ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 31, nº 1 (1995), p. 81-118
<http://www.numdam.org/item?id=AIHPB_1995__31_1_81_0>

Ann. Inst. Henri Poincaré, (In Memoriam C. KIPNIS)

Vol. 31, n° 1, 1995, p. 81-118. *Probabilités et Statistiques*

Brownian fluctuations of the interface in the D=1 Ginzburg-Landau equation with noise

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ABSTRACT. – We consider the Ginzburg-Landau equation in an interval of $\mathbb{R}$, perturbed by a white noise and with Neumann boundary conditions. The initial datum is close to the stationary solution (that we call instanton) of the equation without noise. We prove that, as the variance of the noise goes to zero and the length of the interval is proportional to the inverse of this variance, then, the solution approaches an instanton which moves as a Brownian motion.

Key words: Stochastic PDE's, interface dynamics, invariance principle

RÉSUMÉ. – Nous considérons l'équation de Ginzburg-Landau dans un intervalle de $\mathbb{R}$ avec bruit blanc additif et avec conditions de Neumann à la frontière. La condition initiale est proche de la solution stationnaire (qu'on appelle « instanton ») de l'équation sans le bruit. Nous démontrons que, lorsque la variance du bruit tend vers zéro et la longueur de l'intervalle spatial est proportionnelle à l'inverse de cette variance, la solution est proche d'un « instanton » qui se déplace comme un mouvement Brownien.

1991 *Mathematics Subject Classification :* 45 M 10, 47 H 15, 82 C 24.

The research has been partially supported by CNR, GNFM, by grant SC1-CT91-0695 of the Commission of European Communities.

1. INTRODUCTION

The semi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + [u - u^3] \tag{1.1}$$

and the version with a stochastic force on its right hand side, appear in the physical literature as basic model equations for phase separation and interface dynamics in systems with non conserved order parameter, *see* Hohenberg and Halperin, 1973, [16], and Allen and Cahn, 1979, [2]. We study in this paper the influence on the motion of the interface of a small white noise added to (1.1).

We start by recalling a few results in the deterministic case when (1.1) is defined in the interval $\mathcal{T}_\epsilon := [-\epsilon^{-1}, \epsilon^{-1}]$, $\epsilon > 0$, with Neumann boundary conditions. The solution of the corresponding Cauchy problem defines a flow $T_t^{(\epsilon)}$, $t \geq 0$, in the space of functions that are twice continuously differentiable in the interior of $\mathcal{T}_\epsilon$ and have vanishing derivatives at the endpoints. The flow $T_t^{(\epsilon)}$ has obviously two fixed, stable points, $u_\pm \equiv \pm 1$, and an unstable one, $u_0 \equiv 0$. These are all the stationary, spatially homogeneous solutions of (1.1), but there are many others which are not spatially homogeneous. For small ϵ the most relevant ones, in the sense of stability, are the two "instantons" $\pm u_\epsilon^\star$, with $u_\epsilon^\star(x)$ a strictly increasing, antisymmetric function of x, positive for $x > 0$. In the limit as $\epsilon \to 0$, $u_\epsilon^\star \to \bar{m}$ pointwise, where

$$\bar{m}(x) = \tanh x \tag{1.2}$$

is a stationary solution of (1.1) in the whole $\mathbb{R}$.

Fusco and Hale, 1989, [15], Carr and Pego, 1989, [4], and Fusco, 1990, [14], have studied the stability of $\pm u_\epsilon^\star$ under $T_t^{(\epsilon)}$ and the corresponding problem for flows generated by semi-linear parabolic equations with more general non linear terms. The analysis shows that $\pm u_\epsilon^\star$ are saddle points for the flow $T_t^{(\epsilon)}$ with a one dimensional unstable manifold, $\mathcal{M}_\epsilon$, that connects $u_\epsilon^\star$ to the stable points $u_\pm$. $\mathcal{M}_\epsilon$ is invariant under the flow $T_t^{(\epsilon)}$ and it is locally stable in the sense that it attracts at exponential rate (uniformly as $\epsilon \to 0$) the orbits that start from a neighborhood of $\mathcal{M}_\epsilon$. Analogous conclusions hold for $-u_\epsilon^\star$.

The points in $\mathcal{M}_\epsilon$ can be parametrized by $x \in \mathcal{T}_\epsilon$ and are denoted by $u_{\epsilon,x}^\star$. These are essentially spatial shifts of the instanton $\bar{m}$, (1.2), namely for any $0 < R < 1$, there is $c > 0$ so that

$$\lim_{\epsilon \to 0} e^{c\epsilon^{-1}} \sup_{|x| \leq \epsilon^{-1} R} \sup_{|y| \leq \epsilon^{-1}} \left| u_{\epsilon,x}^\star(y) - \bar{m}(y - x) \right| = 0 \tag{1.3}$$

The flow on $\mathcal{M}_\epsilon$ is represented by two orbits $x_\epsilon^\pm(t)$, $t \in \mathbb{R}$, one, $x_\epsilon^+(t)$, in $\{x > 0\}$ and the other in $\{x < 0\}$. The former, $x_\epsilon^+(t)$, is an increasing function of t and $u_{\epsilon,x}^\star$, $x \equiv x_\epsilon^+(t)$, converges to the stable point u_- as $t \to \infty$ and to $u_\epsilon^\star$ as $t \to -\infty$. Analogous behavior is found in the other half of $\mathcal{M}_\epsilon$ where $x < 0$. The motion along $\mathcal{M}_\epsilon$ is very slow: the speed at $x \in \mathcal{M}_\epsilon$ is bounded by $\exp\{-c(r)\epsilon^{-1}\}$, with $r = \epsilon x$ and $c(r) > 0$. Therefore the points in a neighborhood of $\mathcal{M}_\epsilon$ are "first" attracted by $\mathcal{M}_\epsilon$ and then move, "very slowly", along $\mathcal{M}_\epsilon$.

In this paper we study the stability of the manifold $\mathcal{M}_\epsilon$ under the small random perturbations of the flow T_t^ϵ defined by the stochastic partial differential equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + [m - m^3] + \sqrt{\epsilon}\alpha \tag{1.4}$$

in $\mathcal{T}_\epsilon$ with Neumann boundary conditions. α is a white noise in space and time. The theory of stochastic PDE's applies to (1.4), as briefly recalled in the next Section, in particular we will refer to Faris and Jona-Lasinio, 1982, [11], Walsh, 1984, [19], and Da Prato and Zabczyk, 1992, [8]. Here it suffices to say that for any continuous initial datum m_0 and almost all the realizations of the noise, there is a unique, continuous function m_t, $t \geq 0$, that solves an integral version of (1.4) with Neumann boundary conditions in $\mathcal{T}_\epsilon$. The solution obtained in this way defines a continuous process m_t, $t \geq 0$, with values in $C(\mathcal{T}_\epsilon)$, hereafter referred to as "the Ginzburg-Landau process".

As a consequence of our analysis (but for brevity we will not state a theorem) it can be seen that $\mathcal{M}_\epsilon$ is stable also for the stochastic flow defined by (1.4) in the limit of small ϵ. In particular the points in a neighborhood of $\mathcal{M}_\epsilon$ if not "too close" to the boundaries $\pm\epsilon^{-1}$ are again attracted by $\mathcal{M}_\epsilon$, but then, at variance with the deterministic case, they move "rapidly along" $\mathcal{M}_\epsilon$. In fact on times $\epsilon^{-1}t$, to be compared to the exponential times of the deterministic case, the process becomes supported, in the limit $\epsilon \to 0$, by translates of the instanton $\bar{m}$ which then performs a brownian motion on $\mathcal{M}_\epsilon$. More precisely we consider an initial datum $\bar{m}^{(\epsilon,x_0)}(x) \in C^0(\mathbb{R})$ satisfying N.b.c. in $\mathcal{T}_\epsilon$ and such that

$$\sup_{x\in\mathcal{T}_\epsilon} |\bar{m}^{(\epsilon,x_0)}(x) - \bar{m}_{x_0}(x)| \leq \epsilon^{1/2} \qquad |x_0| \leq (1-\zeta)\epsilon^{-1} \tag{1.5}$$

and call m_t the process that solves (1.4) with initial datum $\bar{m}^{(\epsilon,x_0)}$. We denote by P^ϵ the probability on the basic space, where the noise α and the process m_t are constructed. Our main theorem in this paper is the following:

1.1. THEOREM. – *For any* $0 < \zeta < 1$ *the following holds. Given any* $\epsilon > 0$, $|x_0| \leq (1-\zeta)\epsilon^{-1}$, m_t *and* P^ϵ *as above, there is a continuous process* ξ_t *adapted to* m_t *such that for any* $T > 0$

$$\lim_{\epsilon \to 0} P^\epsilon\Big(\sup_{t \leq \epsilon^{-1} T} \sup_{x \in \mathcal{T}_\epsilon} \Big| m_t(x) - \bar{m}\big(x - \xi_t\big)\Big| > \epsilon^{1/4}\Big) = 0 \tag{1.6}$$

Moreover let $\mathbb{P}^\epsilon$ *be the law on* $C([0,T],\mathbb{R})$ *of the variable* $Y_t \equiv \xi_{\epsilon^{-1}t} - x_0$. *Then* $\mathbb{P}^\epsilon$ *converges weakly to a Brownian motion starting from 0 with diffusion coefficient*

$$D = \frac{3}{4} \tag{1.7}$$

Analogous results have been obtained by Dell'Antonio, 1988, [7], for the small random perturbations of a deterministic flow defined on a finite dimensional manifold and with an attractive submanifold that plays the role of $\mathcal{M}_\epsilon$. The problem in $\mathbb{R}$ with the noise strength a function of x that suitably vanishes as $|x| \to \infty$ has been considered by one of us, S.B., 1993, [3], and, more recently by Funaki, 1993, [13]. In [3] the process is studied for times much smaller than ϵ^{-1}, when the displacements of the instanton are still infinitesimal. In the limit as $\epsilon \to 0$ and with a proper normalization, they are proven to converge to a Brownian motion.

In [13], a paper that we have received when completing the first draft of this one, the analysis concerns much longer times. In our language Funaki studies times $t = \epsilon^{-1-2\delta}$, with δ positive and small. The displacement ξ_t of the instanton is renormalized as $\xi^\epsilon_\tau := \epsilon^\delta \xi_t$. Space is renormalized in the same way so that the instanton becomes in the new coordinates a step function. Funaki proves that this step function moves in the limit as $\epsilon \to 0$ as a Brownian motion with a drift, which comes from the spatial dependence of the noise strength. The different scalings and the different norms used make this paper quite different from ours. We are indebted to T. Funaki for his observations on our paper. In Section 3 we make some more comments on his approach.

2. THE GINZBURG-LANDAU PROCESS

In this section we state the Cauchy problem for (1.4) in a form that is particularly convenient for the proof of Theorem 1.1. The standard way, *see*

Walsh, 1981, [19], and 1984, [18], to give a sense to the Cauchy problem for (1.4) is to consider the corresponding integral equation

$$m_t = H_t^{(\epsilon)} m_0 - \int_0^t ds H_{t-s}^{(\epsilon)} (m_s^3 - m_s) + \sqrt{\epsilon} Z_t^{(\epsilon)} \tag{2.1}$$

$$Z_t^{(\epsilon)}(x) = \int d\alpha(s,y) \chi_{[0,t]}(s) H_{t-s}^{(\epsilon)}(x,y) \tag{2.2}$$

where $m_0 \in C(\mathcal{T}_\epsilon)$ is the initial datum, $H^{(\epsilon)}$ is the Green operator for the heat equation with Neumann boundary conditions (N.b.c.) in $\mathcal{T}_\epsilon$ and χ_A is the characteristic function of the set A. The stochastic integral in (2.2) defines a Gaussian process in space and time for which the following properties hold:

2.1 Lemma. – *For any $\epsilon > 0$ the process $Z_t^{(\epsilon)}(x)$ is continuous in both variables and there are constants b_0 and b_1 positive such that for any $a > 0$*

$$P^\epsilon \{ \sup_{x \in \mathcal{T}_\epsilon, t \le 1} |Z_t^{(\epsilon)}(x)| > \epsilon^{-a} \} \le b_0 e^{-b_1 \epsilon^{-2a}} \tag{2.3}$$

The proof of Lemma 2.1 is given in Section 5. It uses some Gaussian processes results, that can be found in [1]. Some useful estimates on the covariance are taken from [18].

By Lemma 2.1 we can restrict to continuous $Z_t^{(\epsilon)}$ in (2.1), in which case an existence and uniqueness theorem holds, as a straightforward extension of the proof in [11], 1982, for the case of Dirichlet boundary conditions. In the sequel we will consider an equivalent representation of the process obtained by using the reflection symmetry associated to N.b.c.

2.2 Definition. – *For any continuous function m in $\mathcal{T}_\epsilon$ we define its extension $\check{m}$ to $\mathbb{R}$ by reflecting m through ϵ^{-1}, and then extending to $\mathbb{R}$ with period $4\epsilon^{-1}$. That is*

$$\check{m}(x) =$$

$$\begin{cases} m(x - 4k\epsilon^{-1}) \; if \; x \in [(4k-1)\epsilon^{-1}, (4k+1)\epsilon^{-1}] \; for \; some \; k \in \mathbb{Z} \\ m((4k+2)\epsilon^{-1} - x) \; if \; x \in [(4k+1)\epsilon^{-1}, (4k+3)\epsilon^{-1}] \; for \; some \; k \in \mathbb{Z} \end{cases} \tag{2.4}$$

Reciprocally, we say that a function $\check{m} \in C^0(\mathbb{R})$ defined on the whole line satisfies N.b.c. if it is the extension in the above sense of a function in $\mathcal{T}_\epsilon$.

