-l)\eta)\Delta)d\eta. \end{aligned}$$

When we sum over k and l, every term in the expansion except the remainder gives a zero contribution because $\sum_{k,l} a_k a_l (k-l)^r = 0$ for any integer r less than $2M(a)$.

We compute the covariance of the stationary centred discrete time process $(\Delta_a X_i)_{0 \leq i < n-p}$:

$$
\begin{aligned}
&E(\Delta_a X_i \Delta_a X_j) \\
&= E\Big(\sum_{k=0}^{p} a_k (X((i+k)\Delta) \sum_{l=0}^{p} a_l (X((j+l)\Delta) - X((i+k)\Delta))\Big) \\
&= \sum_{k,l} a_k a_l E(X((i+k)\Delta)(X((j+l)\Delta) - X((i+k)\Delta))) \\
&= \sum_{k,l} a_k a_l E(X((j+l)\Delta)(X((i+k)\Delta) - X((j+l)\Delta))) \\
&= -\frac{1}{2} \sum_{k,l} a_k a_l E((X((i+k)\Delta) - X((j+l)\Delta))^2) \\
&= -\sum_{k,l} a_k a_l v((i+k-j-l)\Delta).
\end{aligned}
$$

We use this formula to compute equivalents of $\sigma^2_{a,\Delta}$, $\rho_\Delta(j)$, then of s^2_n.

Equivalent of $\sigma^2_{a,\Delta}$

Assume that a satisfies (A2). Using notation (6) and the previous covariance computation:

$$
\begin{aligned}
\sigma^2_{a,\Delta} &= -\sum_{k,l} a_k a_l v((k-l)\Delta) = \Delta^{2D} R(0, \Delta, 2D, v^{(2D)}) \\
&= \Delta^{2D}(-C(-1)^D R(0, \Delta, 2D, |x|^s) + R(0, \Delta, 2D, r)) \\
&= -C(-1)^D \Delta^{2h} \sum_{k,l} a_k a_l |k-l|^{2h} + o(\Delta^{2h})
\end{aligned}
$$

The first term $-C(-1)^D \Delta^{2h} \sum_{k,l} a_k a_l |k - l|^{2h}$ is non-zero because of assumption (A2).

Equivalent of $\rho_\Delta(j)$

We distinguish three cases depending on the value of j.

a) $1 \leq |j| \leq p$: we use the same expansion as for the calculation of $\sigma^2_{a,\Delta}$:

$$
\begin{aligned}
\rho_\Delta(j) &= \frac{\sum_{k,l} a_k a_l v((j+k-l)\Delta)}{\sum_{k,l} a_k a_l v((k-l)\Delta)} \\
&= \frac{R(j, \Delta, 2D, v^{(2D)})}{R(0, \Delta, 2D, v^{(2D)})} \underset{n\to\infty}{\longrightarrow} \frac{R(j, 1, 2D, |x|^s))}{R(0, 1, 2D, |x|^s)}.
\end{aligned}
$$

b) $p < |j| \leq [\delta/\Delta]$: we split $\rho_\Delta(j)$ into two terms:

$$\rho_\Delta(j) = \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}}(-C(-1)^D R(j,\Delta,2D,|x|^s) - R(j,\Delta,2D,r)).$$

c) $[\delta/\Delta] < |j| < n-p$:

$$\rho_\Delta(j) = -\frac{\Delta^{2D+d}}{\sigma^2_{a,\Delta}} R(j,\Delta,2D+d,v^{(2D+d)}).$$

5.2. Proofs of theorems

Proof of Theorem 1. – In order to apply the Borel-Cantelli lemma, we compute an equivalent of s_n^2.

Equivalent of s_n^2

Case (i). – Denote $g(j) = (n-p-|j|)\, p_\Delta^2(j)$. We split s_n^2 into three parts corresponding to the three preceding intervals: $s_n^2 = A_n + B_n + C_n$, where $A_n = 2\sum_{|j|\leq p} g(j)$, $B_n = 2\sum_{p<|j|\leq[\delta/\Delta]} g(j)$, $C_n = 2\sum_{[\delta/\Delta]<|j|<n-p} g(j)$.

a) $A_n = 2\sum_{|j|\leq p} g(j) = 2n\left(1 + 2\sum_{1\leq j\leq p}\left(\frac{R(j,1,2D,|x|^s)}{R(0,1,2D,|x|^s)}\right)^2 + o(1)\right).$

b) $B_n = \frac{4\Delta^{4D}}{\sigma^4_{a,\Delta}}\sum_{j=p+1}^{[\delta/\Delta]}(n-p-j)(C(-1)^D R(j,\Delta,2D,x^s) + R(j,\Delta,2D,r))^2.$

$\star$ We differentiate d times the first term:

$$R(j,\Delta,2D,x^s) = \Delta^d R(j,\Delta,2D+d,s(s-1)\cdots(s-d+1)x^{s-d}).$$

Denoting $I(j,k,l,2D+d-1;s-d) = \int_0^1 \frac{(1-\eta)^{2D+d-1}}{(2D+d-1)!}(j+(k-l)\eta)^{s-d}d\eta$, we have:

$$R(j,\Delta,2D,x^s) = \Delta^s s(s-1) \cdots (s-d+1)\sum_{k,l} a_k a_l (k-l)^{2D+d} I(j,k,l,2D+d-1,s-d).$$

