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Lévy processes that can creep downwards never increase

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ABSTRACT. – A Lévy process can creep downwards if the probability that it does not jump at the first instant when it passes below a given negative level is positive. We show that in this case, it never increases, where increases is taken in the sense of Dvoretzky, Erdös and Kakutani. In particular, a Lévy process with no negative jumps or with non-zero Gaussian component never increases.

Key words: Lévy process, non-increase, downwards creeping.

RÉSUMÉ. – On dit qu'un processus de Lévy peut ramper vers le bas si la probabilité pour qu'il ne saute pas lors de son premier temps de passage au dessous d'un niveau strictement négatif fixé, est non nulle. On montre que dans ce cas, il ne croît jamais, au sens de Dvoretzky, Erdös et Kakutani.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Consider a real-valued Lévy process $X = (X_s : s \geqq 0)$ started at $X_0 = 0$. That is X has stationary independent increments and its paths, $s \rightarrow X_s$, are right-continuous and have left-limits on $(0, \infty)$, a.s. We say

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that X never increases if the set of positive t's for which there exists $\varepsilon \in (0, t)$ such that

$$X_{t'} \leqq X_t \leqq X_{t''} \quad \text{for all } t' \in [t-\varepsilon, t] \quad \text{and} \quad t'' \in [t, t+\varepsilon], \tag{1.a}$$

is empty a.s. This notion was introduced by Dvoretzky, Erdös and Kakutani [D-E-K] who established that the Brownian motion never increases. Simpler proofs of this remarkable property have been given by Knight [Kn], Adelman [Ad], Aldous [Al] and Burdzy [Bu]. Recently, the author [Be] extended Dvoretzky, Erdös and Kakutani Theorem by characterizing the class of Lévy processes with no positive jumps that can increase (the argument in [Be] relies crucially on the absence of positive jumps and does not apply to general Lévy processes). Observe that the mere existence of positive jumps does not necessarly imply the existence of increase times. More precisely, due to the strong Markov property, the instant of a positive jump is not an increase time a.s. whenever 0 is regular for $(-\infty, 0)$, that is whenever X visits the negative half-line immediately after time 0.

We say that X can creep downwards if the probability that it does not jump at the first instant when it passes strictly below a given negative level is positive. This property has been studied by Millar [Mi-1] (who called it continuous downwards passage) and by Rogers [Ro] (from whom we borrowed the terminology). As announced in the title, the main result of this paper is the following

THEOREM. – *If X can creep downwards, then it never increases.*

Remark. – There exist Lévy processes which cannot creep downwards and which never increase. For instance, sub-section 3.d in [Be] provides an example of a Lévy process which has zero Gaussian coefficient, no positive jumps, and which never increases. Since it has no positive jumps, it creeps upwards, but it cannot creep downwards because its Gaussian coefficient is null (*see* Millar [Mi-1], Theorem 3.5 and Corollary 3.1, or Rogers [Ro], Corollary 3).

Obviously, if X has no negative jumps and is not an increasing process, then it creeps downwards, and therefore it never increases. This should be compared with the results of [Be], where it is shown that "most" Lévy processes with no positive jumps and no Gaussian component possess increase times. It may seem paradoxal: the absence of positive jumps facilitates increase while the absence of negative jumps prevents increase! The intuitive explanation is simple, at least when X has bounded variation and is not monotone. If all the jumps are negative, then the drift is positive, and conversely, if all the jumps are positive, then drift is negative.

Informally, the effect of the drift for increase is much more important than the effect of the jumps. A positive drift facilitates increase, and a negative drift interferes with increase. This argument is only heuristic when X has unbounded variation (because the notion of drift vanishes); nevertheless, it makes the result less surprising.