We define

$$Z_t = \check{Z}_t^{(\epsilon)} \tag{2.5}$$

and refer to Z_t as the "free process". We denote by H_t the Green operator for the heat equation on the whole line, so that for any $m \in C(\mathcal{T}_\epsilon)$,

$$H_t^\epsilon m(x) = \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \check{m}(y) \equiv H_t \check{m}(x) \tag{2.6}$$

2.3 PROPOSITION. – *For any $\epsilon > 0$, for any $m_0 \in C^0(\mathbb{R})$ that satisfies N.b.c. in $\mathcal{T}_\epsilon$ and for any $Z_t(x)$ continuous in both variables and satisfying N.b.c., there is a unique continuous solution m_t of the integral equation*

$$m_t = H_t m_0 - \int_0^t ds H_{t-s}(m_s^3 - m_s) + \epsilon^{1/2} Z_t \tag{2.7}$$

with H_t as in (2.6). *Moreover $m_t = \check{m}_t^{(\epsilon)}$ where $m_t^{(\epsilon)}$ solves* (2.1) *with $Z_t^{(\epsilon)}$ and $m_0^{(\epsilon)}$ obtained by restricting Z_t and m_0 to $\mathcal{T}_\epsilon$.*

Proof. – We know from [10] that (2.7) has a unique continuous solution which therefore satisfies N.b.c. in $\mathbb{R}$. Then by (2.6) its restriction to $\mathcal{T}_\epsilon$ solves (2.1), whose solution is unique. □

We shall hereafter call the solution m_t of (2.7) the Ginzburg-Landau process. By Proposition 2.3, m_t is a process adapted to the basic process Z_t. We study the Ginzburg-Landau process by following the approach proposed in [3] which is based on a perturbative analysis of the Ginzburg-Landau process around the instanton solution (1.2). We write

$$\bar{m}_{x_0}(x) := \bar{m}(x - x_0), \quad \bar{m}(x) = \tanh x; \tag{2.8}$$

then the linearized Ginzburg-Landau evolution operator around $\bar{m}_{x_0}$ is

$$(L_{x_0}\phi)(x) = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(x) - V''\Big(\bar{m}_{x_0}(x)\Big)\phi(x), \tag{2.9}$$

where V'' is the derivative of

$$V'(u) = u^3 - u \tag{2.10}$$

Observe that if m_0 satisfies N.b.c. then $\|m_0 - \bar{m}_{x_0}\|_\infty$ is not small, even if m_0 were equal to an instanton in $\mathcal{T}_\epsilon$. However, by proving a barrier lemma, Proposition 5.3, we will be able to "localize" the analysis and the relevant norm will be a supremum restricted to $\mathcal{T}_\epsilon$. In this way our perturbative scheme will work also for initial data that satisfy N.b.c. in $\mathcal{T}_\epsilon$.

L_{x_0} is a self-adjoint operator in $L^2(dx, \mathbb{R})$ and $\bar{m}'_{x_0}$ an eigenvector of L_{x_0} with eigenvalue 0. The remaining part of the spectrum of L_{x_0} is in the negative axis at non zero distance from the origin, [12]. Therefore the semigroup $g_{t,x_0} = e^{L_{x_0} t}$, is a contraction semigroup whose norm restricted to the subspace orthogonal to $\bar{m}'_{x_0}$, decays exponentially in t. This is true also in $C^0(\mathbb{R})$, as it can be seen by using a Perron-Frobenius argument, based on the strict positivity of the eigenvector $\bar{m}'$. The analysis has been carried through in a paper involving two of the present authors, (A. DM and E. P), and T. Gobron, 1993, [9], for a different case, where (1.1) is replaced by a non local evolution equation. The proofs however extend to the present case and for the sake of brevity are omitted. We thus state without proof the following theorem.

THEOREM 2.4. – *There are $\alpha > 0$ and c so that for any $\phi \in C^0(\mathbb{R})$ and $x_0 \in \mathbb{R}$*

$$\|e^{L_{x_0} t}[\phi - N\tilde{m}'_{x_0}]\|_\infty \le ce^{-\alpha t}\|\phi - N\tilde{m}'_{x_0}\|_\infty \tag{2.11}$$

where $\tilde{m}'_{x_0}$ is the unitary vector in $L^2(dx, \mathbb{R})$ parallel to $\bar{m}'_{x_0}$:

$$\tilde{m}'_{x_0} = \frac{\sqrt{3}}{2}\bar{m}'_{x_0}, \qquad N = \int dx \tilde{m}'_{x_0}(x)\phi(x) \tag{2.12}$$

We will next give a heuristic proof of Theorem 1.1. As the initial state is $\bar{m}_{x_0}$, (forgetting for the time being of the finitiness of $\mathcal{T}_\epsilon$), the main contribution to m_t at the very beginning comes from the noise. We introduce then an orthonormal basis whose first vector is $\tilde{m}'_{x_0}$ and decompose $m_t - \bar{m}_{x_0}$ in this basis. Its coordinates are at short times approximately independent brownian motions. At some later time the drift in the equation becomes significant, but the deviations that are still small are ruled, to leading orders, by the semigroup g_{t,x_0} generated by L_{x_0}. Since all the directions are exponentially damped except along $\tilde{m}'_{x_0}$, after a time T_ϵ, in the successive analysis $T_\epsilon = \epsilon^{-1/10}$, (1/10 should be regarded as much smaller than 1/2) we will have, *see* Proposition 5.4,

$$m_{T_\epsilon} \approx \bar{m}_{x_0} + \epsilon^{1/2} b_{T_\epsilon} \tilde{m}'_{x_0} \tag{2.13}$$

with b_{T_ϵ} the value of a standard brownian motion at time T_ϵ. We then write, to first orders,

$$m_{T_\epsilon} \approx \bar{m}_{x_0 + c\epsilon^{1/2} b_{T_\epsilon}}, \qquad c = \frac{\sqrt{3}}{2} \tag{2.14}$$

By iterating this procedure we eventually reach a time when the displacement from the initial instanton is finite. By (2.14) its law is that of the limit Brownian motion of Theorem 1.1.

There are several points in the argument to be fixed, one for all the error when going from (2.13) to (2.14): the first term neglected is proportional to $\epsilon b_{T_\epsilon}^2 \bar{m}_{x_0}''$. If we simply sum up all these errors and iterate till time ϵ^{-1}, we get a finite contribution. In fact, since $b_{T_\epsilon} \approx T_\epsilon^{1/2}$, according to (2.14) we need N_ϵ iterations to have a finite displacement of the instanton, where $N_\epsilon^{1/2}\epsilon^{1/2}T_\epsilon^{1/2} \approx 1$ (having supposed that the increments are independent). Since the sum of the errors is $N_\epsilon \epsilon b_{T_\epsilon}^2 \approx N_\epsilon \epsilon T_\epsilon$, the sum is also of the order of 1.

The circumstance that helps us in this case is that $\bar{m}_{x_0}''$ is normal to $\bar{m}_{x_0}'$ in $L^2(\mathbb{R}, dx)$, so that the above error is killed by the exponential decay in Theorem 2.4 at the next step of the iteration. We deal with this and the other problems implicit in the above heuristic argument in some unified way, as explained in the next Section, and conclude this one by writing the equation for m_t in terms of the semigroup generated by L_{x_0}.

We denote by $u_t := m_t - \bar{m}_{x_0}$ and we expand $V'(m_s)$ around $\bar{m}_{x_0}$. Then using that $\bar{m}_{x_0}$ is a stationary solution to the deterministic equation in $\mathbb{R}$, we rewrite (2.7) as

$$u_t = H_t u_0 - \int_0^t ds H_{t-s}[V''(\bar{m}_{x_0})u_s + 3\bar{m}_{x_0}u_s^2 + u_s^3] + \epsilon^{1/2} Z_t \quad (2.15)$$

Recalling that $g_{t,x_0} = e^{L_{x_0}t}$, we have

$$u_t = g_{t,x_0}u_0 - \int_0^t ds g_{t-s,x_0}(3\bar{m}_{x_0}u_s^2 + u_s^3) + \epsilon^{1/2}\hat{Z}_{t,x_0} \quad (2.16)$$

where

$$\hat{Z}_{t,x_0} = Z_t + \int_0^t ds g_{t-s,x_0} V''(\bar{m}_{x_0}) Z_s \quad (2.17)$$

The proof of (2.16), that is omitted, makes formal the following heuristic argument: we rewrite (2.15) in differential form

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + V''(\bar{m}_{x_0})\right)(u_t - \epsilon^{1/2}Z_t)$$
$$= -V''(\bar{m}_{x_0})\epsilon^{1/2}Z_t - 3\bar{m}_{x_0}u_s^2 - u_s^3. \quad (2.18)$$

and then observe that it is "solved" by (2.16).

The representation (2.17) shows that $\hat{Z}_{t,x_0}$ is bounded and continuous in any compact time interval in the same set where this holds for Z_t, hence by Lemma 2.1, with probability 1. Some estimates on the distribution of $\hat{Z}_{t,x_0}$, like those in Proposition 5.4, are more easily established by writing an almost surely equal version of $\hat{Z}_{t,x_0}$ as the stochastic integral:

$$\hat{Z}_{t,x_0} = \int d\alpha(s,y)\chi_{[0,t]}(s)\,\chi_{[-\epsilon^{-1},\epsilon^{-1}]}(y)\sum_{k\in\mathbb{Z}}\Big(g_{t-s,x_0}(x,y+4k\epsilon^{-1}) + g_{t-s,x_0}(x,4k\epsilon^{-1}+2\epsilon^{-1}-y)\Big) \quad (2.19)$$

We omit the proof of (2.19) that makes formal the following argument, where we rewrite (2.17) in differential form:

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + V''(\bar{m}_0)\Big)(\hat{Z}_{t,x_0} - Z_t) = -V''_{x_0} Z_t$$

which then becomes

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + V''(\bar{m}_0)\Big)\hat{Z}_{t,x_0} = \Big(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big)Z_t$$

that yields (2.19).

We summarize the above discussion in the following Proposition.

2.5 PROPOSITION. – *Let* $\epsilon > 0$ *and* m_0 *and* Z_t *as in Proposition* 2.3. *Then, given any* $x_0 \in \mathcal{T}_\epsilon$, m_t *solves* (2.7) *if and only if* $u_t = m_t - \bar{m}_{x_0}$ *solves* (2.16) *with* $\hat{Z}_{t,x_0}$ *as in* (2.17), *or, alternatively, as in* (2.19).

3. FLUCTUATIONS OF THE INSTANTON

In this section we reduce the proof of Theorem 1.1 to several propositions, that will be proven in the next sections.

An important role in the whole proof is played by the notion of center: the center of an instanton is the position where the instanton has the value 0. For a general function we set:

3.1 DEFINITION. – *The function* $m \in C^0(\mathbb{R})$ *has a center* x_0 *if, using the notation* (2.8),

$$\int dx[m(x) - \bar{m}_{x_0}(x)]\bar{m}'_{x_0}(x) = 0 \quad (3.1)$$

In the next Proposition we will see that if a function is close to an instanton then it has a center. We will then proceed by showing that the Ginzburg-Landau process satisfies with large probability this closeness condition at all the times involved in Theorem 1.1. The notion of center will then be crucial in the proof of Theorem 1.1 as it is in the proof of Funaki, [13]. This is therefore a good point for some comments on the notion of center and on the Funaki's approach.

Recalling Theorem 2.4 the center x_0 of m can be interpreted as the center of the instanton $\bar{m}_{x_0}$ to which the orbit starting from m converges when the evolution T_t, defined by (1.1) in the whole $\mathbb{R}$, is linearized around $\bar{m}_{x_0}$. From this perspective it looks more natural to define the center directly in terms of the flow T_t: the new center y_0, generally different from x_0, is such that $T_t m \to \bar{m}_{y_0}$. The advantage of working with y_0 is that $y_0(m_t)$ is a martingale when m_t is the the Ginzburg-Landau process (in the whole $\mathbb{R}$), as observed by Funaki. In fact, by its definition, $y_0(m_t)$ does not vary when moving in the direction of the drift. This suggests a proof of Theorem 1.1 based on a martingale convergence theorem, which is indeed accomplished in [13]. The starting points in [13] and in the present paper are quite similar, because the process will be proven to stay always very close to an instanton, hence the two notions of center yield variables very close to each other. It is therefore amazing to see how such a small difference develops rather different strategies and in the end proofs that are essentially different.

3.2 Proposition. – *There are $\delta > 0$ and, given any ζ' and ζ such that $0 < \zeta' < \zeta < 1$, there are c and ϵ_0 so that for any $0 < \epsilon \leq \epsilon_0$ and any $|x_0| \leq (1-\zeta)\epsilon^{-1}$ the following holds. Let $m \in C^0(\mathbb{R})$, $\|m\|_\infty \leq 2$, and*

$$\|m - \bar{m}_{x_0}\|_\epsilon := \sup_{x \in \mathcal{T}_\epsilon} |m(x) - \bar{m}_{x_0}(x)| \leq \delta \tag{3.2}$$

Then

(1) *m has a center ξ in $\mathcal{T}_\epsilon$,*

$$|x_0 - \xi| \leq c\big(\|m - \bar{m}_{x_0}\|_\epsilon + e^{-\epsilon^{-1}\zeta'}\big) \tag{3.3}$$

and ξ is unique in $\{|x| \leq (1-\zeta')\epsilon^{-1}\}$.