Using the Cauchy Schwarz inequality:

$$\left(\sum_{k,l} a_k a_l (k-l)^{2D+d} I(j,k,l,2D+d-1,s-d)\right)^2 \leq \sum_{k,l}(a_k a_l (k-l)^{2D+d})^2 \sum_{k,l}(I(j,k,l,2D+d-1,s-d))^2.$$

$$4\sum_{j=p+1}^{[\delta/\Delta]}(n-p-j)\frac{\Delta^{4D}}{\sigma_{a,\Delta}^4}C^2R^2(j,\Delta,2D,x^s)$$
$$\leq(\mathrm{Const})n\sum_{k,l}\sum_{j=p+1}^{[\delta/\Delta]}(I(j,k,l,2D+d,s-d))^2.$$

Each integral I satisfies $0\leq I(j,k,l,2D+d-1,s-d)\leq(\mathrm{Const})(j-p)^{s-d}$. Because $d-s>1/2$, the sum over j is bounded, so that:

$$4\sum_{j=p+1}^{[\delta/\Delta]}(n-p-|j|)\frac{\Delta^{4D}}{\sigma_{a,\Delta}^4}C^2R^2(j,\Delta,2D,x^s)=O(n).$$

$\star$ We differentiate the second term q times:

$$R(j,\Delta,2D,r)=\Delta^{2D+q}\sum_{k,l}a_ka_l(k-l)^{2D+q}$$
$$\times\int_0^1\frac{(1-\eta)^{2D+q-1}}{(2D+d-1)!}r^{(q)}((j+(k-l)\eta)\Delta)d\eta.$$

Using the assumption on r:

$$|R(j,\Delta,2D,r)|\leq(\mathrm{Const})\Delta^{2D+q}\sum_{k,l}|a_ka_l|\,|k-l|^{2D+q}\frac{j^{\gamma-q}}{(2D+q)!}.$$

$$4\sum_{j=p+1}^{[\delta/\Delta]}(n-p-|j|)\frac{\Delta^{4D}}{\sigma_{a,\Delta}^4}R^2(j,\Delta,2D,r)$$
$$\leq(\mathrm{Const})n\Delta^{2(\gamma-s)}\sum_{j>p}j^{2(\gamma-q)}.$$

Because $q-\gamma>1/2$, the sum over j converges. This term is $O(n\Delta^{2(\gamma-s)})$ and its contribution is negligible. So B_n is $O(n)$.

c) $C_n=\dfrac{4\Delta^{4D+2d}}{\sigma_{a,\Delta}^4}\displaystyle\sum_{j=[\delta/\Delta]+1}^{n-p}(n-p-j)(R(j,\Delta,2D+d,v^{(2D+d)}))^2.$

$$C_n=\frac{4\Delta^{4D+2d}}{\sigma_{a,\Delta}^4}\sum_{j=[\delta/\Delta]+1}^{n-p}(n-p-j)$$
$$\times\left(\sum_{k,l}a_ka_l|k-l|^{2D+d}\int_0^1\frac{(1-\eta)^{2D+d-1}}{(2D+d-1)!}v^{(2D+d)}((j+(k-l)\eta)\Delta)d\eta\right)^2.$$

We apply the Cauchy Schwarz inequality to the square of the summation over k and l. This square is less than the product:

$$\sum_{k,l}(a_k a_l)^2(k-l)^{4D+2d}\sum_{k,l}\left(\int_0^1 \frac{(1-\eta)^{2D+d-1}}{(2D+d-1)!}v^{(2D+d)}((j+(k-l)\eta)\Delta)d\eta\right)^2.$$

The first factor is independent of j. We consider each integral as the value of the function $v^{(2D+d)}$ at a point near $j\Delta$, so that the summation over j of the second factor is a Riemann sum for the definite integral $\int_\delta^T v^{(2D+d)}(x))^2dx$. Thus:

$$C_n \le n\Delta^{2(d-s)-1}\frac{4p^2}{R(0,1,2D,|x|^s)^2}$$
$$\times \Big(\sum_{k,l}(a_k a_l)^2(k-l)^{4D+2d}\Big)\int_\delta^T (v^{(2D+d)}(x))^2dx.$$

So C_n is $o(n)$ and s_n^2 is asymptotically proportional to n.

Case (ii). – We follow the same method of proof. The calculation concerning A_n is the same. Concerning B_n, $v^{(2D+2)}(t) = C(-1)^D s(s-1)|t|^{s-2} + r''(t)$, so that on $[-\delta,\delta]$, $v^{(2D+d)}(t)| = O(|t|^{s-2})$. We have $\int_0^T (v^{(2D+2)}(x))^2dx < \infty$, therefore B_n and C_n are studied together. For this, we use lemma 1, expanding v to the order $2D+2$:

$$\rho_\Delta(j) = -\frac{\Delta^{2D+2}R(j,\Delta,2D+2,v^{(2D+2)})}{\sigma_{a,\Delta}^2}.$$

