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Slow entropy type invariants and smooth realization of commuting measure-preserving transformations

by

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ABSTRACT. – We define invariants for measure-preserving actions of discrete amenable groups which characterize various subexponential rates of growth for the number of "essential" orbits similarly to the way entropy of the action characterizes the exponential growth rate. We obtain above estimates for these invariants for actions by diffeomorphisms of a compact manifold (with a Borel invariant measure) and, more generally, by Lipschitz homeomorphisms of a compact metric space of finite box dimension. We show that natural cutting and stacking constructions alternating independent and periodic concatenation of names produce $\mathbb{Z}^2$ actions with zero one-dimensional entropies in all (including irrational) directions which do not allow either of the above realizations.

RÉSUMÉ. – Nous définissons des invariants pour des actions préservant la mesure de groupes moyennables discrets. Ces invariants caractérisent divers taux de croissance sous-exponentiels du nombre d'orbites « essentielles »

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de la même manière que l'entropie caractérise leur taux de croissance exponentiel.

Nous obtenons des majorations de ces invariants pour des actions par des difféomorphismes sur une variété compacte (avec une mesure invariante Borélienne) ou plus généralement pour des actions par des homéomorphismes Lipschitziens sur un espace métrique compact de dimension « par boîtes » finie.

Nous montrons comment des constructions naturelles par découpages et empilements alternant des concaténations de noms indépendantes et périodiques peuvent produire des actions de $\mathbb{Z}^2$ dont toutes les entropies directionnelles (incluant les directions irrationnelles) sont nulles et dont on ne peut trouver aucun modèle du type précédent.

INTRODUCTION

The smooth realization problem for measure–preserving transformations has many aspects and facets, most of them far from well understood. For the sake of this discussion we will stick to two basic types of questions. Let T be a measure–preserving transformation of a Lebesgue space (X, μ).

1. Does there exists a smooth, say C^∞, diffeomorphism f of a compact differentiable manifold M, preserving a Borel probability measure ν such that f considered as the automorphism of the Lebesgue space (M, ν) is metrically isomorphic to T?

2. The same with an extra requirement that the measure in question is absolutely continuous, or smooth, or given by a smooth positive density.

The basic and well-known restriction on the possibility of smooth realization is finiteness of entropy $h_\mu(T)$. This was originally shown by Kushnirenko for the absolutely continuous measures in his seminal paper [6] and was first proven by Margulis for arbitrary Borel measures in the late sixties (unpublished). This fact follows from the entropy inequalities in [10] or [5] . In fact, the entropy of a Lipschitz homeomorphism of any metric space of finite box dimension is also finite (see e.g. [4], Theorem 3.2.9).

The realization problem 1 for finite entropy transformations has been solved positively by Lind and Thouvenot [7] based on the methods of the Krieger generator theory. In their work $M = \mathbb{T}^2$, the two-dimensional torus. In fact, they showed that the measure ν can be made positive on open sets. Without the latter requirement their method works for an arbitrary compact manifold of dimension ≥ 2. However, with that requirement the smooth realization on an arbitrary manifold or, for that matter, on S^3, is not known. A related fact is that an arbitrary measure-preserving transformation with finite entropy can be realized as a *homeomorphism* of an arbitrary differentiable compact manifold of dimension ≥ 2 preserving a measure given by a smooth positive density ([3], Chapter 15; for the case of $\mathbb{T}^2$ this result was proved earlier in [7]).

The situation with the "genuine" smooth realization (problem 2) is much less understood. There are no known restrictions for realization within the class of all compact manifolds. As is well–known if $M = S^1$ any diffeomomorphism preserving a non-atomic measure is conjugate to a rotation. The only non-trivial restriction follows from Pesin's theory of diffeomorphisms with non-zero Lyapunov exponents which implies that in dimension two any positive entropy diffeomorphism is essentially Bernoulli [9]. No restrictions are known in the zero entropy case in dimension greater than one and in the positive entropy case in dimension greater than two.

When one passes from single automorphisms to groups of measure–preserving transformations the situation changes. Orbit growth characteristics derived from and similar to entropy provide a number of obstructions to smooth realization. Let us consider the simplest interesting case, that of the free abelian groups $\mathbb{Z}^k$, $k \geq 2$. For smooth actions the (l-dimensional) entropy of the restriction of the action to any subgroup Γ of $\mathbb{Z}^k$ of rank $l \geq 2$ must be equal to zero. Furthermore, the entropy with respect to an absolutely continuous invariant measure is given by the Pesin formula [9] as the sum of positive Lyapunov exponents and hence entropy of an element is a subadditive function on the group. Since there are examples of actions for which this is not true (such examples were constructed independently by the second author and by Ornstein and Weiss [8]) we deduce that not every action whose elements have finite entropy allows smooth realization with an absolutely continuous invariant measure. Furthermore, one had to take into account "irrational" directions in the group. The entropy in such a direction can be described as the entropy of an appropriate element of the suspension action of $\mathbb{R}^k$. Particularly interesting examples of that kind constructed by the second author feature an ergodic action of $\mathbb{Z}^2$ whose elements have zero entropy but some (evidently irrational) element of the suspension has positive entropy. Since smooth realization of an action implies a smooth realization of its suspension such actions cannot be realized with an absolutely continuous invariant measure.