(2) *Let*

$$\xi^0 := x_0 - \frac{3}{4} \int dx \bar{m}'_{x_0}(x)[m(x) - \bar{m}_{x_0}(x)] \tag{3.4}$$

then

$$|\xi - \xi^0| \leq c\big(\|m - \bar{m}_{x_0}\|^2_\epsilon + e^{-\epsilon^{-1}\zeta'}\big) \tag{3.5}$$

(3) *Let* $m^\star \in C^0(\mathbb{R})$, $\|m^\star\|_\infty \le 2$ *and*

$$\|m^\star - m\|_\epsilon \le \delta \tag{3.6}$$

Then $m^\star$ *has a unique center* $\xi^\star$ *in* $\{|x| \le (1-\zeta')\epsilon^{-1}\}$ *and*

$$|\xi - \xi^\star| \le c \int dx \bar{m}'_{x_0}(x)|m^\star(x) - m(x)| \tag{3.7}$$

(4) *If* m *satisfies N.b.c. in* $\mathcal{T}_\epsilon$, *then* m *has a unique center in* $\mathcal{T}_\epsilon$.

In Proposition 5.2 we will prove that with large probability the Ginzburg-Landau process that starts from an instanton has sup norm bounded by 2 at all the times t that are involved in our analysis, hence the condition $\|m\|_\infty \le 2$ in Proposition 3.2.

3.3 DEFINITION. – *We define the function* $\xi(m)$ *as the center of* m *whenever* m *satisfies the conditions of Proposition* 3.2 *with some* $\zeta > 0$ *and we then say that* $\xi(m)$ *is "proper". In the other cases we set* $\xi(m) = 0$ *and say that* $\xi(m)$ *is not proper. We will also use the shorthand notation*

$$\xi_t \equiv \xi(m_t) \tag{3.8}$$

A set where $\xi(m)$ is proper that will often appear in the sequel is the set $\mathcal{C}_{\epsilon,\zeta}$, with ζ and ϵ positive, where

$$\mathcal{C}_{\epsilon,\zeta} = \{m \in \mathcal{C}^0(\mathbb{R}) : \ \|m\|_\infty \le 2 \text{ and there is } |x_0| \le (1-\zeta)\epsilon^{-1}, \\ \text{such that } \|m - \bar{m}_{x_0}\|_\epsilon \le \epsilon^{1/4}\} \tag{3.9}$$

Then, according to Proposition 3.2, for any $0 < \zeta' < \zeta$ there is ϵ_0 such that for all $\epsilon \le \epsilon_0$, any $m \in \mathcal{C}_{\epsilon,\zeta}$ has a center $\xi(m)$ in $\{|x| \le (1-\zeta')\epsilon^{-1}\}$.

We prove next that if $m \in \mathcal{C}_{\epsilon,\zeta}$, then, after a "short time", (that only grows as ϵ^{-b}, $b > 0$, that may be chosen arbitrarily small) it gets "much closer than $\epsilon^{1/4}$" to the instanton with center $\xi(m)$.

3.4 PROPOSITION. – *For any* $0 < \zeta < 1$ *and* $0 < a \le 1/4$ *there are positive constants* C *and* b, $b < 1/10$, *and given* n, c_n, *so that the following holds. Suppose that* $m_0 \in \mathcal{C}_{\epsilon,\zeta}$ *and that it satisfies N.b.c. in* $\mathcal{T}_\epsilon$. *Let* x_0 *be the number corresponding to* m_0, see (3.9). *Denote by* m_t *the Ginzburg-Landau process starting from* m_0 *and call* $\xi \equiv \xi_{\epsilon^{-b}}$, *then*

$$P^\epsilon\Big(\sup_{t \le \epsilon^{-b}} \|m_t - \bar{m}_{x_0}\|_\epsilon \le C\epsilon^{1/4};\ \|m_{\epsilon^{-b}} - \bar{m}_\xi\|_\epsilon \le \epsilon^{1/2-a}\Big) \\ \ge 1 - c_n\epsilon^n \tag{3.10}$$

Using the above result we are going to prove the following Lemma:

3.5 LEMMA. – *Let ζ, a, b and m_0 be as in Proposition* 3.4. *Then there is c' and, given n, c_n so that, setting $s_k = k\epsilon^{-b}$, $k \in \mathbb{N}$,*

$$P^\epsilon\Big(\sup_{\epsilon^{-b}\le s_k \le \epsilon^{-1-1/8}} \|m_{s_k} - \bar m_{\xi_{s_k}}\|_\epsilon \le \epsilon^{1/2-a}\Big) \ge 1 - c_n\epsilon^n \tag{3.11}$$

$$P^\epsilon\Big(|\xi_t - x_0| \le c'(1\vee t)\epsilon^{1/4} \ \textit{for all}\ t \le \epsilon^{-1-1/8}\Big) \ge 1 - c_n\epsilon^n \tag{3.12}$$

Proof. – Let $k^\star$ be the first integer such that $k^\star\epsilon^{-b} \ge \epsilon^{-1-1/8}$ and C' and c'_n the values of C and c_n in Proposition 3.4 when ζ is replaced by $\zeta' = 10^{-2}\zeta$, (a, b and ζ as in the statement of the Lemma).

We are going to prove by induction on $k \le k^\star$, that there is c' so that

$$P^\epsilon\Big(|\xi_{s_k} - x_0| \le c'\epsilon^{1/4}k \ ;\ \|m_{s_k} - \bar m_{\xi_{s_k}}\|_\epsilon \le \epsilon^{1/2-a}\Big) \ge 1 - kc'_n\epsilon^n \tag{3.13}$$

that implies (3.11).

By Proposition 3.4

$$P^\epsilon\Big(\|m_{s_1} - \bar m_{x_0}\|_\epsilon \le C'\epsilon^{1/4} \ ;\ \|m_{s_1} - \bar m_{\xi_{s_1}}\|_\epsilon \le \epsilon^{1/2-a}\Big) \ge 1 - c'_n\epsilon^n \tag{3.14}$$

By (3.3) we then get (3.13) with $k = 1$, by choosing properly c'.

We next suppose (3.13) true for $1 \le k < k^\star$ and we prove it for $k+1$. We call $m^\star = m_{s_k}$, $\xi^\star = \xi(m^\star)$ and suppose that $m^\star$ is in the set on the left hand side of (3.13). Then we can apply Proposition 3.4 with ζ' instead of ζ because

$$|\xi^\star| \le |x_0| + c'\epsilon^{1/4}k \le \epsilon^{-1}(1-\zeta) + c'\epsilon^{1/4}k^\star \le \epsilon^{-1}(1-\zeta')$$

for all ϵ small enough. Therefore (3.14) holds with $m^\star_t$ the process starting from $m^\star$ and we have

$$P^\epsilon\Big(\|m^\star_{s_1} - \bar m_{\xi^\star}\|_\epsilon \le C'\epsilon^{1/4} \ ;\ \|m^\star_{\epsilon^{-b}} - \bar m_\xi\|_\epsilon \le \epsilon^{1/2-a},\ \xi \equiv \xi_{\epsilon^{-b}}\Big) \ge 1 - c'_n\epsilon^n \tag{3.15}$$

By (3.13), that holds for k (by the induction assumption), (3.15) and (3.3) we then get

$$P^\epsilon\Big(|\xi_{s_{k+1}} - x_0| \le c'\epsilon^{1/4}(k+1) \ ; \ \|m_{s_{k+1}} - \bar{m}_{\xi_{s_{k+1}}}\|_\epsilon \le \epsilon^{1/2-a}\Big) \ge 1 - (k+1)c'_n\epsilon^n \tag{3.16}$$

(3.13) is thus proved for all $s_k \le \epsilon^{-1-1/8}$.

Recalling the statement below (3.13) we are only left with the proof of (3.12). Using (3.13) and Proposition 3.4 with ζ' instead of ζ we have for $s_k \le \epsilon^{-1-1/8}$

$$P^\epsilon\Big(\sup_{s_k \le t \le s_{k+1}} \|m_t - \bar{m}_{\xi_{s_k}}\|_\epsilon > C'\epsilon^{1/4}\Big) \le (k+1)c'_n\epsilon^n \tag{3.17}$$

We then derive (3.12) from (3.13), (3.17) and (3.3) so that the Lemma is proved. □

By Lemma 3.5 the Ginzburg-Landau process converges to an instanton as $\epsilon \to 0$, thus becoming a one-dimensional process which describes the motion of the center, (this under the assumption that the process starts close enough to an instanton and till times much longer than ϵ^{-1}). (3.12) also gives bounds on the values of ξ_t but they are far from sharp, as shown in the next Theorem:

3.6 THEOREM. – *Given any $\zeta > 0$ and $\epsilon > 0$, let m_t be the process that starts from $\bar{m}^{(\epsilon,x_0)}$*, see (1.5), *with $|x_0| \le (1-\zeta)\epsilon^{-1}$. Define*

$$X_t^\epsilon := \xi_{\epsilon^{-1}t} - x_0 \tag{3.18}$$

and let $\mathcal{P}^\epsilon$ be the law on $C(\mathbb{R}_+, \mathbb{R})$ of X_t^ϵ. Then $\mathcal{P}^\epsilon$ converges weakly as $\epsilon \to 0$ to $\mathcal{P}$ the law of the Brownian motion with diffusion $D = \frac{3}{4}$ that starts from 0.

Theorem 3.6, (3.10) and (3.11) prove Theorem 1.1. To prove Theorem 3.6 it is convenient to work in discrete times. With $b > 0$ as in Proposition 3.4, we define

$$Y_t = \xi_{t_n} - x_0, \quad \text{where } t_n \le t < t_{n+1}, t_n = nT_\epsilon \tag{3.19}$$

and

$$T_\epsilon = n_\epsilon \epsilon^{-b}, \quad n_\epsilon = [\epsilon^{-1/10+b}], \qquad \text{so that } \epsilon^{-1/10} - \epsilon^{-b} \le T_\epsilon \le \epsilon^{-1/10} \tag{3.20}$$

The reason of this choice of T_ϵ will become clear in the proof of Proposition 3.8 below. We set

$$Y^\epsilon_\tau = Y_{\epsilon^{-1}\tau} \tag{3.21}$$

Of course $X^\epsilon_\tau = Y^\epsilon_\tau$ for all $\tau = \epsilon t_n$. We call $\mathbb{P}^\epsilon$ the law on $D(\mathbb{R}_+, \mathbb{R})$ of Y^ϵ_τ and we will first prove that, as $\epsilon \longrightarrow 0$, it converges weakly by subsequences. We will then show that the limit points are supported by $C(\mathbb{R}_+, \mathbb{R})$ and finally that they are all equal to the same measure $\mathcal{P}$. We will also prove that the increments of X^ϵ_t in the single intervals $[\epsilon t_n, \epsilon t_{n+1}]$ vanish as $\epsilon \longrightarrow 0$, thus completing the proof of Theorem 3.6.

We call $\mathcal{F}_t$ the σ-algebra generated by the process Z_s, $s \le t$, recalling that m_t is adapted to $\mathcal{F}_t$, and state the following two key Propositions. In the first one we state criteria for tightness and support properties of the limit measures and in the second one that they are satisfied.

3.7 PROPOSITION. – *Given any $T > 0$, the family of laws $\mathbb{P}^\epsilon$, $\epsilon > 0$, on $D([0,T], \mathbb{R})$, is tight if there is c so that for all ϵ*

$$\sup_{t_n \le \epsilon^{-1}T} E^\epsilon\big(\gamma_i(t_n)^2\big) \le c, \quad i = 1, 2 \tag{3.22}$$

where

$$\gamma_1(t_n) = \epsilon^{-1}T_\epsilon^{-1}E^\epsilon\Big(Y_{t_{n+1}} - Y_{t_n}\Big|\mathcal{F}_{t_n}\Big) \tag{3.23}$$

$$\gamma_2(t_n) = \epsilon^{-1}T_\epsilon^{-1}\Big\{E^\epsilon\Big(Y^2_{t_{n+1}} - Y^2_{t_n}\Big|\mathcal{F}_{t_n}\Big) - 2Y^0_{t_n}E^\epsilon\Big(Y_{t_{n+1}} - Y_{t_n}\Big|\mathcal{F}_{t_n}\Big)\Big\} \tag{3.24}$$

with

$$Y^0_{t_n} = \frac{1}{2}\Big[Y_{t_n} + E^\epsilon\Big(Y_{t_{n+1}}\Big|\mathcal{F}_{t_n}\Big)\Big] \tag{3.25}$$

If (3.22) *holds and if*

$$\lim_{\epsilon \longrightarrow 0} \sup_{t_n \le \epsilon^{-1}T} [\epsilon^{-1}T_\epsilon^{-1}]E^\epsilon\Big([Y_{t_{n+1}} - Y_{t_n}]^4\Big) = 0 \tag{3.26}$$

then any limit point $\mathbb{P}$ of $\mathbb{P}^\epsilon$ is supported by $C([0,T], \mathbb{R})$.

