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Degeneration of effective diffusion in the presence of periodic potential

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ABSTRACT. – The asymptotic behaviour of effective diffusion for a parabolic operator in R^n with smooth periodic potential and small periodic initial diffusion is studied. We obtain logarithmic asymptotics of effective diffusion with respect to the initial diffusion. The answer is given in terms of auxiliary variational problem on the torus, which consists in minimizing a length in certain metric of the curves passing through the maximal point of the potential and having given homological class. The paper generalizes our previous result [3] where the case of piecewise constant potential was investigated.

Key words: Homogenization, large deviations, effective diffusion.

RÉSUMÉ. – On considère le comportement asymptotique de la diffusion effective pour une équation parabolique avec un potentiel périodique,

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lorsque le coefficient de diffusion initial tend vers zéro. On trouve cet asymptotique au sens logarithmique en termes de problèmes variationnels liés au potentiel donné.

INTRODUCTION

In the paper we study a diffusion model of biological particles assuming that the life conditions and diffusive properties of the medium are described by periodic functions. To investigate the behaviour of a population of such particles for large time one can apply the methods of automodel homogenization theory of parabolic equations. As it follows from [1], [2], the long time behaviour of the population could be described in terms of so-called effective diffusion which in our case depends on the initial diffusivity and the potential. The goal of this work is to obtain the asymptotics of the effective diffusion as the initial diffusion goes to zero. Here we adopt the terminology and the main notations from [3] where the similar problem was solved for special case when the potential is piecewise constant and the initial diffusion doesn't depend on the point of medium.

The corresponding parabolic equation for the density $u(x, t)$ of the particles has the following form

$$\left.\begin{aligned} \frac{\partial}{\partial t} u = \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u + v(x)\, u \\ u|_{t=0} = v_0(x), \end{aligned}\right\} \tag{0.1}$$

where $v_0(x)$ is the initial density and μ is small positive parameter characterizing the initial diffusion of the particle, summation over repeated indices is omitted. From now on we suppose that the matrix $(a_{ij}(x))$ is periodic symmetric and uniformly elliptic and the potential $v(x)$ is periodic and has only one global maximum point on each period.

To describe the behaviour of $u(x, t)$ for large time let's introduce the following eigenvalue problem

$$\left(\mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + v(x)\right) p = \lambda p \tag{0.2}$$

and denote the first eigenvalue and eigenfunction of this problem simply by λ and $p(x)$ respectively. We fix the choice of $p(x)$ by the normalisation condition $\langle p \rangle = 1$, where $\langle \cdot \rangle$ means the average of a periodic function over the period. Then according to [2] for any positive μ the asymptotic relation

$$u(x,t) = \exp(\lambda t)\, p(x)\, \hat{u}(x,t)\,(1 + o(1)) \qquad (0.3)$$

holds in the region $\{(x,t) | x^2 < c_1 t + c_2\}$ as $t \to \infty$. The function $\hat{u}(x,t)$ describing the diffusion properties of the solution $u(x,t)$ satisfies the homogenized parabolic equation

$$\frac{\partial}{\partial t}\hat{u} = \frac{\partial}{\partial x_i}\sigma_{ij}\frac{\partial}{\partial x_j}\hat{u}, \qquad \hat{u}|_{t=0} = \langle p^2 \rangle v_0(x) \qquad (0.4)$$

whose matrix of constant coefficients $(\sigma_{ij}) = (\sigma_{ij}(\mu))$ is called the effective diffusion matrix. For reader convenience we outline here the method for constructing of σ_{ij}. Namely let ψ_k, $k = 1, 2, \ldots, n$, be periodic solutions of the equations

$$\frac{\partial}{\partial x_i}\left(p^2(x)\, a_{ij}(x)\frac{\partial}{\partial x_j}(\psi_k + x_k)\right) = 0 \qquad (0.5)$$

where x_k is k-th independent variable. Then the effective diffusion matrix (σ_{ij}) is defined by the formula

$$\sigma(\mu) = \langle (I + \nabla\psi)^T p^2 a (I + \nabla\psi) \rangle / \langle p^2 \rangle, \qquad (0.6)$$

where $\nabla\psi = \left(\frac{\partial}{\partial x_i}\psi_j\right)$, I is the unit matrix and the symbol ^{T} means transposition of the matrix, $a = (a_{ij})$.

The expression in the right hand side of (0.6) depends on μ inexplicitly thus the studying of this expression for small μ is a complicated problem involving resolution of the singularly perturbed PDE. So we use another approach to find the asymptotics of $\sigma(\mu)$ as $\mu \to 0$. Namely we'll transform equation (0.1) to an equation without potential and then applying the rough estimates for effective diffusion replace this equation by another one of divergence form or equivalently with potential drift and isotropic diffusion. To that end we must control the ground state $p(x)$ all over the torus as μ goes to zero, we find it to be of order $\exp(-W(x)/\mu)$ in logarithmic sense where $W(x)$ is a distance from the point x to the maximum point in the certain metric on the torus. Here our analysis relies on Freidlin-Wentzel results [12] where analogues problem for the invariant measure for diffusion operator was considered. After that using the result of [4] we'll show (and this is the main result of the paper) that effective diffusion $\sigma(\mu)$ satisfies the following limiting relation

$$\lim_{\mu \to 0} \mu \ln \sigma(\mu) = -\Theta,$$

where Θ is a positive matrix, whose coefficients will be given in terms of auxiliary variational problems. Here we point out that each its eigenvalue is the minimal length of the closed curve passing through the maximum point of the potential and having fixed homological class, in the metric constructed in terms of the potential and quadratic form of the principal part of the operator. These curves also define a sequence of invariant subspaces of Θ. Detailed construction of limiting matrix is presented in Section 3.

