ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 32, nº 5 (1996), p. 661-667
<http://www.numdam.org/item?id=AIHPB_1996__32_5_661_0>

Ann. Inst. Henri Poincaré,

Vol. 32, n° 5, 1996, p. 661-667 *Probabilités et Statistiques*

Convolution powers of spread-out probabilities

by

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ABSTRACT. – Let G be a locally compact σ-compact group, and let μ be a spread-out probability, adapted and strictly aperiodic. We prove that for any continuous isometric representation $T(t)$ in a uniformly convex Banach space, $||U_\mu^{n+1} - U_\mu^n|| \to 0$ (where $U_\mu = \int T(t)d\mu$).

RÉSUMÉ. – Soit G un groupe localement compact dénombrable à l'infini, et soit μ une probabilité étalée, adaptée et strictement apériodique. Nous prouvons que pour toute représentation continue $T(t)$ par isométries d'un espace de Banach uniformément convexe, $||U_\mu^{n+1} - U_\mu^n|| \to 0$ (où $U_\mu = \int T(t)d\mu(t)$).

1. INTRODUCTION

Let G be a locally compact σ-compact group with right Haar measure λ. For a regular probability μ on G, the convolution operator $\mu * f(t) =$

[1] Research partially supported by the Israel Ministry of Science.

[2] Heisenberg fellow of the Deutsche Forschungsgemeinschaft.

$\int f(ts)d\mu(s)$ is a Markov opeartor with σ-finite invariant measure, which is the μ-average of the translation operators $\delta_s * f(t) = f(ts)$.

Let S be the support of the probability μ. We say that μ is *adapted* if the closed subgroup generated by S is G, and *strictly aperiodic* if the smallest closed normal subgroup, a class of which contains S, is G.

An important property for the study of the asymptotic behaviour of $\{\mu^n\}$ is the *ergodicity* of μ, *i.e.*, that $||\frac{1}{n}\sum_{k=1}^{n}\mu^k * f||_1 \to 0$ for every $f \in L_1(G, \lambda)$ with $\int f d\lambda = 0$. Ergodic probabilities are necessarily adapted [A].

In applications, we often have μ *spread-out* (*i.e.*, for some $n > 0$, μ^n is not singular with respect to λ). Glasner [G] proved that if μ is an ergodic and strictly aperiodic spread-out probability on G, then $||\mu^{n+1} - \mu^n|| \to 0$. (If G is compact we even have $||\mu^n - \lambda|| \to 0$ [M]; *see* [RX] for more results.) If $||\mu^{n+1} - \mu^n|| \to 0$, then for every bounded continuous representation $T(t)$ in a Banach space, $||U_\mu^{n+1} - U_\mu^n|| \to 0$, where $U_\mu x = \int T(t)xd\mu(t)$ is the μ-average of the representation.

Glasner also gave an example for μ adapted, strictly aperiodic and spread-out, with $||\mu^{n+k} - \mu^n|| = 2$ for any $n, k > 0$. Following Jaworski [J], let $\eta = \frac{1}{2}(\mu + \mu^3)$ with μ of Glasner's example. Clearly also η is adapted and strictly aperiodic, and $||\eta^{n+2} - \eta^n|| \to 0$ by [F]. However, $||\eta^{n+1} - \eta^n|| = 2$ for every n, since all the powers of μ are mutually singular. (*See* [LW] for related results.) Nevertheless, it was shown in [DL] that if μ is adapted, strictly aperiodic and spread-out, then for any continuous representation by isometries in a uniformly convex Banach space, the iterates of the μ-average U_μ converge strongly (necessarily to a projection on the common fixed points). In this paper we improve this result, by showing that in fact $||U_\mu^{n+1} - U_\mu^n|| \to 0$.

2. OPERATOR-NORM CONVERGENCE IN UNIFORMLY CONVEX SPACES

PROPOSITION 2.1. – *Let μ be a spread-out probability on a locally compact σ-compact group. Then for every $\varepsilon > 0$ there exist an integer N and neighbourhood A of e, such that for $n \geq N$ and $t^{-1}s \in A$ we have $||\delta_t * \mu^n - \delta_s * \mu^n|| < \varepsilon$, and $||T(t)U_\mu^n - T(s)U_\mu^n|| < \varepsilon$ for any contractive continuous representation.*

Proof. – Let $\mu^n = \nu_n + \eta_n$ be the Lebsegue decomposition of μ^n. Since μ is spread-out, $\nu_{n_0} \neq 0$ for some n_0, so $||\eta_{n_0}|| < 1$. Hence $||\eta_{jn_0}|| \leq ||\eta_{n_0}^j|| \leq ||\eta_{n_0}{}^j|| \to 0$.

