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Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales

Part I: the multiplicative ergodic theory

by

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ABSTRACT. – We consider a class of stochastic linear functional differential systems driven by semimartingales with stationary ergodic increments. We allow smooth convolution-type dependence of the noise terms on the history of the state. Using a stochastic variational technique we construct a compactifying stochastic semiflow on the state space. As a necessary ingredient of this construction we prove a general perfection theorem for cocycles with values in a topological group (Theorem 3.1). This theorem is an extension of a previous result of de Sam Lazaro and Meyer (*cf.* [7], Theorem 1, p. 40). A multiplicative Ruelle-Oseledec ergodic theorem then gives the existence of a discrete Lyapunov spectrum and a saddle-point property in the hyperbolic case.

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Key words: Stochastic f.d.e., measure-preserving flow, semimartingale, helix, stationary increments, multiplicative cocycle, compact cocycle, perfection of cocycles, Lyapunov spectrum, multiplicative ergodic theorem, exponential dichotomy.

RÉSUMÉ. – Nous considérons une classe de systèmes linéaires stochastiques différentiels gouvernés par des semimartingales à accroissements stationnaires ergodiques. Nous permettons une dépendence lisse de type convolutive du bruit du passé de l'état. Utilisant une technique variationnelle stochastique, nous construisons un semi-flot stochastique compactifiant sur l'espace des états. Un ingrédient nécessaire à cette construction est notre preuve d'un théorème général de perfection pour les cocycles à valeur dans un groupe topologique (Théorème 3.1). Ce résultat qui est une extension d'un résultat de Sam Lazaro et Meyer (*cf.* [7], Théorème 1, p. 40). Un théorème multiplicatif ergodique de type Ruelle-Oseledec donne alors l'existence d'un spectre discret de Lyapunov et une propriété de selle dans le cas hyperbolique.

1. INTRODUCTION

In [23] the first author developed a multiplicative ergodic theory for a class of n-dimensional stochastic linear functional differential equations

$$\left.\begin{aligned} dx(t) &= H(x(t-d), x(t), x_t)\,dt + g(x(t))\,dW(t) \\ x_t(s) &:= x(t+s), \quad -r \le s \le 0, \quad t \ge 0, \quad 0 \le d \le r \end{aligned}\right\} \qquad (\star)$$

with state space $M_2 := \mathbb{R}^n \times \mathbb{L}^2([-r, 0], \mathbb{R}^n)$. The analysis in [23] depended crucially on the fact that the diffusion term $g(x(t))$ does *not look into the (past) history* $x(s)$, $s < t$, of the state. The present article is an attempt to relax this limitation. (Note, however, the pathological example in Mohammed [22], pp. 144-148.) Indeed we wish to extend the results of [23] in two directions:

(i) We allow "smooth" convolution-type dependence on the history $x(s)$, $t - r \le s \le t$, in the noise coefficient.

(ii) The driving noise processes consist of a large class of *semimartingales with jointly stationary (ergodic) increments*. Within this context, our results appear to be new *even in the non-delay case* $r = 0$.

More specifically we look at a linear stochastic functional differential equation

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,\,0]} \mu(t)(ds)\,x(t+s)\right\} dt \\ &\quad + dN(t)\int_{-r}^{0} K(t)(s)\,x(t+s)\,ds + dL(t)\,x(t-) \\ &\quad (x(0),\, x_0) = (v,\, \eta) \in M_2. \end{aligned}\right\} \tag{I}$$

In the above stochastic f.d.e. (s.f.d.e.), μ is a stationary measure-valued process such that each $\mu(t,\omega)$ is an $n\times n$-matrix-valued measure on $[-r,\,0]$. The random field $K(t)(s)$ is stationary in t. The process N is a general $n \times n$-matrix valued semimartingale with jointly stationary increments. The second noise process L is also $n \times n$-matrix-valued, has jointly stationary increments but admitting a representation as a *continuous* local martingale plus a right continuous process of locally bounded variation. Assuming that $(\mu,\, K,\, dN,\, dL)$ form an ergodic process and satisfy fairly general moment conditions, we show that (I) has an almost sure Lyapunov spectrum

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t),\, x_t)\|_{M_2}$$

consisting of a discrete non-random set of *Lyapunov exponents* $\{\lambda_i\}_{i=1}^{\infty} \subset \mathbb{R} \cup \{-\infty\}$. If none of the Lyapunov exponents λ_i is zero, we obtain a flow-invariant exponential dichotomy for the stochastic flow X on M_2 associated with the trajectories $\{(x(t),\, x_t) : t \geq 0,\, (x(0),\, x_0) = (v,\, \eta) \in M_2\}$ of (I).

Our approach above requires the construction of a *very robust* and *compactifying* version of the stochastic flow X of (I) on the Hilbert space M_2. As a basic ingredient of our construction we prove a general perfection theorem for cocycles with values in a metrizable second countable topological group. *See* Theorem 3.1. This result is an extension of de Sam Lazaro and Meyer's perfection theorem ([7], Theorem 1, p. 40). Our proof uses techniques developed in [7].

Although the existence of a unique solution of (I) has been known for some time (*cf.* e.g. [19], [24] and [26], p. 197), the problem of constructing a flow $X : \mathbb{R} \times \Omega \times M_2 \to M_2$, with the appropriate cocycle and regularity properties, appears to be difficult because of the pathological example

$$dx(t) = x(t-1)\,dW(t),$$

where W is the one-dimensional Wiener process.

This example is a serious obstruction to the existence of a stochastic flow on M_2 that is measurable and linear, let alone compactifying. This is due to the Gaussian nature of the driving noise and the infinite-dimensionality of the state space M_2 (Mohammed [22]). Needless to say our Theorem 3.1 and the perfection techniques of de Sam Lazaro and Meyer [7] will not apply here.

In order to overcome the above difficulty we develop a new method for constructing a sufficiently robust version of the flow X of (I). The key idea is to show that the s.f.d.e. (I) is equivalent to a *random* integral equation (IV) in § 4. The cocycle property, the compactness of the flow and Ruelle's integrability condition for the stochastic flow (Theorem 5.1) are then read off from the random integral equation. This method of construction of the flow is different from the one used by Mohammed in [23]. It has the added advantage of being conceptually simpler and perhaps more efficient. This technique also points the way towards possible applications to certain types of stochastic linear P.D.E.s.

Once the regular version of the flow X is constructed, the existence of the Lyapunov spectrum (Theorem 5.2) and the stable-manifold theorem (Theorem 5.3) are established using Ruelle's infinite-dimensional multiplicative ergodic theorem (Ruelle [28], [27]). This part of the analysis is closely parallel to the one used by Mohammed in [23].

In order to outline the scope of the theory we indicate below examples of linear stochastic differential equations which are covered by the theorems in this article. The reader may formulate the appropriate conditions under which these results apply to the examples listed below. Note that in all of these examples the state $x(t)$ is a multidimensional process.

Example 1. – *Linear o.d.e.'s driven by white noise*

$$dx(t) = a(t)\,x(t)\,dt + \sum_{i=1}^{p} \sigma_i(t)\,x(t)\,dW_i(t) \tag{1}$$

For each $t > 0$, $a(t)$ and $\sigma_i(t)$, $1 \leq i \leq p$, are $n \times n$-matrices and the processes (a, σ_i, dW_i) are stationary ergodic and non-anticipating; the Brownian motions W_i, $1 \leq i \leq p$, are independent and one-dimensional. The case of constant coefficients $a(t) \equiv a$, $\sigma_i(t) \equiv \sigma_i$, $1 \leq i \leq p$, has been studied by several authors, e.g. Arnold, Kliemann & Oeljeklaus [2], Has'minskii [10], and Baxendale [3]. The Lyapunov spectrum of (1) has been discussed by Arnold and Kliemann [1] when $a(t)$, $\sigma_i(t)$, $i = 1, \ldots, p$, are stationary ergodic processes which are *independent* of W_i, $i = 1, \ldots, p$. Note that our results do not necessarily require that $a(t)$, $\sigma_i(t)$, $i = 1, \ldots, p$, be independent of the noises W_i, $i = 1, \ldots, p$.

Example 2. – *Random delay equations driven by white noise*

$$dx(t) = \sum_{i=1}^{m} a_i(t)\, x(t - d_i(t))\, dt + \sum_{i=1}^{p} \sigma_i(t)\, x(t)\, dW_i(t) \tag{2}$$

The coefficients a_i, σ_i are matrices (possibly stationary) and the delays d_i are non-anticipating stationary bounded processes with non-negative values. The equation is driven by several Wiener processes W_i. The dynamics of (2) was studied in (Mohammed [22] VI § 3, pp. 167-186) within the context of Markov processes on the state space $C([-r, 0], \mathbb{R}^n)$ and under the condition that each d_i is fixed in t and is independent of $(W_1, \ldots, W_p)$. A sufficient condition is given in ([22], Corollary 3.1.2, p. 184) which guarantees asymptotic stability in distribution of the trajectory $\{x_t : t \geq 0\}$ of (2). Observe that (2) reduces to (1) when $d_i \equiv 0$, $1 \leq i \leq m$.

Example 3. – *Diffusions with distributed memory*

$$dx(t) = \sum_{i=1}^{p} \sigma_i(t) \int_{-r}^{0} K(s)\, x(t+s)\, ds\, dW_i(t) \tag{3}$$

The matrix-valued processes $\sigma_i(t)$, $1 \leq i \leq p$, are stationary (ergodic) while $K(s)$ is just a deterministic matrix-valued function. The Brownian motions W_i, $1 \leq i \leq p$, are one-dimensional. Although equations like (3) fall under the class studied by Itô and Nisio [13] and Mohammed [22], so far little is known regarding the *almost sure* asymptotic behavior of the trajectory $(x(t), x_t)$ as $t \to \infty$.

Example 4. – *Linear o.d.e.'s driven by Poisson noise*

$$dx(t) = a(t)\, x(t)\, dt + \sum_{i=1}^{p} \sigma_i(t)\, x(t-)\, dN_i(t) \tag{4}$$

The driving noises $N_i(t)$ are one-dimensional Poisson processes and the coefficients $a(t)$, $\sigma_i(t)$ are stationary ergodic matrix-valued, for $1 \leq i \leq p$. For constant coefficients, $a(t) \equiv a$, $\sigma_i(t) \equiv \sigma_i$ a.s. for all $t \geq 0$, $1 \leq i \leq p$, the Lyapunov exponents of (4) were studied by Li and Blankenship [16] using classical results on random matrix products.

Example 5. – *Linear functional differential equations driven by Poisson noise*

$$dx(t) = \left\{ \int_{[-r, 0]} \mu(t)(ds)\, x(t+s) \right\} dt + \sum_{i=1}^{p} \sigma_i(t)\, x(t-)\, dN_i(t) \tag{5}$$

Here μ is a measure-valued process as in (I), $\sigma_i(t)$ are stationary matrices and $N_i(t)$ Poisson processes, $i = 1, \ldots, p$. Under suitable conditions on the coefficients μ, σ_i, unique solutions to (5) are known to exist (Doléans-Dade [8], Métivier and Pellaumail [19], Protter [24]). However, to our knowledge, issues of almost sure asymptotic stability for solutions of (5) have hitherto not been explored.

Example 6. – *Linear f.d.e.'s driven by white noise*

$$dx(t) = H(t, \cdot, x(t), x_t)\, dt + \sum_{i=1}^{p} g_i(t, \cdot, x(t))\, dW_i(t) \tag{6}$$

The coefficients $H(t, \cdot, \cdot, \cdot)$, $g_i(t, \cdot, \cdot)$ are stationary ergodic processes with values in $L(M_2, \mathbb{R}^n)$ and $L(\mathbb{R}^n)$ respectively. The Brownian motion $W(t) = (W_1(t), \ldots, W_p(t))$ is p-dimensional. The case of constant coefficients corresponds to equations like $(\star)$ whose Lyapunov exponents were studied in the article [23] referred to earlier.

It is evident that the s.f.d.e. (I) also includes as special cases various (finite) "linear combinations" of all the examples mentioned above.