Recall now the Lévy-Khintchine formula

$$\Psi(z) := \log E(\exp z X_1), \qquad \mathrm{Re}(z) = 0, \tag{1.b}$$

where

$$\Psi(z) = az + \frac{1}{2}\sigma_0^2 z^2 + \int (e^{zx} - 1 - zx 1_{\{|x|<1\}})\, d\Pi(x). \tag{1.c}$$

Here, a is a real number, $\sigma_0 \geqq 0$ is the Gaussian component and represents the Brownian part of X, and Π is the Lévy measure. The process X has bounded variation a.s. iff $\sigma_0 = 0$ and $\int (1 \wedge |x|)\, d\Pi(x) < \infty$. In this case, the characteristic exponent can be re-expressed as

$$\Psi(z) = a' z + \int (e^{zx} - 1)\, d\Pi(x), \tag{1.d}$$

where a' is the drift coefficient.

Millar [Mi-1] gave various conditions in terms of the characteristics a, σ_0 and Π which ensure that a Lévy process can creep downwards. We deduce from his results the following

COROLLARY. – *Assume that at least one of the conditions below is fulfilled*

(i) $\sigma_0 > 0$.

(ii) *X has bounded variation and negative drift coefficient.*

(iii) $\int_{-1}^{0} x\, d\Pi(x) > -\infty$ *and* $\int_{0}^{1} x\, d\Pi(x) = \infty$.

Then X never increases.

We refer the reader to Millar [Mi-1], Theorem 3.4, for further more precise (but less simple) sufficient conditions for downwards creeping.

The Theorem is obvious when 0 is not regular for $[0, \infty)$. Indeed, let $T_0, T_1, \cdots$ stand for the successive ascending ladder times, that is $T_0 \equiv 0$, $T_{i+1} = \inf\{t > T_i : X_t \geqq X_s \text{ for all } s \leqq t\}$. Plainly, ladder times are stopping times. By the strong Markov property and the regularity of 0 for $(-\infty, 0)$ (which is implied by downwards creeping), for every fixed $T > 0$

$$\left.\begin{array}{c} P(\exists t \in (0, T) : X_{t'} \leqq X_t \leqq X_{t''} \\ \text{for all } t' \in [0, t] \text{ and } t'' \in [t, T]) = 0. \end{array}\right\} \tag{1.e}$$

This implies that X never increases.

The Theorem is proved in section 3 (in the case when 0 is regular for $[0, \infty)$). The arguments are close to those in Dvoretzky, Erdös and Kakutani [D-E-K]: informally, if X could increase, then there would be "too many" increase times in the neighborhood of the instants when X increases. To make this heuristic idea rigorous, Dvoretzky *et al.* estimate the probability of fairly involved events. Their computations rely on the Gaussian densities of the Brownian law, and cannot be carried out in the case of a general Lévy process. Our approach is based on a reinforcement of the strong Markov property. More precisely, we establish a Markov property for $(\mathcal{G}_t)$-stopping times, where $(\mathcal{G}_t)_{t \geqq 0}$ is the natural filtration of X expanded with the future-infimum process. The first "global" increase time is then a $(\mathcal{G}_t)$-stopping time, and the above informal argument can be made rigorous. The reinforcement of the Markov property is presented in section 2, as a consequence of the decomposition of X at its infimum. The description of this path-decomposition extends a famous result due to Williams [Wi] in the Brownian case, and may be of independent interest.

2. PRELIMINARIES

We use the canonical notation. The probability space Ω is the set of càdlàg functions $\omega : [0, \infty) \to R \cup \{\partial\}$ (where ∂ is a cemetery point), endowed with the topology of Skorohod and the Borel σ-field, $\mathcal{F}$. As usual, $\theta.$ and $k.$ stand respectively for the translation and the killing operators. We denote the coordinate process by $X := (X_t(\omega) = \omega(t), t \geqq 0)$, its lifetime by $\zeta := \inf\{t : X_t = \partial\}$, and its natural filtration by $(\mathcal{F}_t)_{t \geqq 0}$. It will be more convenient for our purposes to work on a random time-interval, so we consider a probability measure P on $(\Omega, \mathcal{F})$ under which X is a Lévy process started at $X_0 = 0$ and killed at unit rate. In particular, $P(\zeta > t) = e^{-t}$.