Fix $\varepsilon > 0$. There exists N with $||\mu^N - \nu_N|| < \varepsilon/3$. Since $\nu_N << \lambda$, by continuity of the translations in $L_1(G, \lambda)$ there exists a neighbourhood A of e such that $||\delta_t * \nu_N - \nu_N|| < \varepsilon/3$ for $t \in A$.

For $n \geq N$ and $t^{-1}s \in A$ we now have

$$\begin{aligned} ||\delta_t * \mu^n - \delta_s * \mu^n|| &\leq ||(\delta_t - \delta_s) * (\mu^N - \nu_N) * \mu^{n-N}|| \\ &\quad + ||(\delta_t - \delta_s) * \nu_N * \mu^{n-N}|| \\ &\leq 2||\mu^N - \nu_N|| + ||\nu_N - \delta_{t^{-1}s} * \nu_N|| < \varepsilon. \end{aligned}$$

For a contractive representation,

$$||T(t)U_\mu^n - T(s)U_\mu^n|| \leq ||\delta_t * \mu^n - \delta_s * \mu^n|| < \varepsilon.$$

THEOREM 2.2. – *Let μ be a spread-out adapted and strictly aperiodic probability on a locally compact σ-compact group G. Then for every continuous representation of G by isometries in a uniformly convex Banach space, we have $||U_\mu^{n+1} - U_\mu^n|| \to 0$.*

Proof. – We may assume $T(e) = I$, so all $T(t)$ are invertible. We denote U_μ by U. Since the theorem is obvious if $U^n = 0$ for some n, we assume that $U^n \neq 0$ for every n.

Let α_m be a sequence of natural numbers increasing to ∞, with $\frac{m}{\alpha_m} \uparrow \infty$ (e.g., $\alpha_m = [\sqrt{m}]$). Let $0 \leq \gamma_m < 1$ with $\gamma_m \uparrow 1$ slowly enough to have $\gamma_m^{m/\alpha_m} \to 0$ (e.g., $\gamma_m = 1 - m^{-\frac{1}{4}}$ for $\alpha_m = [\sqrt{m}]$).

Fix m with $m > 3\alpha_m$, and define $X_m = \{x \in X : \ U^{m-2\alpha_m}x \neq 0\}$. For $x \in X_m$ we have $U^j x \neq 0$ for $j \leq m - 2\alpha_m$, so we can define

$$D(m, x) = \max\left\{\frac{||U^{j+2\alpha_m}x||}{||U^j x||} : \alpha_m \leq j \leq m - 2\alpha_m\right\}.$$

Clearly $D(m, x) \leq 1$. For $x \neq 0$ we define $i(m, x)$ as follows:

(i) If $x \in X_m$ and $D(m, x) \leq \gamma_m$, then $i(m, x) = m - \alpha_m$.

(ii) If $x \in X_m$ and $D(m, x) > \gamma_m$, let

$$i(m, x) = \min\left\{j : \alpha_m \leq j \leq m - 2\alpha_m, \ \frac{||U^{j+2\alpha_m}x||}{||U^j x||} = D(m, x)\right\}$$

(iii) $i(m, x) = m - \alpha_m$ for $x \notin X_m$.

Let $A_m = \{x \in X_m : ||x|| \leq 1, \ D(m, x) \leq \gamma_m\}$. For $x \in A_m$, we have $m - 3\alpha_m + 1$ inequalities

$$||U^{j+2\alpha_m}x|| \leq \gamma_m ||U^j x|| \quad (\alpha_m \leq j \leq m - 2\alpha_m).$$

Starting with $j = m - 2\alpha_m$ and iterating back (with jumps of $2\alpha_m$) we use $\left[\frac{m}{2\alpha_m}\right]-1$ inequalities to obtain

$$(1) \qquad ||U^m x|| \leq \gamma_m^{\left[\frac{m}{2\alpha_m}\right]-1} ||x|| \text{ for } x \in A_m.$$

Let $B_m = \{x \in X_m : ||x|| \leq 1,\ D(m,x) > \gamma_m\}$.

CLAIM. – Let $t \in S^k$, where $S = \operatorname{supp}\mu$. For $m > 3\alpha_m$ let

$$\delta_k(t,m) = \sup\left\{ ||T(t)U^{i(m,x)+j}x - U^{i(m,x)+j+k}x|| : \frac{1}{2}\alpha_m \leq j \leq \alpha_m,\ x \in B_m \right\}.$$

Then $\lim_{m\to\infty} \delta_k(t,m) = 0$.