In the present paper we construct obstructions to smooth realization (even with just a Borel invariant measure) of actions of finitely generated amenable groups which can be non-trivial already for actions of $\mathbb{Z}^k, k \geq 2$ for which all elements of the suspension have zero entropy. In fact, our obstructions work for realization of actions by Lipschitz homeomorphisms of compact metric spaces of finite box dimension. This is different in an essential way from the obstructions used in [8] which relied on much deeper facts from smooth ergodic theory. Our invariants generalize to the group actions the invariants for individual maps whose description is outlined in section 11 of [1]. They represent asymptotics of the growth of the number of balls in the Hamming metrics needed to cover the "names" representing an essential measure of codes with respect to a partition. The key observation (Proposition 1) is that to majorize those asymptotics it is sufficient to consider a single generating partition or, more generally, a family of partitions whose σ–algebras asymptotically generate the σ–algebra of all measurable sets (generating or sufficient families). This property makes the invariants both calculable and useful unlike the asymptotics of the entropy of iterated partitions which are useless in the zero entropy case. For smooth and Lipschitz transformations our invariants can be estimated from above via the box dimension and Lipschitz constants (Proposition 3). More generally, for groups of homeomorphisms of compact metric spaces there is an intimate connection between the invariants and similar asymptotics obtained from the sequences of metrics measuring the maximal distance between finite pieces of orbits (Proposition 2) It is rather remarkable that zero-entropy examples not allowing smooth or Lipschitz realization come from the most natural and "regular" cutting and stacking constructions which alternate random and periodic behavior for increasing time scales (Proposition 4 and Corollary 1). In order to keep notations reasonable and presentation short we describe examples only for $\mathbb{Z}^2$ actions although proper modifications work for finitely generated amenable groups which are not finite extensions of $\mathbb{Z}$.

1. DESCRIPTION OF INVARIANTS

1.1. Preliminaries

Let Γ be a discrete group, $F \subset \Gamma$ its subset. We consider the spaces

$$\Omega_{N,F} = \{\omega = (\omega_\gamma)_{\gamma \in F}; \quad \omega_\gamma \in \{1, \ldots, N\}\}$$

with the natural projections $\pi_{F,F'} : \Omega_{N,F} \to \Omega_{N,F'}$ for $F \supset F'$. In particular, for every $F \subset \Gamma$ there is a projection $\pi_{\Gamma,F} : \Omega_{N,\Gamma} \to \Omega_{N,F}$.

For any finite set $F \subset \Gamma$ we define the *Hamming metric* d_F^H in $\Omega_{N,F}$ as follows

$$d_F^H(\omega, \overline{\omega}) = \frac{1}{\mathrm{card} F} \sum_{\gamma \in F} (1 - \delta_{\omega_\gamma \overline{\omega}_\gamma})$$

In other words, d_F^H measures the average number of coordinates which take different values.

Let $T : (X, \mu) \times \Gamma \to (X, \mu)$ be an action of the group Γ by measure–preserving transformations of a Lebesgue space; let $\xi = (c_1, \ldots, c_N)$ be a finite measurable partition. Our constructions will use an ordering of elements of ξ but the results will be independent of a particular choice of an ordering. We define the "coding map" $\phi_{T,\xi} : X \to \Omega_{N,\Gamma}$ by $(\phi_{T,\xi})_\gamma = \omega_\gamma(x)$ where $T(\gamma)x \in c_{\omega_\gamma(x)}$. Partial coding $\phi_{T,\xi}^F$ for $F \subset \Gamma$ is defined by $\phi_{T,\xi}^F = \pi_{\Gamma,F} \circ \phi_{T,\xi}$. We will call $\phi_{T,\xi}^F x$ the *F–name* of x with respect to ξ. The partial coding $\phi_{T,\xi}^F$ defines the measure $(\phi_{T,\xi}^F)_* \mu$ in $\Omega_{N,F}$.

Consider now a compact metric space with the distance function d and a probability measure λ. For $\epsilon > 0, \delta > 0$ let $S_{d,\lambda}(\epsilon, \delta)$ be the minimal number of balls of radius ϵ whose union has measure $\geq 1 - \delta$.

For the special case $X = \Omega_{F,N}$, $d = d_F^H$, $\lambda = (\phi_{T,\xi}^F)_* \mu$ we adopt the following notation

$$S_{d,\lambda}(\epsilon, \delta) = S_\xi^H(T, F, \epsilon, \delta).$$

1.2. Comparison of codes

For $E, F \subset \Gamma$ let $EF = \{\gamma\gamma' : \gamma \in E, \gamma' \in F\}$. Let us denote by ξ^E the iterated partition

$$\vee_{\gamma \in E} T(\gamma)\xi.$$

Any ordering of elements of ξ induces the lexicographical ordering of elements of ξ^E. We will compare codings via ξ and via ξ^E in terms of the Hamming distance.

