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INTRODUCTION

In this paper, we study the connection between the almost sure convergence of semi-martingales and the asymptotic behaviour of their local times. Our study is motivated by the following example.

Let $h(t)$ be a non-negative decreasing function and (B_t) a Brownian motion. Let $X_t = h(t) B_t$.

If X_t converges as $t \to \infty$, then it must converge to zero, because $\{t : B_t = 0\}$ is unbounded. We expect that this property gets reflected in the behavior of the local time of X at zero, which is easily seen to be equal to the process $\left(\int_0^t h(s) \, dl_s \right)$, where l_t is the local time of B at zero. In section 1, we state necessary and sufficient conditions so that

$$\lim_{t \to \infty} h(t) B_t = 0 \text{ and } \lim_{t \to \infty} \int_0^t h(s) \, dl_s < \infty, \text{ in the case when } h \text{ is a}$$

deterministic function. These results are due to Jeulin and Yor (*see* [6], Proposition 15) and C. Donati-Martin (*see* [3], p. 150) respectively.

Now, the integration by parts formula gives us:

$$X_t = X_0 + \int_0^t h(s)\, dB_s + \int_0^t B_s\, dh(s),$$

and the effect of the drift term is to pull the process towards the origin. We make this notion precise below in Section 2, where we study subsets of the sample space Ω on which the drift term of a given semi-martingale has this property. In the case of a continuous martingale, the almost sure convergence of the martingale, its local time and its quadratic variation processes are all equivalent. In the case of the semi-martingales which we study, the situation is similar and yet there are some subtle differences.

1. BROWNIAN ASYMPTOTICS

Consider a continuous local martingale $(M_t)_{t\geq 0}$, $M_0 = 0$ a.s. Let $(L_t^x)_{t\geq 0;\, x\in\mathbb{R}}$ denote a jointly continuous version of its local times and $(\langle M\rangle_t)$ its quadratic variation process. The following lemma is well known (*see*, e.g. [12], Prop. (1.8), p. 171) and follows by writing M as a time change of Brownian motion.

LEMMA 1.1. – *For every $x\in\mathbb{R}$, the following sets are almost surely equal:*

(i) $\{\omega : \lim\limits_{t\to\infty} M_t\ exists\}$

(ii) $\{\omega : \lim\limits_{t\to\infty} M_t\ exists\ and\ is\ finite\}$

(iii) $\{\omega : \langle M\rangle_\infty < \infty\}$

(iv) $\{\omega : L_\infty^x < \infty\}$.

Now, consider a Brownian motion $(B_t)_{t\geq 0}$ starting from zero. Let $g_t = \sup\{s\leq t : B_s = 0\}$ and $\{H_u,\ u\geq 0\}$ a locally bounded previsible process; let $M_t^H = H_{g_t} B_t$. It follows from the "Balayage formula" (Azéma-Yor [1], Yor [8]) that (M_t^H) is a continuous local martingale and it is easy to see that its local time at 0 is $\left(\int_0^t |H_s|\, dl_s\right)_{t\geq 0}$ where $(l_s)_{s\geq 0}$ is the local time of (B_t) at the origin. We have the following consequence of Lemma 1.1.

PROPOSITION 1.2. – *The following are equivalent:*

(i) $\lim\limits_{t\to\infty} (H_{g_t} B_t) = 0$ a.s.,

(ii) $\int_0^\infty H_{g_s}^2\, ds < \infty$ a.s.,

(iii) $\int_0^\infty |H_s| \, dl_s < \infty$ a.s.

We now specialise to the case when $H_s = h(s)$, where h is a non-negative decreasing deterministic function. It is easy to see that the process $\left(\int_0^t h(s) \, dl_s \right)_{t \geq 0}$ is also the local time at the origin of the semi-martingale $X_t = h(t) B_t$. It is well known that the measure (induced by the increasing process $(l_t)_{t \geq 0}$ is, almost surely, singular w.r.t. the Lebesgue measure on $[0, \infty)$. The following result (first proved by Donati-Martin [3], when h is smooth) is nonetheless true.

THEOREM 1.3. – *The following are equivalent.*

(i) $\int_0^\infty h(s) \, dl_s < \infty$ a.s.;

(ii) $\int_0^\infty h(s) \, \dfrac{ds}{\sqrt{s}} < \infty.$

The proof of this theorem depends on a result of Jeulin [4] (*see also* Jeulin [5] and, for some applications, Xue [7]). We state and prove it for completeness.