2. BASIC SETTING AND HYPOTHESES

We wish to formulate the basic set-up and hypotheses on the stochastic f.d.e.

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,0]} \mu(t)(ds)\, x(t+s)\right\} dt \\ &\quad + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds + dL(t)\, x(t-), \quad t > 0 \\ x(0) &= v \in \mathbb{R}^n, \quad x(s) = \eta(s), \quad -r \le s \le 0; \quad r > 0 \end{aligned}\right\} \tag{I}$$

which will be needed in the sequel.

Throughout the article we will denote by $B(H)$ the Borel σ-algebra of any topological space H.

We will also use the basic framework developed by de Sam Lazaro and Meyer [7] and Protter [25]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose $\mathcal{F}^0$ is a sub-σ-algebra of $\mathcal{F}$, and $\mathcal{F}$ is the completion of $\mathcal{F}^0$ under P. Let $\theta(t, .)$, $t \in \mathbb{R}$ be a group of measure-preserving transformations on Ω such that the map $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(B(\mathbb{R})\otimes\mathcal{F}^0, \mathcal{F}^0)$-measurable. Fix a sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{F}$ such that $\theta(t, .)^{-1}(\mathcal{A}) \subset \mathcal{A}$ for $t \le 0$. We define a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ by setting $\mathcal{F}_0 = \mathcal{A}$, $\mathcal{F}_t = \theta(t, .)^{-1}(\mathcal{A})$, $t \in \mathbb{R}$. Thus $\mathcal{F}_{t+s} = \theta(t, .)^{-1}(\mathcal{F}_s)$ for all t, $s \in \mathbb{R}$.

We assume throughout that $\mathcal{A}$ is complete in $\mathcal{F}$. This implies that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is right continuous (de Sam Lazaro and Meyer [7], p. 4). Furthermore we assume that $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\in\mathbb{R}} \mathcal{F}_t$, the σ-algebra generated by $\bigcup_{t\in\mathbb{R}} \mathcal{F}_t$.

We shall impose two sets of hypotheses on the coefficients of (I). The first set of hypotheses, denoted by (C_i), $i = 1, 2, \dots 5$, guarantees the existence of a continuous linear stochastic flow on the state space $M_2 := \mathbb{R}^n \times \mathbb{L}^2([-r, 0], \mathbb{R}^n)$ with the Hilbert norm

$$\|(v, \eta)\|_{M_2}^2 := |v|^2 + \int_{-r}^{0} |\eta(s)|^2 \, ds, \tag{7}$$

$v \in \mathbb{R}^n$, $\quad \eta \in \mathbb{L}^2([-r, 0], \mathbb{R}^n)$.

Observe that $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^n$. The second set of hypotheses (I_j), $j = 1, 2, 3, 4$, pertains to moment-type restrictions which are designed in order for the stochastic flow to satisfy Ruelle-Oseledec integrability condition (Theorem 5.1, § 5). These integrability hypotheses are spelled out in § 5.

The space of all real $n \times n$ matrices is denoted by $\mathbb{R}^{n\times n}$ and is usually given the Euclidean norm

$$\|A\|^2 := \sum_{i,j=1}^{n} a_{ij}^2, \qquad A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}.$$

The symbol $\mathcal{M}([-r, 0], \mathbb{R}^{n\times n})$ shall stand for the space of all $n \times n$-matrix valued Borel measures on $[-r, 0]$ (or $\mathbb{R}^{n\times n}$-valued functions of bounded variation on $[-r, 0]$). This space will be given the σ-algebra generated by all evaluations.

A *solution* of the stochastic f.d.e. (I) is a stochastic process $x : [-r, \infty)\times\Omega \to \mathbb{R}^n$ such that $x \,|\, \mathbb{R}^+\times\Omega$ has cadlag paths, is $(\mathcal{F}_t)_{t\geq 0}$-adapted and satisfies the stochastic integral equation

$$x(t) = \begin{cases} v + \displaystyle\int_0^t \int_{[-r,0]} \mu(u)(ds)\, x(u+s)\, du \\ \quad + \displaystyle\int_0^t dN(u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds \\ \quad + \displaystyle\int_0^t dL(u)\, x(u-), \quad t \geq 0 \\ \qquad \eta(t), \qquad -r < t < 0 \end{cases} \tag{II}$$

almost surely. Note that in (I) and (II) all n-vectors are column vectors and the products are to be understood in the sense of matrix multiplication. Throughout the article we shall adopt the following terminology (*cf.* Protter [25]). An $\mathbb{R}^{n\times n}$-valued stochastic process $z(t)$, $t \in \mathbb{R}$, is a *semimartingale* (resp. *local martingale*) if $z(t)$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}$, z has cadlag paths, $z(0) = 0$ and $z \mid [0, \infty)$ has the semimartingale (local martingale, resp.) property with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. An $\mathbb{R}^{n\times n}$-valued stochastic process $z(t)$, $t \in \mathbb{R}$, is said to be a *helix* or a *perfect helix* if $z(t)$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}$, z has cadlag paths, $z(0) = 0$ and the following identity holds

$$z(t+h, \omega) - z(s+h, \omega) = z(t, \theta(h, \omega)) - z(s, \theta(h, \omega))$$

for all $s, t, h \in \mathbb{R}$ and all $\omega \in \Omega$. The process z is called a *crude helix* if it satisfies all the above conditions except that the last helix property holds a.s. for each fixed $s, t, h \in \mathbb{R}$, *with the exceptional set possibly depending on* s, t, h.

Hypotheses (C)

(C_1) The process $\mu : \mathbb{R} \times \Omega \to \mathcal{M}([-r, 0], \mathbb{R}^{n\times n})$ has a representation

$$\mu(t, \omega) = \tilde{\mu}(\theta(t, \omega)), \qquad t \in \mathbb{R}, \quad \omega \in \Omega$$

where $\tilde{\mu} : \Omega \to \mathcal{M}([-r, 0], \mathbb{R}^{n\times n})$ is $\mathcal{F}_0 \cap \mathcal{F}^0$-measurable. Note that this implies that μ is stationary, $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted and $B(\mathbb{R}) \otimes \mathcal{F}^0$-measurable.

(C_2) For each $\omega \in \Omega$ and $t \geq 0$, let $\bar{\mu}(t, \omega)$ be the positive measure on $[-r, \infty)$ defined by

$$\bar{\mu}(t, \omega)(A) := |\mu|(t, \omega)\{(A - t) \cap [-r, 0]\}$$

for all Borel subsets A of $[-r, \infty)$. Note that $|\mu|$ denotes the total variation measure of μ with respect to the norm on $\mathbb{R}^{n\times n}$. It is easy to check that, for each $\omega \in \Omega$,

$$\nu(\omega)(\cdot) := \int_0^\infty \bar{\mu}(t, \omega)(\cdot)\, dt$$

is also a positive measure on $[-r, \infty)$. Suppose that $\nu(\omega)$ has a density $\dfrac{d\nu(\omega)}{ds}$ with respect to Lebesgue measure which is locally essentially bounded for each fixed $\omega \in \Omega$. Note that (C_2) is satisfied in the deterministic case $\mu(t, \omega) = \mu^0$, $t \geq 0$, $\omega \in \Omega$, for a fixed $\mu^0 \in \mathcal{M}([-r, 0], \mathbb{R}^{n\times n})$.

(C_3) Assume that K: $[0, \infty) \times [-r, 0] \times \Omega \to \mathbb{R}^{n\times n}$ has a representation

$$K(t, \omega)(\cdot) = \tilde{K}(\theta(t, \omega))$$

where $\tilde{K}: \Omega \to \mathbb{L}^\infty([-r, 0], \mathbb{R}^{n\times n})$ is $(\mathcal{F}_0 \cap \mathcal{F}^0, B(\mathbb{L}^\infty([-r, 0], \mathbb{R}^{n\times n}))$-measurable. Further, assume that $\bar{K}: \{(t, s) \in \mathbb{R}^2 : t \geq 0, -r \leq s - t \leq 0\} \times \Omega \to \mathbb{R}^{n\times n}$ defined by

$$\bar{K}(t, s, \omega) = K(t, \omega)(s - t)$$

is absolutely continuous in t for Lebesgue-a.a. s and all $\omega \in \Omega$, $\frac{\partial \bar{K}}{\partial t}(t, s, \omega)$ and $\bar{K}(t, s, \omega)$ are locally essentially bounded in (t, s) for every $\omega \in \Omega$; and $\frac{\partial \bar{K}}{\partial t}(t, s, \omega)$ is jointly measurable.

(C_4) The process $N: \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ is a helix-semimartingale.

(C_5) The process $L: \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ is a helix-semimartingale admitting a representation

$$L = M + V$$

where M is a *continuous* helix-local martingale and V is a helix whose sample paths are all right continuous and of bounded variation on compact sets of $\mathbb{R}^+$.

3. SOME PRELIMINARIES

Our strategy for a sample-wise analysis of the s.f.d.e. (I) is to *free the equation of stochastic differentials and replace it by an equivalent random family of integral equations.* In order to construct these random integral equations we shall require some preliminaries. These are discussed below.

The main ingredient in our construction of the random integral equation is a very general perfection theorem for cocycles taking values in a metrizable topological group (Theorem 3.1 and its corollary below). This result contains the perfection theorem of de Sam Lazaro and Meyer ([7], Theorem 1, p. 40) as a special case. Our proof, however, uses similar techniques to those in [7]. First we begin with the following definition.

DEFINITION 3.1. – Let the family $(\Omega, \mathcal{F}, \mathcal{F}^0, P, \theta(t, \cdot), t \in \mathbb{R})$ satisfy our general assumptions in Section 2. Suppose $(H, \star, e)$ is a topological group with binary operation $\star$ and identity e. A map $\varphi : \mathbb{R} \times \Omega \to H$ is called a *crude H-valued cocycle* if the following conditions are satisfied:

(i) φ is $(B(\mathbb{R}) \otimes \mathcal{F}^0, B(H))$-measurable.

(ii) For every $s \in \mathbb{R}$ there exists a P-null set $N_s \subset \Omega$ such that

$$\varphi(t + s, \omega) = \varphi(t, \theta(s, \omega)) \star \varphi(s, \omega) \tag{8}$$

for all $\omega \notin N_s$ and all $t \in \mathbb{R}$.

The map φ is said to be a *(perfect) H-valued cocycle* if it is a crude cocycle and in addition (8) holds identically for *all* $\omega \in \Omega$ and *all* $s, t \in \mathbb{R}$. Setting $s = t = 0$ in (8) implies that $\varphi(0, \omega) = e$ for all (a.a.) $\omega \in \Omega$, if φ is a perfect (resp. crude) cocycle.

Remarks. – (i) If H is T_1 (*i.e.* singletons are closed) and $\varphi(\cdot, \omega) : \mathbb{R} \to H$ is either right-continuous for almost all $\omega \in \Omega$ or left-continuous for almost all $\omega \in \Omega$, then Condition (ii) of the above definition is implied by

(ii)′ for every $s, t \in \mathbb{R}$ there exists a P-null set $N_{s,t} \subset \Omega$ such that (8) holds for all $\omega \notin N_{s,t}$.

To see this, set $N_s := \bigcup_{t \in \mathbb{Q}} N_{s,t} \cup \bar{N}$ where $\mathbb{Q}$ is the set of all rationals and $\bar{N}$ is a null set such that $\varphi(\cdot, \omega)$ is right (resp.-left continuous) for all $\omega \notin \bar{N}$. Then N_s is a P-null set. By the right (or left)-continuity of φ and the uniqueness of limits in H, it follows that (8) actually holds for all $t \in \mathbb{R}$ and all $\omega \notin N_s$.

(ii) Suppose H is metrizable. Let $\varphi : \mathbb{R} \times \Omega \to H$ have almost all sample paths continuous (resp. cadlag) and satisfy (8) of Definition 3.1 together with

(i)′ $\varphi(t, \cdot)$ is $(\mathcal{F}, B(H))$-measurable for all $t \in \mathbb{R}$.