Notice that the F–names with respect to ξ^E are completely determined by EF–names with respect to ξ. Furthermore, any disagreement between EF–names with respect to ξ will show up at most $\mathrm{card}E$ times as a disagreement between F–names with respect to ξ^E. Furthermore, if $F' \supset F$ there are at most $(\mathrm{card}F' - \mathrm{card}F)$ more disagreements between F'–names than between F–names. Hence the following inequalities hold:

$$\begin{aligned} d_F^H(\phi_{T,\xi^E}^F x, \phi_{T,\xi^E}^F y) &\leq \frac{\mathrm{card}E \cdot \mathrm{card}EF}{\mathrm{card}F} \cdot d_{EF}^H(\phi_{T,\xi}^{EF} x, \phi_{T,\xi}^{EF} y) \\ &\leq (\mathrm{card}E) d_F^H(\phi_{T,\xi}^F x, \phi_{T,\xi}^F y) + \frac{\mathrm{card}EF - \mathrm{card}F}{\mathrm{card}EF} \end{aligned}$$

and hence

$$S^H_{\xi^E}(T,F,\epsilon,\delta) \leq S^H_\xi(T,F,(\mathrm{card}E)\epsilon + \frac{\mathrm{card}EF - \mathrm{card}F}{\mathrm{card}EF},\delta) \tag{1.1}$$

It is obvious that if $\eta \leq \xi$, $d^H_F(\phi^F_{T,\eta}x, \phi^F_{T,\eta}y) \leq d^H_F(\phi^F_{T,\xi}x, \phi^F_{T,\xi}y)$ and hence

$$S^H_\eta(T,F,\epsilon,\delta) \leq S^H_\xi(T,F,\epsilon,\delta). \tag{1.2}$$

Finally let us consider two partitions $\xi = (c_1, \ldots, c_N)$ and $\eta = (d_1, \ldots, d_N)$ and let
$\alpha = \sum_{i=1}^N \mu(c_i \bigtriangleup d_i)$. Using Chebychev inequality one immediately sees that

$$\mu\{x : d^H_F(\phi^F_{T,\xi}x, \phi^F_{T,\eta}x) > \alpha^{\frac{1}{2}}\} \leq \alpha^{\frac{1}{2}}.$$

This translates into the following inequality

$$S^H_\eta(T,F,\epsilon+\alpha^{\frac{1}{2}},\delta+\alpha^{\frac{1}{2}}) \leq S^H_\xi(T,F,\epsilon,\delta). \tag{1.3}$$

1.3. Asymptotic invariants

Now assume that Γ is an amenable group, $\{F_n\}, n = 1, 2, \ldots$ is a Følner sequence of finite subsets of Γ, *i.e.* $F_1 \subset F_2 \subset \ldots$, $\bigcup_{n=1}^\infty F_n = \Gamma$ and for every finite set $E \subset \Gamma$

$$\frac{\mathrm{card}EF_n}{\mathrm{card}F_n} \to 1, \text{ as } n \to \infty. \tag{1.4}$$

Our invariants are built from the characteristics of the asymptotic growth of the quantities $S^H_\xi(T,F_n,\epsilon,\delta)$ as $n \to \infty$. We then let $\epsilon \to 0$ and $\delta \to 0$ and take the supremum over all finite partitions ξ. Specific characteristics of the asymptotic growth can be produced in a number of ways. For example, fix a sequence of positive numbers a_n increasing to infinity and take

$$\limsup_{n\to\infty} \frac{S^H_\xi(T,F_n,\epsilon,\delta)}{a_n} \text{ or } \liminf_{n\to\infty} \frac{S^H_\xi(T,F_n,\epsilon,\delta)}{a_n}.$$

Alternatively, one can fix a "scale" *i.e.* a family $a_n(t)$ of sequences increasing to infinity and monotone in t such as the power family n^t, $0 < t < \infty$ and define the following characteristic of the asymptotic growth

$$\sup\{t : \limsup_{n\to\infty} S^H_\xi(T,F_n,\epsilon,\delta)a_n(t) > 0\},$$

or similarly with lim inf. The reader will find without much difficulty other convenient characteristics of the asymptotic growth. Whatever specific procedure is chosen, we will denote the resulting quantity which may be either a non-negative real number or infinity by $A(\xi,\epsilon,\delta)$. Since $S_\xi^H(T,F,\epsilon,\delta)$ are by definition non-increasing in ϵ and δ

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} A(\xi,\epsilon,\delta)=\sup_{\substack{\epsilon>0\\ \delta>0}} A(\xi,\epsilon,\delta)=A(\xi)$$

Thus our invariants are defined as

$$\sup_\xi A(\xi),$$

where ξ is an arbitrary finite partition. Let us point out that if the action T is ergodic and we use the exponential scale $a_n(t)=\exp tn$ in our definition the resulting invariant coincides with the entropy of the action (See [2] where the case $\Gamma=\mathbb{Z}$ is treated in detail). Thus if the entropy $h_\mu(T)$ is equal to zero, in order to produce non-trivial invariants we should use slower scales such as n^t or $\exp tn^\alpha$ for some $\alpha<1$. The resulting invariant may then be called the "slow entropy" associated with the given scale.

Similarly to the case of the standard (exponential) entropy we need to develop effective methods of calculating our invariants. Recall that a partition ξ is called *a generator* for the action T if the minimal invariant σ–algebra containing ξ is the σ–algebra of all measurable sets. This fact is symbolically denoted by $\xi^\Gamma=\epsilon$ where ϵ denotes the partition into single points generating the σ–algebra of all measurable sets. More generally we will call a sequence of partitions ξ_m *generating* (or *sufficient*) if $\xi_m^\Gamma\to\epsilon$. This means that for every measurable set A and every $\delta>0$ there exists m_0 such that for every $m>m_0$ one can find a finite set $E_m\subset\Gamma$ and a set A_m measurable with respect to the partition $\xi_m^{F_m}$ such that $\mu(A\triangle A_m)\to 0$ as $m\to\infty$. The following fact generalizes the well-known properties of entropy including the generator theorem (See e.g. [4], Section 4.3.)