LEMMA 1.4. – *Let $(R_t)_{t \geq 0}$ be a positive measurable process such that*

1) *The law ν of R_t does not depend on t.*
2) $\nu(\{0\}) = 0$
3) $E(R_t) < \infty.$

Then, for any positive Radon measure μ on $\mathbb{R}$, $\int_0^\infty d\mu(t) < \infty$ iff

$\int_0^\infty R_t \, d\mu(t) < \infty$ *a.s.*

Proof. – The "if" part is clear: if $\int_0^\infty d\mu(t) < \infty$, then in fact by condition 3), $E \int_0^\infty R_t \, d\mu(t) < \infty.$

Conversely, let $\int_0^\infty R_t \, d\mu(t) < \infty$ a.s. and let n be such that $P(J_n) > 0$ where $J_n = \left\{ \omega : \int_0^\infty R_t \, d\mu(t) \leq n \right\}.$ Then,

$$E(I_{J_n} R_t) = \int_0^\infty du \, E[I_{J_n} I_{\{R_t > u\}}] = \int_0^\infty du \, E([I_{J_n} - I_{\{R_t \leq u\}}]^+)$$

$$\geq \int_0^\infty du \, (P(J_n) - \nu([0, u]))^+.$$

Since $P(J_n) > 0$ and $\nu[0, u] \to 0$ as $u \to 0$ by Condition 2), the last integral is in fact strictly positive, say equal to a_n. It follows that:

$$\infty > n\, P(J_n) \geq \int_0^\infty E(J_n\, R_t)\, d\mu(t) \geq a_n \int_0^\infty d\mu(t)$$

and the proof is complete.

Proof of Theorem 1.3. – Since $E(l_s) = c\sqrt{s}$ where c is a constant not depending on s, it is easy to see (since h is deterministic) that

$E \int_0^\infty h(s)\, dl_s = \dfrac{c}{2} \int_0^\infty h(s)\, \dfrac{ds}{\sqrt{s}}.$ Clearly, (ii)$\Rightarrow$(i).

To go the other way, suppose that (i) holds. We will show that the two functions $h(t)\sqrt{t}$ and $\int_0^t \sqrt{s}\, dh(s)$ converge to a finite limit as $t \to \infty$. It follows from the equation

$$\sqrt{t}\, h(t) = \int_0^t \sqrt{s}\, dh(s) + \int_0^t h(s)\, \frac{ds}{2\sqrt{s}}$$

that $\displaystyle\int_0^\infty h(s)\, \frac{ds}{\sqrt{s}} < \infty.$

We now show: $\displaystyle\lim_{t\to\infty} \int_0^t \sqrt{s}\,(-dh(s)) < \infty.$ We have:

$$h(t)\, l_t + \int_0^t l_s\,(-dh_s) = \int_0^t h(s)\, dl_s.$$

Since both terms in the LHS are non negative and since the RHS converges by assumption, it follows that $\displaystyle\int_0^\infty l_s\,(-dh_s) < \infty$ a.s. Now,

Lemma 1.4 applied to $R_t = \dfrac{l_t}{\sqrt{t}}$ and $d\mu(s) = -\sqrt{s}\, dh(s)$ shows that $\displaystyle\int_0^\infty \sqrt{s}\,(-dh(s)) < \infty.$

We then show: $\displaystyle\lim_{t\to\infty} h(t)\sqrt{t} < \infty.$ Since h is decreasing, $h(t)\,|B_t| \leq h(g_t)\,|B_t|$, where $g_t = \max\{s \leq t : B_s = 0\}$. From Proposition 1.2, it follows that $h(t)\, B_t \to 0$ a.s. Since $(h_t\, B_t)$ are gaussian random variables, this implies that $h_t\, B_t \to 0$ in L^1, *i.e.* $h(t)\sqrt{t} \to 0$ and the proof is complete.

Remark 1.5. – Let H be a bounded previsible process. From Proposition 1.2, $(H_{g_t}\, B_t)_{t\geq 0}$ is a martingale which is not uniformly inte-

grable, unless it is identically zero. Consequently, if $\int_0^\infty |H_s|\,dl_s < \infty$ a.s.

then by Proposition 1.2 $\int_0^\infty H_{g_s}^2\,ds < \infty$ a.s. but $E\left(\int_0^\infty H_{g_s}^2\,ds\right)^{1/2} = \infty$.

Remark 1.6. – There are situations where the conclusions of Lemma 1.4 are true, but $ER_t = \infty$. Let $g_s = \sup\{u \le s : B_u = 0\}$. Consider the pair $(R_s,\,\mu(ds))$ where $R_s = \dfrac{s^{2\alpha}}{g_s^{2\alpha}}$, $\alpha > 1/2$ and $\mu(ds) = \dfrac{ds}{s^{2\alpha}}\,1_{[1,\infty)}(s)$.

Then $\int_0^\infty \mu(ds) < \infty$ and from Proposition 1.2 applied to $H_s = \dfrac{1}{s^\alpha}$, it

follows that $\int_0^\infty R_s\,\mu(ds) < \infty$. But this conclusion cannot be obtained from Lemma 1.4 because $ER_1 = \infty$, since g_1 is arcsine distributed.

In the next Theorem, we reproduce a result of Jeulin and Yor [6] which gives a necessary and sufficient condition for $h(t)\,B_t$ to converge to zero almost surely when h is a deterministic function. Here, we consider the particular case where h is decreasing.

