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Existence of positive harmonic functions on groups and on covering manifolds

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à la mémoire de notre ami Claude Kipnis

ABSTRACT. – It is shown, in particular, that a centered probability measure on a linear group of exponential growth has non-constant positive harmonic functions. The same conclusion holds also for the Laplace-Beltrami operator on a co-compact Riemannian covering, with such a deck group.

Key words: Harmonic functions, random walks, Riemannian coverings.

RÉSUMÉ. – On montre en particulier qu'une probabilité centrée sur un groupe linéaire à croissance exponentielle possède des fonctions harmoniques positives non constantes. Il en est de même de l'opérateur de Laplace Beltrami sur le revêtement d'une variété compacte admettant un tel groupe de revêtement.

Classification A.M.S. : 22 D, 31 A 05, 31 C 12, 53 C 21, 60 J 50, 60 J 15.

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1. INTRODUCTION

During the last years a lot of papers have studied the existence of bounded or positive harmonic functions on locally compact groups and on Riemannian manifolds. The purpose of this article is to show that there are non-constant positive harmonic functions on groups and manifolds of exponential growth, at least if the group can be embedded in an almost connected group and if the manifold is a co-compact covering with such a deck transformation group.

We first consider groups. Let μ be a probability measure on a locally compact group G. A measurable function $f : G \to \mathbf{R}$, either bounded or positive, is said to be harmonic (or μ-harmonic) if

$$\int_G f(xg)\, d\mu(g) = f(x), \qquad \forall x \in G.$$

We suppose that G is compactly generated. Let V be a compact neighborhood of the identity in G such that $G = \bigcup_{n \geq 0} V^n$. The growth of G is the rate of growth of the sequence $m(V^n)$, $n \in \mathbf{N}$, where m is a Haar measure [13]. In particular, G is of exponential growth if $\limsup_{n \to +\infty} m(V^n)^{1/n} > 1$. We set

$$\delta_V(g) = \inf\{n \in \mathbf{N};\ g \in V^n\}, \qquad g \in G.$$

Following Guivarc'h [14], we say that the probability measure μ on G has a moment of order $\alpha > 0$ if $\int \delta_V(g)^\alpha\, d\mu(g) < +\infty$. It is called centered if it has a moment of order 1 and if for any additive character χ of G (*i.e.*, any continuous homomorphism $\chi : G \to \mathbf{R}$), $\int_G \chi(g)\, d\mu(g) = 0$. For instance a symmetric measure with compact support is centered. The probability measure μ is called adapted if there is no proper closed subgroup H of G such that $\mu(H) = 1$.

Let us recall some general results on the existence of non-constant harmonic functions. The first one, due to Azencott [5], is the following (notice that the existence of bounded harmonic functions obviously implies the existence of positive ones).

THEOREM 1.1 [5]. – *If G is non-amenable and if μ is adapted, there exist non-constant bounded harmonic functions.*

On the other hand, since the work of Avez [4], it is known that there is a strong connection between the existence of harmonic functions on a group and its growth. Let us gather some related recent results of Alexopoulos [1], Kaimanovitch [20], Guivarc'h [14], Hebish and Saloff Coste [18]:

THEOREM 1.2 ([14], [1], [20]). – *Let us suppose that μ is adapted, centered, with a continuous compactly supported density. Then, if G is either connected and amenable, or polycyclic, the bounded harmonic functions are constant.*

THEOREM 1.3 [18]. – *We suppose that μ is adapted, symmetric, with a continuous compactly supported density. Then if G has a polynomial growth, the positive harmonic functions are constant.*

Our aim in this paper is to prove the following theorem. It gives a partial converse of Theorem 1.3. An almost connected group is a group that has a co-compact connected subgroup.