For every $x < 0$, we say that X creeps across x if X visits $(-\infty, x)$ and does not jump at its first passage time strictly below x, that is $T(x) < \zeta$ and $X_{T(x)} = x$, where

$$T(x) = \inf\{t : X_t < x\}. \tag{2.a}$$

Note that when 0 is irregular for $(-\infty, 0)$, X cannot creep across any negative level. Indeed, the descending ladder time set (that is the instants when X attains a new infimum) is discrete a.s., and plainly, a discrete-time process cannot creep. Therefore, we may assume with no loss of generality that 0 is regular for the negative half-line.

If the probability that X creeps across a given negative level is positive, then, for every $x < 0$, $P(X$ creeps across $x) > 0$ (*see* Millar [Mi-1], Corollary 3.1). In this case, we say that X can creep downwards under P. This property can be characterized in a simple way in the framework of fluctuation theory. Specifically, denote the (past) infimum process of X by

$$\underline{X}(t) := \inf\{X_s : s \leqq t\}, \qquad t < \zeta. \tag{2.b}$$

Then the reflected process $X - \underline{X}$ is a strong Markov process under P, *see* e.g. Theorem 2.a in Bingham [Bi]. Its zero set, also called the descending ladder set, possesses a local time process, $\underline{L}$. That is $\underline{L}$ is a positive continuous additive functional of $X - \underline{X}$ that increases only (as a matter of fact, exactly) when $X - \underline{X} = 0$ (or equivalently when $\underline{X}$ decreases). The multiplicative constant in the choice of $\underline{L}$ is picked such $E(\underline{L}(\zeta-)) = 1$. Let $\underline{\tau}(t) := \inf\{s : \underline{L}(s) > t\}$ be the right-continuous inverse of $\underline{L}$. The time-changed process

$$S := -X \circ \underline{\tau} = -\underline{X} \circ \underline{\tau} \tag{2.c}$$

is a subordinator (that is an increasing Lévy process), *see* Theorem 9.1 in Fristedt [Fr]. Then X creeps across $x < 0$ iff S hits $-x$, *see* [Mi-1], Proposition 3.1. This event has a positive probability iff S has a positive drift, *see* Kesten [Ke], Proposition 6. In this case, the continuous part of the infimum process, $\underline{X}^c$, is not identically zero, and is proportional to $-\underline{L}$.

Throughout the rest of this paper, we will suppose that X can creep downwards and that 0 is regular for the positive half-line under P. Recall that 0 is also regular for $(-\infty, 0)$.

The probability that S hits $-x$ can be expressed in terms of the potential density of S, *see* Neveu [Ne] and Kesten [Ke], Proposition 6. Specifically, the potential measure given by

$$E\left(\int_0^\infty f(-S_t)\, 1_{\{\underline{\tau}(t)<\infty\}}\, dt\right) = E\left(\int_0^\zeta f(X_t)\, d\underline{L}(t)\right) \tag{2.d}$$

for Borel functions $f \geqq 0$, is absolutely continuous with respect to Lebesgue measure on $(-\infty, 0]$, and has continuous density g. Moreover, $g > 0$, and $g(0) = 1/\delta$, where δ is the drift coefficient of S. Then the probability that S hits $-x$, that is that X creeps across $x < 0$, equals $\delta g(x)$.

Recall that the lifetime of S, $\underline{L}(\zeta-)$, has expectation 1, and that $T(x)$ stands for the first passage time of X below x. Therefore, we have also

$$\begin{aligned} P(\underline{X}(\zeta-) < x) &= P(T(x) < \zeta) \\ &= P(\exists t < \underline{L}(\zeta-) : S_t > -x) \\ &= E\left(\int_0^\infty 1_{\{S_t > -x\}} 1_{\{\underline{\tau}(t) < \infty\}} dt\right) \\ &= \int_{-\infty}^x g(y)\, dy, \end{aligned} \tag{2.e}$$

where the third equality above comes from the strong Markov property and the fact that S has increasing paths. We record these results in the following.