THEOREM 1.7. – *Let $h(t)$ be a non negative decreasing deterministic function. Then, the following are equivalent:*

(i) $\lim\limits_{t\to\infty} h(t)\,B_t = 0$ a.s.;

(ii) *for every $\varepsilon > 0$,* $\int_1^\infty \exp\left(\dfrac{-\varepsilon}{th^2(t)}\right)\dfrac{dt}{t} < \infty$.

Proof. – Let $B_1^* = \sup\limits_{0\le t\le 1} |B_t|$. Then, condition (ii) is equivalent to

(ii′) $\sum\limits_{n=1}^\infty P\left(B_1^* > \dfrac{x\,2^{-n/2}}{h(2^n)}\right) < \infty$ for every $x > 0$.

This follows from the inequalities: for $\varepsilon > 0$,

$$\sum_{k\ge 0} \int_{2^k}^{2^{k+1}} \exp\left(\dfrac{-2\varepsilon\,2^{-(k+1)}}{h^2(2^{k+1})}\right)\dfrac{dt}{t} \le \int_1^\infty \exp\left(-\dfrac{\varepsilon}{th^2(t)}\right)\dfrac{dt}{t}$$

$$\le \sum_{k\ge 0} \int_{2^k}^{2^{k+1}} \exp\left(-\dfrac{\varepsilon}{2}\cdot\dfrac{2^{-k}}{h^2(2^k)}\right)\dfrac{dt}{t}$$

and $\left(\dfrac{2}{\pi}\right)^{1/2}\int_\varepsilon^\infty \exp\left(-\dfrac{1}{2}u^2\right)du \le P(B_1^* > \varepsilon) \le 2\exp\left(-\dfrac{1}{2}\varepsilon^2\right)$.

We now show that (ii′) holds iff (i) holds.

Let $V_n = 2^{-n/2} \sup_{2^n \leq t \leq 2^{n+1}} |B_t - B_{2^n}|$. It is easy to see, by Borel-Cantelli arguments, that (ii′) is equivalent to $\lim_{n \to \infty} 2^{n/2} V_n h(2^n) = 0$ and that (ii′) implies $\lim_{n \to \infty} |B_{2^n}| h(2^n) = 0$. In particular, we remark that:

$$2^{n/2} V_n h(2^n) \overset{\text{(law)}}{=} h(2^n) \sup_{t \leq 2^n} |B_t| \overset{\text{(law)}}{=} h(2^n) 2^{n/2} B_1^*.$$

Now, (ii′)⇒(i) follows from the above observations and from the inequalities

$$\sup_{2^n \leq t \leq 2^{n+1}} h(t) |B_t| \leq 2^{n/2} V_n h(2^n) + |B_{2^n}| h(2^n).$$

Conversely, if (i) holds, then $\lim_{t \to \infty} \sqrt{t}\, h(t) = 0$. Moreover, (V_n) are independent and have the same law as $B_1^* = \sup_{0 \leq t \leq 1} |B_t|$. Hence $\lim_{n \to \infty} 2^{n/2} V_n h(2^n) = 0$ and (ii′) holds.

Section 2. – In this section, we prove a version of Lemma 1.1 for semi-martingales. This is Theorem 2.4. Our analysis will be restricted to the class of semi-martingales (X_t) with $\sum_{s \leq t} |\Delta X_s| < \infty$, a.s. for all $t \geq 0$. These can be written in the form $X_t = X_0 + M_t + V_t$ where (M_t) is a *continuous* local martingale and (V_t) a process of finite variation. For a semi-martingale X, this property will be assumed to hold for the rest of the paper.

We recall the Tanaka formula for the semi-martingale X: for $a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_{s-} > a\}}\, dX_s + \frac{1}{2} \mathbf{L}_t^a \qquad (1)$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{\{X_{s-} \leq a\}}\, dX_s + \frac{1}{2} \mathbf{L}_t^a \qquad (2)$$

where

$$\mathbf{L}_t^a = l_t^a + L_t^a$$

$$l_t^a = 2 \left(\sum_{s \leq t} \left(I_{\{X_{s-} > a\}} (X_s - a)^- + I_{\{X_{s-} \leq a\}} (X_s - a)^+ \right) \right)$$

and $(L_t^a)_{t \geq 0}$ is the local time at a of X, which is an increasing continuous process, supported on the set $\{s : X_{s-} = X_s = a\}$; for a discussion of these formulae, *see* e.g. Yor ([11], p. 20). For the semi-martingales which we consider, there is a version of the process $(\mathbf{L}_t^a, t \geq 0, a \in \mathbb{R})$ which is jointly right continuous in (t, a) and has left limits (*see* Yor [9]). This property is important in what follows and we will always take equations (1) and (2) to be true outside a null set, for all $t \geq 0$ and for all $a \in \mathbb{R}$.