THEOREM 1.4. – *Let G be a compactly group for which there is a continuous homomorphism ϕ from G into an almost connected group L such that the closure of $\phi(G)$ has an exponential growth. Then any centered, adapted probability measure on G with a third moment has non-constant continuous positive harmonic functions.*

A polycyclic group is a countable solvable group that can be realized as a closed subgroup of a connected group. Therefore, we deduce from these theorems that:

COROLLARY 1.5. – *Let G be a polycyclic group and let μ be an adapted symmetric probability measure on G with finite support. Then*

1. *The bounded harmonic functions are constant.*

2. *The positive harmonic functions are constant if and only if G has polynomial growth.*

A simple example of polycyclic group with exponential growth is the semidirect product $\mathbf{Z} \times_\tau \mathbf{Z}^2$ where, for any $n \in \mathbf{Z}$, $\tau(n) = A^n$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This group gives a positive answer to a question of S. Northshields who asked in a conference in Frascati whether there is a discrete group on which for some finitely supported symmetric measure, there are non-trivial positive harmonic functions and no bounded ones. It was already known that a similar phenomenon occurs for the Brownian motion or for measures with a density w.r.t. the Haar measure on the real affine group (*see* Molchanov [26], Elie [11]) and on $\mathbf{R} \times_\tau \mathbf{R}^2$ (Lyons and Sullivan [24]).

Another example of countable solvable group to which Theorem 1.4 can be applied is the group G of affine transformations of $\mathbf{R}$ generated by the two transformations $x \mapsto 2x$ and $x \mapsto x + 1$. This is the group with two generators a, b and the relation $aba^{-1} = b^2$. It is of exponential growth but not polycyclic. The closure of G in the whole affine group is the group of exponential growth of all the transformations $x \mapsto 2^n x + b$, where $n \in \mathbf{Z}$ and $b \in \mathbf{R}$.

On the other hand, a typical class of groups for which our theorem does not apply is given by the countable non-closed subgroups of exponential growth of the group of motions of the plane, that is itself of polynomial growth. Actually, we do not know if there are positive harmonic functions on such a group.

Let us now consider covering manifolds. A harmonic function f on a Riemannian manifold M is a solution of $\Delta f = 0$, where Δ is the Laplacian. We say that M is a regular covering of a compact manifold N, with deck group Γ, if $M \backslash \Gamma = N$ where Γ is a discrete group of isometries of M. We will deduce from Theorem 1.4 the following one. It provides a partial answer to Lyons and Sullivan ([24], p. 305).

THEOREM 1.6. – *Let M be a regular covering of a compact manifold such that the deck transformation group Γ is a closed subgroup of an almost connected group. Then there exist non-constant positive harmonic functions on M if and only if M is of exponential growth.*

The plan of this paper is the following. We first prove Theorem 1.4 in Section 2 for the closed subgroups of the group $\mathbf{S}_d$ of affine similarities of $\mathbf{R}^d$. Our approach is probabilistic. We then deduce in Section 3 this theorem in full generality by a reduction to this particular case. Finally, we make use of a discretization procedure that goes back to Furstenberg and to Lyons and Sullivan in order to prove Theorem 1.6.

After finishing this paper it was observed that Theorem 2.3 can also be derived from Lin [22] (*see* Babillot *et al.* [6]).

2. HARMONIC FUNCTIONS ON THE GROUP OF AFFINE SIMILARITIES

In this section, we prove Theorem 1.4 for closed subgroups of the group of affine similarities $\mathbf{S}_d$. For notational conveniences, we consider the following realization of $\mathbf{S}_d$:

DEFINITION 2.1. – *The group $\mathbf{S}_d$ of affine similarities of $\mathbf{R}^d$ is defined as the semidirect product*

$$\mathbf{S}_d = (\mathbf{R} \times \mathbf{O}(d)) \times_\sigma \mathbf{R}^d$$

where $\mathbf{O}(d)$ *is the orthogonal group, and where the product is given by*

$$(a, k, b)(a', k', b') = (a + a', kk', b + e^{-a} kb'),$$

for all (a, k, b), $(a', k', b') \in \mathbf{R} \times \mathbf{O}(d) \times \mathbf{R}^d$.