Then φ is indistinguishable from a continuous (resp. cadlag) crude cocycle $\varphi' : \mathbb{R} \times \Omega \to H$ satisfying $\varphi'(0, \omega) = e$ for *all* $\omega \in \Omega$. We construct φ' as follows. For every $t \in \mathbb{Q}$ there exists a map $\tilde{\varphi}(t, \cdot) : \Omega \to H$ which is $(\mathcal{F}^0, B(H))$-measurable and satisfies $\tilde{\varphi}(t, \cdot) = \varphi(t, \cdot)$ a.s. Therefore there is a set $\Omega_1 \in \mathcal{F}^0$ such that $P(\Omega_1) = 1$ and $\Omega_1 \subset \{\omega \in \Omega : \tilde{\varphi}(t, \omega) = \varphi(t, \omega)$ for all $t \in \mathbb{Q}$; $\varphi(0, \omega) = e$ and $\varphi(\cdot, \omega)$ is continuous (resp. cadlag)$\}$. Define

$$\varphi'(t, \omega) = \begin{cases} \varphi(t, \omega), & \text{for } \omega \in \Omega_1 \\ e, & \text{for } \omega \notin \Omega_1, \end{cases}$$

and all $t \in \mathbb{R}$.

Since φ' has continuous (resp. cadlag) paths and H is metrizable, it follows that $\varphi'(t, \cdot)$ is $(\mathcal{F}^0, B(H))$-measurable for all $t \in \mathbb{R}$. Furthermore by the same reasoning one gets that φ' is $(B(\mathbb{R}) \otimes \mathcal{F}^0, B(H))$-measurable. Therefore φ' is a continuous (resp. cadlag) crude cocycle, indistinguishable from φ and satisying $\varphi'(0, \omega) = e$ for all $\omega \in \Omega$.

(iii) Suppose H is metrizable and $\varphi : \mathbb{R}^+ \times \Omega \to H$ satisfies (8) for every $s \geq 0$, all $t \geq 0$ and all $\omega \notin N_s$, a P-null set. Then one can extend φ to a map $\varphi'' : \mathbb{R} \times \Omega \to H$, defined on the whole line, in the following manner. Define $\varphi'' : \mathbb{R} \times \Omega \to H$ inductively by

$$\varphi''(t,\omega) := \begin{cases} \varphi(t,\omega), & t \geq 0 \\ \varphi(t+k,\,\theta(-k,\,\omega)) \star \varphi(1,\,\theta(-k,\,\omega))^{-1} \star \varphi''(-k+1,\,\omega), \\ \qquad\qquad -k \leq t < -k+1, \end{cases}$$

for all integers $k \geq 1$ and all $\omega \in \Omega$.

It is easy to check that φ'' satisfies (8). If φ satisfies (8) for *all* $\omega \in \Omega$ and all $t,\, s \geqq 0$, then φ'' satisfies (8) for all $\omega \in \Omega$ and all $t,\, s \in \mathbb{R}$. If φ has (almost) all sample paths continuous (resp. cadlag), then φ'' has the same property. If φ is $(B(\mathbb{R}^+) \otimes \mathcal{F}^0,\, B(H))$-measurable, then φ'' is $(B(\mathbb{R}) \otimes \mathcal{F}^0,\, B(H))$-measurable. As a convention, we shall always extend a crude helix on $\mathbb{R}^+$ to a crude helix on $\mathbb{R}$ in the above manner.

(iv) A helix (resp. crude helix) is a cadlag cocycle (resp. crude cocycle) with values in the additive group of $n \times n$-matrices $(\mathbb{R}^{n \times n},\, +)$. To see this it is sufficient to check the helix property for $s = 0$ only.

We can now state our general perfection theorem.

THEOREM 3.1. – *Let the family* $(\Omega,\, \mathcal{F},\, \mathcal{F}^0,\, P,\, \theta(t,\,\cdot),\, t \in \mathbb{R})$ *satisfy our general assumptions in Section 2. Suppose* $(H,\, \star,\, e)$ *is a metrizable second countable topological group with binary operation* $\star$ *and identity* e. *Let* $\varphi : \mathbb{R} \times \Omega \to H$ *be a crude cocycle which has continuous (resp. cadlag) sample paths.*

Then there exists a map $\tilde{\varphi} : \mathbb{R} \times \Omega \to H$ *with the following properties:*

(a) $\tilde{\varphi}$ *is a perfect cocycle.*

(b) $\tilde{\varphi}$ *has continuous (resp. cadlag) sample paths.*

(c) $\tilde{\varphi}$ *is indistinguishable from* φ, *viz.*

$$P(\{\omega \in \Omega : \tilde{\varphi}(t,\,\omega) = \varphi(t,\,\omega) \quad \textit{for all } t \in \mathbb{R}\}) = 1.$$

Remarks. – (i) If $H = (\mathbb{R},\, +)$, then the above theorem reduces to Theorem 1 (p. 40) of de Sam Lazaro and Meyer [7].

(ii) Let $(\Omega,\, F,\, F^0,\, P,\, \theta(t,\,\cdot),\, t \in \mathbb{R})$, $(H,\, \star,\, e)$ be as in the theorem. Let $(G,\, \cdot)$ be a second countable metrizable subgroup of the group of automorphisms of H, such that the composition map

$$G \times H \to H$$

$$(g,\, h) \to g \circ h$$

is continuous. Assume that $\psi : \mathbb{R} \times \Omega \to G$ is a crude cocycle with continuous (resp. cadlag) sample paths. Let $\varphi : \mathbb{R} \times \Omega \to H$ be a $(B(\mathbb{R}) \otimes F^0, B(H))$-measurable map satisfying the following two conditions:

(ii)″ For every $s \in \mathbb{R}$, there exists a P-null set N_s such that

$$\varphi(t+s, \omega) = (\psi^{-1}(s, \omega) \circ \varphi(t, \theta(s, \omega))) \star \varphi(s, \omega) \tag{9}$$

for all $\omega \notin N_s$ and all $t \in \mathbb{R}$.

(iii) φ has continuous (resp. cadlag) sample paths.

Then φ is indistinguishable from a map $\tilde{\varphi} : \mathbb{R} \times \Omega \to H$ which has continuous (resp. cadlag) sample paths and satisfies (9) identically for all $\omega \in \Omega$ and all $s, t \in \mathbb{R}$.

To prove the above statement we proceed as follows.

Define a second countable metrizable topological group $(\hat{G}, \otimes)$ by $\hat{G} := G \times H$ with multiplication

$$(g_1, h_1) \otimes (g_2, h_2) := (g_1 \cdot g_2, (g_2^{-1} \circ h_1) \star h_2)$$

for all $g_i \in G$, $h_i \in H$, $i = 1, 2$. Let $\hat{\varphi} : \mathbb{R} \times \Omega \to \hat{G}$ be the map

$$\hat{\varphi}(t, \omega) := (\psi(t, \omega), \varphi(t, \omega))$$

for all $(t, \omega) \in \mathbb{R} \times \Omega$. Then $\hat{\varphi}$ is a $\hat{G}$-valued crude cocycle which by Theorem 3.1 is indistinguishable from a perfect cocycle $\tilde{\hat{\varphi}} = (\tilde{\hat{\varphi}}_1, \tilde{\hat{\varphi}}_2)$. It is easy to check that $\tilde{\hat{\varphi}}_2$ is the required version $\tilde{\varphi}$ of φ.

Proof of Theorem 3.1. – By Remark (ii) preceding the statement of Theorem 3.1, we can assume without loss of generality that $\varphi(0, \omega) = e$ for *all* $\omega \in \Omega$.

Define the sets E and Ω_0 by

$$E := \{(s, \omega) \in \mathbb{R} \times \Omega : \text{ (8) holds for all } t \in \mathbb{R}\}$$

and

$$\Omega_0 := \{\omega \in \Omega : \text{ for Lebesgue-}a.a.\ s \in \mathbb{R},\ \text{(8) holds for all } t \in \mathbb{R}\}.$$

Since H is second countable, the measurability hypothesis on φ and θ easily imply that the map

$$(t, s, \omega) \mapsto \varphi(t+s, \omega) \star \varphi(s, \omega)^{-1} \star \varphi(t, \theta(s, \omega))^{-1}$$

is $(B(\mathbb{R}^2) \otimes \mathcal{F}^0,\ B(H))$-measurable. Now, using the facts that H is metrizable and φ has cadlag paths, it is easy to see that $E \in B(\mathbb{R}) \otimes \mathcal{F}^0$. The crude cocycle property (8) and Fubini's Theorem give $\Omega_0 \in \mathcal{F}^0$ and $P(\Omega_0) = 1$. Define

$$\Omega_1 := \{\omega \in \Omega : \ \theta(s, \omega) \in \Omega_0 \text{ for Lebesgue} - \text{almost all } s \in \mathbb{R}\}.$$

Then, again by Fubini's Theorem, it follows that $\Omega_1 \in \mathcal{F}^0$ and $P(\Omega_1) = 1$. Since H is second countable and metrizable, it is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$, where $\mathbb{N}$ is the set of all positive integers (Bourbaki [4], p. 156). So we will assume that $H \subset [0, 1]^{\mathbb{N}}$ and define $\tilde{\varphi} : \mathbb{R}\times\Omega \to [0, 1]^{\mathbb{N}}$ by

$$\tilde{\varphi}(t, \omega) = \begin{cases} \operatorname{ess}\limsup_{s\to 0+} \varphi(t-s, \theta(s, \omega)), & \text{for } \omega \in \Omega_1,\ t \in \mathbb{R} \\ e, & \text{for } \omega \notin \Omega_1,\ t \in \mathbb{R} \end{cases} \tag{10}$$

where the ess lim sup is taken component-wise. *See* Dellacherie and Meyer [5], Chapter IV for the definition of ess lim sup.

If $\omega \in \Omega_0$ and $s \in \mathbb{R}$, then by (8)

$$\varphi(t+s+u, \omega) = \varphi(t, \theta(s+u, \omega)) \star \varphi(s+u, \omega)$$

for a.a. $u \in \mathbb{R}$ and all $t \in \mathbb{R}$. Replacing t by $t-u$, we get

$$\varphi(t+s, \omega) \star \varphi^{-1}(s+u, \omega) = \varphi(t-u, \theta(u, \theta(s, \omega))) \tag{11}$$

for a.a. $u \in \mathbb{R}$ and all $t \in \mathbb{R}$. Now take ess lim sup as $u \to 0+$ in (11) and observe that the left-hand side is right-continuous in u. Then we get

$$\tilde{\varphi}(t, \theta(s, \omega)) = \varphi(t+s, \omega) \star \varphi^{-1}(s, \omega)$$

for *all* $\omega \in \Omega_0 \cap \Omega_1$ and *all* $s, t \in \mathbb{R}$. This implies

$$\tilde{\varphi}(t, \omega) = \begin{cases} \varphi(t-s, \theta(s, \omega)) \star \varphi^{-1}(-s, \theta(s, \omega)), \\ \quad \text{for } \omega \in \Omega_1,\ \theta(s, \omega) \in \Omega_0,\ t \in \mathbb{R} \\ e, \quad \text{for } \omega \notin \Omega_1,\ t \in \mathbb{R}. \end{cases} \tag{12}$$

In particular one obtains from (12) that $\tilde{\varphi}$ takes values in H. If we replace the "ess lim sup" in (10) by "ess lim inf", we still get (12). So the "ess lim sup" in (10) is actually an "ess lim".