Let $V_+ (t) = \int_0^t I_{\{X_{s^-} > 0\}} \, dV_s$ and $V_- (t) = \int_0^t I_{\{X_{s^-} \leq 0\}} \, dV_s$. The finite variation part V of the processes considered in Examples 1 and 2 have the property that V_+ is decreasing and V_- is increasing. For a semi-martingale $X = X_0 + M + V$, we define the following subsets of Ω:

$$\nu_+^- = \{\omega : V_+ \text{ is decreasing and } V_- \text{ is increasing}\}$$

$$\nu_-^+ = \{\omega : V_+ \text{ is increasing and } V_- \text{ is decreasing}\}$$

$$\nu_+^+ = \{\omega : V_+, \ V_- \text{ are increasing}\}$$

$$\nu_-^- = \{\omega : V_+, \ V_- \text{ are decreasing}\}.$$

Of course, these sets do not necessarily partition Ω. We shall use the notation $\underline{X}_\infty$ (resp. $\bar{X}_\infty$) for $\liminf_{t \to \infty} X_t$ (resp. $\limsup_{t \to \infty} X_t$). We shall denote $\lim_{t \to \infty} X_t$ by X_∞ whenever it exists. In this notation, $\{\omega : X_\infty \in (a, b)\}$ means $\{\omega : \lim_{t \to \infty} X_t$ exists and belongs to $(a, b)\}$. The following lemma will be used in the proof of our main result (Theorem 2.4).

LEMMA 2.1. – *Let* $-\infty < a < b < \infty$. *Then:*

(i) *On the set* $\left\{\omega : \int_0^t I_{(a, b]} (X_{s^-}) \, dV_s \text{ is decreasing}\right\}$, *we have, for all* $c \in (a, b]$, *almost surely,*

$$\{\omega : \mathbf{L}_\infty^a < \infty\} \subset \{\omega : \mathbf{L}_\infty^c < \infty\}$$

and $\{\omega : \mathbf{L}_\infty^a < \infty\} \subset \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \bar{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\}$.

(ii) *On the set* $\left\{\omega : \int_0^t I_{(a, b]} (X_{s^-}) \, dV_s \text{ is increasing}\right\}$, *we have for all* $c \in [a, b)$, *almost surely,*

$$\{\omega : \mathbf{L}_\infty^b < \infty\} \subset \{\omega : \mathbf{L}_\infty^c < \infty\}$$

and

$$\{\omega : \mathbf{L}_\infty^b < \infty\} \subset \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \bar{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\}.$$

Proof. – (i) Suppose that $\int_0^t I_{(a,\,b]}\,(X_{s-})\,dV_s$ is decreasing. Then, for $c \in (a,\,b]$ we have, from equation (1), that

$$(X_t - a)^+ - (X_0 - a)^+ - (X_t - c)^+ + (X_0 - c)^+$$

$$= \int_0^t I_{(a,\,c]}\,(X_{s-})\,dM_s + \frac{1}{2}\,\mathbf{L}_t^a - \frac{1}{2}\,\mathbf{L}_t^c$$

$$+ \int_0^t I_{(a,\,c]}\,(X_{s-})\,dV_s. \tag{3}$$

Note that the LHS of equation (3) is bounded by $2(b - a)$. Now if $\mathbf{L}_\infty^a < \infty$, it follows that the decreasing function $\int_0^t I_{(a,\,c]}\,(X_{s-})\,dV_s - \frac{1}{2}\,\mathbf{L}_t^c$ has a finite limit, as also $\int_0^t I_{(a,\,c]}\,(X_{s-})\,dM_s$ (this follows from Lemma 1.1). Thus $\mathbf{L}_\infty^c < \infty$, proving the first inclusion in (i). Now the LHS of equation (3) has a finite limit as $t \to \infty$. In particular, it has a finite limit when $c = b$. But this means precisely that $\underline{X}_\infty \geq b$ or $\bar{X}_\infty \leq a$ or $X_\infty \in (a,\,b)$.

(ii) The proof is similar to that of (i) using the equation

$$(X_t - b)^- - (X_0 - b)^- - (X_t - c)^- + (X_0 - c)^-$$

$$= -\int_0^t I_{(c,\,b]}\,(X_{s-})\,dM_s + \frac{1}{2}\,\mathbf{L}_t^b - \frac{1}{2}\,\mathbf{L}_t^c$$

$$- \int_0^t I_{(c,\,b]}\,(X_{s-})\,dV_s \tag{4}$$

for all $c \in (a,\,b]$. $\square$

COROLLARY 2.2.

$$\left\{ \omega : \int_0^t I_{(a,\,b]}\,(X_{s-})\,dV_s \text{ is monotonic} \right\} \cap \{ \omega : [X,\,X]_\infty < \infty \}$$

$$\overset{\text{a.s.}}{\subset} \{ \omega : \underline{X}_\infty \geq b \} \cup \{ \omega : \bar{X}_\infty \leq a \} \cup \{ \omega : X_\infty \in (a,\,b) \}.$$