Similar problem for a diffusive particle in the presence of a vector field was largely discussed in the physical and mathematical literature (*see* [5], [6] for incompressible vector field, [7] for general discussion and [8] where different from [4] approach to the case of potential vector field is presented). We also mention paper [10] where more realistic case of Fokker-Plank equation was treated.

1. THE ASYMPTOTIC PROPERTIES OF PERIODIC EIGENVALUE PROBLEM

In this paragraph we study the asymptotic behaviour of the first eigenvalue and eigenfunction of (0.2) for small μ. Without loss of generality we'll suppose that $\max_{R^n} v(x) = 0$ and the set of its maximum points $\{x \in R^n | v(x) = 0\}$ coincides with integer lattice Z^n.

PROPOSITION 1. – *The first eigenvalues λ satisfies the estimates*

$$0 \le -\lambda \le c\,\mu \tag{1.1}$$

where the constant c doesn't depend on μ.

Proof. – The first inequality (1.1) is simple consequence of the following variational representation for λ

$$-\lambda = \mu^2 \inf_{\|u\|_{L^2(T^n)}=1} \left\{ \int_{T^n} a_{ij}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u \, dx + \frac{1}{\mu^2} \int_{T^n} |v(x)| u^2 \, dx \right\} \tag{1.2}$$

where $T^n = R^n / Z^n$ is standard n-dimensional torus. To prove the inequality from above let us introduce the test function

$$u(x) = c\,\mu^{-n/4} \exp(-|x|^2/\mu)\, \chi(|x|)$$

where the normalizing constant c is used to satisfy the relation $\|u\|_{L^2(T^n)} = 1$ and $\chi(|x|)$ is smooth positive cutoff, $\chi = 1$ for $|x| < \frac{1}{8}$ and $\chi = 0$ for $|x| > \frac{1}{4}$. Under such choice of χ the constant c is uniformly in μ positive and bounded. We have, substituting this test function into (1.2)

$$\begin{aligned}
-\lambda/\mu^2 &\le \int_{T^n} a_{ij}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u\, dx + \frac{1}{\mu^2} \int_{T^n} |v(x)| u^2\, dx \\
&\le c\mu^{-\frac{n}{2}} \left(\frac{4}{\mu^2} \int_{T^n} a_{ij}(x)\, x_i\, x_j\, e^{-|x|^2/\mu} e^{-|x|^2/\mu} \chi^2(|x|)\, dx \right. \\
&\quad + \frac{4}{\mu} \int_{T^n} a_{ij}(x)\, x_i\, e^{-2|x|^2/\mu} \frac{\partial}{\partial x_j} \chi^2(|x|)\, dx \\
&\quad + \int_{T^n} a_{ij}(x)\, e^{-2|x|^2/\mu} \frac{\partial}{\partial x_i} \chi(|x|) \frac{\partial}{\partial x_j} \chi(|x|)\, dx \\
&\quad \left. + \frac{1}{\mu^2} \int_{T^n} |v(x)| e^{-2x^2/\mu} \chi^2(|x|)\, dx \right).
\end{aligned}$$

The second and third integrals here are bounded because corresponding derivatives of $\chi(|x|)$ are equal to zero in the domain $\left\{ x : |x| < \frac{1}{8} \right\}$. Then taking into account our assumptions we obtain the inequality $|v(x)| \le c_1 |x|^2$ for some positive constant c_1. This yields

$$\begin{aligned}
-\lambda/\mu^2 &\le c + c^2 \mu^{-\frac{n}{2}} \left(\frac{4}{\mu^2} \int_{T^n} a_{ij}(x)\, x_i\, x_j\, e^{-2x^2/\mu} \chi^2(|x|)\, dx \right. \\
&\qquad \left. + \frac{c_1}{\mu^2} \int_{T^n} x^2\, e^{-2x^2/\mu} \chi^2(|x|)\, dx \right) \\
&\le c + \frac{4c^2}{\mu} \int_{R^n} a_{ij}(\sqrt{\mu}\, y)\, e^{-2y^2} y_i\, y_j\, dy \\
&\quad + \frac{c_1 c^2}{\mu} \int_{R^n} y^2\, e^{-2y^2}\, dx \le c + \frac{c_2}{\mu}.
\end{aligned}$$

Now let's establish the main properties of the eigenfunction $u_0(x)$ which solves (1.2).

Proposition 2. – *For any* $\mu > 0$

$$1 \le \max_{T^n} u_0(x) \le c\mu^{-n/2} \tag{1.3}$$

Proof. – The left inequality immediately follows from the relation $\|u_0\|_{L^2(T^n)} = 1$. The right one can be derived from standard estimates for elliptic equations. Indeed, in rescaled coordinates $y = \dfrac{x}{\mu}$ the equation (0.2) takes the sight

$$\frac{\partial}{\partial y_i} a_{ij}(\mu y) \frac{\partial}{\partial y_j} u_0 + v(\mu y) u_0 = \lambda u_0 \tag{1.4}$$

while L^2-norm of $u_0(\mu y)$ over the period is equal to $\mu^{-n/2}$. According to Proposition 1 the coefficients of (1.4) are uniformly in μ bounded so by the internal estimates for the solutions of elliptic equations [11] we obtain the second inequality (1.3).

Denote by x_μ^0 the maximum point of $u_0(x)$.

PROPOSITION 3. – *The maximum point x_μ^0 goes to zero as $\mu \to 0$.*

Proof. – According to maximum principle [11] x_μ^0 is located inside the domain $\{x \in T^n : v(x) - \lambda > 0\}$. By Proposition 1 we find $-v(x_\mu^0) < -\lambda < c\mu$ and required statement is the consequence of our assumption about uniquiness of maximum point of $v(x)$ on T^n.