For any $g \in \mathbf{S}_d$, we write $g = (a(g), k(g), b(g))$ where $a(g) \in \mathbf{R}$, $k(g) \in \mathbf{O}(d)$ and $b(g) \in \mathbf{R}^d$. The group $\mathbf{S}_d$ acts on $\mathbf{R}^d$ by the formula

$$g \cdot x = e^{-a(g)} k(g) x + b(g),$$

for all $g \in \mathbf{S}_d$ and $x \in \mathbf{R}^d$. For any positive measure m on $\mathbf{R}^d$ and $g \in \mathbf{S}_d$, we define $g \cdot m$ as the image of m under the map $x \mapsto g \cdot x$. Let μ be a probability measure μ on $\mathbf{S}_d$. We say that a measure m on $\mathbf{R}^d$ is μ-invariant if, for any Borel non-negative function $\phi : \mathbf{R}^d \to \mathbf{R}$,

$$\int \int \phi(g \cdot x) \, d\mu(g) \, dm(x) = \int \phi(x) \, dm(x).$$

The measure m is a Radon measure (or a regular measure) if the mass of each compact set is finite. The proof of the following lemma is straightforward.

LEMMA 2.2. – *Let us suppose that there exists a* μ*-invariant Radon measure* m *on* $\mathbf{R}^d$. *For any non-negative continuous* $\phi : \mathbf{R}^d \to \mathbf{R}$ *with compact support, the function*

$$f(g) = \int \phi(g \cdot x) \, dm(x), \qquad g \in \mathbf{S}_d,$$

is harmonic and continuous. Therefore, there exist non-constant positive continuous harmonic functions as soon as $\mu \{g \in \mathbf{S}_d;\ g \cdot m = m\} \neq 1$.

For any $g \in \mathbf{S}_d$, let $\delta(g) = |a(g)| + \log(1 + \|b(g)\|)$. Then, for any neighborhood V of the identity in S_d there exist constants $A, B, C, D > 0$ such that

$$A\,\delta_V(g) - B \leq \delta(g) \leq C\,\delta_V(g) + D,$$

for all $g \in \mathbf{S}_d$ (*see* Guivarc'h [14], Elie [11]). Therefore the probability measure μ has a moment of order $\alpha > 0$ if and only if

$$\int |a(g)|^\alpha + (\log(1 + \|b(g)\|))^\alpha \, d\mu(g) < +\infty.$$

The map $\chi : \mathbf{S}_d \to \mathbf{R}$ defined by $\chi(a, k, b) = a$ is an additive character. Thus $\int a(g)\, d\mu(g) = 0$, when μ is centered. Our aim is to prove the following theorem:

THEOREM 2.3. – *Let μ be a centered probability measure on $\mathbf{S}_d$ with a moment of order* 3. *We suppose that $\mu\{g \in \mathbf{S}_d;\ a(g) = 0\} \neq 1$ and that $\mu\{g \in \mathbf{S}_d;\ g \cdot x = x\} \neq 1$, for any $x \in \mathbf{R}^d$. Then there exists a μ-invariant Radon measure m on $\mathbf{R}^d$ such that $\mu\{g \in \mathbf{S}_d;\ g \cdot m = m\} \neq 1$. Therefore, there exist continuous non-trivial positive harmonic functions.*

This theorem will be proved in a series of steps, by probabilistic arguments. We shall see that it is quite easy to construct a candidate to the invariant measure m. The main difficulty is to prove that this measure is actually a Radon measure.

COROLLARY 2.4. – *The Theorem* 1.4 *holds true when G is a closed subgroup of $\mathbf{S}_d$ of exponential growth.*

Proof of the corollary. – Let μ be a centered, adapted probability measure on a closed subgroup G of $\mathbf{S}_d$ with a moment of order 3. Then, considered as a measure on $\mathbf{S}_d$ itself, μ is also centered and with a moment of order 3. If $\mu\{g \in \mathbf{S}_d;\ a(g) = 0\} = 1$, then G is contained in the group of polynomial growth $\mathbf{O}(d) \times \mathbf{R}^d$. If for some $x \in \mathbf{R}^d$, $\mu\{g \in \mathbf{S}_d;\ g \cdot x = x\} = 1$, then G is contained in the group $\{g \in \mathbf{S}_d;\ g \cdot x = x\}$ which is isomorphic to $\mathbf{R} \times \mathbf{O}(d)$ and thus also of polynomial growth. Therefore, the assumptions of Theorem 2.3 are fulfilled when G is of exponential growth.