Proof. – It follows from the results in Yoeurp [10] that

$$[X,\,X]_\infty = \int_{-\infty}^{\infty} \mathbf{L}_\infty^x\,dx.$$

Hence if $[X, X]_\infty (\omega) < \infty$, then $\mathbf{L}_\infty^x (\omega) < \infty$ for all $x \notin N (\omega) \subset \mathbf{R}$, $\lambda (N (\omega)) = 0$, where λ is the Lebesgue measure. It follows from Fubini's theorem that there exists $N \subset \mathbf{R}$, $\lambda (N) = 0$, such that:

$$\forall x \notin N, \quad \{\omega : [X, X]_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}_\infty^x < \infty\}.$$

Choose $a_n, b_n \notin N$, $a_n \downarrow a$, $b_n \uparrow b$. If $\displaystyle\int_0^t I_{(a, b]} (X_{s-}) \, dV_s$ is monotone, then so is $\displaystyle\int_0^t I_{(a_n, b_n]} (X_{s-}) \, dV_s$ for every n. It follows from Lemma 2.1 that if $[X, X]_\infty (\omega) < \infty$, then $\underline{X}_\infty \geq b_n$ or $\bar{X}_\infty \leq a_n$ or $X_\infty \in (a_n, b_n)$ for all n. This means that $\underline{X}_\infty \geq b$ or $\bar{X}_\infty \leq a$ or $X_\infty \in (a, b)$. $\square$

COROLLARY 2.3. – *On the set* ν_+^-,

$$\{\omega : \underline{X}_\infty < \bar{X}_\infty\} \cap \{\omega : \mathbf{L}_\infty^a < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \bar{X}_\infty \leq a\} \quad \text{for } a > 0,$$

and

$$\{\omega : \underline{X}_\infty < \bar{X}_\infty\} \cap \{\omega : \mathbf{L}_\infty^a < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq a\} \quad \text{for } a < 0.$$

In particular, on the set ν_+^-

$$\{\omega : \underline{X}_\infty < a < \bar{X}_\infty\} \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}_\infty^a = \infty\}$$

for every a.

THEOREM 2.4. – *Let* (X_t) *be a semi-martingale with* $X_t = X_0 + M_t + V_t$ *where* (M_t) *is a continuous local martingale and* (V_t) *a process of finite variation. Then, we have the following:*

a) (i) *On the set* ν_+^-, *for all* $x \neq 0$,

$$\{\omega : \lim_{t \to \infty} X_t \text{ exists}\} \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}_\infty^x < \infty\}$$

and

$$\{\omega : \lim_{t \to \infty} X_t \text{ exists}\} = \bigcap_{\substack{x \in Q \\ x \neq 0}} \{\omega : \mathbf{L}_\infty^x < \infty\}$$

(ii) $\{\omega : \lim_{t \to \infty} X_t = X_\infty \text{ exists and } X_\infty \neq 0, \pm\infty\}$

$\overset{\text{a.s.}}{\subset} \{\omega : [X, X]_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t \to \infty} X_t \text{ exists}\}$ *on the set* ν_+^-.

b) (i) *For all* $x \in \mathbf{R}$,

$$\{\omega : \lim_{t \to \infty} X_t \text{ exists}\} \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}_\infty^x < \infty\}$$

$$\{\omega : \lim_{t\to\infty} X_t \ exists\} \overset{\text{a.s.}}{=} \bigcap_{x\in Q} \{\omega : \mathbf{L}^x_\infty < \infty\}$$

holds on each of the sets ν^+_-, ν^+_+, ν^-_-.

(ii) $\{\omega : \lim_{t\to\infty} X_t = X_\infty \ exists, X_\infty \neq \pm\infty\} \overset{\text{a.s.}}{\subset} \{\omega : [X, X]_\infty <$

$\infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t\to\infty} X_t \ exists\}$ holds on each of the sets ν^+_-, ν^+_+, ν^-_-.

Proof. – a) (i) Suppose $0 < a < b$, a, $b \in Q$. Then, by Lemma 2.1 (i),

$$\bigcap_{\substack{x\in Q \\ x\neq 0}} \{\omega : \mathbf{L}^x_\infty < \infty\} \cap \nu^-_+$$

$$\subset \{\omega : \mathbf{L}^a_\infty < \infty\} \cap \left\{\omega : \int_0^\bullet I_{(a, b]}(X_{s-}) \, dV_s \text{ is } \downarrow\right\}$$

$$\overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \bar{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\}.$$

Letting $a \downarrow 0$, $b \uparrow \infty$, we get

$$\bigcap_{\substack{x\in Q \\ x\neq 0}} \{\omega : \mathbf{L}^x_\infty < \infty\} \cap \nu^-_+ \overset{\text{a.s.}}{\subset} \{\omega : \bar{X}_\infty \leq 0\} \cup \{\omega : X_\infty \in (0, \infty]\}.$$