Finally, if (3.22) *and* (3.26) *hold and if*

$$\lim_{\epsilon \longrightarrow 0} \sup_{t_n \le \epsilon^{-1}T} E^\epsilon\big(|\gamma_1(t_n)|\big) = 0 \tag{3.27}$$

$$\lim_{\epsilon\to 0}\sup_{t_n\leq\epsilon^{-1}T} E^\epsilon\left(\left|D-[\epsilon^{-1}T_\epsilon^{-1}]E^\epsilon\left((Y_{t_{n+1}})^2-(Y_{t_n})^2\middle|\mathcal{F}_{t_n}\right)\right|\right)=0 \quad (3.28)$$

$\left(D=\dfrac{3}{4}\right)$, *then any limit point* $\mathbb{P}$ *is equal to* $\mathcal{P}$, *the law of the Brownian motion with diffusion* D *that starts from* 0.

3.8 PROPOSITION. – *Under the same assumptions as in Theorem* 3.6, *the conditions* (3.22), (3.26), (3.27) *and* (3.28) *are satisfied for all* ϵ *small enough.*

Thus Propositions 3.7 and 3.8 prove that $\mathbb{P}^\epsilon$ converges weakly to $\mathcal{P}$. At this point very little is missing for proving Theorem 3.6, namely that the increments of X_t^ϵ in any of the intervals $[\epsilon t_n, \epsilon t_{n+1}]$ are infinitesimal, see the remarks following Theorem 3.6. By (3.11) and (3.12), if $|x_0|\leq(1-\zeta)\epsilon^{-1}$ there is $\zeta'>0$ so that, for any $T>0$,

$$P^\epsilon\Big(\sup_{t_n\leq\epsilon^{-1}T}\|m_{t_n}-\bar{m}_{\xi_{t_n}}\|_\epsilon\leq\epsilon^{1/2-a},$$
$$\sup_{t_n\leq\epsilon^{-1}T}|\xi_{t_n}|\leq(1-\zeta')\epsilon^{-1}\Big)\geq 1-c_n\epsilon^n \quad (3.29)$$

We then fix any interval $[t_k, t_{k+1}]$, $t_k\leq\epsilon^{-1}T$, take m_{t_k} in the set on the left hand side of (3.29) and consider the process for a time T_ϵ starting from such m_{t_k}. By (3.12), the probability that the corresponding increments of ξ_t are bounded by $c'\epsilon^{1/4-1/10}$ is larger than $1-c_n\epsilon^n$. The intersection of all these events with $t_k\leq\epsilon^{-1}T$ has also probability larger than $1-c_n\epsilon^n$, (with different coefficients). Theorem 3.6 and Theorem 1.1 are thus proved, once all the above Propositions are also proved.

4. PROOFS OF THE PROPOSITIONS OF SECTION 3

In this Section we prove the Propositions stated in Section 3 using properties of the Ginzburg-Landau and of the free processes that will be proven in Section 5.

Proof of Proposition 3.2. – By assumption the function

$$h(x):=m(x)-\bar{m}_{x_0}(x) \quad (4.1)$$

is continuous in $\mathbb{R}$ and

$$\|h\|_\infty<3, \quad (4.2)$$

Let

$$C(y) = \int_{-\infty}^{+\infty} dx[m(x) - \bar{m}_{x_0}(x-y)]\bar{m}'_{x_0}(x-y) \tag{4.3}$$

then ξ is a center of m if and only if $C(\xi - x_0) = 0$. Since for all y

$$\int_{-\infty}^{+\infty} dx\, \bar{m}_{x_0}(x-y)\bar{m}'_{x_0}(x-y) = 0$$

we have

$$C(y) = \int_{-\infty}^{+\infty} dx \bar{m}_{x_0}(x)\bar{m}'_{x_0}(x-y) + \int_{-\infty}^{+\infty} dx h(x)\bar{m}'_{x_0}(x-y) =: F(y) + H(y) \tag{4.4}$$

F and H are in $C^\infty(\mathbb{R})$, F is odd, increasing and

$$F'(0) = \frac{4}{3}, \qquad \lim_{y\to\infty} F(y) = \int_{-\infty}^{+\infty} \bar{m}'(x)dx = 2 \tag{4.5}$$

Moreover, there is c so that, for all y such that $|x_0 + y| \le (1-\zeta')\epsilon^{-1}$,

$$|H(y)| \le \int_{\mathcal{T}_\epsilon} dx|h(x)|\bar{m}'_{x_0}(x-y) + \int_{\mathbb{R}\setminus\mathcal{T}_\epsilon} dx|h(x)|\bar{m}'_{x_0}(x-y) \le c[\|h\|_\epsilon + e^{-\epsilon^{-1}\zeta'}] \tag{4.6}$$

and, analogously,

$$|H'(y)| \le c[\|h\|_\epsilon + e^{-\epsilon^{-1}\zeta'}] \tag{4.7}$$

We fix arbitrarily $\gamma > 0$, then there are κ, δ and ϵ_0 positive so that for all $\epsilon \le \epsilon_0$ and all $\|h\|_\epsilon \le 2\delta$ the following holds. Firstly if $|y| \le \gamma$, then $|x_0 + y| \le (1-\zeta')\epsilon^{-1}$, (because $|x_0| \le (1-\zeta)\epsilon^{-1}$ and $\zeta' < \zeta$). Moreover

$$|F(y)| > |H(y)| \text{ for all } |y| \ge \gamma \text{ such that } |x_0 + y| \le (1-\zeta')\epsilon^{-1} \tag{4.8}$$

$$F'(y) - |H'(y)| \ge \kappa \text{ for all } |y| \le \gamma \tag{4.9}$$

We are going to show that for all ϵ small enough $C(y)$ has a zero, y_0, in $|y| \le \gamma$ and no other one in $|x_0 + y| \le (1-\zeta')\epsilon^{-1}$.

By (4.8), $C(\gamma) = F(\gamma) + H(\gamma) > 0$ and since, analogously, $C(-\gamma) < 0$, by the continuity of $C(y)$ there is a zero in $|y| \le \gamma$. By (4.9) it is unique in $|y| \le \gamma$ and by (4.8) in $|x_0 + y| \le (1 - \zeta')\epsilon^{-1}$. To refine the location of the zero we observe that $F(0) = 0$ so that, by (4.9),

$$C(y) \ge H(0) + \kappa y, \quad \text{for } 0 \le y \le \gamma$$

Thus, by (4.6),

$$C(y_-) \le 0 \le C(y_+), \quad \text{where } y_\pm = \pm\frac{c}{\kappa}\left(\|h\|_\epsilon + \epsilon^{-\epsilon^{-1}\zeta'}\right)$$

hence $|y_0| \le y_+$, which proves (3.3). Moreover

$$0 = C(y_0) = C(0) + C'(0)y_0 + \frac{1}{2}C(\hat{y})y_0^2, \qquad |\hat{y}| \le y_+$$

We then have

$$\left|y_0 - \frac{C(0)}{F'(0)}\right| \le \left|\frac{C(0)}{C'(0)} - \frac{C(0)}{F'(0)}\right| + \left|\frac{C(\hat{y})y_0^2}{2C'(0)}\right| = \left|\frac{3H(0)H'(0)}{4C'(0)}\right| + \left|\frac{C(\hat{y})y_0^2}{2C'(0)}\right|,$$

hence (3.5).

Calling $h^\star(x) = m^\star(x) - \bar{m}_{x_0}$, we have $\|h^\star\|_\epsilon \le 2\delta$, hence (4.8) and (4.9) and the conclusions thereafter are also valid for $h^\star$. Since $C(y_0) = 0$, we have

$$|C^\star(y_0)| \le \int_{-\infty}^{+\infty} dx|m(x) - m^\star(x)|\bar{m}'_{x_0}(x - y_0) =: c^\star \tag{4.10}$$

By taking δ small enough we get

$$|y_0| + \frac{c^\star}{\kappa} \le \gamma$$

We can then apply (4.9) and using (4.10)

$$|y_0^\star - y_0| \le \frac{c^\star}{\kappa}$$

so that (3.7) is proven.

The last statement in Proposition 3.2 (about the uniqueness of the center in the whole $\mathcal{T}_\epsilon$) follows after replacing the first inequality in (4.6) by

$$\begin{aligned}|H(y)| \le &\int_{-(1+\zeta')\epsilon^{-1}}^{(1+\zeta')\epsilon^{-1}} dx|h(x)|\bar{m}'_{x_0}(x - y)\\ &+ \int_{\mathbb{R}\setminus(1+\zeta')[-\epsilon^{-1},\epsilon^{-1}]} dx|h(x)|\bar{m}'_{x_0}(x - y)\end{aligned} \tag{4.11}$$

that we need to bound uniformly in $x_0 + y \in \mathcal{T}_\epsilon$. By the assumption that m satisfies N.b.c. in $\mathcal{T}_\epsilon$, when $x = \epsilon^{-1} + x'$, $0 < x' \leq \zeta' \epsilon^{-1}$,

$$h(x) = h(\epsilon^{-1} - x') + [\bar{m}_{x_0}(\epsilon^{-1} - x') - \bar{m}_{x_0}(\epsilon^{-1} + x')] \tag{4.12}$$

so that

$$|h(x)| \leq \delta + c\epsilon^{-1} e^{-(\zeta - \zeta')\epsilon^{-1}} \tag{4.13}$$

Same bound holds when $x = -[\epsilon^{-1} + x']$, $0 < x' \leq \zeta' \epsilon^{-1}$, so that using (4.11) and (4.13) we prove that the last bound in (4.6) holds in the whole $\mathcal{T}_\epsilon$, so that the uniqueness in the whole $\mathcal{T}_\epsilon$ follows and the Proposition is proved. □

Proof of Proposition 3.4. – Setting $\xi_0 = \xi(m_0)$, by (3.3) we have that $|\xi_0 - x_0| \leq c\epsilon^{1/4}$ and by (1.2)

$$\|\bar{m}_x - \bar{m}_y\|_\infty \leq |x - y| \tag{4.14}$$

Then

$$\|m_0 - \bar{m}_{\xi_0}\|_\epsilon \leq \epsilon^{1/4} + c\epsilon^{1/4} = C_0 \epsilon^{1/4}, \qquad C_0 = 1 + c \tag{4.15}$$

We define

$$\hat{m}_0(x) = \begin{cases} m_0(x) & \text{if } \ |x - \xi_0| \leq 10^{-4}\zeta\epsilon^{-1} \\ 1 & \text{if } \ x - \xi_0 > 10^{-4}\zeta\epsilon^{-1} + 1 \\ -1 & \text{if } \ x - \xi_0 < -10^{-4}\zeta\epsilon^{-1} - 1 \end{cases} \tag{4.16}$$

A linear interpolation in the missing intervals completes the definition of $\hat{m}_0$.

Let $\hat{m}_t$ and m_t be the solutions of (2.7) with initial data respectively $\hat{m}_0$ and m_0 (and same noise). By Propositions 5.2 and 5.3, for any n there is c_n so that

$$P^\epsilon\Big(\sup_{t \leq \epsilon^{-b}} \sup_{|x - \xi_0| \leq 10^{-5}\zeta\epsilon^{-1}} |m_t(x) - \hat{m}_t(x)| \leq c_n \epsilon^n \Big) \geq 1 - c_n \epsilon^n \tag{4.17}$$

We will prove that there is $\hat{a} < a$ and, for any n, c_n so that

$$P^\epsilon\Big(\sup_{t \leq \epsilon^{-b}} \sup_{|x - \xi_0| \leq 10^{-5}\zeta\epsilon^{-1}} |\hat{m}_t(x) - \bar{m}_{\hat{x}_0}(x)| < C\epsilon^{1/4};$$

$$\sup_{|x - \xi_0| \leq 10^{-5}\zeta\epsilon^{-1}} |\hat{m}_{\epsilon^{-b}}(x) - \bar{m}_{\hat{x}_0}(x)| < \epsilon^{1/2 - \hat{a}} \Big) \geq 1 - c_n \epsilon^n \tag{4.18}$$

where $\hat{x}_0$ is the center of $\hat{m}_0$. Observe that by (3.7), for any n there is c_n so that

$$|\hat{x}_0 - \xi_0| \le c_n \epsilon^n \tag{4.19}$$

Then from (4.17), (4.18) and (4.19)

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \sup_{|x-\xi_0|\le 10^{-5}\zeta\epsilon^{-1}} |m_t(x) - \bar{m}_{\xi_0}(x)| < 2C\epsilon^{1/4}; \\ \sup_{|x-\xi_0|\le 10^{-5}\zeta\epsilon^{-1}} |m_{\epsilon^{-b}}(x) - \bar{m}_{\xi_0}(x)| < 2\epsilon^{1/2-\hat{a}}\Big) \ge 1 - c_n\epsilon^n \tag{4.20}$$