By putting $s = 0$ in (12), it follows that $\tilde{\varphi}$ and φ agree on $\mathbb{R} \times (\Omega_0 \cap \Omega_1)$. Hence $\tilde{\varphi}$ satisfies conclusions (*b*) and (*c*) of the theorem. For $\omega \in \Omega_1$ and $\theta(u, \omega) \in \Omega_0$, we get from (12) the following relation

$$\tilde{\varphi}(t+s, \omega) = \varphi(t+s-u, \theta(u, \omega)) \star \varphi^{-1}(-u, \theta(u, \omega)).$$

On the other hand, we have

$$\begin{aligned}
&\tilde{\varphi}(t, \theta(s, \omega)) \star \tilde{\varphi}(s, \omega) \\
&\quad = \varphi(t+s-u, \theta(u, \omega)) \star \varphi^{-1}(s-u, \theta(u, \omega)) \\
&\quad\quad \star \varphi(s-u, \theta(u, \omega)) \star \varphi^{-1}(-u, \theta(u, \omega)) \\
&\quad = \varphi(t+s-u, \theta(u, \omega)) \star \varphi^{-1}(-u, \theta(u, \omega)),
\end{aligned}$$

for $\omega \in \Omega_1$ and $\theta(u, \omega) \in \Omega_0$.

Therefore (8) holds identically for all $\omega \in \Omega$ and all $s, t \in \mathbb{R}$. Finally for each $k \in \mathbb{N}$ define

$$\tilde{\varphi}^{(k)}(t, \omega) = \begin{cases} \operatorname{ess} \sup_{s \in [0, 1/k]} \varphi(t-s, \theta(s, \omega)), & \text{for } \omega \in \Omega_1, \ t \in \mathbb{R} \\ e, & \text{for } \omega \notin \Omega_1, \ t \in \mathbb{R}. \end{cases}$$

Here again the ess sup is taken component-wise. It is clear that each $\tilde{\varphi}^{(k)}$ is $(B(\mathbb{R}) \otimes \mathcal{F}^0, B(H))$-measurable. Since

$$\tilde{\varphi}(t, \omega) = \lim_{k \to \infty} \tilde{\varphi}^{(k)}(t, \omega)$$

for all $t \in \mathbb{R}$ and all $\omega \in \Omega$, and H is metrizable, it follows that $\tilde{\varphi}$ is $(B(\mathbb{R}) \otimes \mathcal{F}^0, B(H))$-measurable. This proves requirement (i) of Definition 3.1, and the proof of the theorem is complete. ∎

Remark. – It is easy to see that one could have defined $\tilde{\varphi}$ by (12) rather than (10). The advantage of using (10), however, is that it gives a simple proof of the measurability requirement (i) of Definition 3.1 for the map $\tilde{\varphi}$.

COROLLARY 3.1. – *Let $N_i : \mathbb{R} \times \Omega \to \mathbb{R}$, $i = 1, 2$ be real helix semimartingales. Then there exists a version of $[N_1, N_2]$ which is a helix. If either N_1 or N_2 has all sample paths continuous, then the helix version of $[N_1, N_2]$ can be chosen to also have all sample paths continuous.*

Proof. – It is proved in Protter ([25], p. 131) that $[N_1]$ and $[N_2]$ are crude helices and hence $[N_1, N_2]$ is also a crude helix by polarization. Hence the assertion follows from Theorem 3.1 except for the continuity statement. If N_1 or N_2 is sample continuous, then, $[N_1, N_2]$ is a.s. sample continuous. Picking a sample continuous crude helix version of $[N_1, N_2]$ the assertion follows again from Theorem 3.1. ∎

Recall that the driving cadlag process L splits up in the form $L = M + V$ where M is a continuous helix local martingale. Corollary 3.2 below says that the helix property of M induces a multiplicative cocycle property for the solution of the linear s.d.e.:

$$\left.\begin{aligned} d\varphi(t) &= dM(t)\,\varphi(t), \quad t > 0 \\ \varphi(0) &= I. \end{aligned}\right\} \tag{III}$$

Let $\tilde{\varphi} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ be a solution of the above s.d.e. It is well known that (III) has a unique solution which has a version, also denoted by the same symbol $\tilde{\varphi}$, that is continuous and invertible for all $\omega \in \Omega$, $t \geq 0$.

COROLLARY 3.2. – *Suppose M is a continuous $\mathbb{R}^{n\times n}$-valued helix local martingale. Then there is a process $\varphi : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ such that*

(i) *$\varphi\,|\mathbb{R}^+$ is indistinguishable from the solution $\tilde{\varphi}$ of (III).*

(ii) *φ is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted and $(B(\mathbb{R}) \otimes \mathcal{F}^0,\, B(\mathbb{R}^{n\times n}))$-measurable.*

(iii) *φ is a $GL(n)$-valued perfect cocycle.*

(iv) *For all $\omega \in \Omega$, the paths $\varphi(\cdot,\, \omega)$ are continuous.*

Proof. – Using Remarks (ii), (iii) following Definition 3.1 together with Theorem 3.1, it is enough to show that the solution $\tilde{\varphi}$ of (III) satisfies the crude cocycle property on $\mathbb{R}^+$. Now this follows easily from uniqueness. Indeed we may fix s, $t \geq 0$ and let $\hat{\varphi} : \mathbb{R} \times \Omega \to GL(n)$ satisfy the s.d.e.

$$\begin{aligned} d\hat{\varphi}(u,\, \omega) &= dM(u,\, \theta(s,\, \omega))\,\hat{\varphi}(u,\, \omega), \quad u \geq 0 \\ \hat{\varphi}(0,\, \omega) &= \tilde{\varphi}(s,\, \omega) \end{aligned} \tag{13}$$

with respect to the filtration shifted by s. Using linearity in the initial conditions, the uniqueness of solutions and the helix property of M, it follows that

$$\tilde{\varphi}(t+s,\, \omega) = \hat{\varphi}(t,\, \omega) = \tilde{\varphi}(t,\, \theta(s,\, \omega)) \star \tilde{\varphi}(s,\, \omega)$$

a.s. Now we use the first remark after Definition 3.1 and Theorem 3.1 to get the assertion of the corollary. Observe that φ is adapted, since $\tilde{\varphi}$ is and $\mathcal{F}_t$ is complete for every $t \in \mathbb{R}$. ■

We now consider the stochastic integral $\int_0^t \varphi^{-1}(u)\,dN(u)$. The next result gives a version of this integral which satisfies the additive property (14) below. This fact will be needed in the construction of the flow of the s.f.d.e. (I).

COROLLARY 3.3. – *Assume Hypotheses (C_4) and (C_5). Let φ be the cocycle constructed in Corollary 3.2. Define the process $\tilde{Z} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ by*

$$\tilde{Z}(t,\, \cdot) := \int_0^t \varphi^{-1}(u)\,dN(u)$$

for $t \geq 0$ a.s.

Then there is a process $Z : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ with the following properties:

(i) $Z \,|\, \mathbb{R}^+$ *and* $\tilde{Z}$ *are indistinguishable.*

(ii) Z *is* $(\mathcal{F}_t)_{t\in\mathbb{R}}$*-adapted and* $(B(\mathbb{R}) \otimes \mathcal{F}^0, B(\mathbb{R}^{n\times n}))$*-measurable.*

(iii) $$Z(t+s, \omega) - Z(s, \omega) = \varphi^{-1}(s, \omega) Z(t, \theta(s, \omega))$$

for all $s, t \in \mathbb{R}$ and all $\omega \in \Omega$. (14)

(iv) *For all* $\omega \in \Omega$, *the paths* $Z(\cdot, \omega)$ *are cadlag.*

Proof. – It is enough to show that $\tilde{Z}$ satisfies (14) for every fixed $s, t \in \mathbb{R}^+$ a.s. *with the exceptional set possibly depending on* s, t. To see this, observe that by the proof of Remark (iii) following Definition 3.1, we conclude that $\tilde{Z}$ can be extended to a process with a.a. sample paths cadlag, defined on the *whole* of $\mathbb{R}$ and such that (14) holds for fixed $s, t \in \mathbb{R}$, a.s. Now apply Remark (ii) following the statement of Theorem 3.1 with $(H, \star) = (\mathbb{R}^{n\times n}, +)$ and $G = GL(n)$, the general linear group of all $n \times n$-invertible matrices with matrix multiplication. This will give the required "perfect" version Z of $\tilde{Z}$ which satisfies all the conclusions of the corollary.

It remains now to check that

$$\tilde{Z}(t+s, \omega) - \tilde{Z}(s, \omega) = \varphi^{-1}(s, \omega) \tilde{Z}(t, \theta(s, \omega)) \qquad (14')$$

for all $s, t \in \mathbb{R}^+$, a.s. Following Sharpe and Protter [25], define the "big shift" $(\Theta_h)_{h\geq 0}$ acting on any process $y(t)$, $t \geq 0$, by setting

$$(\Theta_h y)(t) := y(t-h, \theta(h, \omega)) \cdot 1_{[h,\infty)}(t), \quad t \geq 0$$

where $1_{[h,\infty)}$ is the indicator function of $[h, \infty)$. If $y(t)$, $t \geq 0$, is a semimartingale and $H : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is predictable and y-integrable, then it follows from Protter ([25], Theorem 3.1 (vi)) that, for every $s \geq 0$, $\Theta_s H$ is $\Theta_s y$-integrable and

$$(\Theta_s H) \cdot (\Theta_s y) = \Theta_s (H \cdot y) \text{ a.s.}, \qquad (15)$$

where $H \cdot y$ denotes the stochastic integral

$$(H \cdot y)(t) = \int_0^t H(u)\, dy(u) \quad t \geq 0.$$

This result obviously extends to matrix-valued processes. Now set $H = \varphi^{-1}$, $y = N \,|\, [0, \infty)$ and use the fact that $\tilde{Z} = \varphi^{-1} \cdot N$ to deduce that the right-hand side of (15) evaluated at $s+t$ is $\tilde{Z}(t, \theta(s, \omega))$. On the other hand, the left-hand side becomes

$$\int_s^{s+t} \varphi^{-1}(u-s, \theta(s, \omega))\, dN(u-s, \theta(s, \omega)). \qquad (16)$$

Using the cocycle property for φ, we can replace the integrand in (16) by $\varphi(s, \omega)\varphi^{-1}(u, \omega)$. Since N is a helix, then the integral (16) is equal to

$$\varphi(s, \omega)(\tilde{Z}(t+s, \omega) - \tilde{Z}(s, \omega)),$$

a.s. This completes the proof of the corollary. ■

4. THE RANDOM INTEGRAL EQUATION

We are now in a position to formulate the random integral equation which we advertised in Section 1. We shall first show that this integral equation is pathwise equivalent to our s.f.d.e. (I). We then establish the existence of a unique solution to the integral equation which depends *linearly* and *continuously* on the initial data $(v, \eta) \in M_2$. The cocycle property for the trajectory $X(t) := (x(t), x_t)$, $t \geq 0$, then follows directly from uniqueness of the solution to the integral equation.

Throughout this section we assume Hypotheses (C_i), $i = 1, 2, 3, 4, 5$, and take $\varphi : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n \times n}$, $Z : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n \times n}$ to be the processes constructed in Corollaries 3.2, 3.3 of the last section.

Let $[M, N]$ denote the $\mathbb{R}^{n \times n}$-valued mutual variation process of M and N, viz. $[M, N] = ([M, N]_{ij})$ where

$$[M, N]_{ij} := \sum_{m=1}^{n} [M_{im}, N_{mj}],$$

$M = (M_{ij})_{i,j=1}^n$, $N = (N_{ij})_{i,j=1}^n$. From Hypothesis (C_4), (C_5) and Corollary 3.1, it follows that there is a $B(\mathbb{R}) \otimes \mathcal{F}^0$-measurable, continuous helix version of $[M, N]$. From now on, this version will be denoted by the same symbol $[M, N]$.