Similarly, from Lemma 2.1 (ii), it follows that

$$\bigcap_{\substack{x\in Q \\ x\neq 0}} \{\omega : \mathbf{L}^x_\infty < \infty\} \cap \nu^-_+ \overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq 0\} \cup \{\omega : X_\infty \in [-\infty, 0)\}.$$

It follows that on the set ν^-_+

$$\bigcap_{\substack{x\in Q \\ x\neq 0}} \{\omega : \mathbf{L}^x_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t\to\infty} X_t \ exists\}.$$

Conversely, suppose $\omega \in \{\lim_{t\to\infty} X_t \ exists\} \cap \nu^-_+$. Since $\{\lim_{t\to\infty} X_t$ exists$\} = \{X_\infty \in (0, \infty)\} \cup \{X_\infty \in (-\infty, 0)\} \cup \{X_\infty = 0, \pm\infty\}$, it is sufficient to show that on each of the sets in the RHS, $\mathbf{L}^a_\infty(\omega) < \infty$ for $\omega \in \nu^-_+$ and $a \neq 0$.

If $\omega \in \{X_\infty \in (0, \infty)\} \cap \nu^-_+$, then from the decomposition $X_t = X_0 + M_t + V_+(t) + V_-(t)$, it follows that V_-, V_+ and hence M converge to a finite limit and hence from equation (1) that $\mathbf{L}^a_\infty < \infty$,

i.e. $\{\omega : X_\infty \in (0, \infty)\} \cap \nu^-_+ \overset{\text{a.s.}}{\subset} \{\omega : X, M, V_+, V_- \text{ converge}\}$

$$\overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}^a_\infty < \infty\}.$$

Similarly, from equation (2), it follows that

$$\{\omega : X_\infty \in (-\infty, 0)\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega : \mathbf{L}_\infty^a < \infty\}.$$

If $\omega \in \{X_\infty = 0, \pm\infty\} \cap \nu_+^-$, then since $a \neq 0$, the process does not visit a after a finite time and there are no jumps of X which cross a. Then, since $\mathbf{L}_t^a = L_t^a + l_t^a$, it follows that $\mathbf{L}_t^a$ does not increase after a finite time *i.e.* $\mathbf{L}_\infty^a < \infty$. Thus $\{X_\infty = 0, \pm\infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega : \mathbf{L}_\infty^a < \infty\}$ and the proof of a) (i) is complete.

a) (ii) To prove the first inclusion in a) (ii), note that as in the proof of a) (i),

$$\{\omega : X_\infty \neq 0, \pm\infty\} \overset{a.s.}{\subset} \{\omega : M, V_+, V_- \text{ converge to a finite limit}\}.$$

If $\omega \in$ RHS above, then of course $\langle X^c \rangle_\infty = \langle M \rangle_\infty < \infty$ and

$$\sum_s \Delta X_s^2 \leq \sup_s |\Delta X_s| \sum_s |\Delta X_s|$$

$$\leq \sup_s |\Delta X_s| \left(\sum_s (|\Delta V_+ (s)| + |\Delta V_- (s)|) \right) < \infty.$$

Thus, $\{\omega : X_\infty \neq 0, \pm\infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega : [X, X]_\infty < \infty\}$.

To prove the second inclusion in a) (ii), let first $0 < a < b$. From Corollary 2.2 we get,

$$\{\omega : [X, X]_\infty < \infty\} \cap \nu_+^-$$

$$\overset{a.s.}{\subset} \{\omega : \bar{X}_\infty \leq a\} \cup \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : X_\infty \in (a, b)\}.$$

Now letting $a \downarrow 0$, $b \uparrow \infty$, we get

$$\{\omega : [X, X]_\infty < \infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega : \bar{X}_\infty \leq 0\} \cup \{\omega : X_\infty \in (0, \infty]\}. \quad (5)$$

Similarly, by applying Corollary 2.2 to the case $a < b < 0$, we can prove,

$$\{\omega : [X, X]_\infty < \infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega : \bar{X}_\infty \geq 0\} \cup \{\omega : X_\infty \in [-\infty, 0)\}. \quad (6)$$

The second inclusion in a) (ii) follows from (5) and (6) and completes the proof of a).

b) (i). The proof that $\bigcap_{b \in Q} \{\omega : \mathbf{L}_\infty^b < \infty\} \overset{a.s.}{\subset} \{\omega : \lim_{t \to \infty} X_t \text{ exists}\}$ on the sets ν_-^+, ν_+^+, ν_-^- is similar to the proof for the set ν_+^-, described in a) (i) using Lemma 2.1. We will prove the reverse inclusion on the set ν_-^+, the other two cases being similar. Fix $b \in \mathbb{R}$.