Now let δ be an arbitrary positive number. We set $Q_\delta = \{x : |x| < \delta\}$. Our next aim is to estimate $u_0(x)$ at the boundary ∂Q_δ from below.

PROPOSITION 4. – *There exist positive c_0 and c_1 such that*

$$u_0(x)|_{\partial Q_\delta} \geq c_1 e^{-c_0 \delta/\mu}.$$

Proof. – Applying Harnack inequality [11] to the solution $u_0(\mu y)$ of (1.4) one can obtain that

$$c^{-1} \leq u_0(\mu y_1)/u_0(\mu y_2) \leq c \tag{1.5}$$

for any y_1, y_2 such that $|y_1 - y_2| \leq 1$, where the constant $c \geq 1$ doesn't depend on μ. In coordinates $x = \mu y$ we find

$$c^{-1} \leq u_0(x_1)/u_0(x_2) \leq c \tag{1.6}$$

for any x_1, x_2 such that $|x_1 - x_2| < \mu$. By Proposition 3 $x_\mu^0 \in Q_\delta$ for sufficiently small μ therefore the distance between x_μ^0 and any point x of ∂Q_δ is less than 2δ. Connecting x_μ^0 with an arbitrary point of ∂Q_δ by the sequence of $[2\delta/\mu]$ points ($[\cdot]$ is the integer part) lying on the same line and iterating (1.6) we find

$$u_0(x)/u_0(x_\mu^0) \geq c^{-[2\delta/\mu]} \geq c_1 e^{2\delta \ln c/\mu} = c_1 e^{-c_0 \delta/\mu}$$

for any $x \in \partial Q_\delta$. To complete the proof it suffices to note that $u_0(x_\mu^0) \geq 1$.

Remark 1. – One can easily establish the exact asymptotics of λ, which is $\lambda = c\,\mu\,(1+O\,(\mu))$ for the potential with nondegenerate maximum point, but the result of the Proposition 1 is sufficient for our purposes.

Now from Propositions 2, 4 we have

$$\int_{T^n} u_0\,(x)\,dx \geq \mu^n$$

so for

$$p\,(x) = u_0/\langle\, u_0\,(x)\,\rangle$$

we have the following

PROPOSITION 5. – *The function* $p\,(x)$ *satisfies the estimates:*

$$\max_{T^n} p\,(x) \leq c_2\,\mu^{\frac{-3n}{2}}, \qquad p\,(x)|_{\partial Q_\delta} \geq c_3\,e^{-c_4\,\delta/\mu}.$$

Now we are going to describe the logarithmic asymptotics of $p\,(x)$. For this purpose we define the function $\bar{W}\,(x)$ as the solution of the following variational problem

$$\bar{W}^2\,(x) = \inf_{\substack{x\,(t)\\ x\,(0)=x,\, x\,(1)=0}} \int_0^1 (-v\,(x\,(t)))\,a^{ij}\,(x\,(t))\,\dot{x}_i\,(t)\,\dot{x}_j\,(t)\,dt$$

where $(a^{ij}\,(x)) = (a_{ij}\,(x))^{-1}$ and inf is taken over all smooth paths connecting x with 0. We will also use the function $W\,(x) = \bar{W}\,(x)/2$ which is more convenient in probabilistic interpretation.

LEMMA 1. – *Uniformly in* $x \in T^n$

$$\lim_{\mu\to 0} \mu \ln p\,(x) = -2\,W\,(x) = -\bar{W}\,(x).$$

Proof. – Let's fix arbitrary $\delta > 0$ and devide the equation (0.1) by the function $(-\mu\,v\,(x))$ in the domain $T^n\backslash Q_\delta$:

$$-\mu\,\frac{a_{ij}\,(x)}{v\,(x)}\,\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_j}\,p\,(x) - \mu\,\frac{1}{v\,(x)}\left(\frac{\partial}{\partial x_i}\,a_{ij}\,(x)\right)\frac{\partial}{\partial x_j}\,p\,(x)$$
$$-\frac{1}{\mu}\left(\frac{-v\,(x)+\lambda}{v\,(x)}\right)p\,(x) = 0.$$

Denote by ξ_t^x the diffusion process issuing from the point x and corresponding to the operator

$$B_\mu = -\mu\,\frac{a_{ij}\,(x)}{v\,(x)}\,\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_j} - \mu\,\frac{1}{v\,(x)}\left(\frac{\partial}{\partial x_i}\,a_{ij}\,(x)\right)\frac{\partial}{\partial x_j}.$$

Let $\tau(x)$ be the exit time from $T^n \backslash Q_\delta$. Then the solution $p(x)$, $x \in T^n \backslash Q_\delta$ can be represented in the following probabilistic form (*see* [12]):

$$p(x) = M\left[\exp\left(-\frac{1}{\mu}\int_0^{\tau(x)} \frac{v(\xi_s^x) - \lambda}{v(\xi_s^x)}\, ds\right) p(\xi_{\tau(x)}^x)\right]$$
$$= M\left[\exp\left(-\frac{\tau(x)}{\mu} + \frac{\lambda}{\mu}\int_0^{\tau(x)} \frac{ds}{v(\xi_s^x)}\right) p(\xi_{\tau(x)}^x)\right]$$
$$= M\left[\exp\left(-\frac{\tau(x)}{\mu}(1 + O(\mu))\right) p(\xi_{\tau(x)}^x)\right].$$