Then summing over j:

$$B_n + C_n = \frac{4\Delta^{4D+4}}{\sigma_{a,\Delta}^4}\sum_{j=[\delta/\Delta]+1}^{n-p-1}(n-p-j)(R(j,\Delta,2D+2,v^{(2D+2)}))^2$$
$$= \frac{4\Delta^{4D+4}}{\sigma_{a,\Delta}^4}\Bigg(\sum_{j=[\delta/\Delta]+1}^{n-p-1}(n-p-j)$$
$$\times\left(\sum_{k,l}a_k a_l|k-l|^{2D+2}\int_0^1\frac{(1-\eta)^{2D+1}}{(2D+d-1)!}v^{(2D+d)}((j+(k-l)\eta)\Delta)d\eta\right)^2\Bigg).$$

Using the Cauchy Schwarz inequality for the summation over k and l:

$$B_n + C_n \le \frac{4\Delta^{4D+4}}{\sigma_{a,\Delta}^4}\Bigg(\sum_{j=[\delta/\Delta]+1}^{n-p-1}(n-p-j)\sum_{k,l}(a_k a_l)^2|k-l|^{4D+4}$$
$$\times\sum_{k,l}\left(\int_0^1\frac{(1-\eta)^{2D+1}}{(2D+1)!}v^{(2D+2)}((j+(k-l)\eta)\Delta)d\eta\right)^2\Bigg).$$

Now we consider the sum over j as a Riemann sum for the definite integral $\int_0^T (v^{(2D+2)}(x))^2 dx$.

$$B_n + C_n \leq n\Delta^{3-2s} \frac{4p^2}{R(0,1,2D,|x|^s)^2} \times (\sum_{k,l} (a_k a_l)^2 (k-l)^{4D+4})^2 \int_0^T (v^{(2D+2)}(x))^2 dx.$$

Thus s_n^2 is $O(n\Delta^{3-2s})$.

Almost sure convergence

We compute $E(V(a,n,\Delta)^4)$, using decomposition (5):

$$V(a,n,\Delta) = \frac{1}{n} \sum_{j=1}^{n-p} \lambda_{j,n}(\xi_{j,n}^2 - 1).$$

Then: $E(V(a,n,\Delta)^4) = \frac{1}{n^4}\left(M_4 \sum_{j=1}^{n-p} \lambda_{j,n}^4 + M_2^2 \sum_{j=1}^{n-p}\sum_{j'=1}^{n-p} \lambda_{j,n}^2 \lambda_{j',n}^2 \right)$ where $M_k = E(\xi_{j,n}^2 - 1)^k$.

$$E(V(a,n,\Delta)^4) \leq \frac{M_4}{n^4} \left(\sum_{j=1}^{n-p} \lambda_{j,n}^2 \right)^2 \leq \frac{M_4 s_n^n}{4n^4}.$$

In case (i), s_n^2 is asymptotically of order n, so $\sum_{n=1}^{\infty} E(V(a,n,\Delta)^4) < \infty$. We use Markov inequality and Borel-Cantelli lemma to conclude that $V(a,n,\Delta)$ converges almost surely to zero.

In case (ii), s_n^2 is $O(n\Delta^{3-2s})$. Because $\sum_{n\geq 1} \frac{1}{n^2 \Delta^{4s-6}} < \infty$, $\sum_{n=1}^{\infty} E(V(a,n,\Delta)^4) < \infty$, $V(a,n,\Delta)$ converges almost surely to zero.

Proof of Corollary 1. – We consider the centred process $Y(t) = X(t) - m(t)$. The variation corresponding to X is the sum of three terms:

$$V(a,n,\Delta) = \frac{1}{n}\left(\sum_{j=1}^{n-p} \left(\frac{\Delta_a Y_j^2}{\sigma_{a,\Delta}^2} - 1 \right) + 2\sum_{j=1}^{n-p} \left(\frac{\Delta_a Y_j \Delta_a m_j}{\sigma_{a,\Delta}^2} \right) + \sum_{j=1}^{n-p} \left(\frac{\Delta_a m_j^2}{\sigma_{a,\Delta}^2} \right) \right).$$

For an integer $l \le M$ and for a l-Hölderian function m, the Taylor expansion shows that $\Delta_a m_j$ is $O(\Delta^l)$.

The term $\displaystyle\frac{1}{n}\sum_{j=1}^{n-p}\left(\frac{\Delta_a m_j^2}{\sigma^2_{a,\Delta}}\right)$ tends to zero as n tends to infinity. Applying the Cauchy Schwarz inequality, the second term tends (a.s.) to zero.

Proof of Theorem 2. – We use a version of Lindeberg condition (Czörgo and Révész, 1981):

LEMMA 2. – *Consider the sequence of variable V_n defined by $V_n = \displaystyle\sum_{j=1}^{n-p}\lambda_{j,n}(\xi^2_{j,n}-1)$, where the $\xi_{j,n}$ are i.i.d. centred normalised Gaussian variables and the $\lambda_{j,n}$ are positive. Let s_n^2 be the variance of V_n and λ_n the maximum of the $\lambda_{j,n}$. If $\lambda_n = o(s_n)$, then V_n/s_n tends in distribution to a centred normalised Gaussian variable.*

Now we have to give bound to the largest eigenvalue λ_n of the correlation matrix. We use a linear algebra lemma (Luenberger, 1979, ch. 6.2, p. 194):

LEMMA 3. – *Let C be a positive defined symmetric matrix and λ its largest eigenvalue.*

Then $\lambda \ge \max\limits_i \sum\limits_j |C_{i,j}|$.