PROPOSITION 1. – *Let ξ_m be a generating sequence of partitions for the action T of an amenable group Γ. Then*

$$\sup_\xi A(\xi)=\lim_{m\to\infty} A(\xi_m).$$

COROLLARY 1. – *If η is a generating partition for T then*

$$\sup_\xi A(\xi)=A(\eta).$$

Proof of Proposition 1.1. – The proof follows the same general scheme as the corresponding proof for the entropy of a single transformation (See e.g. [4], Theorem 4.3.12). We will use inequalities (1.1) – (1.3); (1.1) will be used to show that $A(\xi) = A(\xi^E)$ for any finite set $E \subset \Gamma$; this is the only place in the proof where amenability of the group Γ is used; (1.2) immediately implies that for $\eta < \xi$, $A(\eta) \leq A(\xi)$. Hence for any finite set $E \subset \Gamma$ and for any partition $\eta < \xi^E$, $A(\eta) \leq A(\xi)$. In the case of entropy the last step in the proof is showing the continuity of the entropy with respect to a partition in the Rokhlin metric $d_R(\xi, \eta) = H(\xi|\eta) + H(\eta|\xi)$. This continuity follows form the Rokhlin inequality ([4], Proposition 4.3.10(4)). In our situation we will use (1.3) for a similar purpose.

To prove the proposition we need to compare $A(\eta)$ for a fixed finite partition $\eta = (c_1, \ldots, c_N)$ with $\limsup_{m\to\infty} A(\xi_m)$ for a generating sequence ξ_m. In fact, it is enough to show that $A(\eta) \leq \sup A(\xi_m)$. For, then in particular $A(\xi_{m_0}) \leq \limsup_{m\to\infty} A(\xi_m)$ for every m_0, hence

$$\sup_m A(\xi_m) = \limsup_{m\to\infty} A(\xi_m) = \lim_{m\to\infty} A(\xi_m).$$

It follows from the definition of a generating sequence of partitions that for any positive integer m_0, and any $\epsilon > 0$ one can find $m = m(\epsilon) > m_0$, a finite set $E = E(\epsilon)$ and a partition $\zeta = (d_1, \ldots, d_N)$ such that

$$\zeta < \xi_m^E \text{ and } \sum_{i=1}^N \mu(c_i \triangle d_i) < \frac{\epsilon^2}{2}.$$

By (1.3)

$$S_\eta^H(T, F_n, \epsilon, \epsilon) \leq S_\zeta^H\left(T, F_n, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$$

and hence

$$A(\eta, \epsilon, \epsilon) \leq A\left(\zeta, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \leq A\left(\xi_m^E, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \tag{1.5}$$

where the second inequality follows from (1.2).

Now we can use the fact that F_n is a Følner sequence (see (1.4)) and pick n_0 such that for any $n \geq n_0$

$$\frac{\mathrm{card} EF_n - \mathrm{card} F_n}{\mathrm{card} EF_n} < \frac{\epsilon}{4}.$$

Using (1.1) we obtain for any such n the inequality

$$S_{\xi_m^E}^H\left(T, F_n, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \leq S_{\xi_m}^H\left(T, F_n, \frac{\epsilon}{4\mathrm{card} E}, \frac{\epsilon}{2}\right)$$

and hence with $n \to \infty$

$$A(\xi_m^E, \frac{\epsilon}{2}, \frac{\epsilon}{2}) \leq A(\xi_m, \frac{\epsilon}{4\text{card}E}, \frac{\epsilon}{2}).$$

Combining this inequality with (1.5) we obtain

$$A\left(\eta, \epsilon, \epsilon\right) \leq A\left(\xi_m, \frac{\epsilon}{4\text{card}E}, \frac{\epsilon}{2}\right).$$

Taking supremum over the second and third arguments completes the proof: $A(\eta) \leq A(\xi_m)$. □

2. ABOVE ESTIMATES FOR SMOOTH AND LIPSCHITZ ACTIONS

2.1. Comparison with d_F^T metrics

Now let Γ act on a compact metric space X with the distance function d by homeomorphisms preserving a non-atomic Borel probability measure μ. For any finite set $F \subset \gamma$ we define an equivalent metric $d_F^T = \max_{\gamma \in F} d \circ T(\gamma)$. Obviously if $F \supset F'$, $d_F^T \geq d_{F'}^T$.

We now specify the quantities $S_{d,\lambda}(\epsilon, \delta)$ defined in section 1.1 for the case $d = d_F^T$, $\lambda = \mu$. The corresponding numbers are denoted by $S_d(T, F, \epsilon, \delta)$. Pick a finite partition $\xi = (c_1, \ldots, c_N)$ such that $\mu(\partial\xi) = 0$, and a number $\delta > 0$. Let $\alpha = \alpha(\delta) > 0$ be such that $\mu(U_\alpha(\partial\xi)) < \delta^2$, where $U_\alpha(A)$ denotes the open α–neighborhood of the set A.

Proposition 2. – *For any $\beta > 0$, $S_\xi^H(T, F, \delta, \beta + \delta) \leq S_d(T, F, \frac{\alpha}{2}, \beta)$.*

Proof. – Denote the neighborhood $U_\alpha(\partial\xi)$ simply by U. Consider the set

$$C = \left\{x \in X : \frac{1}{\text{card}F} \sum_{\gamma \in F} \chi_U(T(\gamma)x) < \delta\right\}$$

By the Chebychev inequality $\mu(C) \geq 1 - \delta$. Consider a cover of the set C by the minimal number of balls of radius $\frac{\alpha}{2}$ in the metric d_F^T.