Proposition 1. – *Assume that X can creep downwards under P. Then*

(i) *the distribution function $G(x) := P(\underline{X}(\zeta-) \geqq x)$, $x \leqq 0$, has a continuous derivative, $G' := -g < 0$ on $(-\infty, 0]$, and $g(0) = 1/\delta > 0$,*

(ii) *for every $x \leqq 0$, P (X creeps across x) $= \delta g(x) = g(x)/g(0)$,*

(iii) *for every Borel function $f \geqq 0$, we have*

$$E\left(\int_0^\zeta f(X_t)\, d\underline{L}(t)\right) = \int_{-\infty}^0 f(x) g(x)\, dx.$$

Now, we apply the preceding Proposition to condition the Lévy process on its infimum. First, we condition X to stay positive on $(0, \zeta)$ as follows. For every $x < 0$, we introduce the conditional law $P_{-x}(\cdot | X > 0)$ in the usual sense: for every nonnegative random variable Z, we set

$$E_{-x}(Z | X > 0) := \frac{1}{G(x)} E_{-x}(Z, \underline{X}(\zeta-) > 0), \tag{2.f}$$

where P_{-x} stands for the law of $X - x$ under P. However, since $G(0) = 0$, the above definition makes no sense for $x = 0$. So, following Millar [Mi-2], we introduce

$$\rho := \sup\{t < \zeta : X_t = \underline{X}(t)\}, \tag{2.g}$$

the (P-a.s. unique) instant when X attains its infimum on $[0, \zeta)$, and we split the path at time ρ. Then under P, the post-infimum process $X \circ \theta_\rho - X_\rho$ is independent of $\mathcal{F}_\rho$, and we denote its law by $P(\cdot | X > 0)$. Since 0 is regular for the positive half-line, X does not jump at time ρ P-a.s. (*see* [Mi-2], Lemma 3.2), $X_\rho = \underline{X}(\rho) = \underline{X}(\rho-)$, and the post-infimum process stays positive. Then, according to Proposition 4.1 in [Mi-2], X is a Markov process in its own filtration under $P(\cdot | X > 0)$, and its semigroup can be expressed as an h-transform of that of the Lévy process killed as it enters the negative half-line. Observe that the latter is a right process and according to Sharpe [Sh] on pages 298-9, so are its h-transforms. This implies that X is a strong Markov process under $P(\cdot|X>0)$. Specifically, for every $(\mathcal{F}_t)$-stopping time $T>0$ and for every nonnegative random variable Z, we have

$$E(Z\circ\theta_T|\mathcal{F}_T,\, X>0)=E_{X_T}(Z|X>0), \tag{2.h}$$

where the left-hand-side represents a conditional expectation with respect to $\mathcal{F}_T$ under the probability measure $P(\cdot|X>0)$ and the right-hand-side is given by (2.f). Recall also that the zero-one law of Millar (Theorem 3.1 of [Mi-2]) implies that $P(T=0|X>0)=0$ or 1 for every stopping time T, and *see* Greenwood and Pitman [G-P] for a simple approach of these results based on excursion theory.

For every $x<0$, we define the probability measure $P(\cdot|\underline{X}(\zeta-)=x)$ as follows. First, we denote by $P^{\downarrow x}$ the law of the Lévy process conditioned to creep across x and then killed as it hits x . That is

$$E^{\downarrow x}(Z):=\frac{1}{\delta g(x)}E(Z\circ k_{T(x)},\, T(x)<\zeta,\, X_{T(x)}=x), \tag{2.i}$$

where $Z\geqq 0$ is a random variable and k stands for the killing operator. Then we introduce $P(\cdot|\underline{X}(\zeta-)=x)$ as the law of the process obtained after pasting together two independent processes, the first with the law $P^{\downarrow x}$, and the second with the law $P(\cdot|X>0)$. Plainly, under $P(\cdot|\underline{X}(\zeta-)=x)$, the infimum of X is $\underline{X}(\zeta-)=x$ a.s. The family of laws $P(\cdot|\underline{X}(\zeta-)=x)$ serves as a regular version of P conditioned on its infimum, an it is amplified in Proposition 2 below (which completes Proposition 3.1 in [Mi-2]).

PROPOSITION 2. – *For every random variable $Z\geqq 0$.*

$$E(Z)=\int_{-\infty}^{0}E(Z|\underline{X}(\zeta-)=x)\,g(x)\,dx.$$

In particular, under P, the post-infimum process $X\circ\theta_\rho-X_\rho$ is independent of $\mathcal{F}_\rho$ and has the law $P(\cdot|X>0)$. Moreover, the law of the pre-infimum process $X\circ k_\rho$ conditionally on $\underline{X}(\zeta-)=x$ is $P^{\downarrow x}$.