In order to prove Theorem 2.3, let us introduce some notation. Let μ be a probability measure on $\mathbf{S}_d$. We consider the product space $\Omega = \mathbf{S}_d^{\mathbf{N}}$. Let $X_n = (A_n, K_n, B_n)$, $n \in \mathbf{N}$, be the coordinate maps and let $\mathcal{F}$ be the σ-algebra on Ω generated by these coordinates. For any probability measure α on $\mathbf{S}_d$, we let $\mathbf{P}_\alpha$ be the probability measure on $(\Omega, \mathcal{F})$, for which the random variables $X_{n+1} X_n^{-1}$, $n \in \mathbf{N}$, are independent with the same distribution μ and independent of X_0 and for which the distribution of X_0 is α. Under each $\mathbf{P}_\alpha$, the process X_n, $n \in \mathbf{N}$, is a Markov chain on $\mathbf{S}_d$ called the left random walk of law μ. Its transition probability P is given by $P(g, A) = \mu(A g^{-1})$, for all Borel set A of $\mathbf{S}_d$.

We set $\mathbf{P} = \mathbf{P}_\alpha$ when α is the Dirac mass at the unit element $(0, \mathrm{Id}, 0)$ of $\mathbf{S}_d$, where Id is the identity matrix of order d. If λ is a probability measure on $\mathbf{R} \times \mathbf{R}^d$, we set $\mathbf{P}_\lambda = \mathbf{P}_\alpha$ when α is the probability measure on $\mathbf{S}_d$ defined by

$$\int f(g)\, d\alpha(g) = \int f(a, \mathrm{Id}, b)\, d\lambda(a, b)$$

for any bounded Borel function $f : \mathbf{S}_d \to \mathbf{R}$. If ν is a probability measure on $\mathbf{R}^d$, we set $\mathbf{P}_\nu = \mathbf{P}_\alpha$ when α is the probability measure on $\mathbf{S}_d$ equal to $\delta_{(0,\,\mathrm{Id})} \otimes \nu$.

Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. If $g_{n+1} = X_{n+1} X_n^{-1}$, then

$$B_{n+1} = g_{n+1} \cdot B_n = e^{-a(g_{n+1})} k(g_{n+1}) B_n + b(g_{n+1}).$$

Therefore, for any Borel function $f : \mathbf{R}^d \to \mathbf{R}^+$, and any probability measure ν on $\mathbf{R}^d$,

$$\mathbf{E}_\nu (f(B_{n+1})/\mathcal{F}_n) = Q f(B_n),$$

where $Q f(b) = \displaystyle\int f(g \cdot b)\, d\mu(g)$. This implies that $\{B_n,\, n \geq 0\}$ is a Markov chain on $\mathbf{R}^d$ with transition kernel Q. A μ-invariant measure on $\mathbf{R}^d$ is also an invariant measure for this chain.

In what follows, we assume that the assumptions of Theorem 2.3 hold. Let

$$\tau = \inf\{n \in N;\ A_n > 0\}.$$

As μ is centered, the random walk (A_n) is recurrent and therefore τ is finite a.s.

LEMMA 2.5. – *There exists a probability measure ν on $\mathbf{R}^d$ such that, for any Borel set D in $\mathbf{R}^d$,*

$$\mathbf{P}_\nu (B_\tau \in D) = \mathbf{P}_\nu (B_0 \in D) = \nu(D).$$

Proof. – This lemma is contained in Elie [11], but for the sake of completeness we give the proof. Let $\tilde{\mu}$ be the distribution of X_τ under $\mathbf{P}$. We shall construct ν as a $\tilde{\mu}$-invariant probability measure on $\mathbf{R}^d$. Let $\{g_n,\, n \in \mathbf{N}\}$ be a sequence of independent random variables with the distribution $\tilde{\mu}$. Let us show that the sequence $(g_1 g_2 \cdots g_n) \cdot 0$ converges almost surely. Since