Lemma 1. – *For any ball B from the cover the diameter of the image $\phi_{T,\xi}^F(B)$ in the metric d_F^H is $\leq \delta$.*

Proof of the Lemma. – Since the cover is minimal $B \cap C \neq \emptyset$. Let $x \in C$. That means that among points $T(\gamma)x$, $\gamma \in F$ less than $\delta\text{card}F$ belong to the set $U_\alpha(\partial\xi)$. Let

$$\overline{F}_x = \{\gamma \in F : T(\gamma)x \notin U_\alpha(\partial\xi)\}.$$

For every $\gamma \in \overline{F}_x$, the $\frac{\alpha}{2}$ ball around $T(\gamma)x$ in the standard metric d lies in the same element of the partition ξ. Hence for any $y \in B_{d^T_F}(x, \frac{\alpha}{2})$ and $\gamma \in \overline{F}_x$, $\omega_\gamma(y) = \omega_\gamma(x)$ and consequently

$$d^F_H(\phi^F_{T,\xi}x, \phi^F_{T,\xi}y) \leq \frac{\text{card}(F \setminus \overline{F}_x)}{\text{card}F} < \delta.$$

To finish the proof of the proposition let us consider any cover $\mathfrak{A}$ of a set A with $\mu(A) \geq 1 - \beta$ by balls in the d^T_F metric of radius $\frac{\alpha}{2}$. For each element $B \in \mathfrak{A}$ let $B^H = \phi^F_{T,\xi}(B \cap C)$. By the lemma the set B^H can be covered by a d^H_F ball of radius δ. Thus, we constructed a cover of the set $\phi^F_{T,\xi}(A \cap C)$ by $S_d(T, F, \frac{\alpha}{2}, \beta)$ balls in the metric d^H_F. Since $\mu(A \cap C) \geq \mu(A) - \mu(C) \geq 1 - \beta - \delta$ we proved the proposition. □

2.2. Estimates of d^T_F covers for smooth and Lipschitz actions

Now we assume that the group Γ is generated by a finite set Γ_0 and that the Følner set F_n consists of elements whose word–length norm with respect to Γ_0 does not exceed a_n. Furthermore, we assume that the compact space X has finite box dimension D with respect to the metric d and that Γ acts on X by bi-Lipschitz homeomorphisms. Let L be a common Lipschitz constant for all elements $T(\gamma)$, $\gamma \in \Gamma_0 \bigcup \Gamma_0^{-1}$. Then obviously all elements of $T(F_n)$ have a common Lipschitz constant L^{a_n}.

PROPOSITION 3. – *For any partition ξ with $\mu(\partial\xi) = 0$*

$$S^H_\xi(T, F_n, \epsilon, \epsilon) \leq c(\epsilon)\exp(D \log La_n).$$

Proof 1.1. – By our assumption about dimension for any given $\epsilon > 0$ the space X can be covered by $c(\epsilon)\exp(D\log La_n)$ balls of radius $\frac{\alpha(\epsilon)}{2}L^{-a_n}$ where the choice of α has been explained before Proposition 2. By our assumption on the Lipschitz constants every such ball lies inside an $\frac{\alpha}{2}$ ball in the metric $d^T_{F_n}$. Hence $S_d(T, F_n, \frac{\alpha(\epsilon)}{2}, 0) \geq c(\epsilon)\exp(D\log La_n)$. But by Proposition 2 $S^H_\xi(T, F_n, \epsilon, \epsilon) \geq S_d(T, F_n, \frac{\alpha(\epsilon)}{2}, 0)$. □

Since one can always find a generating sequence of partitions satisfying the assumption of Proposition 3 one can apply Proposition 1 to show that a necessary condition for realization of a measure-preserving action of Γ by Lipschitz homeomorphisms in a space of finite box dimension, e.g. by diffeomorphisms of a compact manifold, is that the slow entropy should be no more than exponential as measured against the *diameter* of the Følner sets in the word–length metric. Naturally, zero entropy of the action corresponds to the subexponential growth measured against the *number of* *elements* in the Følner sets. For any group which is not a finite extension of $\mathbb{Z}$ there is a gap between the two asymptotics which allows to construct actions by zero entropy maps whose slow entropy is superexponential in terms of the diameter of Følner sets. Hence such actions do not allow smooth or Lipschitz realization. For example, let $\Gamma = \mathbb{Z}^k$, $k \geq 2$; F_n, the standard symmetric n-cubes around the origin, Γ_0 the standard set of generators. In this case $a_n = kn$, $\mathrm{card} F_n = (2n+1)^k$. In the next section we will construct actions of $\mathbb{Z}^2$ such that every element of the suspension $\mathbb{R}^2$ action has zero entropy but the growth of $S_\xi^H(T, F_n, \epsilon, \delta)$ is faster than exponential in n and hence in a_n. By the results of the present section such actions do not allow smooth or Lipschitz realization.