Bound of λ_n

Lemma 3 implies that $\lambda_n \ge \max\limits_i \displaystyle\sum_{j=1}^{n-p}|\rho_\Delta(j-i)| \le 2\sum_{j=0}^{n-p-1}|\rho_\Delta(j)|$.

Case (i). – As in the proof of Theorem 1, we split the sum $2\sum_{j=1}^{n-p-1}|\rho_\Delta(j)|$ into three parts A'_n, B'_n and C'_n:

a) $A'_n = 2\displaystyle\sum_{j=0}^{p}|\rho_\Delta(j)|$ is bounded by $2p+2$.

b) $B'_n = 2\displaystyle\sum_{j=p+1}^{[\delta/\Delta]}|\rho_\Delta(j)| \le 2\sum_{j=p+1}^{[\delta/\Delta]}\frac{\Delta^{2D}}{\sigma^2_{a,\Delta}}C|R(j,\Delta,2D,x^s)| + |R(j,\Delta,2D,r)|$

$\star$ We differentiate twice the first term $R(j,\Delta,2D,x^s)$:

$$\begin{aligned}|R(j,\Delta,2D,x^s)| &= \Delta^2|R(j,\Delta,2D+2,s(s-1)x^{s-2})|\\ &\le \Delta^s\frac{s|s-1|\sum_{k,l}|a_k a_l|(k-l)^{2D+2}}{(2D+1)!}(j-p)^{s-2}.\end{aligned}$$

– If $s < 1$, the summation over j of these terms converges. Then:

$$2 \sum_{p<j<[\delta/\Delta]} \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}} C|R(j,\Delta,2D,x^s)| \\ \leq \frac{2Cs|s-1|\sum_{k,l}|a_k a_l|(k-l)^{2D+2}}{|R(0,1,2D,|x|^s)|(2D+1)!} \sum_{j=p+1}^{\infty}(j-p)^{s-2}.$$

– If $s = 1$, the term is zero.

– If $s > 1$, $\Delta \sum_{j=p+1}^{[\delta/\Delta]} (j-p)^{s-2}\Delta^{s-2}$ is a Riemann sum for the converging integral $\int_0^\delta x^{s-2}dx$, so that:

$$2 \sum_{j=p+1}^{[\delta/\Delta]} \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}} C|R(j,1,2D,x^s)| \\ \leq \Delta^{1-s}\frac{2Cs|s-1|\sum_{k,l}|a_k a_l|(k-l)^{2D+2}}{|R(0,1,2D,|x|^s)|(2D+1)!} \int_0^\delta x^{s-2}dx.$$

⋆ The second term depending on r satisfies:

$$|R(j,\Delta,2D,r)| = \Delta^q|R(j,\Delta,2D+q,r^{(q)})| \\ \leq \Delta^q \frac{G}{q!}\sum_{k,l}|a_k a_l|\,|k-l|^{2D+2}(j-p)^{\gamma-q}.$$

If $\gamma - q < -1$, this summation over j converges. Then:

$$2 \sum_{j=p+1}^{[\delta/\Delta]} \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}} |R(j,1,2D,r)| \\ \leq \Delta^{\gamma-s}\frac{2G\sum_{k,l}|a_k a_l|\,|k-l|^{2D+q}}{q!|R(0,1,2D,|x|^s)|} \sum_{j=p+1}^{\infty}(j-p)^{\gamma-s} = o(1).$$

If $\gamma - q = -1$,

$$2 \sum_{j=p+1}^{[\delta/\Delta]} \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}} |R(j,1,2D,r)| \\ \leq \Delta^{\gamma-s}\frac{2G\sum_{k,l}|a_k a_l|\,|k-l|^{2D+q}}{q!|R(0,1,2D,|x|^s)|} \sum_{j=p+1}^{\infty}(j-p)^{-1} \\ = O(-\Delta^{\gamma-s}\log(\Delta)) = o(1).$$

If $\gamma - q > -1$, $\Delta \sum_{j=p+1}^{[\delta/\Delta]} (j-p)^{\gamma-s}\Delta^{\gamma-s}$ is a Riemann sum for the converging integral $\int_0^\delta x^{\gamma-s}dx$, so

$$2\sum_{j=p+1}^{[\delta/\Delta]} \frac{\Delta^{2D}}{\sigma^2_{a,\Delta}}|R(j,1,2D,r)| \leq \Delta^{q-s-1}\frac{2G\sum_{k,l}|a_k a_l|\,(k-l)^{2D+q}}{q!|R(0,1,2D,|x|^s)|}\int_0^\delta x^{\gamma-s}dx.$$

Because $q \geq 2$, this part is $O(\Delta^{1-s})$. So B'_n is $O(\max(1,\Delta^{1-s}))$.

c) $C'_n = 2\sum_{j=[\delta/\Delta]+1}^{n-p-1} |\rho_\Delta(j)| = 2\sum_{j=[\delta/\Delta]+1}^{n-p-1} \frac{\Delta^{2D+d}}{\sigma^2_{a,\Delta}}|R(j,\Delta,2D,v^{(2D+d)})|$.