Remark. – In the Brownian case, this result was discovered by Williams [Wi] and is the origin of a huge literature in path decompositions.

Proof. – Since we know already that under P, the post-infimum process is independent of $\mathcal{F}_\rho$ and has the law $P(\cdot|X>0)$, it is sufficient to show that for every left-continuous adapted process $Z\geqq 0$, we have

$$\begin{aligned}E(Z_\rho)&=\int_{-\infty}^{0}E^{\downarrow x}(Z_{\zeta-})\,g(x)\,dx\\&=\frac{1}{\delta}\int_{-\infty}^{0}E(Z_{T(x)},\, T(x)<\zeta,\, X_{T(x)}=x)\,dx\end{aligned} \tag{2.j}$$

[where the second equality comes from the definition (2.i) of $P^{\downarrow x}$]. As well-known [*see* e.g. Sharpe [Sh], (35.8), (35.11) and (68.1)], the $(\mathcal{F}_t, P)$ dual predictable projection of the increasing process $1_{\{\rho<t\}}$ is $\underline{L}(t)$. Therefore,

$$\begin{aligned} E(Z_\rho) &= E\left(\int_0^\zeta Z_t \, d\underline{L}(t)\right) \\ &= E\left(\int_0^\infty Z_{\underline{T}(t)} \, 1_{\{\underline{T}(t)<\infty\}} \, dt\right) \\ &= \left(\int_0^\infty Z_{\underline{T}(t)} \, 1_{\{\underline{T}(t)<\infty\}} 1_{\{S(t)=S(t-)\}} \, dt\right), \end{aligned} \tag{2.k}$$

where the last equality comes from the fact that the set of the jump times of $S = -X_{T(\cdot)}$ is countable. Recall that the drift coefficient of S is $\delta > 0$, so we can re-express the above quantity as

$$\frac{1}{\delta} E\left(\int_0^\infty Z_{\underline{T}(t)} \, 1_{\{\underline{T}(t)<\infty\}} 1_{\{S(t)=S(t-)\}} \, dS_t\right). \tag{2.l}$$

By the substitution $S_t = -x$, this equals

$$\frac{1}{\delta} \int_{-\infty}^0 E\left(Z_{T(x)}, \, T(x) < \zeta, \, X_{T(x)} = x\right) dx, \tag{2.m}$$

which establishes (2.j). $\Diamond$

Our next result can be viewed as a strong Markov properties in an expanded filtration. First, introduce the future-infimum process

$$\underline{\underline{X}}(t) := \inf\{X_s : t \leqq s < \zeta\}, \qquad \text{for} \quad t < \zeta, \tag{2.n}$$

and

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\underline{\underline{X}}(t)), \tag{2.o}$$

the enlargement of $\mathcal{F}_t$ by the σ-field generated by the future-infimum of the path at time t. Clearly, $\underline{\underline{X}}(s) = \inf\{X_u : s \leqq u \leqq t\} \wedge \underline{\underline{X}}(t)$ for $s \leqq t$, and so $\mathcal{G}_t = \sigma(X(s), \underline{\underline{X}}(s) : s \leqq t)$. In particular, $(\mathcal{G}_t)_{t \geqq 0}$ is an increasing family of σ-fields (that is a filtration).

PROPOSITION 3. – *Let T be a $(\mathcal{G}_t)$-stopping time such that $X(T) = \underline{\underline{X}}(T)$ on $\{T < \zeta\}$, P-a.s. Then under $P(\cdot | T < \zeta)$, $X \circ \theta_T - X_T$ is independent of $\mathcal{G}_t$, and has the law $P(\cdot | X > 0)$.*

Remark 1. – Plainly, the instant of the infimum, ρ, is a finite $(\mathcal{G}_t)$-stopping time with $X_\rho = \underline{\underline{X}}_\rho$. So, Proposition 3 agrees with Proposition 3.1 in [Mi-2] (*see* also the present Proposition 2).