The last equality here follows from Proposition 1 which states that λ/μ is bounded. We also use the fact $\xi_s^x \in T^n \backslash Q_\delta$ for $s < \tau(x)$ and uniqiness of the maximum point. The function $O(\mu)$ depends on δ. By Proposition 5

$$c_1\, e^{-c\delta/\mu} \le p(x)|_{\partial Q_\delta} \le c_2\, \mu^{-\frac{3n}{2}}$$

Therefore

$$c_1\, e^{-c\delta/\mu}\, M\, e^{-\frac{\tau(x)}{\mu}(1+O(\mu))}$$
$$\le p(x) \le c_2\, \mu^{-\frac{3n}{2}}\, M\, e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} \tag{1.7}$$

for $x \in T^n \backslash Q_\delta$. Now let δ_1 be positive sufficiently small number. We set $\delta_0 = \inf_{\partial Q_{\delta_1}} W(x)$ and suppose that δ satisfies the relation $\delta < \delta_0^2$. For $x \in T^n \backslash Q_\delta$ we define the function $W_\delta(x)$ as follows

$$4\, W_\delta^2(x) = \inf_{\{x(t)\,:\, x(0)=x,\, x(1)\in\partial Q_\delta\}}$$
$$\times \int_0^1 (-v(x(t)))\, a^{ij}(x(t))\, \dot{x}_i(t)\, \dot{x}_j(t)\, dt$$

According to the definitions of $W(x)$ and $W_\delta(x)$ and relation for δ and δ_1 we have for $x \in T^n \backslash Q_{\delta_1}$

$$0 \le W(x) - W_\delta(x) \le c\,\delta \tag{1.8}$$

$$W(x) \ge \delta_0 \tag{1.9}$$

First we are going to estimate $M\, e^{-\frac{\tau(x)}{\mu}(1+O(\mu))}$ from below. For this purpose let's construct the smooth path $\tilde{\varphi}(t)$, $\tilde{\varphi}(0) = x \in T^n \backslash Q_{\delta_1}$, $|\tilde{\varphi}(1)| = \delta/2$, such that

$$\frac{1}{4}\int_0^1 (-v(\tilde{\varphi}(t)))\, a^{ij}(\tilde{\varphi}(t))\, \dot{\tilde{\varphi}}_i(t)\, \dot{\tilde{\varphi}}_j(t)\, dt \le W^2(x).$$

Then the function $\varphi(t) = \tilde{\varphi}\left(\frac{t}{W(x)}\right)$ satisfies the following relations:

$$\frac{1}{4}\int_0^{W(x)} (-v(\varphi(t)))\, a^{ij}(\varphi(t))\, \dot{\varphi}_i(t)\, \dot{\varphi}_j(t)\, dt \leq W(x),$$

$$\varphi(0) = x, \qquad |\varphi(W(x))| = \delta/2.$$

According to [12, chapter 3] this implies

$$\mathbf{P}\{\rho_{0, W(x)}(\xi_t^x, \varphi(t)) < \delta/4\} \geq c \exp\left(-\frac{W(x)+\delta}{\mu}\right)$$

where $\rho_{0, W(x)}(\xi_t^x, \varphi(t)) = \sup_{0 \leq t \leq W(x)} |\xi_t^x - \varphi(t)|$. Hence taking into account the implication $\{\rho_{0, W(x)}(\xi_t^x, \varphi(t)) < \delta/4\} \subset \{\tau(x) \leq W(x)\}$ we have

$$\mathbf{P}\{\tau(x) \leq W(x)\} \geq c \exp\left(-\frac{W(x)+\delta}{\mu}\right)$$

At last by Chebysheff inequality

$$M e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} \geq \mathbf{P}\{\tau(x) \leq W(x)\}\, e^{-\frac{W(x)}{\mu}(1+O(\mu))}$$

$$\geq e^{-\frac{W(x)}{\mu}(1+O(\mu))}\, e^{-\frac{W(x)+\delta}{\mu}(1+O(\mu))}$$

$$\geqq c(\delta)\, e^{-2\frac{W(x)+\delta}{\mu}} \tag{1.10}$$

To prove a similar upper bound we rewrite the quantity $M e^{-\frac{\tau(x)}{\mu}(1+O(\mu))}$ in the following form

$$\begin{aligned}
M e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} &= M(\chi_{\{\tau(x)<3\max W\}}\, e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} \\
&\quad + \chi_{\{\tau(x)\geq 3\max W\}}\, e^{-\frac{\tau(x)}{\mu}(1+O(\mu))}) \\
&\leq e^{3c\max W} M e^{-\tau(x)/\mu} + ce^{-2\max W/\mu} \\
&= c_1 M e^{-\tau(x)/\mu} + ce^{-2\max W/\mu} \\
&= c_1 \int_0^\infty e^{-s/\mu}\, d\theta(s) + ce^{-2\max W/\mu} \\
&= \frac{c_1}{\mu}\int_0^\infty \theta(s)\, e^{-s/\mu}\, ds + ce^{-2\max W/\mu}
\end{aligned}$$

where $\theta(s)$ is the distribution function of $\tau(x)$. Let's devide the last integral into three parts:

$$M e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} \leq \frac{c_1}{\mu}\left(\int_0^{W(x)/2} \theta(s) e^{-s/\mu} ds + \int_{W(x)/2}^{2W(x)} \theta(s) e^{-s/\mu} ds + \int_{2W(x)}^{\infty} \theta(s) e^{-s/\mu} ds\right) + ce^{-2\max W/\mu} \quad (1.11)$$