$$\begin{aligned}\sum_{j=[\delta/\Delta]+1}^{n-p-1} &|R(j,\Delta,2D,v^{(2D+d)})| \\ &\leq \sum_{j=[\delta/\Delta]+1}^{n-p-1}\sum_{k,l}|a_k a_l|\,(k-l)^{2D+d} \\ &\quad\times\left|\int_0^1 \frac{(1-\eta)^{2D+d-1}}{(2D+d-1)!}v^{(2D+d)}((j+(k-l)\eta)\Delta)d\eta\right|.\end{aligned}$$

As before, we consider the integral as a value of v at a point near $j\Delta$ and the summation over j as a Riemann sum for the integral $\int_\delta^T |v^{(2D+d)}(x)|dx$.

$$C'_n \leq \Delta^{d-s-1}\frac{2}{|R(0,1,2D,|x|^s)|}\sum_{k,l}|a_k a_l|\,(k-l)^{2D+2}\int_\delta^T |v^{(2D+d)}(x)|dx.$$

Because $d \leq 2$, C'_n is $O(\Delta^{1-s})$.

Then λ_n is $O(\max(1,\Delta^{1-s}))$. Because $n\Delta^{2(s-1)} \to \infty$, λ_n is $o(s_n)$.

Case (ii). – A'_n is bounded by $2p+2$. Function v is $2D+2$ times differentiable on $]0,T]$, and $\int_0^T |v^{(2D+2)}(x)|dx < \infty$.

So we compute directly $B'_n + C'_n$.

We use lemma 1, expanding v up to order $2D+2$:

$$\rho_\Delta(j) = -\frac{\Delta^{2D+2}R(j,\Delta,2D+2,v^{(2D+d)})}{\sigma^2_{a,\Delta}}.$$

Thus:

$$|B'_n + C'_n| \leq 2 \sum_{j=p+1}^{n-p-1} |\rho_\Delta(j)| = 2\frac{\Delta^{2D+2}}{\sigma^2_{a,\Delta}} \sum_{j=p+1}^{n-p-1} \sum_{k,l} |a_k a_l| (k-l)^{2D+2}$$
$$\times \int_0^1 \frac{(1-\eta)^{2D+1}}{(2D+1)!} |v^{(2D+2)}((j+(k-l)\eta)\Delta)| d\eta$$
$$\leq \Delta^{1-s} \frac{2}{|R(0,1,2D,|x|^s)|} \sum_{k,l} |a_k a_l| (k-l)^{2D+2} \int_0^T |v^{(2D+d)}(x)| dx.$$

Then λ_n is $O(\Delta^{1-s})$.

Using that $M(a) = D+1$, we compute a lower bound for s_n^2.

$B_n \geq \frac{4\Delta^{4D}}{\sigma^2_{a,\Delta}} \sum_{j=p+1}^{[\delta/\Delta]} (n-p-j) C^2 R^2(j,\Delta,2D,x^s)$, because the part depending on function r gives negligible contribution.

$$\frac{4\Delta^{4D}}{\sigma^2_{a,\Delta}} \sum_{j=p+1}^{[\delta/\Delta]} (n-p-j) C^2 R^2(j,\Delta,2D,x^s)$$
$$\geq n\Delta^{2-2s} \frac{4C^2 s|s-1|}{R^2(0,1,2D,|x|^s)}$$
$$\times \sum_{j=p+1}^{[\delta/\Delta]} (\sum_{k,l} a_k a_l (k-l)^{2D+2} I(j,k,l,2D+1,s-2))^2.$$

Developing the square:

$$\sum_{j=p+1}^{[\delta/\Delta]} (\sum_{k,l} a_k a_l (k-l)^{2D+2} I(j,k,l,2D+1,s-2))^2$$
$$= (\sum_{k,l,k',l'} a_k a_l a_{k'} a_{l'} ((k-l)(k'-l'))^{2D+2}$$
$$\times \sum_{j=p+1}^{[\delta/\Delta]} I(j,k,l,2D+1,s-2) I(j,k',l',2D+1,s-2)$$
$$\geq \Delta |(\sum_{k,l,k',l'} a_k a_l a_{k'} a_{l'} ((k-l)(k'-l'))^{2D+2}| \int_0^T |v^{(2D+2)}(x))^2 dx.$$

Because

$$M(a) = D + 1,$$

$$B_n \geq n\Delta^{3-2s} \frac{4C^2 s(s-1)}{R^2(0,1,2D,|x|^s)} (\sum_k a_k k^{D+1})^4 \int_0^T (v^{(2D+2)}(x))^2 dx.$$

Thus s_n^2 is asymptotically proportional to $n\Delta^{3-2s}$. Because $n\Delta \to \infty$, λ_n is $o(s_n)$.

Proof of Corollary 2. – The proof of corollary 1 and conditions on the index l shows that the bias induced by m vanishes.

Calculation of the covariance

We consider two sequences a and b with length p and q and the associated differences $\Delta_a X$ and $\Delta_b X$. Then:

$$\mathrm{Cov}(\Delta_a X_i, \Delta_b X_j)) = -\sum_{k=0}^{p} \sum_{l=0}^{q} a_k b_l v((i+k-j-l)\Delta).$$

It is a simple matter to modify the previous calculation concerning s_n^2 to find the claimed expression for the covariance of the corresponding quadratic variations.

Proof of Theorem 3.