Proof. – Fix $\rho > 0$. By uniform convexity, there exists $1 > \varepsilon > 0$, such that $||y|| \leq 1,\ ||z|| \leq 1,\ ||y+z|| \geq 2(1-\varepsilon)$ imply $||y-z|| < \rho$.

By Proposition 2.1, there exist N, and a neighbourhood A of e, such that $s^{-1}s' \in A \Rightarrow ||T(s)U^n - T(s')U^n|| < \varepsilon$ for $n \geq N$. Define $V = tA$. Since $t \in S^k$, $\mu^k(V) > 0$.

There exists m_0 such that for $m \geq m_0$, we have (i) $\beta_m < \frac{1}{2}\varepsilon\mu^k(V)$ where $\beta_m = 1 - \gamma_m$. (ii) $\frac{1}{2}\alpha_m \geq N$. (iii) $\alpha_m \geq k$. (iv) $m > 3\alpha_m$.

Fix $m \geq m_0$. Let $x \in B_m$. Denote $i(m,x)$ by i, since x and m are now fixed. Then $\alpha_m \leq i \leq m - 2\alpha_m$ by definition, and satisfies $||U^{i+2\alpha_m}x|| > \gamma_m ||U^i x||$. Since $k \leq \alpha_m$, for $j \leq \alpha_m$ we have

$$||U^{i+j+k}x|| \geq ||U^{i+2\alpha_m}x|| > \gamma_m ||U^i x||.$$

Hence, for $j \leq \alpha_m$,

$$2\gamma_m ||U^i x|| < 2||U^{i+j+k}x|| \leq \int ||T(s)U^{i+j}x + U^{i+j+k}x|| d\mu^k(s).$$

The integrand (and hence the integral) is bounded above by $2||U^i x||$. We show that for some $s_j \in V$ ($j \leq \alpha_m$) we have

$$||T(s_j)U^{i+j}x + U^{i+j+k}x|| > 2||U^i x|| \left(1 - \frac{\beta_m}{\mu^k(V)}\right).$$

Indeed, if not, we obtain, by integrating over V and over V^c,

$$\begin{aligned} 2\gamma_m ||U^i x|| &< \mu^k(V) 2||U^i x|| \left(1 - \frac{\beta_m}{\mu^k(V)}\right) + \mu^k(V^c) 2||U^i x|| \\ &= 2||U^i x||(1 - \beta_m) \end{aligned}$$

and the strict inequality yields a contradiction.

Hence, for fixed j with $\frac{1}{2}\alpha_m \leq j \leq \alpha_m$, we have

$$\begin{aligned}&|\ ||T(t)U^{i+j}x + U^{i+j+k}x|| - ||T(s_j)U^{i+j}x + U^{i+j+k}x||\ | \\ &\quad \leq ||T(t)U^{i+j}x - T(s_j)U^{i+j}x|| \\ &\quad \leq ||T(t)U^j - T(s_j)U^j||\ ||U^i x|| < \varepsilon ||U^i x||\end{aligned}$$

since $s_j \in tA$, and $j \geq \frac{1}{2}\alpha_m \geq N$. Hence

$$\begin{aligned}||T(t)U^{i+j}x + U^{i+j+k}x|| &\geq ||T(s_j)U^{i+j}x + U^{i+j+k}x|| - \varepsilon ||U^i x|| \\ &\geq 2(1-\varepsilon)||U^i x||\end{aligned}$$

since $\beta_m < \frac{1}{2}\varepsilon\mu^k(V)$.

By the uniform convexity choice of ε,

$$||T(t)U^{i+j}x - U^{i+j+k}x|| < \rho ||U^i x|| \leq \rho ||x|| \leq \rho.$$

This yields $\delta_k(t,m) \leq \rho$ for $m \geq m_0$, which proves the claim.

Proof of the Theorem. – Fix $t \in S^k$. Let $\beta_k(t,m) = \max\{\gamma_m^{[\frac{m}{2\alpha_m}]-1}, \delta_k(t,m)\}$ so $\beta_k(t,m) \underset{m\to\infty}{\longrightarrow} 0$ by the claim.