The last term in brackets is evidently less then $\exp(-2W(x)/\mu)$. Then from [12, chapter 4] it follows that uniformly in $s \in [\delta_0, 2\max W]$ and $x \in T^n \backslash Q_{\delta_1}$

$$\theta(s) \leq c(\delta) e^{-\frac{W_\delta^2(x)-c_1\delta}{\mu s}} \leq c(\delta) e^{-\frac{W^2(x)-c\delta}{\mu s}}$$

where we also use (1.8). Thus for any $s \leq W(x)/2$

$$\theta(s) \leq \theta(W(x)/2) \leq c(\delta) e^{-\frac{2W(x)-c\delta}{\mu}}.$$

So the first integral in (1.11) is not greater then $c(\delta) e^{-2\frac{W(x)-c\delta}{\mu}}$. Let's estimate the second one

$$\begin{aligned}\int_{W(x)/2}^{2W(x)} \theta(s) e^{-s/\mu} ds &\leq \int_{W(x)/2}^{2W(x)} c(\delta) e^{-\frac{W^2(x)-c\delta}{\mu s}} e^{-s/\mu} ds \\ &= c(\delta) \int_{W(x)/2}^{2W(x)} e^{\frac{c\delta}{\mu s}} e^{-\frac{1}{\mu}\left(\frac{W^2(x)}{s}+s\right)} ds \\ &\leq c(\delta) e^{c\delta/\mu} \int_{W(x)/2}^{2W(x)} e^{-\frac{1}{\mu}\min_s\left(\frac{W^2(x)}{s}+s\right)} ds \\ &= c(\delta) e^{c\delta/\mu} \int_{W(x)/2}^{2W(x)} e^{-2W(x)/\mu} ds \\ &\leq c(\delta) e^{-2\frac{W(x)-c\delta}{\mu}}.\end{aligned}$$

Combining the estimates for all terms in (1.11) we find that

$$M e^{-\frac{\tau(x)}{\mu}(1+O(\mu))} \le c(\delta)\, e^{-2\frac{W(x)-c\delta}{\mu}}$$

uniformly in $x \in T^n \backslash Q_{\delta_1}$. By (1.9), (1.10) and last inequality we obtain for $x \in T^n \backslash Q_{\delta_1}$

$$\lim_{\mu\to 0} \mu \ln p(x) = -\bar{W}(x) \tag{1.12}$$

uniformly over this set. Then according to the definition of $W(x)$ and Propositions 5 the following relations

$$0 \le \bar{W}(x) \le c\,\delta_1,$$

$$c\,e^{-c\delta_1/\mu} \le p(x) \le c\,\mu^{-3n/2}$$

hold uniformly in $x \in Q_{\delta_1}$. Hence

$$\max_{Q_{\delta_1}} |\mu \ln p(x) - \bar{W}(x)| \le c\,\delta_1.$$

With (1.12) this implies

$$-\delta_1 - \bar{W}(x) \le \liminf_{\mu\to 0} \mu \ln p(x) \le \limsup_{\mu\to 0} \mu \ln p(x) \le \delta_1 - \bar{W}(x)$$

uniformly on T^n and since δ_1 is arbitrary number the lemma is proved.

2. REDUCTION TO THE EQUATION WITHOUT POTENTIAL

This section is devoted to a transformation of (0.1) to the equation without potential. For this aim we introduce new unknown function $q(x, t)$ by the relation $u(x, t) = e^{\lambda t} p(x)\, q(x, t)$. Substituting this expression into (0.1) instead of $u(x, t)$ and taking into account equation (0.2) we find the equation for the function $q(x, t)$:

$$p(x)\frac{\partial}{\partial t} q(x, t) = p(x)\,\mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} q(x, t) + 2\,\mu^2 a_{ij}(x)\left(\frac{\partial}{\partial x_i} p(x)\right)\frac{\partial}{\partial x_j} q(x, t)$$

Let's multiply the last equation by $p(x)$:

$$p^2(x)\frac{\partial}{\partial t} q(x, t) = p^2(x)\,\mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} q(x, t) + 2\,p(x)\,\mu^2 a_{ij}(x)\left(\frac{\partial}{\partial x_i} \ln p(x)\right)\frac{\partial}{\partial x_j} q(x, t)$$

After simple transformation we have

$$p^2(x)\frac{\partial}{\partial t}q(x,t)=\mu^2\frac{\partial}{\partial x_i}p^2(x)\,a_{ij}(x)\frac{\partial}{\partial x_j}q(x,t)$$
$$q|_{t=0}=v_0(x)\,p^{-1}(x)$$

This equation can be studied by the methods of homogenization theory. The homogenized equation takes the sight (*see* [2])

$$\langle p^2\rangle\frac{\partial}{\partial t}\hat{q}(x,t)=\mu^2\frac{\partial}{\partial x_i}\hat{a}_{ij}\frac{\partial}{\partial x_j}\hat{q}(x,t)$$
$$\hat{q}|_{t=0}=v_0(x)/\langle p^2\rangle \tag{2.1}$$

where $\hat{a}_{ij}$ is effective diffusion matrix for the elliptic part of the equations. This matrix differs from σ_{ij} only by factor $\langle p^2\rangle$ [*see* (0.4)-(0.6)]. Using the equality $\langle p\rangle=1$ and Proposition 5 it is easy to see that

$$1\le\langle p^2\rangle\le c\,\mu^{-3n}.$$

Thus it is enough to study the asymptotic behaviour of the effective diffusion matrix for the elliptic operator of the form $\frac{\partial}{\partial x_i}p^2(x)\,a_{ij}(x)\frac{\partial}{\partial x_j}$.

3. ASYMPTOTICS OF EFFECTIVE DIFFUSION

In order to study the properties of the function $\bar{W}(x)$ more carefully we define the function $W(x,\delta)$ as follows

$$W^2(x,\delta)=\inf_{\{x(t),\,x(0)=x,\,x(1)=0\}}\int_0^1(-v(x)+\delta)\,a^{ij}(x)\,\dot{x}_i\,\dot{x}_j\,dt$$

The following statement is a simple consequence of the definitions for $\bar{W}(x)$ and $W(x,\delta)$.