Almost sure convergence

Following Theorem 1 and the calculation of the equivalent of $\sigma_{a,\Delta}^2$, vector Z is almost surely asymptotically equivalent to vector $(C(i\Delta)^{2h})_{i=1\ldots p}$. Estimators $(\hat{h}, \hat{C})$ are defined by a continuous function of the co-ordinates of Z, so they converge almost surely to (h, C) .

Distribution of a couple of quadratic a-variations

We study the joint distribution P_n of $(V(a,n,\Delta),\ V(b,n,\Delta))$ corresponding to two sequences a and b.

Consider two positive reals λ and μ, and define $V_n(\lambda,\mu) = \lambda V(a,n,\Delta) + \mu V(b,n,\Delta)$. Then:

$$V_n(\lambda,\mu) = \frac{1}{n} \sum_{j=1}^{2(n-p)} (Z_j^2 - E(Z_j^2))$$

where $Z_j = \sqrt{\lambda}\frac{\Delta_a X_i}{\sigma_{a,\Delta}^2}$ if $j \leq n-p$ and $Z_j = \sqrt{\lambda}\frac{\Delta_b X_i}{\sigma_{b,\Delta}^2}$ if $j > n-p$. The correlation matrix of Z_j is:

$$\rho_\Delta(i,j) = \frac{-\lambda \sum_{k,l} a_k a_l v((i+k-j-l)\Delta)}{\sigma_{a,\Delta}^2} \quad \text{if } i,j \leq n-p,$$

$$\rho_\Delta(i,j) = \frac{-\sqrt{\lambda\mu} \sum_{k,l} a_k b_l v((i+k-j-l)\Delta)}{\sigma_{a,\Delta}\sigma_{b,\Delta}} \quad \text{if } i \leq n-p \text{ and } j > n-p,$$

$$\rho_\Delta(i,j) = \frac{-\mu \sum_{k,l} b_k b_l v((i+k-j-l)\Delta)}{\sigma_{b,\Delta}^2} \quad \text{if } i,j > n-p.$$

The variance of $V_n(\lambda,\mu)$ is computed using the preceding expression of the covariance. The calculation of the bound of the largest eigenvalue requires the bound of ρ_Δ in the case $i \leq n-p$ and $j > n-p$. The calculations and results are the same. Then $N(n,\Delta)V_n(\lambda,\mu)$ converges in distribution to a Gaussian variable, where $N(n,\Delta)$ is the normalisation factor of both $V(a,n,\Delta)$ and $V(b,n,\Delta)$.

Denote $\Psi_n(\lambda,\mu)$ the Laplace transform of the distribution P_n. Because of the convergence of $N(n,\Delta)V_n(\lambda,\mu)$, we know that $\Psi_n(\lambda,\mu)$ tends to $\Psi(\lambda,\mu)$ when λ and μ are both non-negative, where $\Psi(\lambda,\mu)$ is the Laplace transform of a Gaussian vector distribution Φ. The same is true when λ and μ are both non-positive.

Assume that the sequence P_n converges in distribution to a probability distribution P. The Laplace transform of P is equal to $\Psi(\lambda,\mu)$ on two quarters of the plan. This Laplace transform is defined on a convex set, so this set is the whole plane. It is analytic in its support, so it is equal to $\Psi(\lambda,\mu)$ on the whole plane. The unique possible limit distribution is Φ. But the sequence P_n is tight, because its marginals are tight, so it converges to the jointly Gaussian distribution Φ.

Asymptotical bias

Conditions involving parameter τ ensure that

$$\sigma_{a,\Delta}^2 = -\Delta^{2h} C(-1)^D \sum_{k,l} a_k a_l |k-l|^{2h} + o(\Delta^{2h+\tau}).$$

The choice of α with respect to τ implies that $N(n,\Delta)\Delta^\tau$ tends to 0, so that the bias $N(n,\Delta)(E(\hat{h})-h)$ tends asymptotically to zero.

Central limit theorem

Vector Z is a linear function of a Gaussian vector. Note that estimator $\hat{h}$ does not directly depend on Δ and is defined as a differentiable function of vector Z. We apply theorem 3.3.11 of Dacunha-Castelle and Duflo (1983) and conclude that this estimator is asymptotically Gaussian with the claimed rate of convergence.

The rate of convergence for estimator $\hat{C}$ is lower. Consider:

$$\begin{aligned}\log \hat{C} - \log C =& \frac{1}{p}\sum_{j=1}^{p} \log\left(\frac{\hat{Z}_j}{Cj\Delta^{2\hat{h}}}\right) = \frac{1}{p}\sum_{j=1}^{p} \log\left(\frac{\hat{Z}_j}{Cj\Delta^{2h}}\right)\\ &+ 2(h-\hat{h})\frac{1}{p}\sum_{j=1}^{p}\log(j) + 2(h-\hat{h})\log(\Delta).\end{aligned}$$

Because of the last term, the rate of convergence for $\log(\hat{C})$ is $N(n,\Delta)/\log(1/\Delta)$.