3. CONSTRUCTION OF EXAMPLES

As before, we consider a Lebesgue space (X, μ) and a finite measurable partition $\xi = (c_1, \ldots, c_N)$. For a measurable set $A \subset X$ we denote by $H(\xi|A)$ the entropy of the trace of the partition ξ on A, *i.e.* the entropy of ξ with respect to the conditional measure $\mu|A$. If $\eta = (d_1, \ldots, d_M)$ is another finite measurable partition of X we denote by $H(\xi|\eta)$ entropy of ξ relative to η. It is a standard fact that $H(\xi|\eta) = H(\xi \vee \eta) - H(\eta) = \sum_{i=1}^M \mu(d_i) H(\xi|d_i)$ (See e.g. [4], Section 4.3.). For $A \subset X$ we define the numbers $S_{\xi,A}^H(T, F, \epsilon, \delta)$ similarly to our definition of $S_\xi^H(T, F, \epsilon, \delta)$ (see the end of Section 1.1) with the only difference that we substitute the measure $(\phi_{T,\xi}^F)_*\mu$ by the image $(\phi_{T,\xi}^F)_*(\mu|A)$ of the conditional measure $\mu|A$. Let $C_n = [0, n-1] \times [0, n-1] \subset \mathbb{Z}^2$. Let T_1 and T_2 be two commuting measure–preserving transformations generating the action T of $\mathbb{Z}^2$ on the space X, μ.

LEMMA 2. – *Suppose A_n is a measurable set such that the images $T_1^i T_2^j A_n$ for $(i, j) \in C_n$ are disjoint and their union is the whole space X. Then for a partition ξ as before and for a positive integer k*

$$\frac{1}{n^2} H(\xi^{C_n}|A_n) \leq \frac{1}{k^2} H(\xi^{C_k}) + \frac{2k \log N}{n} \tag{3.1}$$

and

$$\frac{1}{n^2} \log S_{\xi,A_n}^H(T, C_n, \epsilon + \frac{2k}{n} + \delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}) \leq \frac{1}{k^2} \log S_\xi^H(T, C_k, \epsilon, \delta). \tag{3.2}$$

Proof. – Let ζ_n be the partition of X into the images of the set A_n under $T_1^i T_2^j$, $(i,j) \in C_n$. We have

$$H(\xi^{C_k}) \geq H(\xi^{C_k}|\zeta_n) = \frac{1}{n^2} \sum_{(i,j)\in C_n} H(\xi^{C_k}|T_1^i T_2^j A_n)$$
$$\geq \frac{k^2}{n^2} \sum_{(a,b)\in C_l} H(\xi^{C_k}|T_1^{ak+p_0} T_2^{bk+q_0} A_n)$$

for some $(p_0, q_0) \in C_k$ and $l = [\frac{n}{k}]$. Let $C' = [0, n-p_0-1] \times [0, n-q_0-1]$. By subadditivity of entropy the right–hand expression in the previous inequality is greater or equal than $\frac{k^2}{n^2} H(\xi^{C'}|A_n)$. To finish the proof of (3.1) it is enough to notice that

(3.3)
$$H(\xi^{C_n}|A_n) \leq H(\xi^{C'}|A_n) + (\log N)(\text{card}(C_n \backslash C') \leq H(\xi^{C'}|A_n) + 2kn \log N.$$

The proof of (3.2) uses essentially the same argument. We begin with a cover of a set B of measure $\geq 1-\delta$ by ϵ–balls in the metric $d^H_{C_k}$. Using Chebychev inequality one can find $(p_0, q_0) \in C_k$ such that among the sets $T_1^{ak+p_0} T_2^{bk+q_0} A_n$, $(a,b) \in C_l$ the proportion of those which intersect B by a set of conditional measure $\geq 1 - \delta^{\frac{1}{2}}$ is at least $1 - \delta^{\frac{1}{2}}$. After that it remains to compare C_n-names and C'-names using (3.3). □

For a single measure-preserving transformation $T: (X,\mu) \to (X,\mu)$ we denote the iterated partition $\vee_{i=0}^{n-1} T^{-i}\xi$ by ξ^n.

LEMMA 3. – *Let T be a measure–preserving transformation of a Lebesgue space.Given a partition ξ as before, a positive integer k and $\epsilon > 0$ there exists an integer n_o and $\delta > 0$ such that if $n \geq n_0$ is an even number and A_n is a measurable set disjoint with its first $n-1$ images and such that*

$$\mu\left(\bigcup_{i=0}^{n-1} T^i A_n\right) \geq 1 - \delta, \tag{3.4}$$

then

$$|H\left(\xi^k | X \setminus \bigcup_{i=0}^{\frac{n}{2}} T^i A_n\right) - H(\xi^k)| < \epsilon.$$

Proof. – Let r be the smallest positive measure of an element of the partition ξ^k. Pick $\delta_1 > 0$ so small that for any probability measure ν such that $\sum_{c\in\xi^k} |\mu(c) - \nu(c)| < \delta_1$ one has $|H_\mu(\xi^k) - H_\nu(\xi^k)| < \epsilon$. Let E be the projection to the the algebra of T-invariant functions. We do not assume that T is ergodic; in the latter case E corresponds to taking average. Let us pick n_0 so big that for any $n > n_0$ and for every element $c \in \xi^k$

$$\left\| \frac{2}{n} \sum_{i=0}^{\frac{n}{2}-1} \chi_c \circ T^{-i} - E(\chi_c) \right\| < (\delta_1 r)^3.$$

Therefore using Chebychev inequality one can find for each $c \in \xi^k$ from a set of conditional measure $\geq 1-\delta_1 r$ a natural number s, $\frac{n}{2} \leq s \leq \frac{n}{2}(1+\delta_1 r)$ (depending on c) such that on the set $T^s A_n$ the following inequality holds

$$|\frac{2}{n-2s} \sum_{i=0}^{\frac{n}{2}-s-1} (\chi_c T^{-i} - E(\chi_c))| < \delta_1 r. \tag{3.5}$$