After proving (4.18) and consequently (4.20) we will extend the result by lifting the condition on x in (4.20) thus having a sup over the whole $\mathcal{T}_\epsilon$. Then, with the help of Proposition 3.2 we will easily conclude the proof of the proposition. We next prove (4.18). Given $a \le 1/4$, let

$$a' < a/4, \quad b < 1/4 - 2a', \quad \hat{a} = a' + b/2 < a \tag{4.21}$$

and let

$$\mathcal{B}_\epsilon = \mathcal{G}_\epsilon(a', \hat{x}_0) \cap \{ \sup_{t\le\epsilon^{-b}} \|\hat{m}_t\|_\infty \le 2\} \tag{4.22}$$

with $\mathcal{G}_\epsilon$ as in (5.27). Then, by Propositions 5.2 and 5.4, for any n there is c_n so that

$$P^\epsilon(\mathcal{B}_\epsilon) \ge 1 - c_n\epsilon^n \tag{4.23}$$

Let

$$u_t = \hat{m}_t - \bar{m}_{\hat{x}_0} \tag{4.24}$$

then, by (2.11) and (2.16), there are constants C_1 and C_2 such that, in $\mathcal{B}_\epsilon$,

$$\begin{aligned} \|u_t\|_\infty &\le C_1 e^{-\alpha t}\|u_0\|_\infty + C_2\int_0^t ds\|u_s\|_\infty^2 + \epsilon^{1/2}\|\hat{Z}_{t,\hat{x}_0}\|_\epsilon \\ &\le C_1 e^{-\alpha t}\|u_0\|_\infty + C_2\int_0^t ds\|u_s\|_\infty^2 + \epsilon^{1/2-a'-b/2} \end{aligned} \tag{4.25}$$

for all $t \le \epsilon^{-b}$. By (4.21) the last term is bounded by $\epsilon^{1/4}$. By (4.19)

$$\begin{aligned} \|u_0\|_\infty \le& \|\bar{m}_{x_0} - \bar{m}_{\hat{x}_0}\|_\infty + \|\hat{m}_0 - \bar{m}_{x_0}\|_\infty \le |x_0 - \hat{x}_0| \\ &+ \|\hat{m}_0 - \bar{m}_{x_0}\|_\infty \le c_n\epsilon^n + C_0\epsilon^{1/4} \\ &+ \sup_{|x-\xi_0|\le 10^{-4}\zeta\epsilon^{-1}} |\hat{m}_0(x) - \bar{m}_{\xi_0}(x)| \le \epsilon^{1/4}C_3/C_1 \end{aligned} \tag{4.26}$$

where $C_3 \geq 1$ is a suitable constant.

Then from (4.25) we get that, in $\mathcal{B}_\epsilon$,

$$\|u_t\|_\infty \leq 2C_3\epsilon^{1/4} + C_2 \int_0^t ds \|u_s\|_\infty^2 \qquad \text{for all } t \leq \epsilon^{-b}$$

Let

$$T = \inf \{t \geq 0 : \|u_t\|_\infty \geq 3C_3\epsilon^{1/4}\}$$

We next prove by contradiction that $\epsilon^{-b} \leq T$, in $\mathcal{B}_\epsilon$. We thus suppose that $T < \epsilon^{-b}$, then,

$$\|u_t\|_\infty < 3C_3\epsilon^{1/4}$$

for all $t < T$ and, by the continuity of $\|u_t\|_\epsilon$,

$$\|u_T\|_\infty = 3C_3\epsilon^{1/4}$$

Hence

$$3C_3\epsilon^{1/4} = \|u_T\|_\infty \leq 2C_3\epsilon^{1/4} + C_2(3C_3\epsilon^{1/4})^2 T$$

that is

$$C_3\epsilon^{1/4} \leq [9C_3C_2\epsilon^{1/4-b}]C_3\epsilon^{1/4}$$

which cannot hold for all ϵ small enough because, by (4.21), $b < 1/4$. We have thus proven that $\epsilon^{-b} \leq T$.

By (4.25) and (4.26) we then get for all $t \leq \epsilon^{-b}$, in $\mathcal{B}_\epsilon$

$$\|u_t\|_\infty \leq C_3 e^{-\alpha t}\epsilon^{1/4} + C_2 \int_0^t ds[3C_3\epsilon^{1/4}]^2 + \epsilon^{1/2-a'-b/2}$$

hence setting $t = \epsilon^{-b}$:

$$\|u_{\epsilon^{-b}}\|_\infty \leq C_3 e^{-\alpha\epsilon^{-b}}\epsilon^{1/4} + C_2\epsilon^{-b}(3C_3)^2\epsilon^{1/2} + \epsilon^{1/2-\hat{a}} \leq C_4\epsilon^{1/2-\hat{a}}$$

By (4.23) we have thus completed the proof of (4.18) hence also that of (4.20).

In $\{|x - \xi_0| > 10^{-5}\zeta\epsilon^{-1}\}$ we use Proposition 5.3 to reduce to the case with the initial datum close to a function identically equal either to 1 or -1. By symmetry we may just consider the interval

$$I = (\ell_-, \ell_+), \qquad \ell_- = \xi_0 + 10^{-6}\zeta\epsilon^{-1}, \;\; \ell_+ = (1 + 10^{-6}\zeta)\epsilon^{-1}$$

Recalling (4.15), we get

$$\sup_{x\in I} |m_0(x) - 1| \le C_0\epsilon^{1/4} + \sup_{x\in I} |\bar{m}_{\xi_0}(x) - 1| \le 2C_0\epsilon^{1/4} \tag{4.27}$$

for all ϵ small enough. We define a new $\hat{m}_0$ as

$$\hat{m}_0(x) = \begin{cases} m_0(x) & \text{if } x \in I \\ m_0(\ell_\pm) & \text{if } x \gtrless \ell_\pm \end{cases} \tag{4.28}$$

Then, by Proposition 5.3, for any n there is c_n so that, calling $x^\star := x_0 + 10^{-5}\zeta\epsilon^{-1}$,

$$P^\epsilon\Big(\sup_{x^\star \le x \le \epsilon^{-1}} |m_{\epsilon^{-b}}(x) - \hat{m}_{\epsilon^{-b}}(x)| \le c_n\epsilon^n \Big) \ge 1 - c_n\epsilon^n \tag{4.29}$$

To estimate $\hat{m}_{\epsilon^{-b}}$ we should study the equation linearized around $m \equiv 1$. The analysis is very similar to the previous one since $m \equiv 1$ is linearly stable for the deterministic evolution (1.1). We then obtain the same estimate we had before, details are omitted. We have thus proved that (4.20) holds with the sup over $x \in \mathcal{T}_\epsilon$.

We now recall what stated right at the beginning of this proof, namely that

$$\|\bar{m}_{\xi_0} - \bar{m}_{x_0}\|_\infty \le c\epsilon^{1/4}$$

We then use this inequality to replace $\bar{m}_{\xi_0}$ by $\bar{m}_{x_0}$ in the first sup in the improved version of (4.20). For the second sup we use Proposition 3.2 to conclude that

$$\|m_{\epsilon^{-b}} - \bar{m}_{\xi_0}\|_\epsilon \le 2\epsilon^{1/2-\hat{a}} \quad \text{implies that} \quad |\xi_{\epsilon^{-b}} - \xi_0| \le c2\epsilon^{1/2-\hat{a}}$$

We then get for the improved version of (4.20), the one with the sup over $x \in \mathcal{T}_\epsilon$,

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \|m_t - \bar{m}_{x_0}\|_\epsilon \le (2C + c)\epsilon^{1/4}; \|m_{\epsilon^{-b}} - \bar{m}_{\epsilon^{-b}}\|_\epsilon \le (2 + c2)\epsilon^{1/2-\hat{a}} \Big) \ge 1 - c_n\epsilon^n$$

hence (3.10). Proposition 3.4 is therefore proved. □

Proof of Proposition 3.7. – We call $m^\epsilon_\tau = m_{\epsilon^{-1}\tau}$ and $\mathcal{F}^\epsilon_\tau$ the σ-algebra generated by $m_{\epsilon^{-1}\tau'}$ for $\tau' \le \tau$. $\mathcal{F}_t$ is instead the σ-algebra associated to m_t, as usual. We will first prove that

$$Y^\epsilon_\tau - \int_0^\tau d\tau' \gamma^\epsilon_{1,\tau}(\tau') =: M^\epsilon_\tau \text{ is a } P^\epsilon\text{-martingale w.r.t. } \mathcal{F}^\epsilon_\tau \tag{4.30}$$

and that

$$(M^\epsilon_\tau)^2 - \int_0^\tau d\tau' \gamma^\epsilon_{2,\tau}(\tau') =: N^\epsilon_\tau \text{ is a } P^\epsilon\text{-martingale w.r.t. } \mathcal{F}^\epsilon_\tau \tag{4.31}$$

with

$$\gamma^\epsilon_{i,\tau}(\tau') = \begin{cases} \gamma_i(t_n) & \text{if } \epsilon t_n \le \tau' < \epsilon t_{n+1} \le \tau \\ 0 & \text{if } \epsilon t_n \le \tau' \le \tau < \epsilon t_{n+1} \end{cases} \tag{4.32}$$

where $i = 1, 2$.

Proof of (4.30) *and* (4.31). – We shorthand

$$y_n = Y_{t_n}, \quad \mathcal{F}^{(n)} = \mathcal{F}_{t_n}, \quad E^\epsilon_i(g) = E^\epsilon\big(g \big| \mathcal{F}_{t_i}\big) \tag{4.33}$$

Then, obviously,

$$y_n - \sum_{i=0}^{n-1} E^\epsilon_i(y_{i+1} - y_i) = M_n \qquad \text{is a } P^\epsilon\text{-martingale w.r.t. } \mathcal{F}^{(n)} \tag{4.34}$$

Moreover

$$(M_n)^2 - \sum_{i=0}^{n-1} h_i = N_n \qquad \text{is a } P^\epsilon\text{-martingale w.r.t. } \mathcal{F}^{(n)} \tag{4.35}$$

where

$$h_n = E^\epsilon_n\Big(y^2_{n+1} - y^2_n - [y_n - E^\epsilon_n(y_{n+1})][E^\epsilon_n\big(y_{n+1} - y_n\big)]\Big) \tag{4.36}$$

Recall the formula $\gamma_2 = Lf^2 - 2fLf$ for the "opérateur carré du champ" valid for the "second compensator" in time-continuous processes, [17].

To prove (4.35) it is enough to check that

$$E^\epsilon_n\big((M_{n+1})^2\big) = (M_n)^2 + h_n \tag{4.37}$$

whose proof is omitted. Then (4.30) and (4.31) are proven since they are just (4.34) and (4.35) with different notation.

Tightness. – Having now the representation (4.30) and (4.31), we can use the following sufficient condition for tightness, see for instance § 2.7.6 of [6]:

$$\sup_{\tau' \le \tau \le T} E^\epsilon\Big((Y^\epsilon_0)^2 + (\gamma^\epsilon_{1,\tau}(\tau'))^2 + (\gamma^\epsilon_{2,\tau}(\tau'))^2\Big) < \infty \tag{4.38}$$

which is implied by (3.22).

Support properties. – A sufficient condition (*see* for instance the proof of Theorem 2.7.8 in [6]) for the support property stated in Proposition 3.7 is

$$\lim_{\epsilon \to 0} E^\epsilon \Big(\sup_{\tau \le T} \big| Y^\epsilon_{\tau+} - Y^\epsilon_{\tau-} \big| \Big) = 0 \tag{4.39}$$

with $Y^\epsilon_{\tau+}$ and $Y^\epsilon_{\tau-}$ respectively the right and left limits of $Y^\epsilon_{\tau'}$ at τ. Recalling the definition of Y^ϵ_τ, (4.39) is equivalent to

$$\lim_{\epsilon \to 0} E^\epsilon \Big(\sup_{t_n \le \epsilon^{-1} T} \big| Y_{t_{n+1}} - Y_{t_n} \big| \Big) = 0 \tag{4.40}$$

which follows from

$$\lim_{\epsilon \to 0} E^\epsilon \Big(\sup_{t_n \le \epsilon^{-1} T} \big| Y_{t_{n+1}} - Y_{t_n} \big|^4 \Big) = 0 \tag{4.41}$$

Since

$$E^\epsilon \Big(\sup_{t_n \le \epsilon^{-1} T} \big| Y_{t_{n+1}} - Y_{t_n} \big|^4 \Big) \le \frac{\epsilon^{-1} T}{T_\epsilon} \sup_{t_n} E^\epsilon \Big(\big| Y_{t_{n+1}} - Y_{t_n} \big|^4 \Big) \tag{4.42}$$

(4.39) follows from (3.26).

Identification of the limit law. – For what already proven, any limit law $\mathbb{P}$ of $\mathbb{P}^\epsilon$ is supported by $C([0,T], \mathbb{R})$. It is then enough to show that Y_τ and $Y_\tau^2 - D\tau$, $D = \frac{3}{4}$, are $\mathbb{P}$ martingales, because, by the Levy's theorem, $\mathbb{P}$ is then equal to $\mathcal{P}$.

Let ϕ be a bounded continuous function on $C([0,T], \mathbb{R}_+)$, measurable with respect to the coordinate process till time σ, $0 \le \sigma < T$. We need to show that along a converging subsequence, for any ϕ as above and any τ such that $\sigma < \tau \le T$

$$\lim_{\epsilon \to 0} E^\epsilon \Big(\phi [Y^\epsilon_\tau - Y^\epsilon_\sigma] \Big) = 0 \tag{4.43}$$

In fact, since the functions in the expectation are $\mathbb{P}$- a.s. continuous in $D([0,T], \mathbb{R}_+)$, because of the support properties of $\mathbb{P}$, then

$$\mathbb{E} \Big(\phi [Y_\tau - Y_\sigma] \Big) = 0 \tag{4.44}$$

with $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

By (4.30)

$$\begin{aligned} \Big| E^\epsilon \Big(\phi [Y^\epsilon_\tau - Y^\epsilon_\sigma] \Big) \Big| &= \Big| E^\epsilon \Big(\phi \int_\sigma^T d\tau' \gamma^\epsilon_{1,\tau}(\tau') \Big) \Big| \\ &\le \|\phi\|_\infty \int_\sigma^T d\tau' E^\epsilon \Big(\big| \gamma^\epsilon_{1,\tau}(\tau') \big| \Big) \end{aligned} \tag{4.45}$$

Using (4.32) and (3.27) we then obtain (4.43).