$$(g_1 g_2 \cdots g_n) \cdot 0 = \sum_{r=0}^{n-1} e^{-(a(g_1)+\cdots+a(g_r))} (k(g_1) \cdots k(g_r))\, b(g_{r+1}),$$

it suffices to prove that

$$\limsup_{r \to +\infty} \frac{1}{r} \log \| e^{-(a(g_1)+\cdots+a(g_r))} (k(g_1) \cdots k(g_r))\, b(g_{r+1}) \| < 0$$

almost surely. By the law of large numbers, this expression is equal to

$$-\mathbf{E}(a(g_1)) + \limsup_{r \to +\infty} \frac{1}{r} \log \| b(g_{r+1}) \|$$

Using the fact that μ has a third moment, it is proved in Elie [11] that $\mathbf{E}(\log^+ \| b(g_1) \|)$ is finite. This implies that $\limsup_{r \to +\infty} \frac{1}{r} \log \| b(g_{r+1}) \| \leq 0$. Since $\mathbf{E}(A_\tau) = \mathbf{E}(a(g_1))$ is positive, we see that the serie converges a.s. Let ν be the distribution of $Z = \lim_{n \to +\infty} g_1 g_2 \cdots g_n \cdot 0$. The relation

$$g_0 \cdot Z = \lim_{n \to +\infty} g_0 g_1 g_2 \cdots g_n \cdot 0,$$

ensures that the distribution of g_0. Z is also ν. Thus, for any Borel set D in $\mathbf{R}^d$,

$$\mathbf{P}_\nu (B_\tau \in D) = \int \mathbf{1}_D (g \cdot x) \, d\tilde{\mu}(g) \, d\nu(x) = \mathbf{P}(g_0 \cdot Z \in D) = \nu(D).$$

For any $t \geq 0$, let

$$\tau_t = \inf \{ n \geq 0; \, A_n > t \},$$

notice that $\tau = \tau_0$. It is known that $\mathbf{E}(A_\tau)$ is finite since μ has a second moment (*see* Feller [12], Theorem 18.5.1). Let ν be the probability measure on $\mathbf{R}^d$ given by Lemma 2.5. Since $E_\nu(A_\tau) = \mathbf{E}(A_\tau)$ is finite, we can define a probability measure λ on $\mathbf{R}^+ \times \mathbf{R}^d$ by the formula

$$\lambda(F) = \frac{1}{\mathbf{E}_\nu(A_\tau)} \mathbf{E}_\nu \left[\int_0^{A_\tau} \mathbf{1}_F (t, B_\tau) \, dt \right],$$

for any Borel set F in $\mathbf{R}^+ \times \mathbf{R}^d$.

PROPOSITION 2.6. – *The probability measure λ is an invariant measure of the continuous time Markov process $Z_t = (A_{\tau_t} - t, B_{\tau_t})$, $t \in R^+$, on $\mathbf{R}^+ \times \mathbf{R}^d$.*

Proof. – We remark that for any Borel set C in $\mathbf{R}$ and D in $\mathbf{R}^d$,

$$\begin{aligned} &\mathbf{P}(A_1 \in a + C, \, B_1 \in D / A_0 = a, \, B_0 = b) \\ &= \mathbf{P}(A_1 \in C, \, B_1 \in D / A_0 = 0, \, B_0 = b). \end{aligned}$$

This means that (A_n, B_n), $n \in \mathbf{N}$, is a semi-Markov chain which implies that (Z_t) is a strong Markov process (*see* Cinlar [10], Jacod [19]). Let $\zeta = A_\tau$, $\eta = \tau_\zeta$ and $(\tilde{A}, \tilde{K}, \tilde{B}) = X_\eta X_\tau^{-1}$. We remark that $Z_0 = (A_\tau, B_\tau)$ and that $Z_\zeta = (A_\eta - A_\tau, B_\eta)$. Moreover, τ and η are the first two ladder indices of the random walk (A_n). Since (X_n) is a left random walk on $\mathbf{S}_d$, it follows from the Markov property that, under $\mathbf{P}_\nu$, $(\tilde{A}, \tilde{K}, \tilde{B})$ is independent of X_τ and has its distribution $\tilde{\mu}$, used in the proof of Lemma 2.5. Since