2) More generally, it can be shown that for every $(\mathcal{G}_t)$-stopping time S, the law of $X \circ \theta_S - X_S$ under P conditionally on $\mathcal{G}_s$, is $P(\cdot | \underline{X}(\zeta-) = x)$, where $x = \underline{\underline{X}}(S) - X(S)$, but we will not use this in the sequel.

Proof. – Recall that for every fixed $t \geqq 0$, $X \circ \theta_t - X_t$ is independent of $\mathcal{F}_t$ and has the law P under $P(\cdot | t < \zeta)$. Observe also that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X(t) - \underline{\underline{X}}(t))$, and that $\mathcal{F}_t$ and $\sigma(X(t) - \underline{\underline{X}}(t))$ are independent under $P(\cdot | t < \zeta)$. We deduce from Proposition 2 that under $P(\cdot | t < \zeta)$, conditionally on $\mathcal{G}_t$, the law of $X \circ \theta_t - X_t$ is $P(\cdot | \underline{X}(\zeta-) = x)$, where $x = \underline{\underline{X}}(t) - X(t)$. Obviously, the above assertion still holds when we replace the fixed time t by an elementary $(\mathcal{G}_s)$-stopping time (that is a stopping time that takes only a finite number of values).

Let T be as in the statement. There exists a decreasing sequence of elementary $(\mathcal{G}_t)$-stopping times that converge to T, P-*a.s.* on $\{T < \zeta\}$. By the right-continuity of the paths, all what we need is to check that the family of probability measures $P(\cdot | \underline{X}(\zeta-) = x)$ converges, say, in the sense of the finite dimensional distributions, to $P(\cdot | X > 0)$ as x goes to 0. For this, recall (from its definition) that the path decomposition at the instant of the infimum, ρ, under $P(\cdot | \underline{X}(\zeta-) = x)$, yields two independent processes. The pre-ρ process has the law $P^{\downarrow x}$, and the post-ρ process has the law $P(\cdot | X > 0)$. It follows from (2.i), the regularity of 0 for the negative half-line and the equality $\delta g(0) = 1$ (according to Proposition 1) that for every $\varepsilon > 0$

$$\lim_{x \uparrow 0} P(\rho < \varepsilon | \underline{X}(\zeta-) = x) = \lim_{x \uparrow 0} P^{\downarrow x}(\zeta < \varepsilon) = 1. \tag{2.p}$$

Now fix $\varepsilon > 0$ and consider $\varepsilon < t_1 < \cdots < t_n$ and a bounded continuous function $F : \mathbb{R}^n \to \mathbb{R}$. We have

$$\begin{aligned} &E(F(X_{t_1}, \cdots, X_{t_n}), \rho < \varepsilon | \underline{X}(\zeta-) = x) \\ &\quad = \int_{(0,\varepsilon)} P^{\downarrow x}(\zeta \in dt) \\ &\quad \quad \times E(F(X_{t_1-t} + x, \cdots, X_{t_n-t} + x) | X > 0). \end{aligned} \tag{2.q}$$

It follows from (2.p) and the quasi-left continuity of the paths that the left-hand-side of (2.q) converges to $E(F(X_{t_1}, \cdots, X_{t_n}) | X > 0)$. $\Diamond$

Remark. – A related weaker result follows easily from time-reversal arguments. More precisely, recall that the processes $(X_t : t < \zeta)$ and $(X_{\zeta-} - X_{(\zeta-t)-} : t < \zeta)$ have the same law under P. Reversing the strong Markov process $\overline{X} - X$, where $\overline{X}_t := \sup\{X_s : s \leqq t\}$, we get that under P, $X - \underline{\underline{X}}$ is a *moderate* Markov process in its own filtration. Here, moderate means that the Markov property holds at predictable stopping times. However, this result is less useful than Proposition 3, because in general the natural filtration of $X - \underline{\underline{X}}$ is strictly included in $(\mathcal{G}_t)$ (the reason being that one cannot recover $\underline{\underline{X}}$ from $X - \underline{\underline{X}}$, except when X has no positive jumps). We refer to Chung and Walsh [C-W] for the moderate Markov property of reversed strong Markov processes.