Analogously, to prove that $Y_\tau^2 - D\tau$ is a $\mathbb{P}$ martingale we write, shorthanding y_n for Y_{τ_n},

$$y_n^2 - \sum_{i=0}^{n-1} E^\epsilon\Big(y_{i+1}^2 - y_i^2 \big| \mathcal{F}_{t_i}\Big) = \mathcal{M}_n$$

which is a P^ϵ martingale with respect to $\mathcal{F}_{t_n}$. Analogously to (4.32) we define

$$\gamma_{3,\tau}^\epsilon(\tau') = \begin{cases} [\epsilon T_\epsilon]^{-1} E^\epsilon\big(y_{i+1}^2 - y_i^2 \big| \mathcal{F}_{t_i}\big) & \text{if } \epsilon t_n \le \tau' < \epsilon t_{n+1} \le \tau \\ 0 & \text{if } \epsilon t_n \le \tau' \le \tau < \epsilon t_{n+1} \end{cases}$$

It is then enough to prove that for all ϕ, σ, τ as before,

$$\lim_{\epsilon \to 0} E^\epsilon\Big(\phi\Big[\int_\sigma^\tau d\tau' [\gamma_{3,\tau}^\epsilon(\tau') - D]\Big]\Big) = 0$$

which is implied by (3.28). The Proposition is therefore proved. □

Proof of Proposition 3.8. – By (3.11), (3.12) and the definition of T_ϵ, we get

$$P^\epsilon\Big(|\xi_{t_k}| \le (1 - \zeta/2)\epsilon^{-1} \ ; \ \|m_{t_k} - \bar{m}_{\xi_{t_k}}\|_\epsilon \le \epsilon^{1/2-a}, \ \text{for all } t_k \le \epsilon^{-1} T\Big) \ge 1 - c_n \epsilon^n \tag{4.46}$$

By definition, *see* Definition 3.3, $|\xi_t| \le \epsilon^{-1}$. It thus suffices to prove that

$$E^\epsilon\Big(\gamma_i(T_\epsilon)^2\Big) \le c, \quad i = 1, 2 \tag{4.47}$$

$$\lim_{\epsilon \to 0} E^\epsilon\Big((\epsilon T_\epsilon)^{-1}[\xi_{T_\epsilon} - \xi_0]^4 + \gamma_1(T_\epsilon) + \big|D - (\epsilon T_\epsilon)^{-1}[\xi_{T_\epsilon} - \xi_0]^2\big|\Big) = 0 \tag{4.48}$$

where the process m_t starts from m_0 and the latter satisfies N.b.c in $\mathcal{T}_\epsilon$ and moreover $\|m_0 - \bar{m}_{x_0}\|_\epsilon \le \epsilon^{1/2-a}$ with $|x_0| \le (1 - \zeta/2)\epsilon^{-1}$. c in (4.47) and the convergence in (4.48) must be uniform in m_0 (provided it satisfies the above conditions). Given m_0, $\xi_0 = \xi(m_0)$, let $m_t^\star$ be the process that starts from $\bar{m}_{\xi_0}$ and denote by $\gamma_i^\star(T_\epsilon)$, $\xi_{T_\epsilon}^\star$ the variables $\gamma_i(T_\epsilon)$ and ξ_{T_ϵ} in this new process. We will prove below that there are c' and, for any n, c_n so that for any m_0 as above

$$P^\epsilon\Big(\|m_{T_\epsilon} - m_{T_\epsilon}^\star\|_\epsilon \le c'(\epsilon^{1/2-a})^2\Big) \ge 1 - c_n \epsilon^n \tag{4.49}$$

Then, by (3.7) there are c'' and, for any n, c_n so that

$$P^\epsilon\Big(|\xi_{T_\epsilon} - \xi^\star_{T_\epsilon}| \le c''(\epsilon^{1/2-a})^2\Big) \ge 1 - c_n\epsilon^n \tag{4.50}$$

It then follows that (4.47) and (4.48) hold if they are verified with $\gamma_i^\star(T_\epsilon)$ and $\xi^\star_{T_\epsilon}$, i.e. when the process starts from an instanton centered at x_0 with $|x_0| \le (1-\zeta/2)\epsilon^{-1}$. The proof of Proposition 3.8 is thus reduced to that of (4.49) and of (4.47)-(4.48) with $\gamma_i^\star(T_\epsilon)$ and $\xi^\star_{T_\epsilon}$, that will be carried out in the sequel.

4.1 LEMMA. – *For any a, ζ and ϵ positive, let $|x_0| \le (1-\zeta)\epsilon^{-1}$, $\|m_0 - \bar m_{x_0}\|_\epsilon \le \epsilon^{1/2-a}$ and suppose that m_0 satisfies N.b.c. in $\mathcal{T}_\epsilon$ and that $\xi(m_0) = x_0$. Denote by m_t and $m_t^\star$ the solutions of* (2.11) *with initial conditions respectively m_0 and $\bar m_{x_0}$. Then there is c and, for any n, c_n so that* (4.49) *holds.*

Proof. – We start by proving that for any n there is c_n so that

$$P^\epsilon\Big(\|m_t - \bar m_{x_0}\|_\epsilon \le 3\epsilon^{1/2-a}t^{1/2}, \quad \text{for all } t \le \epsilon^{-1/10}\Big) > 1 - c_n\epsilon^n \tag{4.51}$$

and that the same inequality holds for $m_t^\star$.

We define $\hat m_0$, $\hat m_t$, u_t as in the proof of Proposition 3.4, *see* (4.16)-(4.25). By (4.19) and the second inequality in (4.26)

$$\|u_0\|_\infty \le c_2'\epsilon^2 + \epsilon^{1/2-a} \le \epsilon^{1/2-a}c_2/C_1 \tag{4.52}$$

with c_2' and c_2 suitable constants. We set

$$\beta_t = 3\,\max\{\epsilon^{1/2-a}c_2, \epsilon^{1/2-a'}\sqrt{t}\} \tag{4.53}$$

and we are going to prove by contradiction that $\|u_t\|_\infty \le \beta_t$ for all $t \le \epsilon^{-1/10}$.

Suppose that $T < \epsilon^{-1/10}$, with

$$T = \inf\{t : \|u_t\|_\infty \ge \beta_t\} \tag{4.54}$$

By the continuity of $\|u_t\|_\infty$

$$\|u_t\|_\infty < \beta_t \text{ for all } t < T \quad \text{and} \quad \|u_T\|_\infty = \beta_T \tag{4.55}$$

Thus

$$\beta_T \le \frac{2}{3}\beta_T + C_2 T\beta_T^2; \quad \beta_T \le 3C_2T\beta_T^2$$

hence, letting $a' < a$,

$$1 \leq 3C_2 T\beta_T \leq 9C_2\epsilon^{1/2-a-3/20} \tag{4.56}$$

which is a contradiction for ϵ small enough and $a < 7/20$. We have therefore proved that if m_t is in the set on the left hand side of (4.17) and in $\mathcal{B}_\epsilon$, then for $|x - x_0| \leq 10^{-5}\zeta\epsilon^{-1}$

$$\left|m_{\epsilon^{-1/10}}(x) - \bar{m}_{x_0}(x)\right| \leq 3\epsilon^{1/2-a-1/20} + |\bar{m}_{x_0}(x) - \bar{m}_{\hat{x}_0}(x)| + c_n\epsilon^n \tag{4.57}$$

where the first term on the right hand side of (4.57) bounds β_t, $t = \epsilon^{-1/10}$, and the last term comes from (4.17). Recalling (4.19), this would prove (4.51) if the sup were taken over $|x - x_0| \leq 10^{-5}\zeta\epsilon^{-1}$. When x does not verify this condition, we proceed as in (4.27)-(4.29) and prove (4.51), we omit the details.

The proof of (4.51) with $m_t^\star$ in the place of m_t is very similar and simpler, it is thus omitted.

Let

$$\begin{aligned} u_t(x) &= m_t(x) - m_t^\star(x) \\ &= [m_t(x) - \bar{m}_{x_0}(x)] - [m_t^\star(x) - \bar{m}_{x_0}(x)] \\ &=: v_t^{(1)}(x) - v_t^{(2)}(x) \end{aligned} \tag{4.58}$$

Recalling (2.16), writing $g_{t,x_0} \equiv g_t$ and $F(w) = -3\bar{m}_{x_0}w^2 - w^3$, we have

$$u_t(x) = (g_t u_0)(x) + \int_0^t ds\Big(g_{t-s}\big(F(v_s^{(1)}) - F(v_s^{(2)})\big)\Big)(x) \tag{4.59}$$

We write

$$F(v_s^{(1)}) - F(v_s^{(2)}) = h_s(x)u_s(x) - 3v_s^{(2)}(x)u_s^2(x) - u_s^3(x) \tag{4.60}$$

$$h_s(x) = -3\bar{m}_{x_0}(x)\big[v_s^{(1)} + v_s^{(2)}\big] - 3v_s^{(2)}(x)^2 \tag{4.61}$$

By (4.51) there is c so that

$$\begin{aligned} P^\epsilon\Big([\|h_t\|_\epsilon + \|v_t^{(1)}\|_\epsilon + \|v_t^{(2)}\|_\epsilon] \leq c\epsilon^{1/2-a}t^{1/2} \text{ for all } t \leq \epsilon^{-1/10}\Big) \\ > 1 - c_n\epsilon^n \end{aligned} \tag{4.62}$$

From (4.59) we get

$$u_t(x) = (g_t u_0)(x) + \int_0^t ds \big(g_{t-s} h_s u_s\big)(x) - D_t(x) \tag{4.63}$$

$$D_t(x) := \int_0^t ds \big(g_{t-s}[3v_s^{(2)} u_s^2 + u_s^3(x)]\big)(x) \tag{4.64}$$

Using (4.63) and (4.64) we write

$$\begin{aligned}\int_0^t ds\big(g_{t-s}h_s u_s\big)(x) &= \int_0^t ds\big(g_{t-s}h_s g_s u_0\big)(x)\\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2 \big(g_{t-s_1} h_{s_1} g_{s_1-s_2} h_{s_2} u_{s_2}\big)\\ &+ \int_0^t ds\big(g_{t-s} h_s D_s\big)\end{aligned} \tag{4.65}$$

In the set on the left hand side of (4.62), calling κ the constant c in (2.11) and κ_1 a bound uniform in t of the sup norm of g_t, we have that there is a constant C so that for $t \le \epsilon^{-1/10}$ and

$$\begin{aligned}\Big|\int_0^t ds\big(g_{t-s}h_s g_s u_0\big)(x)\Big| &\le \kappa\kappa_1 c[\epsilon^{1/2-a}]^2 \int_0^t ds\sqrt{s}e^{-\alpha s}\\ &\le C[\epsilon^{1/2-a}]^2\end{aligned} \tag{4.66}$$

$$\begin{aligned}\Big|\int_0^t ds_1 \int_0^{s_1} ds_2 \big(g_{t-s_1} h_{s_1} g_{s_1-s_2} h_{s_2} u_{s_2}\big)\Big| &\le t^2\kappa_1^2 2c^3[\epsilon^{1/2-a}t^{1/2}]^3\\ &\le C\epsilon^{-[(4+3)/20]+3(1/2-a)}\end{aligned} \tag{4.67}$$

$$\begin{aligned}\Big|\int_0^t ds\big(g_{t-s}h_s D_s\big)\Big| &\le t\kappa_1[ct^{1/2}\epsilon^{1/2-a}]\big[t\kappa_1 20(c\epsilon^{1/2-a}t^{1/2})^3\}\big]\\ &\le C\epsilon^{-[8/20]+4(1/2-a)}\end{aligned} \tag{4.68}$$

Using (4.66)-(4.68) we get from (4.63)

$$\|u_t\| \le e^{-\alpha t}\|u_0\| + \bar{C}[\epsilon^{1/2-a}]^2$$

so that Lemma 4.1 is proved. □

4.2 LEMMA. – *Given any* $\zeta > 0$, *for any* $\epsilon > 0$ *let* $x_0 \in \mathcal{T}_\epsilon$, $|x_0| \le (1-\zeta)\epsilon^{-1}$, *and let* $m_t^\star$ *be the process that starts from* $\bar{m}_{x_0}$. *Set*

$$Y^\star = [\xi(m^\star_{T_\epsilon}) - x_0] \tag{4.69}$$

Then for any n *there is* c_n *so that*

$$|E^\epsilon(Y^\star)| \le c_n \epsilon^n \tag{4.70}$$

and for any positive p *there is* C_p *so that*

$$E^\epsilon(|Y^\star|^p) \le C_p[\epsilon^{1/2}(T_\epsilon)^{1/2}]^p \tag{4.71}$$

Proof of (4.70). – We fix $\epsilon > 0$ and x_0. We define

$$\tilde{H}_t(x,y) = \begin{cases} H_t(x,y) & \text{if } |y-x_0| \le 10^{-4}\zeta\epsilon^{-1} \\ 0 & \text{otherwise} \end{cases} \tag{4.72}$$

$$\tilde{Z}_t(x) = \int d\alpha(s,y)\chi_{[0,t]}(s)\tilde{H}_{t-s}(x,y) \tag{4.73}$$

$$\tilde{m}_t = \tilde{H}_t\bar{m}_{x_0} + \int_0^t ds\tilde{H}_{t-s}f(\tilde{m}_s) + \epsilon^{1/2}\tilde{Z}_t \qquad f(m) = m - m^3 \tag{4.74}$$

Observe that in the above only the law of $\tilde{Z}_t$ depends on the boundary conditions.