PROPOSITION 6. – *Uniformly in* $x\in T^n$

$$\lim_{\delta\to 0}W(x,\delta)=\bar{W}(x).$$

PROPOSITION 7. – $W(x)$ *is Lipschitz function on* T^n.

Proof. – For any $\delta > 0$ the function $W(x, \delta)$ is the distance of x from 0 in the riemannien metrics $((-v(x) + \delta) a_{ij}(x))$ on T^n. Therefore $W(x, \delta)$ is Liphisch function and by triangle axiom $|W(x, \delta) - W(y, \delta)| \leq \max_{T^n} ((-v(x) + \delta) \nu(x))|x - y|$ where the notation $\nu(x)$ is used for the maximal eigenvalue of the matrix $a^{ij}(x)$. Passing to the limit in both sides of this inequality as $\delta \to 0$ and considering Proposition 6 we obtain

$$|\bar{W}(x) - \bar{W}(y) \leq \max_{T^n} (-v(x) \nu(x))|x - y|.$$

PROPOSITION 8. – *The only local minimum point of $\bar{W}(x)$ on T^n is the origin.*

Proof. – Let's suppose that there exists another local minimum point say $x_0 \neq 0$ of $\bar{W}(x)$ on T^n. Then for some $\rho > 0$ the inequality $\bar{W}(x_0) < \min_{|x - x_0| = \rho} \bar{W}(x)$ holds. According to Proposition 6 for sufficiently small δ the same inequality holds for the functions $W(x, \delta)$. This contradicts the triangle axiom. The proposition is proved.

Now having the function $\bar{W}(x)$ continued by periodicity to all R^n, we construct the matrix $A_{\bar{W}}$ as follows. Let β_1 be the solution of the following variational problem

$$\beta_1 = \inf_{i \in Z^n \backslash 0} F(i), \qquad F(i) = \inf_{\{x(t), x(0)=0, x(1)=i\}} \sup_t \bar{W}(x(t))$$

where inf is taken over all smooth paths connecting 0 with $i \in Z^n \backslash 0$. For every sequence $\{i_k\}_{k=1}^{\infty}$, $i_k \in Z^n \backslash 0$, denote by $\Lambda(\{i_k\})$ the set of the limiting points for the normalized sequence $\{i_k/|i_k|\}$. Then we define Λ_{β_1} as a union of $\Lambda(\{i_k\})$ over all the sequences $\{i_k\}$ satisfying the relation $\lim_{k \to \infty} F(i_k) = \beta_1$. According to the definition of β_1 and compactness of the unit sphere the set Λ_{β_1} is closed and not empty. We also set $\Gamma_{\beta_1} = \{x \in R^n : x/|x| \in \Lambda_{\beta_1}\}$.

PROPOSITION 9. – *Γ_{β_1} is a linear subspace of R^n.*

Proof. – By periodicity of W it is easy to see that Γ_{β_1} is symmetric with respect to the origin. Suppose that Λ_{β_1} contains two linearly independent vectors z_1 and z_2. Then there exist two sequences $\{i_k^1\}$ and $\{i_k^2\}$, $F(i_k^j) \to \beta_1$, $j = 1, 2$, such that $(i_k^j/|i_k^j|) \to z_j$. The inequality $F(l^1 i_1 + l^2 i_2) \leq \max(F(i_1), F(i_2))$ is evidently true for any integer l^1 and l^2, $|l^1| + |l^2| > 0$, therefore any sequence of the form $(l_k^1 i_k^1 + l_k^2 i_k^2)$, $|l_k^1| + |l_k^2| > 0$, satisfies the relation $\lim_{k \to \infty} F(l_k^1 i_k^1 + l_k^2 i_k^2) = \beta_1$. At last arbitrary unit vector from the span of z_1 and z_2 can be approximated by the sequence of the form $((l_k^1\, i_k^1 + l_k^2\, i_k^2)/|(l_k^1\, i_k^1 + l_k^2\, i_k^2)|)$ with integer l_k^1 and l_k^2.

PROPOSITION 10. – *There exists* $\delta_0 > 0$ *such that the inequality* $F(i) \geq \beta_1 + \delta_0$ *holds for all* $i \in Z^n \backslash \Gamma_{\beta_1}$.

Proof. – Let us suppose that the statement is false. Then one can find the sequence $\{ i_k \}$, $i_k \in Z^n \backslash \Gamma_{\beta_1}$, to satisfy the equality $\lim_{k \to \infty} F(i_k) = \beta_1$. Denote by i'_k the projection of i_k onto Γ_{β_1}. By definition of Γ_{β_1} for each k there exists the sequence $\{ i_{km} \}_{m=1}^{\infty}$ of integer vectors such that $F(i_{km}) \to \beta_1$ and $(i_{km}/|i_{km}|) \to (i'_k/|i'_k|)$ as $m \to \infty$. This implies the convergence $\left(i_k - \frac{|i'_k|}{|i_{km}|}\, i_{km} \right) \to (i_k - i'_k)$ as $m \to \infty$. Approximating $(|i'_k|/|i_{km}|)$ by the rational numbers we obtain the sequence $\{ j_{km} \}_{m=1}^{\infty}$, $j_{km} \in Z^n \backslash 0$ satisfying the following relations $F(j_{km}) \leq F(i_k)$ and $\lim_{m \to \infty} (j_{km}/|j_{km}|) = (i_k - i'_k)/|i_k - i'_k|$. Under the proper choice of $m = m(k)$ any limiting point of the sequence $\{ j_{km(k)}/|j_{km(k)}| \}$ is orthogonal to Γ_{β_1}. On the other hand $F(j_{km(k)}) \to \beta_1$, so this limiting point lies in Γ_{β_1}.