$$B_\eta = \tilde{B} + e^{-\tilde{A}} \tilde{K} B_\tau, \qquad A_\eta - A_\tau = \tilde{A},$$

and since the distribution of B_τ is ν under $\mathbf{P}_\nu$, we see that, for any bounded Borel $f : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$,

$$\begin{aligned}\mathbf{E}_\nu\left(f\left(A_\eta - A_\tau,\, B_\eta\right)\right) &= \mathbf{E}_\nu\left(f\left(\tilde{A},\, \tilde{B} + e^{-\tilde{A}}\,\tilde{K}\,B_\tau\right)\right)\\ &= \int f\left(a\left(g\right),\, g \cdot x\right) d\tilde{\mu}\left(g\right) d\nu\left(x\right)\\ &= \mathbf{E}_\nu\left(f\left(A_\tau,\, B_\tau\right)\right).\end{aligned}$$

This proves that Z_0 and Z_ζ have the same distribution under $\mathbf{P}_\nu$. Since ζ is a stopping time of the Markov process (Z_t), it is well known and easy to show that this implies that the formula

$$\lambda_0\left(F\right) = \mathbf{E}_\nu\left(\int_0^\zeta \mathbf{1}_F\left(Z_t\right) dt\right), \quad F \text{ Borel set in } \mathbf{R}^+ \times \mathbf{R}^d,$$

defines an invariant measure of this process (*see*, e.g., Asmussen [3]). When $t < \zeta$, $Z_t = (A_\tau - t,\, B_\tau)$, therefore,

$$\lambda_0\left(F\right) = \mathbf{E}_\nu\left[\int_0^{A_\tau} \mathbf{1}_F\left(t,\, B_\tau\right) dt\right].$$

This implies that λ is an invariant probability measure.

COROLLARY 2.7. – *Under $\mathbf{P}_\lambda$, for all $t \geq 0$, the law of the process $\{(A_{\tau_t+n} - t,\, B_{\tau_t+n}),\ n \in \mathbf{N}\}$ is the same as the law of the process $\{(A_n,\, B_n),\ n \in \mathbf{N}\}$.*

Proof. – Under $\mathbf{P}_\lambda$, $A_0 > 0$, a.s., hence $\tau_0 = 0$. This implies that $Z_0 = (A_0,\, B_0)$ and therefore that the distribution of Z_0 is λ. Since λ is an invariant measure of the Markov process (Z_t), we see that the distribution of Z_t is λ for all $t \geq 0$. We now use the fact that $X_n = (A_n,\, K_n,\, B_n)$, $n \in \mathbf{N}$, is a Markov chain and that τ_t is a stopping time. Let ϕ be a bounded measurable function on $(\mathbf{R} \times \mathbf{R}^d)^{\mathbf{N}}$. For any $t \geq 0$, let

$$R_t = \phi\left(\left(A_n - t,\, B_n\right),\, n \geq 0\right).$$

It follows from the strong Markov property that, if θ is the shift on Ω,

$$\mathbf{E}_\lambda\left(\phi\left(\left(A_{\tau_t+n} - t,\, B_{\tau_t+n}\right),\, n \geq 0\right)\right) = \mathbf{E}_\lambda\left(R_t \circ \theta^{\tau_t}\right) = \mathbf{E}_\lambda\left(\mathbf{E}_{X_{\tau_t}}\left(\mathbf{R}_t\right)\right).$$

On the other hand, as R_t depends only on the process $(A_n,\, B_n)$, $n \geq 0$,

$$\mathbf{E}_{X_{\tau_t}}\left(R_t\right) = \mathbf{E}_{(A_{\tau_t},\, \mathrm{Id},\, B_{\tau_t})}\left(R_t\right) = \mathbf{E}_{(A_{\tau_t} - t,\, \mathrm{Id},\, B_{\tau_t})}\left(R_0\right).$$

Since the distribution of Z_t is λ, this entails that

$$\mathbf{E}_\lambda\left(\phi\left(\left(A_{\tau_t+n} - t,\, B_{\tau_t+n}\right),\, n \geq 0\right)\right) = \mathbf{E}_\lambda\left(R_0\right),$$

which proves the corollary.