Let $\mathcal{R}$ be the map from $C^0(\mathbb{R})$ into itself defined as

$$(\mathcal{R}g)(x) = -g\big(x_0 - (x - x_0)\big) \tag{4.75}$$

Thus $\mathcal{R}$ reflects around x_0 and then changes the sign.

The processes $\tilde{Z}_t$ and $\mathcal{R}\tilde{Z}_t$ are equal in distribution, hence the same is true for $\tilde{m}_t$ and $\mathcal{R}\tilde{m}_t$. Setting $\tilde{Y}_t := \xi(\tilde{m}_t) - x_0$, we have that $\tilde{Y}_t$ and $-\tilde{Y}_t$ are equal in distribution, hence

$$E^\epsilon\big(\tilde{Y}_{T_\epsilon}\big) = 0 \tag{4.76}$$

There is $\delta > 0$ and for any n c_n so that

$$P^\epsilon\Big(\sup_{t \le T_\epsilon} \sup_{|x-x_0| \le 10^{-5}\zeta\epsilon^{-1}} \big|Z_t(x) - \tilde{Z}_t(x)\big| \le e^{-\delta\epsilon^{-1}}\Big) \ge 1 - c_n\epsilon^n \tag{4.77}$$

There is $\delta_1 > 0$ so that, in the set appearing on the left hand side of (4.77),

$$\sup_{t \le T_\epsilon} \sup_{|x-x_0| \le 10^{-6}\zeta\epsilon^{-1}} \big|m_t^\star(x) - \tilde{m}_t(x)\big| \le e^{-\delta_1\epsilon^{-1}} \tag{4.78}$$

Using the same argument as in the proof of Proposition 3.2, we obtain

$$\begin{aligned}&\big|\xi(m^\star_{T_\epsilon}) - \xi(\tilde{m}^\star_{T_\epsilon})\big|\\&\le \int dx\bar{m}'_{x_0}(x)[e^{-\delta_1\epsilon^{-1}}\mathbf{1}_{|x-x_0|\le 10^{-6}\zeta\epsilon^{-1}} + 4\mathbf{1}_{|x-x_0|>10^{-6}\zeta\epsilon^{-1}}]\\&\le e^{-\delta_2\epsilon^{-1}}\end{aligned} \tag{4.79}$$

with $\delta_2 > 0$ a suitable constant.

Thus, for any n there is c_n so that

$$\big|E^\epsilon\big(\xi(m^\star_{T_\epsilon}) - x_0\big)\big| \le E^\epsilon\big(\big|\xi(m^\star_{T_\epsilon}) - \xi(\tilde{m}_{T_\epsilon})\big|\big) \le c_n\epsilon^n \tag{4.80}$$

We have therefore proved (4.70).

Proof of (4.71). – Let $\hat{Z}_{t,x_0}$ be as in (2.17) and suppose that $\hat{Z}_{t,x_0} \in \mathcal{G}_\epsilon(a, x_0)$, *see* (5.27) and (5.28). By the first inequality in (4.25) with $u_t := m^\star_t - \bar{m}_{x_0}$, $u_0 = 0$, we have

$$\|u_t - \epsilon^{1/2}\hat{Z}_{t,x_0}\|_\infty \le C_2\int_0^t ds\|u_s\|^2_\infty \tag{4.81}$$

It then follows that for all ϵ small enough, since $\hat{Z}_{t,x_0} \in \mathcal{G}_\epsilon(a, x_0)$,

$$\|u_t\|_\infty \le 2\epsilon^{1/2-a}(t\vee 1)^{1/2}, \qquad \text{for all } t \le T_\epsilon \tag{4.82}$$

hence

$$\|u_t - \epsilon^{1/2}\hat{Z}_{t,x_0}\|_\infty \le C_2 4\epsilon^{1-2a}T^2_\epsilon, \qquad \text{for all } t \le T_\epsilon \tag{4.83}$$

Then, recalling (5.24) we have

$$\|m^\star_{T_\epsilon} - (\bar{m}_{x_0} + \epsilon^{1/2}B_{T_\epsilon}\tilde{m}'_{x_0})\|_\infty \le \|\epsilon^{1/2}R_{T_\epsilon,x_0}\|_\infty + C_2 4\epsilon^{1-2a-1/5} \tag{4.84}$$

Since $\tilde{m}'_{x_0} = \sqrt{D}\bar{m}'_{x_0}$, $D = 3/4$, there is $\bar{c}$ so that

$$\|\bar{m}_{x_0} + \epsilon^{1/2}B_{T_\epsilon}\tilde{m}'_{x_0} - \bar{m}_{x_0+\epsilon^{1/2}\sqrt{D}B_{T_\epsilon}}\|_\infty \le \bar{c}\epsilon B^2_{T_\epsilon}$$

Then by (3.7), the center $\xi^\star_{T_\epsilon}$ of $m^\star_{T_\epsilon}$ and the center $x_0 + \epsilon^{1/2}\sqrt{D}B_{T_\epsilon}$ of $\bar{m}_{x_0+\epsilon^{1/2}\sqrt{D}B_{T_\epsilon}}$ satisfy the following inequality

$$\begin{aligned}&\big|\xi^\star_{T_\epsilon} - (x_0 + \epsilon^{1/2}\sqrt{D}B_{T_\epsilon})\big|\\&\le c\Big(\|\epsilon^{1/2}R_{T_\epsilon,x_0}\|_\infty + \epsilon B^2_{T_\epsilon} + C_2 4\epsilon^{1-2a-1/5}\Big)\end{aligned} \tag{4.85}$$

for a suitable constant c. Recalling (4.69), in $\mathcal{G}_\epsilon(a, x_0)$,

$$\begin{aligned} &\left|(\epsilon T_\epsilon)^{-1/2} Y^\star - T_\epsilon^{-1/2}\sqrt{D} B_{T_\epsilon})\right| \\ &\quad \le c\big(T_\epsilon^{-1/2}\epsilon^{-a} + \epsilon^{1/2} T_\epsilon^{-1/2} B_{T_\epsilon}^2 + C_2 4\epsilon^{1-2a-1/10}\big) \end{aligned} \tag{4.86}$$

By (5.28) and because $|Y^\star| \le \epsilon^{-1}$, the contribution to the expectation of $Y^\star$ outside $\mathcal{G}_\epsilon(a, x_0)$ is negligible, then (4.71) is proven, recalling that the distribution of $T_\epsilon^{-1/2} B_{T_\epsilon}$ is a normal with 0 average and variance D_ϵ given in (5.25). □

Proof of Proposition 3.8 (Conclusion). – As observed after (4.50) the proof of Proposition 3.8 is reduced to that of (4.47) and (4.48) with the process starting from an instanton, since (4.50) is proven in Lemma 4.1. (4.47) follows from (4.71). The first term on the left hand side of (4.48) vanishes by (4.71), the second one by (4.70). The proof that the third term vanishes follows from (4.86) and (5.26), recalling that by (4.69) $Y^\star \equiv \xi_{T_\epsilon} - x_0$. □

5. ESTIMATES ON THE GINZBURG-LANDAU AND THE FREE PROCESSES

In this section we prove some basic properties of the Ginzburg-Landau and the free processes that have been used in the previous sections.

Proof of Lemma 2.1. – The continuity is proved by Walsh, 1981, [18] (Section 4). To prove (2.3), we denote by

$$\|Z^{(\epsilon)}\| = \sup_{x \in \mathcal{T}_\epsilon, t \le 1} |Z_t^{(\epsilon)}(x)|$$

and by

$$\sigma^2 := \sup_{t \le 1,\ x \in \mathcal{T}_\epsilon} E^\epsilon\{Z_t^{(\epsilon)}(x)^2\} \tag{5.1}$$

We then use Theorem 2.1 of [1], which states that for any Gaussian, centered process $Z^{(\epsilon)}$ the following inequality holds, provided that both σ^2 and $E^\epsilon(\|Z^{(\epsilon)}\|)$ are finite,

$$P^\epsilon\big(\|Z^{(\epsilon)}\| > \lambda\big) \le 2\exp\left\{-\frac{\big[\lambda - E^\epsilon(\|Z^{(\epsilon)}\|)\big]^2}{2\sigma^2}\right\} \tag{5.2}$$

where λ is any number such that $\lambda > E^\epsilon(\|Z^{(\epsilon)}\|)$.

By means of the explicit form of the covariance of $Z^{(\epsilon)}$ (*see* [18] (Section 4)) it is not difficult to prove that

$$\sigma^2 \leq 4 \tag{5.3}$$

We will also prove below that there is a positive constant k_1 such that

$$E^\epsilon(\|Z^{(\epsilon)}\|) \leq k_1(\log \epsilon^{-1})^{\frac{1}{2}} \tag{5.4}$$

Lemma 2.1 then easily follows from (5.4) and (5.2), with $\lambda = \epsilon^{-a}$.

Inequality (5.4) follows from standard results on Gaussian processes (*see* [1] and references therein) as we will briefly explain. From Corollary 4.15 of [1], there exists a universal constant K such that

$$E^\epsilon\|Z^{(\epsilon)}\| \leq K \int_0^\infty \{H(\delta)\}^{\frac{1}{2}} d\delta \tag{5.5}$$

where $H(\delta)$ is the metric entropy of our parameter space $S =: \{(x,t) : x \in \mathcal{T}_\epsilon, 0 \leq t \leq 1\}$. In other words, $H(\delta) = \log N(\delta)$, with $N(\delta)$ the minimal number of balls of radius δ needed to cover S, where the metric considered is

$$d\Big((x,t),(y,s)\Big) = \Big[E^\epsilon\Big(Z_t^{(\epsilon)}(x) - Z_s^{(\epsilon)}(y)\Big)^2\Big]^{1/2}$$

After some computations (that may be found in [18], Prop 4.2), one sees that there exist constants k_2 and k_3 such that, for any $h \leq 1$, $x \in \mathcal{T}_\epsilon$, $0 \leq t \leq 1$

$$E^\epsilon\Big(\{Z_t^{(\epsilon)}(x+h) - Z_t^{(\epsilon)}(x)\}^2\Big) \leq k_2 h \tag{5.6}$$

$$E^\epsilon\Big(\{Z_{t+h}^{(\epsilon)}(x) - Z_t^{(\epsilon)}(x)\}^2\Big) \leq k_3 h^{\frac{1}{2}} \tag{5.7}$$

Using (5.6) and (5.7) it is possible to show that $N(\delta) \leq k_4 \epsilon^{-1}\delta^{-8}$ for some positive constant k_4 and from this (5.4) follows. □

From now on we use the integral representation (2.7) for the process.

We next prove a comparison theorem that follows from the parabolic structure of the equation.

5.1 PROPOSITION. – *Let $m_0 \geq m_0^-$ be continuous functions. Let m_t and m_t^- solve* (2.7) *with initial data m_0 and m_0^- and with the same noise Z_t. Then, almost surely, $m_t \geq m_t^-$, for all $t \geq 0$.*

Proof. – Since Z_t is bounded on the compacts, by standard arguments on the reaction diffusion equations it follows that both m_t and m_t^- are bounded for all $t \geq 0$.