Let $t, s \in S^k$, and fix m with $\frac{m}{3} > \alpha_m \geq 2k$. Then

$$\begin{aligned}(2)\quad &\sup\left\{ ||T(t)U^{i(m,x)+j}x - T(s)U^{i(m,x)+j}x|| : \frac{1}{2}\alpha_m \leq j \leq \alpha_m,\ x \in B_m\right\} \\ &\quad \leq \delta_k(t,m) + \delta_k(s,m) \leq \beta_k(t,m) + \beta_k(s,m)\end{aligned}$$

$$\begin{aligned}(3)\quad &\sup\Big\{ ||T(t^{-1})U^{i(m,x)+j+k}x - T(s^{-1})U^{i(m,x)+j+k}x|| : \\ &\quad \frac{1}{2}\alpha_m \leq j \leq \alpha_m,\ x \in B_m\Big\} \leq \beta_k(t,m) + \beta_k(s,m).\end{aligned}$$

Taking $j = \alpha_m$ in (2), and $j = \alpha_m - k$ in (3) (since $\alpha_m - k \geq \frac{1}{2}\alpha_m$), we obtain for any $x \in B_m$

$$(4)\quad ||T(t^{-1}s)U^{i(m,x)+\alpha_m}x - U^{i(m,x)+\alpha_m}x|| \leq \beta_k(t,m) + \beta_k(s,m)$$

$$(5)\quad ||T(ts^{-1})U^{i(m,x)+\alpha_m}x - U^{i(m,x)+\alpha_m}x|| \leq \beta_k(t,m) + \beta_k(s,m).$$

Since $i(m,x) + \alpha_m = m$ for $x \in A_m$, we obtain from (1) that (4) and (5) hold for $x \in X_m$ with $||x|| \leq 1$. Since $U^m x = 0$ for $x \notin X_m$ and

$i(m,x) + \alpha_m = m$, we conclude that (4) and (5) hold for *every* $x \in X$ with $||x|| \leq 1$.

From $\lim_{m\to\infty} \beta_k(t,m) = 0$ it now follows that $\bigcup_{k=1}^{\infty}(S^{-k}S^k \cup S^kS^{-k})$ is contained in

$$G' = \{t \in G : \lim_{m\to\infty} [\sup_{||x||\leq 1} ||T(t)U^{i(m,x)+\alpha_m}x - U^{i(m,x)+\alpha_m}x||] = 0\}.$$

We show that G' is a closed subgroup. It is trivially closed under inversion. If $s, t \in G'$, then $st \in G'$ since

$$\begin{aligned} ||T(st)U^jx - U^jx|| &\leq ||T(st)U^jx - T(s)U^jx|| + ||T(s)U^jx - U^jx|| \\ &= ||T(t)U^jx - U^jx|| + ||T(s)U^jx - U^jx|| \end{aligned}$$

holds for $j = j(m,x) = i(m,x) + \alpha_m$.

We show that G' is closed. Let $t_0 \in \overline{G'}$. By Proposition 2.1, for $\varepsilon > 0$ there exist N and a neighbourhood A of e, such that for $j \geq N$ and $t^{-1}s \in A$ we have $||T(t)U^j - T(s)U^j|| < \varepsilon$. Let $t' \in G'$ be in t_0A. Then, since $j(m,x) \geq \alpha_m$ and $t_0^{-1}t^1 \in A$, for sufficiently large m we have

$$\begin{aligned} &||T(t_0)U^{j(m,x)}x - T(t')U^{j(m,x)}x|| \\ &\leq ||T(t_0)U^{j(m,x)} - T(t')U^{j(m,x)}||\,||x|| < \varepsilon||x||. \end{aligned}$$

Hence

$$\begin{aligned} &\sup_{||x||\leq 1} ||T(t_0)U^{j(m,x)}x - U^{j(m,x)}x|| \\ &\leq \sup_{||x||\leq 1} ||T(t')U^{j(m,x)}x - U^{j(m,x)}x|| + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $t_0 \in G'$, so G' is a closed subgroup. By strict aperiodicity, $G' = G$.

Define $f_m(t) = \sup_{||x||\leq 1} ||T(t)U^{j(m,x)}x - U^{j(m,x)}x||$. Then $f_m(t) \to 0$ everywhere on G. Strong continuity of the representation yields that $f_m(t)$ is lower semi-continuous, so is Borel measurable. By Lebesgue's theorem, $\int f_m(t)d\mu(t) \to 0$.

Fix $\varepsilon > 0$, and let m_0 be such that $\int f_m(t)d\mu(t) < \varepsilon$ for $m > m_0$. For such m, we obtain for *every* $||x|| \leq 1$, (since $j(m,x) = i(m,x) + \alpha_m \leq m$ by construction), that

$$\begin{aligned} ||U^{m+1}x - U^mx|| &\leq ||U^{j(m,x)+1}x - U^{j(m,x)}x|| \\ &= \left|\left|\int [T(t)U^{j(m,x)}x - U^{j(m,x)}x]d\mu(t)\right|\right| \leq \int f_m(t) < \varepsilon. \end{aligned}$$

Hence $||U^{m+1} - U^m|| < \varepsilon$ for $m > m_0$. Hence $||U^m(U-I)|| \underset{m\to\infty}{\longrightarrow} 0$.

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(Manuscript accepted May 30, 1995.)