Pick a $\delta > 0$ so small that for every $c \in \xi^k$, $|\int_{A_n} E\chi_c d\mu - \mu(c)| < \delta_1 r$. We have

$$\begin{aligned} \int_{\bigcup_{i=s}^{frac\frac{n}{2}} T^i A_n} \chi_c d\mu &= \int_{T^s A_n} \left(\sum_{i=0}^{\frac{n}{2}-s} \chi_c \circ T^{-i} \right) d\mu \\ &= 2\frac{n-2s}{2} \int_{T^s A_n} \frac{\sum_{i=0}^{\frac{n}{2}-s} \chi_c \circ T^{-i}}{n-2s} d\mu. \end{aligned}$$

By (3.5) the last integral is within $2\delta_1 r$ from $\frac{\mu(c)}{2}$, and hence (3.4) implies the assertion of the Lemma. □

LEMMA 4. – *Given a measure-preserving transformation $T : (X, \mu) \to (X, \mu)$ and a finite partition ξ assume that for every positive integer n_0 and $\epsilon > 0$ there exist $n > n_0$ and a set A_n such that*

(1) *The sets $T^i A_n$, $0 \leq i \leq n-1$ are disjoint;*

(2) $\mu(\bigcup_{i=0}^{n-1} T^i A_n) \geq 1 - \epsilon$;

(3) $H(\xi^n | A_n) - H(\xi^{[\frac{n}{2}]} | A_n) < n\epsilon$.

Then $H(T, \xi) = 0$.

Proof. – First we show that if n is large enough and $\epsilon > 0$ is small enough $\frac{1}{n} H(\xi^n | A_n)$ is close to $H(T, \xi)$. To see this we use the argument of Lemma 2 for the case of a single transformation instead of a $\mathbb{Z}^2$-action to obtain

$$\frac{1}{k} H(\xi^k) \geq \frac{1}{n} H(\xi^n | A_n) - \left(\frac{k}{n} + \epsilon k \right) \log N.$$

Consider the partition η which consists of intersections of the elements of ξ^n with A_n and the complement of A_n. The invariant σ-algebra generated by η approximates ξ up to ϵ. Hence $H(\eta) \geq H(T, \eta) \geq H(T, \xi) - \epsilon$, and the statement follows since $|H(\eta) - H(\xi^n | A_n)| \leq \frac{\log n}{n} + (1 - \frac{1}{n}) \log(1 - \frac{1}{n})$.

Now we show that under our assumptions $\frac{1}{n} H(\xi^{[\frac{n}{2}]} | A_n) \rightarrow \frac{H(T,\xi)}{2}$ as $n \rightarrow \infty$ and $\epsilon \rightarrow 0$. This together with the previous statement and assumption (3) imply the statement of the lemma.

Let ζ_n be the partition into the sets $T^i A_n$, $i = 0, \ldots, \frac{n}{2} - 1$ and $X \setminus \bigcup_{i=0}^{\frac{n}{2}-1} T^i A_n$. We have for a suitably chosen k_0, $0 \leq k_0 \leq k$,

$$H(\xi^k) \geq H(\xi^k | \zeta_n) \geq \frac{k}{n} \sum_{l=1}^{[\frac{n}{2k}]} H(\xi^k | T^{k_0 + lk} A_n) + \frac{1}{2} H\left(\xi^k | X \setminus \bigcup_{i=0}^{\frac{n}{2}-1} T^i A_n\right).$$

Lemma 3 implies that the last term converges to $\frac{1}{2} H(\xi^k)$. The first term is bigger that $\frac{k}{n} H(\xi^{\frac{n}{2}} | A_n) - \frac{C(k,\xi)}{n}$ so the statement follows. □

Now we are ready to describe the construction of our example of a $\mathbb{Z}^2$-action.

PROPOSITION 4. – *Let ϵ_n, $n = 1, 2, \ldots$ be a sequence of positive numbers decreasing to 0. There exists a measure-preserving ergodic $\mathbb{Z}^2$-action T on the unit interval with Lebesgue measure generated by transformations T_1, T_2 and a generating partition ξ such that*

(1) *for some $\epsilon > 0, \delta > 0$*

$$\limsup_{n \rightarrow \infty} \frac{\log(S_\xi^H(T, C_n, \epsilon, \delta)}{n^2 \epsilon_n} \geq 1;$$

(2) *the entropy $h_\mu(T) = 0$;*

(3) *every element of the suspension action of $\mathbb{R}^2$ has zero entropy.*

Proof. – First, notice that due to the subadditivity of entropy of iterated partitions condition (2) follows from the fact that generators of the action have zero entropy, which in turn follows from (3). The generating partition ξ will consist of two intervals of equal Lebesgue measure $1/2$. The construction will be determined by a sequence $k(n)$, $n = 0, 1, \ldots$ where either $k(n) = 0$ or $k(n) = 1 + k(n')$ where $n' = \{\sup p : \ p < n, \ k(p) \neq 0\}$.

At the step n there will be an integer $h(n)$ and a set A_n such that the sets $T_1^i T_2^j A_n$, $(i, j) \in C_{h(n)}$ are disjoint and cover the whole interval $[0, 1]$.

We will alternate two types of construction. When $k(n) = 0$ we set $h(n+1) = 2h(n)$ and the set A_{n+1} is partitioned into $C_{h(n+1)}$ names in such a way that four blocks of $C_{h(n)}$ names occur independently inside the $C_{h(n+1)}$ names. This is achieved by the standard cutting and stacking construction. At the base of the induction we put $h(0) = 2,\ \lambda(A_0) = 1/4$ and all 16 elements of the partition ξ^{C_2} are intervals of equal measure.