PROPOSITION 11. – *In* Γ_{β_1} *there exists a basis of integer vectors.*

Proof. – It is enough to choose arbitrary orthonormal basis $z_1, \ldots, z_k$ in Γ_{β_1}, $k = \dim \Gamma_{\beta_1}$, and approximate each z_i by the sequence $\{ z_{im}/|z_{im}| \}$ such that $F(z_{im}) \to \beta_1$ as $m \to \infty$. Indeed for sufficiently large m vectors $z_{1m}, \ldots, z_{km}$ are linearly independent and by Proposition 10 lie in Γ_{β_1}. The proposition is proved.

Now we set $\beta_2 = \inf_{i \in Z^n \backslash \Gamma_{\beta_1}} F(i)$. According to Proposition 10 $\beta_2 > \beta_1$. Like Λ_{β_1} above Λ_{β_2} is defined as a union of $\Lambda(\{ i_k \})$ over all the sequences $\{ i_k \}$, $i_k \in Z^n \backslash 0$, such that $\limsup_{k \to \infty} F(i_k) \leq \beta_2$. Let $\Gamma_{\leq \beta_2} = \left\{ x \in R^n : \frac{x}{|x|} \in \Lambda_{\beta_2} \right\}$ and let Γ_{β_2} be the orthocomplement to Γ_{β_1} in $\Gamma_{\leq \beta_2}$. The following assertion can be proved in the same way as Propositions 9, 10 and 11.

PROPOSITION 12. – $\Gamma_{\leq \beta_2}$ *is the linear subspace of* R^n. *There exists* $\delta_0 > 0$ *such that* $F(i) > \beta_2 + \delta_0$ *for any* $i \in Z^n \backslash \Gamma_{\leq \beta_2}$. *In* Γ_{β_2} *there exists the basis of integer vectors.*

The next step gives β_3, $\Gamma_{\leq \beta_3}$ and Γ_{β_3} and so on. Continuing the process we find $\beta_1, \ldots, \beta_s$ and $\Gamma_{\beta_1}, \ldots, \Gamma_{\beta_s}$, $\Gamma_{\beta_1} \oplus \ldots \oplus \Gamma_{\beta_s} = R^n$, where $1 \leq s \leq n$. Let $z_1, \ldots, z_n$ be the orthonormal basis in R^n consisting of the basises of $\Gamma_{\beta_1}, \ldots, \Gamma_{\beta_s}$. We introduce the symmetric operator $A_{\bar{W}}$ to be diagonal in the basis $z_1, \dots, z_n$ with eigenvalues β_i in the corresponding subspaces Γ_{β_i}.

Let us now consider the matrix A_W as a function of $W(\cdot)$.

LEMMA 2. – *A_W is the continuous monotonic function of W in the functional space $C(T^n)$.*

Proof. – To prove continuity let us fix arbitrary continuous function $W_0(x)$ on T^n and construct corresponding $\beta_1, \dots, \beta_s$ and $\Gamma_{\beta_1}, \dots, \Gamma_{\beta_s}$. If we set $\kappa = \frac{1}{10} \min_{1 \le i < s} (\beta_{i+1} - \beta_i)$ then for any $W(x) \in \{W(x) \in C(T^n) : |W_0 - W|_{C(T^n)} < \delta < \kappa\}$ and for any $i \in Z^n \backslash 0$ we evidently have $|F_{W_0}(i) - F_W(i)| < \delta$ where indexes W and W_0 are used to indicate the function $F(i)$ defined as above for W and W_0 respectively. Now let $\gamma_1 \le \gamma_2 \le \dots \le \gamma_n$ and $e_1, \dots, e_n$ be the eigenvalue and eigenvectors of A_W respectively. It is easy to see that $|\limsup_{k \to \infty} F_W(i_k) - \beta_1| < \delta$ for any sequence $\{i_k\}$ satisfying the relation $\lim_{k \to \infty} F_{W_0}(i_k) = \beta_1$ and that $|F_W(i) - \beta_1| \ge 2\kappa$ for $i \in Z^n \backslash \Gamma_{\beta_1}$. This means that $|\gamma_i - \beta_1| < \delta$ for $i = 1, 2, \dots, k_1$, $k_1 = \dim \Gamma_{\beta_1}$, and a span of $e_1, \dots, e_{k_1}$ coincides with Γ_{β_1}. Similarly $|\gamma_{k_1+i}, -\beta_2| < \delta$ for $i = 1, 2, \dots, k_2$, $k_2 = \dim \Gamma_{\beta_2}$ and a span of $e_{k_1+1}, \dots, e_{k_1+k_2}$ coincides with Γ_{β_2} and so on. As a result we obtain $|A_{W_0} - A_W| < \delta$ in appropriate matrix norm. Monotonicity can be proved in the similar way.

The next statement is the main result of the paper.