Denote by $\mathcal{E}$ the vector space of all Borel-measurable maps $g : [-r, \infty) \to \mathbb{R}^n$ such that $g \mid [-r, 0]$ belongs to $\mathbb{L}^2([-r, 0], \mathbb{R}^n)$ and $g \mid [0, \infty)$ is cadlag. For each $\omega \in \Omega$ define the linear map $I(\omega) : \mathcal{E} \to \mathcal{E}$ as follows: For any $g \in \mathcal{E}$ set

$$I(\omega)(g)(t) := g(t) \quad \text{a.e. } t \in [-r, 0] \tag{17}$$

and

$$
\begin{aligned}
I(\omega)(g)(t) := \varphi(t,\omega)\Big[& g(0) - \int_0^t Z(s,\omega)\bar{K}(s,s,\omega)g(s)\,ds \\
& + \int_{-r}^{t-r} Z(s+r,\omega)\bar{K}(s+r,s,\omega)g(s)\,ds \\
& - \int_{-r}^{t}\int_{s\vee 0}^{t\wedge(s+r)} Z(u,\omega) \\
& \times \frac{\partial}{\partial u}\bar{K}(u,s,\omega)\,du\; g(s)\,ds \\
& + Z(t,\omega)\int_{t-r}^{t}\bar{K}(t,s,\omega)g(s)\,ds \\
& + \int_{-r}^{t}\int_{[(s-t)\vee(-r),\,0\wedge s]} \varphi^{-1}(s-u,\omega)\,\mu(s-u,\omega)(du)\; g(s)\,ds \\
& + \int_0^t \varphi^{-1}(s,\omega)\,dV(s,\omega)\,g(s-) \\
& - \int_{-r}^{t}\int_{s\vee 0}^{t\wedge(s+r)} \varphi^{-1}(u,\omega)\,d[M,N](u,\omega) \\
& \times \bar{K}(u,s,\omega)g(s)\,ds\Big]
\end{aligned}
\tag{18}
$$

for $t \in \mathbb{R}^+$.

Our first result in this section (Theorem 4.1 below) shows that the random family of integral equations

$$
\left.\begin{aligned}
x = I(\omega)(x), \qquad x \in \mathcal{E}, \quad \omega \in \Omega \\
(x(0),\, x_0) = (v,\,\eta) \in M_2
\end{aligned}\right\}
\tag{IV}
$$

is equivalent to the s.f.d.e. (I).

The existence of a unique (cadlag $(\mathcal{F}_t)_{t\geq 0}$-adapted) solution to the above integral equation will be established in Theorem 4.2. We now prove:

THEOREM 4.1. – *The s.f.d.e. (I) and the random integral equation (IV) are equivalent: Every cadlag $(\mathcal{F}_t)_{t\geq 0}$-adapted solution of (IV) is a solution of (I). Conversely, every solution of (I) has a version which satisfies (IV).*

Proof. – Fix $(v, \eta) \in M_2$ and let $x : [-r, \infty) \times \Omega \to \mathbb{R}^n$ be a solution of the s.f.d.e. (I) starting off at (v, η).

The $\mathbb{R}^n$-valued process

$$H(t) := v + \int_0^t \int_{[-r,\,0]} \mu(u)(ds)\, x(u+s)\, du + \int_0^t dV(u)\, x(u-) + \int_0^t dN(u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds, \quad t \in \mathbb{R}^+ \tag{19}$$

is clearly an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale because x is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Denote by $[M, H]$ the B.V. process

$$[M, H] := ([M, H]_i)_{i=1}^n, \quad [M, H]_i(t) := \sum_{j=1}^n [M_{ij}, H_j](t), \qquad t \in \mathbb{R}^+.$$

Applying the integration by parts formula (Métivier [20], Equation (26.9.3), p. 185) to the process

$$\tilde{x}(t) := \varphi(t)\left\{v + \int_0^t \varphi^{-1}(u)\, dH(u) - \int_0^t \varphi^{-1}(u)\, d[M, H](u)\right\},$$

$t \in \mathbb{R}^+$, it is easy to see that $\tilde{x}$ satisfies the linear s.d.e.

$$\left.\begin{aligned} d\tilde{x}(t) &= dH(t) + dM(t)\,\tilde{x}(t), \quad t \in \mathbb{R}^+ \\ \tilde{x}(0) &= v \end{aligned}\right\} \tag{V}$$

a.s.

Now our s.f.d.e. (I) says that x also satisfies the above s.d.e. (V). So by uniqueness of solutions to (V) we get that

$$\tilde{x}(t) = x(t) = \varphi(t)\left\{v + \int_0^t \varphi^{-1}(u)\, dH(u) - \int_0^t \varphi^{-1}(u)\, d[M, H](u)\right\} \tag{20}$$

for all $t \in \mathbb{R}^+$, a.s. (*see* also Jacod [14], Theorem 2).

Inserting H from (19) into (20) and using the definition of Z (Corollary 3.3), we get

$$\begin{aligned} x(t) = \varphi(t)\Big\{ v &+ \int_0^t \varphi^{-1}(u) \int_{[-r,\,0]} \mu(u)(ds)\, x(u+s)\, du \\ &+ \int_0^t \varphi^{-1}(u)\, dV(u)\, x(u-) \\ &+ \int_0^t dZ(u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds \\ &- \int_0^t \varphi^{-1}(u)\, d[M,\, N](u) \\ &\times \int_{-r}^0 K(u)(s)\, x(u+s)\, ds \Big\}, \quad t \in \mathbb{R}^+. \end{aligned} \tag{21}$$

Note that the last term in the above relation is obtained via the equality

$$\begin{aligned} &\left[M,\, \int_0^{(\cdot)} dN(u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds \right](t) \\ &\quad = \int_0^t d[M,\, N](u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds, \quad t \in \mathbb{R}^+. \end{aligned} \tag{22}$$

Now in (21) integration by parts (Métivier [20], p. 192) and Hypothesis (C_3) yield:

$$\begin{aligned} &\int_0^t dZ(u) \int_{-r}^0 K(u)(s)\, x(u+s)\, ds \\ &\quad = Z(t) \int_{-r}^0 K(t)(s)\, x(t+s)\, ds \\ &\qquad - \int_0^t Z(u) \frac{d}{du} \int_{-r}^0 K(u)(s)\, x(u+s)\, ds\, du \\ &\quad = Z(t) \int_{-r}^0 K(t)(s)\, x(t+s)\, ds \\ &\qquad - \int_0^t Z(u) \frac{d}{du} \int_{u-r}^u \bar{K}(u,\, s')\, x(s')\, ds'\, du \\ &\quad = Z(t) \int_{-r}^0 K(t)(s)\, x(t+s)\, ds - \int_0^t Z(u) \{ K(u)(0)\, x(u) \end{aligned}$$

$$- K(u)(-r)\,x(u-r)\}\,du$$
$$- \int_0^t Z(u) \int_{u-r}^{u} \frac{\partial}{\partial u} (\bar{K}(u, s'))\,x(s')\,ds'\,du, \qquad t \in \mathbb{R}^+. \quad (23)$$

Substituting the above relation into (21) and changing variables and the order of integration implies that x satisfies the integral equation (IV) a.s. for all $t \in \mathbb{R}^+$.

Conversely, let x be a cadlag $(\mathcal{F}_t)_{t \geq 0}$-adapted process which solves the integral equation (IV). Using (23) it is easy to see that x satisfies (21). If we define H by (19) as before, then (20) holds. The latter relation implies that x fulfills (V) and is therefore a solution of our s.f.d.e. (I). This completes the proof of the theorem. ■

The following result is the main theorem of this section. It is crucial for the existence of the Lyapunov spectrum of our s.f.d.e. (I). Basically it says that the random integral equation (IV) has a unique solution which yields a robust version of the trajectory $(x(t), x_t)$ of (I).

THEOREM 4.2. – *Let Hypotheses* (C) *be satisfied. Then for each* $\omega \in \Omega$ *and* $(v, \eta) \in M_2$, *the integral equation* (IV) *has a unique cadlag solution* $x(\cdot, \omega, (v, \eta)) : [-r, \infty) \to \mathbb{R}^n$. *Define the map* $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ *by*

$$X(t, \omega, (v, \eta)) := (x(t, \omega, (v, \eta)), x_t(\cdot, \omega, (v, \eta))) \qquad (24)$$

for $t \in \mathbb{R}^+$, $\omega \in \Omega$, $(v, \eta) \in M_2$. *Then the following is true:*

(i) *For each* $(v, \eta) \in M_2$, $\{x(t, \cdot, (v, \eta)) : t \in \mathbb{R}^+\}$ *is the unique* $(\mathcal{F}_t)_{t \geqq 0}$*-adapted solution of the s.f.d.e. (I) starting off at* (v, η).

(ii) *For every* $\omega \in \Omega$ *and* $(v, \eta) \in M_2$, *the path* $X(\cdot, \omega, (v, \eta))$: $\mathbb{R}^+ \to M_2$ *is cadlag.*

(iii) *The map* $X(t, \omega, \cdot) : M_2 \to M_2$ *is continuous linear for all* $t \in \mathbb{R}^+$ *and* $\omega \in \Omega$.

(iv) *The map* $(t, \omega) \mapsto \|X(t, \omega, \cdot)\|_{L(M_2)}$ *from* $\mathbb{R}^+ \times \Omega$ *to* $\mathbb{R}$ *is* $(B(\mathbb{R}^+) \otimes \mathcal{F}^0, B(\mathbb{R}))$*-measurable and locally bounded in t for each* $\omega \in \Omega$.

(v) *The map* $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ *is* $(B(\mathbb{R}^+) \otimes \mathcal{F}^0 \otimes B(M_2), B(M_2))$*-measurable.*

(vi) *For each* $t \geq r$ *and* $\omega \in \Omega$, $X(t, \omega, \cdot) : M_2 \to M_2$ *is compact.*

(vii) $$X(t_2, \theta(t_1, \omega), \cdot) \circ X(t_1, \omega, \cdot) = X(t_1 + t_2, \omega, \cdot) \qquad (25)$$

for all $\omega \in \Omega$ *and* $t_1, t_2 \in \mathbb{R}^+$.

(viii) *For every* $\omega \in \Omega$, $t \in \mathbb{R}^+$*and* $(v, \eta) \in M_2$ *the map* $s \mapsto X(t, \theta(s, \omega), (v, \eta))$ *from* $\mathbb{R}$ *to* M_2 *is right-continuous.*

Remark. – The map $t \mapsto X(t, \omega, \cdot)$ from $\mathbb{R}^+$ to $L(M_2)$ will *not* be right-continuous in general.

Proof of Theorem 4.2. – We establish a unique cadlag solution $x(\cdot, \cdot, (v, \eta)) : [-r, \infty) \times \Omega \to \mathbb{R}^n$ for the integral equation (IV) using the classical technique of successive approximations.

Fix $\omega \in \Omega$ and $0 < T < \infty$ till further notice. Define a sequence of successive approximations

$$\{x^k(t, \omega, (v, \eta)) : t \in [-r, \infty), (v, \eta) \in M_2\}, \qquad k = 1, 2, \ldots$$

as follows:

$$x^1(t, \omega, (v, \eta)) := \begin{cases} \eta(t) & \text{a.e.}\, t \in [-r, 0) \\ v & t \geq 0 \end{cases} \tag{26}$$

and

$$x^{k+1}(t, \omega, (v, \eta)) = I(\omega)(x^k(\cdot, \omega, (v, \eta)))(t), \qquad t \geq -r, \tag{27}$$

$(v, \eta) \in M_2$, $k \geq 1$. It is clearly seen, by induction on k, that $x^k(\cdot, \omega, (v, \eta)) \in \mathcal{E}$ for $k \geq 1$ and $(v, \eta) \in M_2$.

Using Hypotheses (C_1)-(C_3) it is easy to see from (18) that there exist positive numbers C_1, C_2, C_3 (depending on ω, T, μ, K, N and L) such that

$$|I(\omega)(g)(t| \leq C_1|g(0)| + C_2 \int_{-r}^{t} |g(u)|du + C_3 \int_0^t |g(u-)| d|V|(u) \tag{28}$$

for all $g \in \mathcal{E}$ and $0 \leq t \leq T$.