3. PROOF OF THE THEOREM

Recall that we assume that X can creep downwards under P and that 0 is regular for the positive half-line. The basic idea of the proof consists of considering the sum of the increments of the past-supremum process of X made when X is close to its future infimum. Lemma 2 below shows that the expectation of this quantity under P is much smaller than under $P(\cdot|X > 0)$. It is then easy to deduce from Proposition 3 the absence of increase times.

Introduce for $\varepsilon > 0$ and $0 \leqq x < y$

$$c_\varepsilon(x, y) = E\left(\int_0^\zeta 1_{\{x \leqq \overline{X}(s) < y\}} 1_{\{X(s) - \underline{\underline{X}}(s) \leqq \varepsilon\}} d\overline{X}(s) \,|X > 0\right), \tag{3.a}$$

where

$$\overline{X}(t) := \sup\{X_s : s \leqq t\}, \qquad t < \zeta, \tag{3.b}$$

stands for the past-supremum process of X. First, we establish two technical results.

LEMMA 1. – *For every* $y' > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{c_\varepsilon(x, y)}{c_\varepsilon(0, y')} = \begin{cases} 0 & if \quad x > 0, \\ 1 & if \quad x = 0. \end{cases}$$

Proof. – According to (2.f) and (2.h), the $(\mathcal{F}_t, P(\cdot|X > 0))$ optional projection of the process $1_{\{X(s) - \underline{\underline{X}}(s) \leqq \varepsilon\}}$ is given on $\{s < \zeta\}$ by

$$P(\underline{\underline{X}}(s) \geqq X(s) - \varepsilon \,|\mathcal{F}_s, X > 0) = \begin{cases} 1 & \text{if} \quad X(s) < \varepsilon \\ G(-\varepsilon)/G(-X(s)) & \text{otherwise.} \end{cases} \tag{3.c}$$

Since $X = \overline{X}$ when $\overline{X}$ increases, we can re-express $c_\varepsilon(x, y)$ as

$$c_\varepsilon(x, y) = E\left(\int_0^\zeta 1_{\{x \leqq \overline{X}(s) < y\}} \times \left(\frac{G(-\varepsilon)}{G(-\overline{X}(s))} \wedge 1\right) d\overline{X}(s) \,|X > 0\right). \tag{3.d}$$

It follows that

$$\limsup_{\varepsilon \downarrow 0} \frac{c_\varepsilon(x, y)}{G(-\varepsilon)} \leqq \frac{y}{G(-x)} \quad \text{for} \quad x > 0. \tag{3.e}$$

On the other hand, we know from Proposition 1-i that $\lim_{x \uparrow 0} x/G(x) = -\delta < 0$. In particular, for every $\eta > 0$ small enough

$$\int_0^\eta \frac{1}{G(-\overline{X}(s))} d\overline{X}(s) \geqq \frac{\delta}{2} \int_0^\eta \frac{1}{(\overline{X}(s))} d\overline{X}(s),$$

and the right-hand-side is greater than or equal to $\delta/2$ for every path $\omega \in \Omega$ started at $\omega(0) = 0$ and such that ω visits $(0, \infty)$ immediately after time 0 (recall that ω is right-continuous). Therefore

$$\int_0^\zeta 1_{\{\overline{X}(s) < y\}} \frac{1}{G(-\overline{X}(s))} d\overline{X}(s) = \infty \qquad P(\cdot|X > 0)\text{-}a.s. \tag{3.f}$$

We deduce from (3.d) that

$$\lim_{\varepsilon \downarrow 0} c_\varepsilon(0, y)/G(-\varepsilon) = \infty. \tag{3.g}$$

The lemma follows from (3.e), since obviously $c_\varepsilon(0, y) = c_\varepsilon(0, x) + c_\varepsilon(x, y)$. $\Diamond$

LEMMA 2. – *For every* m, $n > 0$*:*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{c_\varepsilon(0, 1)} E\left(\int_0^\zeta 1_{\{\overline{X}(s) < m, \underline{\underline{X}}(0) > -n\}} 1_{\{X(s) - \underline{\underline{X}}(s) \leqq \varepsilon\}} d\overline{X}(s)\right) = 0.$$