Since $\{\omega : \lim_{t\to\infty} X_t \text{ exists}\} = \{\omega : X_\infty = 0, \pm\infty\} \cup \{X_\infty \neq 0, \pm\infty\}$,
it suffices to show that each of the sets in the RHS is contained in
$\{\omega : \mathbf{L}^b_\infty < \infty\}$ on ν^+_-. For the set $\{\omega : X_\infty \neq 0, \pm\infty\}$, the proof
is similar to the proof given in a) (i). We will show the inclusion
$\{\omega : X_\infty = 0, \pm\infty\} \cap \nu^+_- \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}^b_\infty < \infty\}$ for every $b \in \mathbb{R}$.

Now, if $X_\infty = \pm\infty$ or if $X_\infty = 0, \pm\infty$ and $b \neq 0$, then $\mathbf{L}^b_t$ does not
increase after a finite time. In particular, $\mathbf{L}^b_\infty < \infty$. If $X_\infty = 0$ and $b = 0$,
then since on ν^+_-, $V_+(t) + \mathbf{L}^0_t$ is increasing, it follows, by letting $t \to \infty$
in equation (3) that $\mathbf{L}^0_\infty < \infty$.

b) (ii) It is easy to see, using the decomposition $X_t = X_0 + M_t + V_+(t) + V_-(t)$, that if $X_\infty \neq 0, \pm\infty$ then M, V_+, V_- converge to a finite limit on
the set ν^+_-. If $X_\infty = 0$, then we argue as in b) (i) and conclude that M,
V_+, V_- converge to a finite limit on the set ν^+_-. Thus, on ν^+_-,

$$\{\omega : X_\infty \neq \pm\infty\} \overset{\text{a.s.}}{\subset} \{\omega : V_+, V_-, M \text{ converge to a finite limit}\}.$$

Now exactly as in case a) (ii), we can show that the RHS set is contained
in $\{\omega : [X, X]_\infty < \infty\}$. The inclusion $\{\omega : [X, X]_\infty < \infty\} \cap \nu^+_- \overset{\text{a.s.}}{\subset}$
$\{\omega : \lim_{t\to\infty} X_t \text{ exists}\}$ is also exactly as in case a) (ii). This completes the
proof of b) and the proof of the theorem. $\square$

Remark 2.5. – The inclusions in Theorem 2.4 viz:

$$\{\omega : X_\infty \neq 0, \pm\infty\} \subset \{\omega : [X, X]_\infty < \infty\} \subset \{\omega : \lim_{t\to\infty} X_t \text{ exists}\}$$

can be strict. Consider for example the process $X_t = h(t) B_t$ where (B_t)
is a Brownian motion and $h(t)$ non negative decreasing, $h(t) \sim \dfrac{1}{\sqrt{t \log t}}$
as $t \to \infty$. Then, by the law of the iterated logarithm, $X_t \to 0$ a.s., but
$[X, X]_\infty = \langle X^c \rangle_\infty = \displaystyle\int_0^\infty h^2(s)\, ds = \infty$. On the other hand, if $h(t) \sim \dfrac{1}{t}$
as $t \to \infty$, then $\{\omega : [X, X]_\infty < \infty\} = \Omega$, $\{\omega : X_\infty \neq 0\} = \varnothing$.

We now consider the local time $\mathbf{L}^0_t$ on the set ν^-_+. The following lemma
shows that in this situation, $\mathbf{L}^0_t$ plays a special role.

LEMMA 2.6. – *On the set ν^-_+,*

$$\{\omega : \mathbf{L}^0_\infty < \infty\}$$
$$\overset{\text{a.s.}}{\subset} \{\omega : \lim_{t\to\infty} X_t \text{ exists and is finite}\} \cap \{\omega : [X, X]_\infty < \infty\}.$$

Proof. – From Lemma 2.1, it follows that for all $b \in \mathbf{R}_+$, on the set ν_+^-, $\{\omega : \mathbf{L}_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \mathbf{L}_\infty^b < \infty\}$. From Theorem 2.4 a) (i), it follows that on ν_+^-,

$$\{\omega : \mathbf{L}_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t \to \infty} X_t \text{ exists}\}.$$

Now, equations (1) and (2) imply that in fact on ν_+^-,

$$\{\omega : \mathbf{L}_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t \to \infty} X_t \text{ exists and is finite}\}.$$

But the RHS set is the union of the sets $\{\omega : X_\infty = 0\}$ and $\{\omega : X_\infty \neq 0, \pm\infty\}$. The latter set is already contained in $\{\omega : [X, X]_\infty < \infty\}$, by Theorem 2.4 a) (ii). If $\omega \in \{X_\infty = 0\}$, we argue as follows: firstly, it follows from equations (3) and (4) that $\int_0^\infty I_{(0, a)}(X_{s-}) \, d\langle M \rangle_s$ (for $a > 0$)

and $\int_0^\infty I_{(a, 0)}(X_{s-}) \, d\langle M \rangle_s$ (for $a < 0$) are both finite.