If $k(n) = L > 0$, we set $h(n+1) = Lh(n)$. In this case A_{n+1} is partitioned into $C_{h(n+1)}$-names in such a way that for every $C_{h(n+1)}$ name the $L^2 C_{h(n)}$ names, which are obtained by tiling it, are identical.

Thus we see that the names in $\xi^{C_n}|A_n$ are obtained from an independent distribution extended periodically in a fixed pattern determined by the sequence $k_(n)$. In order to guarantee the zero entropy condition (3) we need to assume that $k(n) \neq 0$ for infinitely many values of n. On the other hand, by taking long enough blocks of zeroes in $k(n)$ we can guarantee condition (1). In fact, even a stronger condition with $\liminf$ instead of $\limsup$ would hold. This can be seen as follows. Due to the independent nature of cutting and stacking the numbers $S^H_{\xi,A_n}(T, C_{k(n)}, \epsilon, \delta)$ can be calculated from the number of different $C_{k(n)}$ names in the set A_n and can be made to grow faster than any given speed which is subexponential in n^2. Then lemma 2 implies (1).

Now we will prove that if $k(n) \neq 0$ for infinitely many values of n condition (3) is satisfied.The phase space Y of the suspension action can be identified mod 0 with the product of the phase space of the $\mathbb{Z}^2$ action T (the unit interval in our case) and the unit square. Fix a positive integer m and consider the Cartesian product of the partition ξ^{C_m} with the standard partition of the unit square into squares of size $1/m$. Denote this partition η_m. Denote the suspension action $S:\ Y \times \mathbb{R}^2 \to Y$. Since the entropies of the elements of a one-parameter subgroup of a $\mathbb{R}^2$ action are determined by the entropy of a generator it is sufficient to consider the entropies of the elements $S(\cos\theta, \sin\theta)$. Without loss of generality we may assume $0 \leq \theta < \frac{\pi}{2}$. There are infinitely many rationals pq such that $|\tan\theta - pq| < \frac{1}{q^2}$. Fix $\epsilon > 0$ and a positive integer m and pick pq such that $|p - q\tan\theta| < \epsilon$. Since the numbers $k(n)$ are unbounded we can find an n such that $\frac{pq}{k(n)} < \epsilon^2,\ \frac{m}{k(n)} < \epsilon$. Now define positive integers L and M as follows: $L = [k(n)h(n) - 2qh(n)\tan\theta],\ \ M = sk(n)$ where s is found from $k(n) = 2sq + r,\ \ 0 \leq r < 2q$. Consider the following set $G \in Y$:

$$\bigcup_{t=0}^{1} S_{t\cos\theta, t\sin\theta}((\bigcup_{j=0}^{M} T_2^{2qj}(\bigcup_{i=0}^{L} T_1^i A_{n+1})) \times (0,0))$$

It is easy to check that the conditions of Lemma 4 are satisfied with $T = S(\cos\theta, \sin\theta),\ n = [\frac{2h(n)q}{\cos\theta}],\ \xi = \eta_m,\ A_n = G$. Hence $h(S(\cos\theta, \sin\theta), \eta_m) = 0$ for all m and since the $\eta_m,\ m = 1, 2, \ldots$ is obviously a generating sequence of partitions $h(S(\cos\theta, \sin\theta)) = 0$ $\square$

Propositions 1, 3, and 4 immediately imply

COROLLARY 2. – *If* $\lim_{n\to\infty} n\epsilon_n = \infty$, *then the action constructed in Proposition 4 is not isomorphic to any action by Lipschitz homeomorphisms on a compact metric space of finite box dimension preserving a Borel probability measure. In particular, it is not isomorphic to an action by diffeomorphisms of a compact differentiable manifold.*

REFERENCES

[1] A. KATOK, Monotone equivalence in ergodic theory, *Math. USSR, Izvestija*, Vol. **11**, 1977, pp. 99-146.

[2] A. KATOK, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math IHES*, Vol **51**, 1980, pp. 137-173.

[3] A. KATOK, Constructions in ergodic theory, preprint.

[4] A. KAKOT and B. HASSELBLATT, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, New York, 1995.

[5] S. KATOK, An estimate from above for the topological entropy of a diffeomorphism, Global theory of Dynamical systems, *Lecture Note Math.*, Vol. **819**, Springer Verlag, 1980, pp. 258-266.

[6] A. G. KUSHNIRENKO, An upper bound of the entropy of classical dynamical systems, *Sov. Math. Dokl.*, Vol. **6**, 1965, pp. 360-362.

[7] D. LIND and J.-P. THOUVENOT, Measure-preserving homeomorphisms of the torus represent all finite entropy ergodic transformations, *Math. Systems Theory*, Vol. **11**, 1977, pp. 275-285.

[8] D. ORNSTEIN and B. WEISS, Isomorphism theorem for amenable group actions, *J. Anal. Math.*, Vol. **48**, 1987, p. 1-141.

[9] Ya. B. PESIN, Characteristic Lyapunov exponents and smooth ergodic theory, *Russian Math. Surveys*, Vol. **32**, 1977, pp. 55-114.

[10] D. RUELLE, An inequality for the entropy of differentiable maps, *Bol. Soc. Brasil. Math.*, Vol. **9**, 1978, pp. 83-87.

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