Proof. – Recall that ρ is the first hitting time of 0 by $X - \underline{\underline{X}}$, and that X does not jump at time ρ. By the Markov type property at time ρ (*see* Proposition 2), we have

$$E\left(\int_0^\zeta 1_{\{\overline{X}(s) < m, \underline{\underline{X}}(0) > -n\}} 1_{\{X(s) - \underline{\underline{X}}(s) \leqq \varepsilon\}} d\overline{X}(s)\right)$$
$$= E\left(\int_0^\rho 1_{\{\overline{X}(s) < m, \underline{\underline{X}}(0) > -n\}} 1_{\{X(s) - \underline{\underline{X}}(s) \leqq \varepsilon\}} d\overline{X}(s)\right)$$
$$+ E\left(c_\varepsilon(\overline{X}_\rho - X_\rho, m - X_\rho) 1_{\{\overline{X}_\rho < m, X_\rho > -n\}}\right). \tag{3.h}$$

On the one hand, by the very definition of ρ, we have

$$\int_0^\rho 1_{\{X(s)-\underline{\underline{X}}(s)\leqq\varepsilon\}}\, d\overline{X}(s) = \int_0^\rho 1_{\{X(s)-\underline{\underline{X}}(0)\leqq\varepsilon\}}\, d\overline{X}(s) \leqq \varepsilon. \qquad (3.i)$$

Notice also that, since $\lim_{x\uparrow 0} G(x)/x = -1/\delta$, (3.g) implies that $\lim_{\varepsilon\downarrow 0} \varepsilon/c_\varepsilon(0, 1) = 0$. Therefore we have

$$\lim_{\varepsilon\downarrow 0} \frac{1}{c_\varepsilon(0, 1)} \times E\left(\int_0^\rho 1_{\{\overline{X}(s)<m,\, \underline{\underline{X}}(0)>-n\}}\, 1_{\{X(s)-\underline{\underline{X}}(s)\leqq\varepsilon\}}\, d\overline{X}(s)\right) = 0. \qquad (3.j)$$

On the other hand, on $\{\overline{X}_\rho < m,\, X_\rho > -n\}$, it holds that $c_\varepsilon(\overline{X}_\rho - X_\rho,\, m - X_\rho) \leqq c_\varepsilon(0, n+m)$. Recall that $\overline{X}_\rho - X_\rho > 0$ a.s., and thus by dominated convergence and Lemma 1,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{c_\varepsilon(0, 1)} E\left[c_\varepsilon(\overline{X}_\rho - X_\rho,\, m - X_\rho)\, 1_{\{\overline{X}_\rho<m,\, X_\rho>-n\}}\right] = 0. \qquad (3.k)$$

We deduce now the lemma from (3.h), (3.j) and (3.k). ◊

Now, we prove the Theorem. Denote by $T = \inf\{t : \overline{X}(t) = \underline{\underline{X}}(t)\}$, the first "global" increase time of X. Note that T is a $(\mathcal{G}_t)$-stopping time and that $X_T = \overline{X}_T = \underline{\underline{X}}_T$ on $\{T < \zeta\}$. Splitting the integral below at T, we get by Proposition 3

$$E\left(\int_0^\zeta 1_{\{\overline{X}(s)<m,\, \underline{\underline{X}}(0)>-n\}}\, 1_{\{X(s)-\underline{\underline{X}}(s)\leqq\varepsilon\}}\, d\overline{X}(s)\right) \geqq E\left[c_\varepsilon(0,\, m - \overline{X}(T)),\, \overline{X}_T < m,\, \underline{\underline{X}}(0) > -n,\, T < \zeta\right]. \qquad (3.l)$$

We deduce now from Lemmas 1 and 2 that

$$P(\overline{X}(T) < m,\, \underline{\underline{X}}(0) > -n,\, T < \zeta) = 0. \qquad (3.m)$$

This implies that X has no "global" increase times P-a.s., and it is immediate to deduce that X never increases under P. ◊

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