THEOREM 1. – *Effective diffusion matrix $\sigma(\mu)$ satisfies the following limiting relation*

$$\lim_{\mu \to 0} \mu \ln \sigma(\mu) = -2 A_{\bar{W}}.$$

Proof. – As was mentionend in Section 2 it suffices to find the asymptotics of homogenized matrix for the operator $\frac{\partial}{\partial x_i} p^2(x) a_{ij}(x) \frac{\partial}{\partial x_j}$. For this purpose we approximate $\bar{W}(x)$ on T^n by the smooth function $W_\delta(x)$ with finite number of degenerate points and with the only global minimum point at the origin in such a way that the estimate $|\bar{W} - W_\delta|_{C(T^n)} < \delta/3$ holds. Under this choice of W_δ we have for sufficiently small μ the following matrix inequality

$$e^{-\frac{W_\delta(x)+2\delta}{\mu}} \le p^2(x) a_{ij}(x) \le e^{-\frac{W_\delta(x)+2\delta}{\mu}} \tag{3.1}$$

Further the homogenized operators keep the order relation of original operators [1] hence (3.1) implies the inequality

$$e^{-\frac{2\delta}{\mu}}\,\tilde a_{ij} \le \hat a_{ij}\,(\mu) \le e^{\frac{2\delta}{\mu}}\,\tilde a_{ij} \tag{3.2}$$

here $\tilde a_{ij}$ is the homogenized matrix of the operator $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e^{-\frac{W_\delta(x)}{\mu}} \frac{\partial}{\partial x_j}$. The properties of $W_\delta(x)$ allow us to apply the results of [8], [4] to find the asymptotics of $\tilde a_{ij}$. This yields the following relation

$$\lim_{\mu\to\infty} \mu \ln \tilde a_{ij} = -2\,A_{W_\delta}. \tag{3.3}$$

Taking into account (3.2) we have

$$-2\,\delta I - 2\,A_{W_\delta} \le \liminf_{\mu\to 0}\, \mu \ln \hat a_{ij} \le \limsup_{\mu\to 0}\, \mu \ln \hat a_{ij} \le 2\,\delta I - 2\,A_{W_\delta}.$$

At last by Lemma 2 and choice of $W_\delta(x)$ the matrix A_{W_δ} tends to $A_{\bar W}$ as $\delta \to 0$ thus

$$\lim_{\mu\to 0} \mu \ln \hat a_{ij} = \lim_{\mu\to 0} \mu \ln \sigma_{ij} = -2\,A_{\bar W}.$$

In conclusion let us derive some consequences from Theorem 1. Consider the symmetrical case when the matrix $a_{ij}(x)$ and the potential $v(x)$ are invariant with respect to any motion of R^n preserving the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Then the matrix $a_{\bar W}$ takes the form $a_{\bar W} = \alpha_{\bar W} I$ where I is the unit matrix and regarding $\alpha_{\bar W}$ Theorem 1 states that $\lim_{\mu\to\infty} \mu \ln \alpha_{\bar W}$ is equal to the distance of 0 from $Z^n \backslash 0$ in the metric $(v(x)\, a^{ij}(x))$. This metric degenerates in the points of Z^n.

In our previous work [3] the similar asymptotics was found for the effective diffusion of the operator $(\mu^2 \Delta + v(x))$ with piecewise constant potential. The methods developed here allow these results to be generalized. Namely let us consider the equation of the form (0.1) with $v(x)$ given by the formula

$$v(x) = \begin{cases} 1, & x \in \bigcup_{j\in Z^n} (Q + j) \\ 0, & \text{otherwise} \end{cases}$$

where Q is simply connected domain with piecewise smooth boundary such that Q and $(Q + j)$ don't intersect for any $j \in Z^n$. Without loss of generality we can assume that $0 \in Q$. Then under the same assumption as above we have

THEOREM 2. – $\lim_{\mu \to 0} \mu \ln \sigma_{ij}(\mu) = -2 A_W$ *where* $W(x)$ *is equal to the distance of* x *from* Q *on* T^n *in the metric* $(a^{ij}(x))$.

With appropriate simplifications the proof of the theorem is quite similar to the proof of Theorem 1 (*see* also [3]).

REFERENCES

[1] A. BENSOUSSAN, J.-L. LIONS and G. PAPANICOLAOU, *Asymptotic analysis for periodic Structures*, North-Holland, Amsterdam, 1978.

[2] S. M. KOZLOV, Reducibility of quasiperiodic operators and homogenization, *Transactions of Moscow Math. Soc.*, Vol. **46**, 1983, pp. 99-123.

[3] S. M. KOZLOV and A. L. PIATNITSKI, Effective diffusion for a parabolic operator with periodic potential, *S.I.A.M. Journal Appl. Math.*, Vol. **53**, 1, 1993.

[4] S. M. KOZLOV and A. L. PIATNITSKI, Averaging on a background of vanishing viscosity, *Math. U.S.S.R. Sbornik*, Vol. **70**, 1, 1991, pp. 241-261.

[5] Ya. B. ZELDOVICH, Exact solution of a diffusion problem in a periodic velocity field and turbulent diffusion, *Dokl. Akad. Nauk U.S.S.R.*, Vol. **266**, 1982, pp. 821-826.

[6] M. AVELLANEDA, Enhanced diffusivity and intercell transition layers in 2-D models of passive advection, *J. Math. Phys.*, Vol. **32**, 11, 1991, pp. 3209-3212.

[7] A. FANNJIANG and G. PAPANICOLAOU, Convection enhanced diffusion for periodic flows, *S.I.A.M. Journal of Appl. Math.* (to appear).

[8] S. M. KOZLOV, Geometric aspects of homogenization, *Russ. Math. Surveys*, Vol. **44**, 2, 1989, pp. 91-144.

[9] S. M. KOZLOV, Asymptoptics of Laplace-Dirichlet integrals, *Functional Analisis and its applications*, Vol. **24**, 2, 1990, pp. 37-49.

[10] S. M. KOZLOV, Effective diffusion in the Fokker-Plank equation, *Math Notes U.S.S.R. Acad. of Science*, Vol. **45**, 5, 1989, pp. 19-31.

[11] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[12] A. D. WENTZELL and M. I. FREIDLIN, *Fluctuations in Dynamical Systems Caused by Small Random Perturbations*, Nauka, Moscow, 1979 (Russian).

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