Denoting by

$$f(u) = u - u^3$$

after an integration by parts from (2.7) we have that for any real a

$$\begin{aligned} m_t =& e^{-at} H_t m_0 + \int_0^t e^{-a(t-s)} ds H_{t-s}[f(m_s) + a m_s] \\ &+ \epsilon^{1/2}[Z_t - a \int_0^t ds e^{-a(t-s)} H_{t-s} Z_s] \end{aligned} \tag{5.8}$$

An analogous expression obviously holds for m_t^-. By subtracting one equation from the other we get for $w_t := e^{-at}[m_t - m_t^-]$,

$$w_t = H_t w_0 + \int_0^t ds H_{t-s}[F_s + a] w_s, \qquad F_s := \frac{f(m_s) - f(m_s^-)}{m_s - m_s^-} \tag{5.9}$$

We fix $T > 0$. Then, for each ω in the probability space there is $a > 0$ so that $F_s + a \geq 0$ for all $s \leq T$. We then solve (5.9) by iteration with F_s as known. Recalling that H_t is positive we obtain a series of non negative terms. By the arbitrarity of T the proposition is therefore proved. □

5.2 PROPOSITION. – *Let* $m_0 \in C^0(\mathbb{R})$, $\|m_0\|_\infty \leq 1 + 1/32$. *Then there are* c' *and* $c > 0$ *so that if* m_t *solves* (2.7), *then*

$$P^\epsilon\Big(\sup_{t \leq \epsilon^{-2}} \|m_t\|_\infty > 2\Big) \leq c' e^{-c\epsilon^{-1}} \tag{5.10}$$

Proof. – By Proposition 5.1, $m_t^- \leq m_t \leq m_t^+$ where $m_t^\pm$ solve (2.7) starting from $m_0^\pm \equiv \pm(1 + 1/32)$. By the symmetry under change of sign, it will thus suffice to show that

$$P^\epsilon\Big(\sup_{t \leq \epsilon^{-2}} \|m_t^+\|_\infty > 2\Big) \leq c' e^{-c\epsilon^{-1}}\Big) \tag{5.11}$$

We denote by $v_t = m_t^+ - 1$, and we write (1.4) in terms of v_t:

$$\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + 2v = -3v^2 - v^3 + \sqrt{\epsilon}\alpha, \tag{5.12}$$

The corresponding integral equation for v_t, in terms of the operator $\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + 2Id\right)^{-1} = e^{-2t}H_t^{(\epsilon)}$ is

$$v_t(x) = e^{-2t}H_t^{(\epsilon)}v_0 + \int_0^t dse^{-2(t-s)}H_{t-s}[-3v_s^2 - v_s^3] + \epsilon^{1/2}V_t(x) \quad (5.13)$$

where

$$V_t(x) = \int_0^t e^{-2(t-s)}H_{t-s}^{(\epsilon)}(x,y)d\alpha(y,s) \quad (5.14)$$

Recall that $v_0 \equiv 1/32$, so

$$e^{-2t}H_t^{(\epsilon)}v_0 = e^{-2t}1/32 \quad (5.15)$$

The process V_t is studied in detail by Walsh in [18], 1981, for fixed L. Using his estimates and the results on Gaussian processes as in the proof of Lemma 2.1, it is not difficult to obtain inequalities analogous to (5.3) and (5.4) for V_t, for $t \leq \epsilon^{-2}$ instead of $t \leq 1$. Then, inequality (5.2) yields that given any $b > 0$ there are positive constants k_1' and k_2' such that

$$P^\epsilon\Big(\sup_{t\leq\epsilon^{-2}, x\in\mathbb{R}} |\sqrt{\epsilon}V_t(x)| > b\Big) \leq k_1'e^{-k_2'\epsilon^{-1}} \quad (5.16)$$

Let us consider, for b small that will be fixed conveniently,

$$T = \inf\{t \geq 0 : \|v_t(\cdot)\|_\infty \geq 2(b + 1/32)\} \quad (5.17)$$

Then,

$$\begin{aligned} P^\epsilon\Big(\sup_{t\leq\epsilon^{-2}} \|m_t^+\|_\infty > 2\Big) &\leq P^\epsilon\Big(\sup_{t\leq\epsilon^{-2}} \|v_t\|_\infty > 1\Big) \\ &\leq P^\epsilon\Big(T \leq \epsilon^{-2}, \sup_{t\leq\epsilon^{-2}} \|\sqrt{\epsilon}V_t\|_\infty \leq b\Big) + k_1'e^{-k_2'\epsilon^{-1}}, \quad (5.18)\end{aligned}$$

where the last inequality follows from (5.16). But, if $T \leq \epsilon^{-2}$, and $\|\sqrt{\epsilon}V_t\|_\infty \leq b$ for all $t \leq \epsilon^{-2}$, from the definition of T, (5.13) and (5.15) we have

$$\begin{aligned} 2(b + 1/32) &= \|v_T\|_\infty \\ &\leq [12(b + 1/32)^2 + 8(b + 1/32)^3] \\ &\quad \sup_x \int_0^T dse^{-2(T-s)} \int dyH_{T-s}(x,y) + b + 1/32 \\ &\leq [12(b + 1/32)^2 + 8(b + 1/32)^3] + b + 1/32 \quad (5.19)\end{aligned}$$

If we take $b = 1/32$, we get a contradiction in the above inequality, what means that the last probability in (5.18) equals 0 and (5.11) follows. □

5.3 PROPOSITION. – *(The barrier Lemma.) There are $V > 0$ and c so that the following holds. Let m_t and $m_t^\star$ both solve* (2.7), *with initial conditions respectively m_0 and $m_0^\star$. Suppose that for some $T > 0$ their sup norms for $t \le T$ are bounded by 2 and that $m_0(x) = m_0^\star(x)$ for all $|x| \le VT$. Then*

$$\sup_{t\le T} |m_t(0) - m_t^\star(0)| \le ce^{-T} \tag{5.20}$$

Proof. – Calling $u_t(x) := |m_t(x) - m_t^\star(x)|$ we have

$$u_t(x) \le \big(H_t u_0\big)(x) + c\int_0^t ds\big(H_{t-s}u_s\big)(x) \tag{5.21}$$

Iterating N times (5.21) we get

$$u_t(0) \le \big(H_t u_0\big)(0) + \sum_{n=1}^{N} \frac{(ct)^n}{n!}\big(H_t u_0\big)(0) + 2\frac{(ct)^N}{N!} \tag{5.22}$$

from which (5.20) follows. □

Observe that as a corollary of Proposition 5.3 we get that for any $L > 0$

$$\sup_{t\le T}\sup_{|x|\le L} |m_t(x) - m_t^\star(x)| \le ce^{-T} \tag{5.23}$$

provided that $m_0(x) = m_0^\star(x)$ for all $|x| \le L + VT$.

5.4 PROPOSITION. – *Given any $\zeta > 0$, for any $\epsilon > 0$ and any $|x_0| \le (1-\zeta)\epsilon^{-1}$, the process $\hat{Z}_{t,x_0}$ has the representation*

$$\hat{Z}_{t,x_0} =: B_t \tilde{m}'_{x_0} + R_{t,x_0} \tag{5.24}$$

with the following properties.

B_t is a process adapted to Z_t, its law is the law of a Brownian motion with diffusion coefficient

$$D_\epsilon = \int_{-\epsilon^{-1}}^{\epsilon^{-1}} dy\Big[\sum_{k\in\mathbb{Z}}\Big(\tilde{m}'_{x_0}(y + 4k\epsilon^{-1}) + \tilde{m}'_{x_0}(4k\epsilon^{-1} + 2\epsilon^{-1} - y)\Big)\Big]^2 \tag{5.25}$$

and there is a constant c such that

$$|D_\epsilon - 1| \le ce^{-\zeta\epsilon^{-1}} \tag{5.26}$$

For any $a > 0$ *let*

$$\mathcal{G}_\epsilon(a, x_0) := \{\|\hat{Z}_{t,x_0}\|_\infty \le \epsilon^{-a}(t \vee 1)^{1/2},\ \|R_{t,x_0}\|_\infty \le \epsilon^{-a}, \quad \textit{for all } t \le \epsilon^{-2}\} \tag{5.27}$$

Then for any $n \ge 1$ *there is* c_n *so that*

$$P^\epsilon\big(\mathcal{G}_\epsilon(a, x_0)\big) \ge 1 - c_n\epsilon^n \tag{5.28}$$

Proof of Proposition 5.4. – We drop x_0 from the subscripts and write

$$\tilde{m}' = \tilde{m}'_{x_0}, \quad \hat{Z}_t = \hat{Z}_{t,x_0}, \quad G_t(x, y) = g_{t,x_0}(x, y) \tag{5.29}$$

We start from the identity

$$\hat{Z}_t = \tilde{m}' \int dy \hat{Z}_t(y)\tilde{m}'(y) + \Big\{\hat{Z}_t - \tilde{m}' \int dy \hat{Z}_t(y)\tilde{m}'(y)\Big\} \tag{5.30}$$

By (2.19), P^ϵ almost surely,

$$\int dy \tilde{m}'(y)\hat{Z}_t(y) = \int dy \tilde{m}'(y) \int d\alpha(s, z)\chi_{[0,t]}(s)\chi_{[-\epsilon^{-1},\epsilon^{-1}]}(z)G^{(\epsilon)}_{t-s}(y, z) \tag{5.31}$$

where

$$G^{(\epsilon)}_t(y, z) := \sum_{k\in\mathbb{Z}} \Big(G_t(y, z + 4k\epsilon^{-1}) + G_t(y, 4k\epsilon^{-1} + 2\epsilon^{-1} - z)\Big) \tag{5.32}$$

By the Fubini theorem, *see* [8], the right hand side of (5.31) is equal to

$$\int d\alpha(s, z)\chi_{[0,t]}(s)\chi_{[-\epsilon^{-1},\epsilon^{-1}]}(z) \int dy \tilde{m}'(y)G^{(\epsilon)}_{t-s}(y, z) \tag{5.33}$$

Since

$$\int dy G_{t-s}(y, x)\tilde{m}'(y) = \tilde{m}'(x) \tag{5.34}$$

because the semigroup G_t is symmetric and $\tilde{m}'$ is the eigenvector of its generator with eigenvalue 1. By (5.31), (5.32), (5.33) and (5.34)

$$\int dy \hat{Z}_t(y)\tilde{m}'(y) = \int d\alpha(s, z)\chi_{[0,t]}(s)\chi_{[-\epsilon^{-1},\epsilon^{-1}]}(z) \times \Big[\sum_{k\in\mathbb{Z}} \big(\tilde{m}'_{x_0}(z + 4k\epsilon^{-1}) + \tilde{m}'_{x_0}(4k\epsilon^{-1} + 2\epsilon^{-1} - z)\big)\Big] \tag{5.35}$$

The right hand side is a Gaussian process with covariance $D_\epsilon t$. It is therefore a Brownian motion which is identified to the Brownian B_t in the statement of the Proposition. (5.26) follows from the exponential decay of $\bar{m}'$. R_{t,x_0} is identified to the curly brackets term on the right hand side of (5.30) and we are left with the proof of (5.28).

For any $t' < t$ and $y \in \mathbb{R}$ we denote by

$$\phi_{t,t'}(x) = \int d\alpha(s,z)\chi_{[t',t]}(s)\chi_{[-\epsilon^{-1},\epsilon^{-1}]}(z)G^{(\epsilon)}_{t-s}(y,z) \tag{5.36}$$

and observe that

$$\phi_{t,t'}(x) \text{ has the same law as } \phi_{t-t',0}(x) \tag{5.37}$$

Furthermore, given t, we let

$$t^\star = \text{ integer part of } t$$

We also observe that for any $t' < t$, any $t' - 1 \le s \le t'$ and any $y, z \in \mathbb{R}$,

$$G_{t-s}(y,z) = \int dx G_{t-t'}(y,x)G_{t'-s}(x,z) \tag{5.38}$$

By the Fubini theorem we then have

$$\hat{Z}_t(y) = \phi_{t,t^\star}(y) + \sum_{h=1}^{t^\star} \int dx G_{t-h}(y,x)\phi_{h-1,h}(x). \tag{5.39}$$

and by (5.34)

$$\begin{aligned}
&\int dy \hat{Z}_t(y)\tilde{m}'(y) \\
&= \int dy \phi_{t,t^\star}(y)\tilde{m}'(y) + \sum_{h=1}^{t^\star} \int dx \int dy G_{t-h}(y,x)\phi_{h-1,h}(x)\tilde{m}'(y) \\
&= \int dy \phi_{t,t^\star}(y)\tilde{m}'(y) + \sum_{h=1}^{t^\star} \int dx \phi_{h-1,h}(x)\tilde{m}'(x)
\end{aligned} \tag{5.40}$$

From (5.39) and (5.40) we then get

$$\begin{aligned}
&\hat{Z}_t(u) - \tilde{m}'(u) \int dy \hat{Z}_t(y)\tilde{m}'(y) \\
&\qquad = \phi_{t,t^\star}(u) - \tilde{m}'(u) \int dy \phi_{t,t^\star}(y)\tilde{m}'(y) - \sum_{h=1}^{t^\star} I_h(u)
\end{aligned} \tag{5.41}$$

where

$$I_h(u) = \left\{ \int dx G_{t-h}(u,x)\phi_{h,h-1}(x) - \tilde{m}'(u) \int dx \phi_{h,h-1}(x)\tilde{m}'(x) \right\} \tag{5.42}$$

Going back to the original notation with x_0 explicited and using (5.34) we have, *see* Theorem 2.4,

$$I_h(u) = e^{L_{x_0} t}\big[\phi_{h,h-1}(x) - N\tilde{m}'_{x_0}(x)\big] \tag{5.43}$$

Then, by (2.11)

$$\|I_h\|_\infty \leq c\|\phi_{h,h-1}\|_\infty e^{-\alpha(t-h)} \tag{5.44}$$

Recalling (5.37) and (2.17) we have that the law of $\phi_{h,h-1}$ is the same as that of

$$Z_1 + \int_0^1 ds g_{1-s,x_0} V''_{x_0} Z_s \tag{5.45}$$

Analogously, the law of $\phi_{t,t^\star}$ is the same as that of

$$Z_\tau + \int_0^\tau ds g_{\tau-s,x_0} V''_{x_0} Z_s, \qquad \tau = t - t^\star \tag{5.46}$$

Then, by (5.44) and (2.3), for any $a > 0$ and for any n there is c_n so that

$$P^\epsilon\Big(\sup_{t\leq\epsilon^{-2}} \|R_{t,x_0}\|_\infty \leq \epsilon^{-a}\Big) \geq 1 - c_n\epsilon^n \tag{5.47}$$

Proposition 5.4 is thus proved. □

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(Manuscript received April 5, 1994;
revised August, 1994.)