Since $X_\infty = 0$, it implies in fact that $\langle M \rangle_\infty < \infty$ and hence that the martingale part M_t converges. Now, from Tanaka's formula, it follows that V_+ and V_- converge to a finite limit and hence that $\sum_s \Delta X_s^2 < \infty$. $[X, X]_\infty < \infty$ now follows.

Remark 2.7. – We list below the mutually exclusive possibilities that can occur on the set ν_+^- with respect to the variables $\mathbf{L}_\infty^0$, $[X, X]_\infty$ and X_∞. Now with the help of Theorems 1.3 and 1.7, we can explicitly compute functions $h(t)$ so that these possibilities are in fact realised by the semi-martingale $X_t = h(t) B_t$ where B_t is a Brownian motion.

a) $\mathbf{L}_\infty^0 < \infty$, $[X, X]_\infty < \infty$ and $X_\infty \neq 0, \pm\infty$. This case does not arise for the semi-martingale $X_t = h(t) B_t$.

b) $\mathbf{L}_\infty^0 < \infty$, $[X, X]_\infty < \infty$ and $X_\infty = 0$ $\left(h(t) \sim \dfrac{1}{t^\alpha}, \; \alpha > \dfrac{1}{2} \right)$

c) $\mathbf{L}_\infty^0 = \infty$, $[X, X]_\infty < \infty$ and $X_\infty = 0$

$$\left(h(t) \sim \frac{1}{\sqrt{t} \, (\log t)^\beta} \quad \text{for} \quad \frac{1}{2} < \beta \leq 1 \right)$$

d) $\mathbf{L}_\infty^0 = \infty$, $[X, X]_\infty = \infty$ and $X_\infty = 0$

$$\left(h(t) \sim \frac{1}{\sqrt{t} \, (\log t)^\beta} \quad \text{for} \quad 0 < \beta \leq \frac{1}{2} \right)$$

e) $\mathbf{L}_\infty^0 = \infty$, $[X, X]_\infty = \infty$ and $-\infty < \underline{X}_\infty < \bar{X}_\infty < \infty$

$$\left(h\left(t\right) \sim \frac{1}{\sqrt{t\left(\log \log t\right)^\gamma}} \text{ for } \gamma = 1 \right)$$

f) $\mathbf{L}_\infty^0 = \infty$, $[X, X]_\infty = \infty$, $\bar{X}_\infty = \underline{X}_\infty = \infty$ $\left(h\left(t\right) \sim \dfrac{1}{\sqrt{t}} \right)$

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Nonetheless, we would like to mention that the inhomogeneous Markov process (Y_t) studied in Chan and Williams [2] and which satisfies the stochastic differential equation $dY_t = -F\left(Y_t\right) dt + \varepsilon\left(t\right) dB_t$ was one of the original motivations of our study.

REFERENCES

[1] J. AZÉMA and M. YOR, En guise d'introduction. Astérisque 52-53. Soc. Math. France, 1978.

[2] C. T. CHAN and D. WILLIAMS, An excursion approach to an annealing problem, *Math. Proc. Camb. Phil. Soc.*, 105, 1989.

[3] C. DONATI-MARTIN, Transformation de Fourier et Temps d'Occupation Browniens. *Proba. Th. and Rel. Fields*, **88**, 1991, pp. 137-166.

[4] T. JEULIN, Semi-martingales et grossissement de filtrations, *Lecture Notes in Maths*, 833, Springer-Verlag, 1980.

[5] T. JEULIN, Sur la convergence absolue de certaines intégrales. Sém Proba XVI. *Lect. Notes in Maths*, Springer-Verlag, **920**, 1982, p. 248-255.

[6] T. JEULIN and M. YOR, Filtration des ponts browniens et équations différentielles stochastiques linéaires. Sém. Proba. XXIV, *Lect. Notes in Maths*, 1426, Springer-Verlag, 1990.

[7] X. X. XUE, A zero-one law for integral functionals of the Bessel process. Sém. Probas XXIV, *Lect. Notes in Math*, Springer–Verlag, 1426, 1990, p. 137-153.

[8] M. YOR, Sur le balayage des semi-martingales continues, Sém. de Prob. XIII, *Lecture Notes in Maths*, Springer-Verlag, 721, 1979.

[9] M. YOR, Sur la continuité des temps locaux associés à certaines semi-martingales. Astérisque, 52-53, Soc. Math. France, 1978.

[10] Ch. YOEURP, Compléments sur les temps locaux et les quasi-martingales, Astérisque 52-53, Soc. Math. France, 1978.
[11] M. YOR, Rappels et préliminaires généraux. Astérisque 52-53, Soc. Math. France, 1978.
[12] D. REVUZ and M. YOR, Continuous martingales and Brownian motion. Springer, Second edition, 1994.

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