Now let $\alpha : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ stand for the non-negative cadlag increasing process

$$\alpha(t, \omega) := t + |V|(t, \omega), \qquad t \geq 0.$$

Suppressing $\omega \in \Omega$ and $(v, \eta) \in M_2$ for the time being, we shall show that

$$|x^{k+1}(t) - x^k(t)| \leq \frac{C_4^{k-1}(\alpha(t))^{k-1}}{(k-1)!} \sup_{0 \leq s \leq T} |x^2(s) - x^1(s)|, \tag{29}$$

$k \geq 1$, for all $0 \leq t \leq T$, where $C_4 := C_4(T, \omega) := C_2 + C_3$. To prove (29) we use induction on $k \geq 1$. Note first that it holds trivially for $k = 1$.

Suppose now that (29) is true for some $k \geq 1$. Then it follows from (28) that, for $0 \leq t \leq T$,

$$\begin{aligned} & |x^{k+2}(t) - x^{k+1}(t)| \\ & \leq C_4 \int_0^t |x^{k+1}(u-) - x^k(u-)| d\alpha(u) \\ & \leq C_4 \frac{C_4^{k-1}}{(k-1)!} \sup_{0 \leq s \leq T} |x^2(s) - x^1(s)| \int_0^t \alpha(u-)^{k-1} d\alpha(u) \\ & \leq C_4 \frac{\alpha(t)^k}{k!} \sup_{0 \leq s \leq T} |x^2(s) - x^1(s)| \end{aligned} \tag{30}$$

where we have used the inequality

$$\int_0^t \alpha(u-)^{k-1} d\alpha(u) \leq \frac{1}{k} \alpha(t)^k, \qquad t \geq 0, \quad k \geq 1. \tag{31}$$

Note that (31) is easily checked by using integration by parts and the fact that α is non-negative and increasing. This proves (30).

Now let $B := \{(v, \eta) \in M_2 : \|(v, \eta)\|_{M_2} \leq 1\}$ be the closed unit ball in M_2. Furthermore, let E be the space of all bounded maps $f : [0, T] \times B \to \mathbb{R}^n$ such that for each $(v, \eta) \in B$, $f(\cdot, (v, \eta))$ is cadlag and for each $t \in [0, T]$, $f(t, \cdot)$ is continuous on B. We equip E with the Banach norm

$$\|f\|_E = \sup_{0 \leq t \leq T} \sup_{(v, \eta) \in B} |f(t, (v, \eta))|.$$

We no longer suppress (v, η), but rather think of x^k as a function $x^k(\cdot, \omega, \cdot)$ of $(t, (v, \eta)) \in \mathbb{R}^+ \times M_2$ into $\mathbb{R}^n$. It follows immediately from (28) and (29) that for each $k \geq 1$

$$\|x^k(\cdot, \omega, \cdot)\|_E < \infty.$$

It follows from (30) that

$$\begin{aligned} & \sum_{k=1}^{\infty} \|x^{k+1}(\cdot, \omega, \cdot) - x^k(\cdot, \omega, \cdot)\|_E \\ & \quad \leq \exp(C_4(T, \omega)\, \alpha(T, \omega)) \|x^2(\cdot, \omega, \cdot) - x^1(\cdot, \omega, \cdot)\|_E < \infty. \end{aligned} \tag{32}$$

Hence $\{x^k(\cdot, \omega, \cdot)\}_{k=1}^{\infty}$ converges to a limit $x(\cdot, \omega, \cdot) \in E$. This limit extends by linearity to a map $x(\cdot, \omega, \cdot) : \mathbb{R}^+ \times M_2 \to \mathbb{R}^n$ such that for each $t \in \mathbb{R}^+$, $x(t, \omega, \cdot) : M_2 \to \mathbb{R}^n$ is continuous linear; and for each $(v, \eta) \in M_2$, $x(\cdot, \omega, (v, \eta)) : \mathbb{R}^+ \to \mathbb{R}^n$ is cadlag. Clearly x solves the integral equation (IV).

To prove uniqueness of the solution of (IV), fix $\omega \in \Omega$ and let y and $z \in \mathcal{E}$ be two solutions such that $(y(0), y_0) = (z(0), z_0)$ and put $x := y - z$. By a similar computation to the one used to get (30), we obtain

$$|x(t)| \leq C_4^k \frac{\alpha(t)^k}{k!} \sup_{0 \leq s \leq T} |x(s)|,$$

for all $k \geq 1$. Letting $k \to \infty$, we get $x \equiv 0$.

Let the flow $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ be defined by (24). From the definition of M_2 and the fact that $x(\cdot, \omega, \cdot) \in E$, it follows that

$$\sup_{0 \leq t \leq T} \sup_{(v, \eta) \in B} \|X(t, \omega, (v, \eta))\|_{M_2} < \infty. \tag{33}$$

Hence (iii) and the local boundedness statement in (iv) follow. Assertion (ii) of the theorem follows from the fact that x is cadlag and the map $t \mapsto x_t \in \mathbb{L}^2([-r, 0], \mathbb{R}^n)$ is continuous.

Let us now prove all measurability assertions. Looking at (18) it follows by induction that for fixed t, (v, η) the map $\omega \mapsto x^k(t, \omega, (v, \eta))$ is $(\mathcal{F}^0 \cap \mathcal{F}_t, B(\mathbb{R}^n))$-measurable. Since x is the pointwise limit of the x^k, x enjoys the same measurability property. In particular, (i) follows. Using the fact that $t \mapsto x(t, \omega, (v, \eta))$ is right-continuous, it follows that $(t, \omega) \mapsto x(t, \omega, (v, \eta))$ is $(B(\mathbb{R}^+) \otimes \mathcal{F}^0, B(\mathbb{R}^n))$-measurable for every $(v, \eta) \in M_2$.

Observe first that for fixed $t \in \mathbb{R}^+$, f, $(v, \eta) \in M_2$, the map $\omega \mapsto \|X(t, \omega, (v, \eta)) - f(\cdot)\|_{M_2}$ is $(\mathcal{F}^0, B(\mathbb{R}))$-measurable. This follows from the definition of the M_2-norm and Fubini's theorem. In view of the separability of M_2, the above statement implies that $\omega \mapsto X(t, \omega, (v, \eta))$ is $(\mathcal{F}^0, B(M_2))$-measurable.

Since $X(t, \omega, (v, \eta))$ is right-continuous in t (and M_2 is metric), it follows that $(t, \omega) \mapsto X(t, \omega, (v, \eta))$ is $(B(\mathbb{R}^+) \otimes \mathcal{F}^0, B(M_2))$-measurable for fixed $(v, \eta) \in M_2$. Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of M_2. Then

$$X(t, \omega, (v, \eta)) = \sum_{i=1}^{\infty} < (v, \eta), e_i > X(t, \omega, e_i)$$

implies (v) since every term in the sum is $(B(\mathbb{R}^+) \otimes \mathcal{F}^0 \otimes B(M_2), B(M_2))$-measurable.

The measurability statement in (iv) follows from the above and the fact that M_2 is separable.

To prove (vi) we fix $T \geq r$ and $\omega \in \Omega$. If we define $X^k : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ by

$$X^k (u, \omega, (v, \eta)) := (x^k (u, \omega, (v, \eta)), x^k_u (\cdot, \omega, (v, \eta)))$$

for $u \in \mathbb{R}^+$, $(v, \eta) \in M_2$, then it is clear that each $X^k (u, \omega, \cdot) : M_2 \to M_2$ is continuous linear and

$$\lim_{k \to \infty} \|X^k (T, \omega, \cdot) - X (T, \omega, \cdot)\|_{L (M_2)} = 0.$$

Hence it suffices to show that $X^k (T, \omega, \cdot)$ is compact for every $k \geq 1$. We will even prove the stronger result that

$$(v, \eta) \to x^k (\cdot, \omega, (v, \eta))$$

is compact from M_2 to $D ([0, T], \mathbb{R}^n)$, equipped with the sup-norm. We prove this by induction on k. The case $k = 1$ is obvious. Suppose that the inductive hypothesis holds for some $k \geq 1$. Pick a sequence $\{ (v_i, \eta_i) \}_{i \in \mathbb{N}}$ in B . By the inductive hypothesis we can choose a subsequence (which we denote by the same symbols) such that $x^k (t, \omega, (v_i, \eta_i))$ converges uniformly on $[0, T]$ as $i \to \infty$. Define $f_i := x^k (\cdot, \omega, (v_i, \eta_i)) | [0, \infty)$ and $\bar{I} (\eta, f) := I (\omega) (g)$ in case $\eta \in \mathbb{L}_2 ([-r, 0), \mathbb{R}^n)$, $f \in D ([0, \infty), \mathbb{R}^n)$ and $g \in \mathcal{E}$ is equal to η on $[-r, 0)$ and equal to f on $\mathbb{R}^+$. Then

$$\begin{aligned} I (\omega) (x^k (\cdot, \omega, (v_i, \eta_i))) (t) - I (\omega) (x^k (\cdot, \omega, (v_j, \eta_j))) (t) \\ = \bar{I} (0, f_i - f_j) (t) + \bar{I} (\eta_i - \eta_j, 0) (t). \end{aligned} \tag{34}$$

By (28) and the inductive hypothesis the first term on the right-hand side of (34) converges to zero uniformly on $[0, T]$ as $i, j \to \infty$.

It is easy to see from (18) and (C_2), (C_3) that for $\eta \in \mathbb{L}^2 ([-r, 0), \mathbb{R}^n)$, $t \geq 0$,

$$\begin{aligned} \bar{I} (\eta, 0) (t) = \varphi (t, \omega) \int_{-r}^0 F_1 (t, s, \omega) \eta (s) \, ds \\ + \varphi (t, \omega) Z (t, \omega) \int_{-r}^0 F_2 (t, s, \omega) \eta (s) \, ds \end{aligned}$$

where $\sup_{0 \leq t \leq T} \operatorname{esssup}_{-r \leq s \leq 0} |F_i (t, s, \omega| < \infty$ and

$$\lim_{h \downarrow 0} \sup_{0 \leq t \leq T} \operatorname{esssup}_{-r \leq s \leq 0} |F_i (t + h, s) - F_i (t, s)| = 0$$

for $i = 1, 2$.

This implies

$$\lim_{h \downarrow 0} \sup_{\|\eta\|_{L^2} \leq 1} \sup_{0 \leq t \leq T} \left| \int_{-r}^{0} (F_i(t+h, s) - F_i(t, s)) \eta(s) \, ds \right| = 0$$

for $i = 1, 2$; and so Arzela-Ascoli's Theorem implies that there exists a subsequence of $\{\eta_i\}_{i \in \mathbb{N}}$ such that $\bar{I}(\eta_i, 0)(t)$ converges uniformly on $[0, T]$. This proves (vi).

Next we prove the cocycle property (vii) for (X, θ). Fix $t_1 \geq 0$, $\omega \in \Omega$ and $(v, \eta) \in M_2$. Let $y^i(\cdot, \omega) : [-r, \infty) \to \mathbb{R}^n$, $i = 1, 2$, denote the paths

$$y^1(t, \omega) := x(t, \theta(t_1, \omega), X(t_1, \omega, (v, \eta))), \qquad t \geq -r \tag{35}$$

$$y^2(t, \omega) := x(t + t_1, \omega, (v, \eta)), \qquad t \geq -r. \tag{36}$$

Note that the cocycle property (25) will follow immediately if we show that

$$y^1(t, \omega) = y^2(t, \omega), \qquad t \geq -r. \tag{37}$$

To prove the above relation (37), first observe that

$$(y^1(0, \omega), y^1_0(\cdot, \omega)) = X(t_1, \omega, (v, \eta)) = (y^2(0, \omega), y^2_0(\cdot, \omega)). \tag{38}$$

We shall next prove that y^2 satisfies the integral equation (IV) with ω replaced by $\theta(t_1, \omega)$, viz:

$$y^2(t, \omega) = I(\theta(t_1, \omega))(y^2(\cdot, \omega))(t), \qquad t \geq 0. \tag{39}$$

Since y^1 also satisfies (39) with the same initial condition $X(t_1, \omega, (v, \eta))$, uniqueness of the solution to the integral equation will give (37) and hence (25). Because of the relation

$$y^2(t, \omega) = I(\omega)(x(\cdot, \omega, (v, \eta)))(t + t_1), \qquad t \geq 0$$

(39) will follow from

$$I(\theta(t_1, \omega))(y^2(\cdot, \omega))(t) = I(\omega)(x(\cdot, \omega, (v, \eta)))(t + t_1), \qquad t \geq 0. \tag{40}$$

Checking (40) is a routine calculation using (18) and Hypotheses (C_1), (C_3), (C_4) and (C_5). We leave it to the reader. This proves (vii).