6. SIMULATIONS

6.1. Choice of the sequences

We use a simple version of the estimators defined in theorem 3. Considering a sequence a^1 of length p, we define the sequence a^2 with "double time mesh", *i.e.* the sequence defined by $a^2_{2i} = a^1_i$ and $a^2_{2i+1} = 0$ for $0 \leq i \leq p$. Then $\sigma^2_{a^1,\Delta}$ tends to a linear combination of $(C(i\Delta)^{2h})_{i=1\ldots p}$ and $\sigma^2_{a^2,\Delta}$ tends to the same combination of $(C(2i\Delta)^{2h})_{i=1\ldots p}$. Estimator $\hat{h}$ is equal to:

$$\hat{h} = \frac{1}{2}\log_2\left(\frac{U(a^2,n,\Delta)}{U(a^1,n,\Delta)}\right).$$

Because of the relation between the two sequences, estimator $\hat{C}$ is equal to:

$$\hat{C} = -\frac{U(a^1,n,\Delta)}{(-1)^D\sum_{k,l} a^1_k a^1_l |k-l|^{2\hat{h}}\Delta^{2\hat{h}}}.$$

For each sequence a, we consider these estimators. The simple quadratic variation corresponds to the sequence $(a_0 = -1,\ a_1 = 1)$. The sequence Db04 corresponds to the wavelet built in Daubechies (1988), with two first zero moments: $a_0 = 0.48296$, $a_1 = -0.83652$, $a_2 = 0.24144$, $a_3 = 0.12941$.

We simulate discrete samplings of three processes P1, P2, P3, using a method based on the spectral representation and the Fast Fourier Transform algorithm (Azaïs 1994).

P1 is the stationary centred Gaussian process with covariance $r(t) = \exp(-|t|)$.

P2 is the stationary centred Gaussian process with covariance $r(t) = \exp(-|t|^{1.7})$.

P3 is the process $t^2 + B(t)$ where B is the standard brownian motion.

For each process, we simulated 100 sample paths. For P1 and P3, we choose the asymptotical framework $\Delta = 64/n$ $(T = 64)$ and verify assumptions of case (i) in theorem 3.

For P2, we have $s = \tau = 1.7$. We choose the asymptotical framework $\Delta(n) = n^{-\alpha}$ with $\alpha = 2/3$, which satisfies $1/2\tau < \alpha < 1/(2s-2)$, $\alpha < 1/(4s-6)$ and $1/(2\tau + 2s - 3) < \alpha < 1$. The observation interval is not bounded. Process P2 and the simple quadratic variation $(M(a) = 1)$ satisfy assumptions (ii) and (iv) in theorem 3; the rate of convergence is $n^{11/30}$. Process P2 and the Db04 quadratic variation $(M(a) = 2)$ satisfy assumptions (i) and (iii) in theorem 3; the rate of convergence is $n^{1/2}$.

6.2. Estimation of parameter h

Results

Simple quadratic variation

TABLE 1. – Table of MSE for the estimate of h by the simple quadratic variation. Column "estimated" represents the empirical MSE computed on the sample, Column "asymptotical" represents the theoretical asymptotical MSE.

	P1		P2		P3	
n	Estimated	Asymptotical	Estimated	Asymptotical	Estimated	Asymptotical
512	0.00318	0.00102	0.00064	0.00068	0.250	0.00102
1024	0.00122	0.00051	0.00033	0.00041	0.248	0.00051
2048	0.00045	0.00026	0.00020	0.00025	0.248	0.00026
4096	0.00017	0.00013	0.00011	0.00015	0.245	0.00013
8192	0.00007	0.00006	0.00008	0.00009	0.243	0.00006

Variation of wavelet Db04.

TABLE 2. – *Table of MSE for the estimate of h by the variation of wavelet Db04.*

	P1		P2		P3	
n	Estimated	Asymptotical	Estimated	Asymptotical	Estimated	Asymptotical
512	0.00271	0.00232	0.00214	0.00209	0.00461	0.00232
1024	0.00131	0.00116	0.00096	0.00104	0.00151	0.00116
2048	0.00065	0.00058	0.00048	0.00052	0.00074	0.00058
4096	0.00029	0.00029	0.00028	0.00026	0.00025	0.00029
8192	0.00014	0.00015	0.00012	0.00013	0.00013	0.00015

Comments

The empirical mean square error agrees with its theoretical value. Biases are not significant except for the simple quadratic variation for P3, where the estimate is biased by the differentiable trend. This trend vanishes with wavelet Db04 which gives satisfactory results. Process P2 shows a different behaviour for the estimator of the simple quadratic variation and the estimator of wavelet Db04. The last one converges faster, but the limiting variance is larger. The Kolmogorov- Smirnov test of normality is positive at level 5% in every case.

6.3. Estimation of constant C

We give here the table of the mean square errors for estimator $\hat{C}$ with the previous simulations.

Results

Simple quadratic variation

TABLE 3. – *Table of MSE for the estimate of C by the simple quadratic variation.*

	P1		P2		P3	
n	Estimated	Asymptotical	Estimated	Asymptotical	Estimated	Asymptotical
512	0.0616	0.0215	0.1258	0.0222	5.53	0.0215
1024	0.0362	0.0175	0.0759	0.0217	291	0.0175
2048	0.0206	0.0131	0.0514	0.0192	>1000	0.0131
4096	0.0112	0.0093	0.0293	0.0161	>1000	0.0093
8192	0.0057	0.0062	0.0245	0.0126	>1000	0.0062

Variation of wavelet Db04.

TABLE 4. – Table of MSE for the estimate of C by the variation of wavelet Db04.

	P1		P2		P3	
n	Estimated	Asymptotical	Estimated	Asymptotical	Estimated	Asymptotical
512	0.0579	0.0574	3.65	0.1936	0.2523	0.0574
1024	0.0442	0.0463	0.5958	0.1262	0.0847	0.0463
2048	0.0409	0.0342	0.1471	0.0797	0.0495	0.0342
4096	0.0268	0.0237	0.1070	0.0492	0.0195	0.0237
8192	0.0153	0.0157	0.0395	0.0298	0.0145	0.0157

Comments

Biases are important. Only the estimation for P1 gives satisfactory results. The estimation is worse than for h because the estimator of C is defined using the estimator of h. The Kolmogorov-Smirnov test of normality is positive at level 5% in every case except for wavelet Db04 applied to process P2.

6.4. Conclusion

Consider the results of the simulations of process P2. The asymptotic study proves that the rate of convergence is greater for the Db04 variations than for the simple variations. But the constant of the variance is greater for the Db04 variations, so that the results of the simple variations are better for the simulated lengths n. The constant of the variance depends on the sequence a^i through the matrix A. It is not a simple matter to find optimising sequences a^i. Using other wavelet sequences, we observed (but did not prove) that the longer the sequences are, the greater the constant is. The length rises with the parameter $M(a)$, so that we have to choose this parameter as little as possible.

REFERENCES

R. J. ADLER, An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes, I.M.S. *Lecture Notes-Monograph Series*, Vol. **12**, 1990.

R. J ADLER and R. PYKE, Uniform quadratic variation for Gaussian processes, *Stoch. Proc. Appl.*, Vol. **48**, 1993, pp. 191-209.

J-M. AZAÏS, Exact error evaluation in Gaussian process simulation, preprint, 1994.

P. BRUGIÈRE, Estimation de la variance d'une diffusion, *C. R. Acad. Sci.*, Tome **312**, Série I, 1991, pp. 999-1004.

H. CRAMER and M. R. LEADBETTER, Stationary and related stochastic processes, Wiley, New York, 1967.

M. CZÖRGO and P. RÉVÉSZ, Strong approximation in probability and statistics, Academic Press, 1981.

D. DACUNHA-CASTELLE and M. DUFLO, Probabilités et statistiques, Tome **2**, Masson, 1983.

I. DAUBECHIES, Orthonormal bases of compactly supported wavelets, *Commun. Pure Appl. Math.*, Vol. **41**, Nov 1988, pp. 909-996.

V. GENON-CATALOT, C. LAREDO and D. PICARD, Non-parametric estimation of the diffusion coefficient by wavelets methods, *Scand. J. of Statist.*, Vol. **19**, 1992, p. 317-335.

U. GRENANDER, Abstract inference, Wiley, 1981.

X. GUYON, Variations des champs gaussiens stationnaires: applications à l'identification, *Proba. Th. Rel. Fields*, n° 75, 1987, pp. 179-193.

X. GUYON and J.R. LEON, Convergence en loi des H-variations d'un processus gaussien stationnaire, *Ann. Inst. Henri Poincaré*, Vol. **25**, n° 3, 1989, pp. 265-282.

P. HALL and A. WOOD, On the performance of box-counting estimators of fractal dimension, *Biometrika*, , Vol. **80**, 1993, pp. 246-252.

P. HALL, A. WOOD and A. FEUERVERGER, Estimation of fractal index and fractal dimension of a Gaussian process by counting the number of level crossings, *J. of Time Ser. Anal.*, n° 6, 15, 1994, pp. 587-606.

P. HUBERT and J.-P. CARBONNEL, Caractérisation fractale de la variabilité et de l'anisotropie des précipations intertropicales, *C. R. Acad. Sci.*, Tome **307**, Série II, 1988, pp. 909-914.

I. A. IBRAGIMOV and Y. A. ROZANOV, Gaussian random processes, Springer Verlag, 1978.

J. ISTAS, Wavelet coefficients of a Gaussian process and applications, Ann. Inst. Henri Poincare, Vol. **28**, n° 4, 1992, pp. 537-556.

J. ISTAS and C. LAREDO, Estimation de la fonction de covariance d'un processus gaussien stationnaire par méthodes d'échelles, *C. R. Acad. Sci.*, Tome **316**, Série I, 1993, pp. 495-498.

G. LANG, Estimation de la régularité et de la longue dépendance de processus gaussiens. Thèse de l'université Toulouse III, 1994.

J. R. LEON and J. ORTEGA, Weak convergence of different types of variation for biparametric Gaussian processes, *Colloquia Math. Soc. J. Bolyai*, n° 57, 1989, Limit theorems in Proba. and Stat., Pecs Hungary.

L. LUENBERGER, Introduction to dynamical systems: theory and applications, Wiley, 1979.

B. MANDELBROT and J. VAN NESS Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Review*, 1968, 10, pp. 422-437.

A. P. PENTLAND, Fractal-based description of natural scenes, IEEE transactions on pattern analysis and machine intelligence, Vol. **PAMI-6**, n° 6, 1984, pp. 661-674.

P. SOULIER, Déviation quadratique pour des estimateurs de la variance d'une diffusion, *C. R. Acad. Sci.*, Tome **313**, Série I, 1991, pp. 783-786.

(Manuscript received October 24, 1994;
Revised December 2, 1996.)