Finally we prove (viii). Using (28) and (32) we get:

$$\begin{aligned}
&|x(t, \omega, (v, \eta))| \\
&\quad \leq \sup_{0\leq s\leq t} |x^2(s, \omega, (v, \eta)) - x^1(s, \omega, (v, \eta))| \\
&\quad\quad \times \exp(C_4(T, \omega)\,\alpha(T, \omega)) + |x^1(t, \omega, (v, \eta))| \\
&\quad \leq (C_1(T, \omega)|v| + C_2(T, \omega) \int_{-r}^{0} |\eta(u)| du \\
&\quad\quad + C_4(T, \omega)\,\alpha(T, \omega)|v| + |v|) \\
&\quad\quad \times \exp(C_4(T, \omega)\,\alpha(T, \omega)) + |v| \qquad \text{for} \quad 0 < t < T. \qquad (41)
\end{aligned}$$

Therefore

$$\|X(t, \omega, (v, \eta))\|_{M_2} \leq C_5(T, \omega)\|(v, \eta)\|_{M_2} \tag{42}$$

for some $C_5(T, \omega) < \infty$. From (18) one can deduce that C_1, C_2 and C_4 can be chosen such that

$$\sup_{0\leq s\leq T} C_i(T, \theta(s, \omega)) < \infty \qquad \text{for} \quad i = 1, 2, 4. \tag{43}$$

We will provide explicit upper bounds in Section 5. Obviously $\sup_{0\leq s\leq T} \alpha(t, \theta(s, \omega)) < \infty$ by the helix property of α. Therefore C_5 can also be chosen such that (43) holds for $i = 5$. Now for $t \geq 0$, $s \in \mathbb{R}$, $y \in M_2$, $\omega \in \Omega$ and $h \geq 0$, we have

$$\begin{aligned}
&\|X(t, \theta(s+h, \omega), y) - X(t, \theta(s, \omega), y)\|_{M_2} \\
&\quad \leq \|X(t, \theta(s+h, \omega), y) - X(t, \theta(s+h, \omega), X(h, \theta(s, \omega), y))\|_{M_2} \\
&\quad\quad + \|X(t, \theta(s+h, \omega), X(h, \theta(s, \omega), y)) - X(t, \theta(s, \omega), y)\|_{M_2}.
\end{aligned} \tag{44}$$

Using the cocycle property (25) and right-continuity of $X(\cdot, \theta(s, \omega), y)$ we see that the last term goes to zero as $h \downarrow 0$. By (42) and (43) for $i = 5$ and the fact that $\|X(h, \theta(s, \omega), y) - y\|_{M_2} \to 0$ as $h \downarrow 0$, the first term on the right-hand side of (44) also goes to zero as $h \downarrow 0$. This prove (viii), and the proof of Theorem 4.2 is complete.

Remark. – The continuity of $X(t, \omega, \cdot) : M_2 \to M_2$ in the norm $\|\cdot\|_{M_2}$ is guaranteed by Hypothesis (C_2). On the other hand, if the state space M_2 is replaced by the space $D := D([-r, 0], \mathbb{R}^n)$ of all cadlag paths

$\eta : [-r, 0] \to \mathbb{R}^n$ with the supremum norm $\|\eta\|_\infty := \sup_{-r \le s \le 0} |\eta(s)|$, then Hypothesis ($C_2$) may be considerably relaxed and Theorem 4.2 will hold with M_2 replaced by D.

5. LYAPUNOV EXPONENTS

In this section we prove the existence of a countable set of Lyapunov exponents

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, (v, \eta))\|_{M_2} \tag{45}$$

for the stochastic flow of the s.f.d.e. (I) which we constructed in Section 4 (Theorem 4.2). Such a Lyapunov spectrum corresponds to almost sure exponential growth rates for trajectories $\{(x(t), x_t) : t \ge 0, (x(0), x_0) = (v, \eta)\}$ of (I) starting off at possibly *random* initial states (v, η). The existence of the Lyapunov spectrum is achieved using Ruelle's infinite-dimensional discrete version of Oseledec's multiplicative ergodic theorem (Ruelle [28], [27]). In the hyperbolic case, when all the Lyapunov exponents are non-zero, we establish an exponential dichotomy for the flow which is invariant under the cocycle (X, θ). The *continuous-time* limit (45) is shown to exist by noting the compactness of $X(r, \omega, \cdot)$ (Theorem 4.2) and then discretizing (45) using multiples of the delay r:

$$\lim_{k \to \infty} \frac{1}{kr} \log \|X(kr, \cdot, (v, \eta))\|_{M_2}. \tag{46}$$

A key step in identifying the limits (45) and (46) is to establish the integrability property

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \|X(t_1, \theta(t_2, \cdot), \cdot)\|_{L(M_2)} < \infty \tag{47}$$

where $\|\cdot\|_{L(M_2)}$ is the uniform operator norm on $L(M_2)$ (*cf.* Lemma 4, § 4 in [23]). Much of the work in this section is directed towards realizing the above integrability property. Observe that Theorem 4.2 (ii), (iii), (v) and (viii) imply that the sup in (47) is $\mathcal{F}^0$-measurable since the sup can be taken over a countable subset of $[0, r] \times [0, r] \times B$, where B is the unit ball in M_2. To begin with, we shall impose the following moment hypotheses on the driving processes in the s.f.d.e. (I):

Hypotheses (I)

(I_1) If ν is the measure defined in Hypothesis (C_2), suppose that

$$E \operatorname*{esssup}_{-r \leq s \leq 2r} \left| \frac{d\nu(s)}{ds} \right|^2 < \infty.$$

(I_2) $E \sup_{0 \leq t \leq 2r} \operatorname*{esssup}_{t-r \leq s \leq t} \|\bar{K}(t, s, \cdot)\|^3 < \infty$

$$E \operatorname*{esssup}_{0 \leq t \leq 2r} \operatorname*{esssup}_{t-r \leq s \leq t} \left\| \frac{\partial}{\partial t} \bar{K}(t, s, \cdot) \right\|^3 < \infty$$

$E|V|^4(2r) < \infty$

(I_3) Write the semimartingale N in the form $N = N^0 + V^0$ where the local $(\mathcal{F}_t)_{t \geq 0}$- martingale $N^0 = (N_{ij}^0)_{i,j=1}^n$ and the locally bounded variation process $V^0 = (V_{ij}^0)_{i,j=1}^n$ satisfy

$$E[N_{ij}^0]^2(2r) < \infty \qquad E|V_{ij}^0|^4(2r) < \infty$$

for all $1 \leq i, j \leq n$. Note that $|V_{ij}^0|(2r)$ is the total variation of V_{ij}^0 over $[0, 2r]$.

(I_4) $[M_{ij}](1) \in \mathbb{L}^\infty(\Omega; \mathbb{R})$ for all $1 \leq i, j \leq n$.

Note that the helix property of M and (I_4) imply $[M_{ij}](t) \in \mathbb{L}^\infty(\Omega; \mathbb{R})$ for *all* $t \in \mathbb{R}$. Our moment conditions (I) are certainly not best possible. Also it will be clear from the proof of (47) that one can relax condition (I_4) at the expense of sharpening any or all of (I_1), (I_2) and (I_3).

Our first goal is to establish the integrability property (47) under Hypotheses (C) and (I). Note that (47) is implied by

$$\int_\Omega \log^+ \sup_{\substack{0 \leq t_1, t_2 \leq r \\ \|(v, \eta)\| \leq 1}} |x(t_1, \theta(t_2, \omega), (v, \eta))| dP(\omega) < \infty. \tag{48}$$

By slight abuse of notation we will write $\alpha(\omega) := \alpha(r, \omega)$ and $C_i(\omega) := C_i(r, \omega)$ for $i = 1, 2, 3, 4$, where $\alpha(t, \omega) := t + |V|(t, \omega)$ as before and C_1, C_2, C_3, C_4 are the "constants" appearing in (28) and (29). We assume that C_1, C_2, C_3 are chosen "best possible". Recall that $C_4(\omega) = C_2(\omega) + C_3(\omega)$. It is clear from (18) that $C_1(\omega) \leq \sup_{0 \leq t \leq r} \|\varphi(t, \omega)\|$. It follows from (41) that a sufficient condition for (48) to hold is

$$E \sup_{0 \leq t_1, t_2 \leq r} \|\varphi(t_1, \theta(t_2, \cdot))\| < \infty \tag{49}$$

and

$$E \sup_{0 \leq t_2 \leq r} |C_4(\theta(t_2, \cdot))\, \alpha(\theta(t_2, \cdot))| < \infty. \tag{50}$$

There is no need to worry about the measurability of the sup in (50). All we really need to show is that the sup can be estimated from above by an L^1-function.

Our first lemma asserts that Hypotheses (I_4) and (C_5) are sufficient to guarantee the existence of all higher order moments for the stochastic flows $\{\varphi(t) : t \geq 0\}$, $\{\varphi^{-1}(t) : t \geq 0\}$.

LEMMA 5.1. – *Let M satisfy Hypotheses* (C_5) *and* (I_4). *Then for each* $0 < T < \infty$ *and every integer* $p \geq 1$,

$$E \sup_{0 \leq t \leq T} \|\varphi(t, \cdot)\|^p < \infty, \tag{51}$$

$$E \sup_{0 \leq t \leq T} \|\varphi^{-1}(t, \cdot)\|^p < \infty, \tag{52}$$

and

$$E \sup_{0 \leq t_1, t_2 \leq T} \|\varphi(t_1, \theta(t_2, \cdot))\|^p < \infty. \tag{53}$$

Proof. – The existence of the moments (51) is proved in (Protter [26], Lemma 2, p. 196). Using the fact that φ^{-1} is the unique solution of the matrix s.d.e.

$$\left.\begin{aligned} d\varphi^{-1}(t) &= -\varphi^{-1}(t)\, dM(t) + \varphi^{-1}(t)\, d[M](t), \qquad t > 0 \\ \varphi^{-1}(0) &= I, \end{aligned}\right\} \tag{VI}$$

(Leandre [17]), it also follows from ([26], Lemma 2, p. 196) that the moments (52) exist for all $p \geq 1$.

The finiteness of the moments in (53) follows immediately from (51), (52), the cocycle property for φ and Hölder's inequality. ■

LEMMA 5.2. – *Suppose* $E\{|V|(2r, \cdot)\}^p < \infty$ *for a fixed* $p \geq 1$. *Then*

$$E \sup_{0 \leq t_2 \leq r} \{\alpha(\theta(t_2, \cdot))\}^p < \infty. \tag{87}$$

Proof. – The lemma follows directly from

$$\alpha(\theta(t_2, \omega)) = r + |V|(r, \theta(t_2, \omega))$$

and the helix property of $|V|$. ■

LEMMA 5.3. – *Let M, N satisfy Hypotheses* (C_4), (C_5) *and* (I_4) *and let $p \geq 1$ be such that*

$$E\,[N_{ij}]^p\,(1) < \infty \quad \text{for all} \quad 1 \leq i,\, j \leq n. \tag{55}$$

Then

$$E \sup_{0 \leq t_2 \leq r} \{\, \|[M,\, N]\|^{2p}(r,\, \theta\,(t_2,\, \cdot))\,\} < \infty. \tag{56}$$

Proof. – By Kunita-Watanabe's inequality (Protter [26], p. 61) and the helix property of $[M_{ik}]$, $[N_{kj}]$ we have a.s.

$$\|[M_{ik},\, N_{kj}]\|(r,\, \theta\,(t_2,\, \omega)) \leq [M_{ik}]^{1/2}\,(2\,r,\, \omega)\,[N_{kj}]^{1/2}\,(2\,r,\, \omega)$$

for $1 \leq i,\, j,\, k \leq n$ and $0 \leq t_2 \leq r$. Using (55), (I_4) and the helix property of M and N this implies (56). ■

The following lemma gives an integrability property for the process Z given in Corollary (3.3):

LEMMA 5.4. – *Suppose M satisfies Hypotheses* (C_5) *and* (I_4). *Let N^0, V^0 be the processes mentioned in* (I_3). *Suppose that*

$$E\,\{\,[N_{ij}^0]\,(2\,r,\, \cdot)\,\}^p < \infty \tag{57}$$

$$E\,\{\,\|V_{ij}^0\|(2\,r,\, \cdot)\,\}^{2p} < \infty \tag{58}$$

for all $1 \leq i,\, j \leq n$ and a given $p \geq 1$. Then

$$E \sup_{0 \leq t_1,\, t_2 \leq r} \|Z\,(t_1,\, \theta\,(t_2,\, \cdot))\|^q < \infty \tag{59}$$

for $1 \leq q < 2p$.

Proof. – From Corollary (3.3).

$$Z\,(t_1,\, \theta\,(t_2,\, \omega)) = \varphi\,(t_2,\, \omega)\,\{\,Z\,(t_1 + t_2,\, \omega) - Z\,(t_2,\, \omega)\,\}, \tag{60}$$

$\omega \in \Omega, \quad t_1,\, t_2 \geq 0.$

Taking $E \sup_{0 \leq t_1,\, t_2 \leq r} \|\cdot\|^q$ on both sides of (60), applying Hölder's inequality and using Lemma (5.1), it is easy to see that (59) will follow from

$$E \sup_{0 \leq t \leq 2r} \|Z\,(t,\, \cdot)\|^{p + \frac{q}{2}} < \infty. \tag{61}$$

So all we have to show is

$$E \sup_{0 \leq t \leq 2r} \left| \int_0^t \varphi_{ik}^{-1}\,(u)\, dN_{kj}^0\,(u) \right|^{p + \frac{q}{2}} < \infty \tag{62}$$

and

$$E \sup_{0 \leq t \leq 2r} \left| \int_0^t \varphi_{ik}^{-1}(u) \, dV_{kj}^0(u) \right|^{p+\frac{q}{2}} < \infty \tag{63}$$

for $1 \leq i, j, k \leq n$. Applying Burkholder's inequality to the left-hand side of (62) (*see* Dellacherie-Meyer [5], Chapter VII) and then using Hölder's inequality and Lemma 5.1, we see that (62) follows from (57). Similarly (63) follows from (58). This proves (61) and hence (59). ■

The next lemma establishes the integrability condition (50).

LEMMA 5.5. – *Assume Hypotheses* (C) *and* (I). *Then*

$$E\{ \sup_{0 \leq t_2 \leq r} \mathrm{C}_4(\theta(t_2, \cdot)) \, \alpha(\theta(t_2, \cdot)) \} < \infty. \tag{50}$$

Proof. – Let $\omega \in \Omega$. Substituting $\theta(t_2, \omega)$ for ω in (18) and using the cocycle property for φ (Corollary 3.2), the stationarity of μ, K (Hypotheses (C_1), (C_3)) and Hypothesis (C_2), the reader may check that

$$\sup_{0 \leq t_2 \leq r} \mathrm{C}_2(\theta(t_2, \omega)) \leq \sum_{i=1}^{4} C_2^i(\omega) \tag{64}$$

where

$$\begin{aligned} C_2^1(\omega) := 6 & \sup_{0 \leq t_1, t_2 \leq r} \|\varphi(t_1, \theta(t_2, \omega))\|^2 \sup_{0 \leq t \leq 2r} \|Z(t, \omega)\| \\ & \times \sup_{0 \leq t \leq 2r} \operatorname*{esssup}_{t-r \leq s \leq t} \|\bar{K}(t, s, \omega)\| \end{aligned}$$

$$\begin{aligned} C_2^2(\omega) := 2 & \sup_{0 \leq t_1, t_2 \leq 2r} \|\varphi(t_1, \theta(t_2, \omega))\|^2 \sup_{0 \leq t \leq 2r} \|Z(t, \omega)\| \\ & \times \operatorname*{esssup}_{0 \leq t \leq 2r} \operatorname*{esssup}_{t-r \leq s \leq t} \left\| \frac{\partial}{\partial t} \bar{K}(t, s, \omega) \right\| \end{aligned}$$

$$C_2^3(\omega) := \sup_{0 \leq t_1, t_2 \leq 2r} \|\varphi(t_1, \theta(t_2, \omega))\| \cdot \sup_{-r \leq s \leq 2r} \left| \frac{d\nu(\omega)}{ds}(s) \right|$$

$$\begin{aligned} C_2^4(\omega) := & \sup_{0 \leq t_1, t_2 \leq 2r} \|\varphi(t_1, \theta(t_2, \omega))\| \\ & \times \sup_{0 \leq t \leq 2r} \operatorname*{esssup}_{t-r \leq s \leq t} \|\bar{K}(t, s, \omega)\| \sup_{0 \leq t_2 \leq r} |[M, N]|(r, \theta(t_2, \omega)). \end{aligned}$$

Using the integrability conditions (I_3) and (I_4), we see that the assumptions of Lemma 5.1, Lemma 5.2 for $p = 4$, Lemma 5.3 for $p = 2$ and Lemma 5.4

for $p = 2$ are satisfied. Now multiply (64) by $\sup_{0 \le t_2 \le r} \alpha(\theta(t_2, \cdot))$ and apply Hölder's inequality to see that $E \sup_{0 \le t_2 \le r} \{ C_2(\theta(t_2, \cdot)) \alpha(\theta(t_2, \cdot)) \} < \infty$.

Further, using (18) and (28), we get

$$\sup_{0 \le t_2 \le r} C_3(\theta(t_2, \omega)) \le \sup_{0 \le t_1, t_2 \le 2r} \|\varphi(t_1, \theta(t_2, \omega))\|, \qquad \omega \in \Omega. \quad (65)$$

Applying Lemma 5.1 and Lemma 5.2 for $p = 4$ we get

$$E\{ \sup_{0 \le t_2 \le r} C_3(\theta(t_2, \omega)) \alpha(\theta(t_2, \omega)) \} < \infty.$$

This proves (50). ■

Observe that (49) follows from Lemma 5.1, and (50) follows from Lemma 5.5. So we have proved the following theorem.

THEOREM 5.1. – *Assume Hypotheses* (C) *and* (I). *Then*

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \|X(t_1, \theta(t_2, \cdot), \cdot)\|_{L(M_2)} < \infty. \quad (47)$$

Once the integrability property (47) is established we can now state the following multiplicative ergodic theorem for the stochastic flow (X, θ) of (I). The proof of the theorem is analogous to that of Theorem 4 in Mohammed ([23], § 4) for the white noise case $L = W$, $N = 0$. In the case when θ is ergodic, the theorem gives a *discrete* set of *non-random* Lyapunov exponents for X. The reader may supply the details of the argument by consulting the proof of Theorem 4 in [23] (pp. 117-122). *See* also Lemmas 6 and 7 ([23], pp. 113-117).

THEOREM 5.2. – *Suppose θ is ergodic and let the stochastic f.d.e.* (I) *satisfy Hypotheses* (C) *and* (I). *Then there exist*

(*a*) *a set $\Omega^* \in \mathcal{F}$ such that $P(\Omega^*) = 1$ and $\theta(t, \cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \in \mathbb{R}^+$,*

(*b*) *a fixed* (i.e. *non-random*) *sequence $\{ \lambda_i \}_{i=1}^{\infty}$ of real numbers,*

(*c*) *a random family $\{ E_i(\omega) : i \ge 1, \omega \in \Omega^* \}$ of closed finite-codimensional subspaces of M_2,*

satisfying the following properties:

(i) *if the* Lyapunov spectrum *$\{ \lambda_i \}_{i=1}^{\infty}$ is infinite, then $\lambda_{i+1} < \lambda_i$ for all $i \ge 1$ and $\lim_{i \to \infty} \lambda_i = -\infty$; otherwise the spectrum is a finite set $\{ \lambda_i \}_{i=1}^{N}$ with $N \ge 1$ a non-random integer and $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$.*

(ii) *for each* $\omega \in \Omega^*$,

$$\ldots \subset E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) := M_2, \qquad i \geq 1.$$

(iii) *for each* $\omega \in \Omega^*$ *and* $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|_{M_2} = \lambda_i$$

and

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} = \lambda_1,$$

(iv) *for each* $i \geq 1$, *the family* $\{ E_i(\omega) : \omega \in \Omega^* \}$ *is* $\mathcal{F}$*-measurable into the Grassmannian of* M_2 *and is invariant under the cocycle* (X, θ), i.e.

$$X(t, \omega, \cdot)(E_i(\omega)) \subseteq E_i(\theta(t, \omega)), \qquad \omega \in \Omega^*, \qquad t \geq 0,$$

(v) *for each* $i \geq 1$, $\operatorname{codim} E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$.

As in [23] we say that the s.f.d.e. (I) is *hyperbolic* if its Lyapunov spectrum does not contain 0. By a straightforward adaptation of the argument in Corollary 2 of [23] (pp. 126–130) we get the following version of the stable-manifold theorem (viz. an exponential dichotomy) in the hyperbolic case:

THEOREM 5.3 (Exponential Dichotomy). – *Let Hypotheses* (C) *and* (I) *hold and* θ *be ergodic. Assume that the s.f.d.e.* (I) *is hyperbolic. Then there exist*

(*a*) *a set* $\hat{\Omega}^* \in \mathcal{F}$ *such that* $P(\hat{\Omega}^*) = 1$ *and* $\theta(t, \cdot)(\hat{\Omega}^*) = \hat{\Omega}^*$ *for all* $t \in \mathbb{R}$,

(*b*) *a measurable splitting*

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \hat{\Omega}^*$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$ $\omega \in \hat{\Omega}^*$, *are closed linear subspaces of* M_2, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \hat{\Omega}^*$.

(ii) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* M_2.

(iii) *For each* $\omega \in \hat{\Omega}^*$ *and* $(v, \eta) \in \mathcal{U}(\omega)$, *there exist* $t_1 = t_1(\omega, v, \eta) > 0$ *and a positive* δ_1, *independent of* (ω, v, η), *such that*

$$\|X(t, \omega, (v, \eta))\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_1 t}, \qquad t \geq t_1.$$

(iv) *For each* $\omega \in \hat{\Omega}^*$ *and* $(v, \eta) \in S(\omega)$, *there exist* $t_2 = t_2(\omega, v, \eta) > 0$ *and a positive* δ_2, *independent of* (ω, v, η), *such that*

$$\|X(t, \omega, (v, \eta))\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_2 t}, \qquad t \geq t_2.$$

(v) *For each* $t \geq 0$ *and* $\omega \in \hat{\Omega}^*$,

$$X(t, \omega, \cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

$$X(t, \omega, \cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)).$$

Remark. – Under the hypotheses of Theorem 5.2, the Lyapunov spectrum of (I) does not change if the state space M_2 is replaced by $D := D([-r, 0], \mathbb{R}^n)$ with the supremum norm $\|\cdot\|_\infty$. In fact the existence of the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|_{M_2}, \qquad \omega \in \Omega^*$$

implies the existence of

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|_{\infty}, \qquad \omega \in \Omega^*$$

and both limits agree for $(v, \eta) \in M_2$. To see this the reader may note the inequalities:

$$\|X(t, \omega, (v, \eta))\|_{M_2} \leq (r+1)^{1/2} \|X(t, \omega, (v, \eta))\|_{\infty}, \qquad t \geq 0,$$

$$\|X(t, \omega, (v, \eta))\|_{\infty} \leq \sup_{-r \leq s \leq 0} \|X(t+s, \omega, (v, \eta))\|_{M_2}, \qquad t \geq r,$$

for $\omega \in \Omega^*$ and $(v, \eta) \in M_2$.

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