LEMMA 2.8. – *For any Borel set C in $\mathbf{R}^d$ let* $m(C) = \mathbf{E}_\lambda \left[\sum_{n=0}^{\tau_1 - 1} \mathbf{1}_C(B_n) \right]$. *Then m is a μ-invariant positive measure.*

Proof. – The proof is classical: since τ_1 is a stopping time of the filtration $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$, and since B_0 and B_{τ_1} have the same distribution under $\mathbf{P}_\lambda$, we can write for any Borel function $f : \mathbf{R}^d \to \mathbf{R}^+$,

$$\mathbf{E}_\lambda \left[\sum_{n=0}^{\tau_1 - 1} f(B_n) \right] = \mathbf{E}_\lambda \left[\sum_{n=0}^{\tau_1 - 1} f(B_{n+1}) \right]$$
$$= \mathbf{E}_\lambda \left[\sum_{n=0}^{\tau_1 - 1} \mathbf{E}_\lambda f(B_{n+1}) / \mathcal{F}_n \right] = \mathbf{E}_\lambda \left[\sum_{n=0}^{\tau_1 - 1} Q f(B_n) \right].$$

Therefore $\int f\, dm = \int Q f\, dm$ and m is invariant.

The next results will be used to prove that the measure m defined above is a Radon measure, *i.e.*, that the measure of each compact set is finite.

PROPOSITION 2.9. – *For all Borel set C in $\mathbf{R}^d$,*

$$m(C) = \mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(0,\,1] \times C}(A_i,\, B_i) \right].$$

Proof. – Making use of the fact that $\mathbf{P}_\lambda(\tau_0 = 0) = 1$, we have

$$\mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(0,\,1] \times C}(A_i,\, B_i) \right]$$
$$= \sum_{n=0}^{+\infty} \mathbf{E}_\lambda \left[\sum_{i=\tau_n}^{\tau_{n+1} - 1} \mathbf{1}_{(0,\,1] \times C}(A_i,\, B_i) \right]$$
$$= \sum_{i=0}^{+\infty} \mathbf{E}_\lambda \left[\sum_{i=0}^{\tau_{n+1} - \tau_n - 1} \mathbf{1}_{(0,\,1] \times C}(A_{\tau_n + i},\, B_{\tau_n + i}) \right]$$
$$= \sum_{n=0}^{+\infty} \mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(-n,\,1-n] \times C}(A_{\tau_n + i} - n,\, B_{\tau_n + i})\, \mathbf{1}_{\{i < \tau_{n+1} - \tau_n\}} \right]$$
$$= \sum_{n=0}^{+\infty} \mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(-n,\,1-n] \times C} \right.$$
$$\left. \times\, (A_{\tau_n + i} - n,\, B_{\tau_n + i})\, \mathbf{1}_{\{A_{\tau_n + p} - n \le 1,\, \forall\, p \in [0,\, i]\}} \right].$$

Thus, it follows from Corollary 2.7 that

$$\begin{aligned}
&\mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(0,\,1]\times C}(A_i,\, B_i)\right] \\
&= \sum_{n=0}^{+\infty} \mathbf{E}_\lambda \left[\sum_{i=0}^{+\infty} \mathbf{1}_{(-n,\,1-n]\times C}(A_i,\, B_i)\, \mathbf{1}_{\{A_p \le 1,\, \forall\, p \in [0,\, i)\}}\right] \\
&= \mathbf{E}_\lambda \left[\sum_{i=0}^{\tau_1 - 1} \mathbf{1}_{(-\infty,\,1]\times C}(A_i,\, B_i)\right] = m(C),
\end{aligned}$$

since $A_i \le 1$ when $i \le \tau_1$.

LEMMA 2.10. – *Let G be a closed subgroup of $\mathbf{S}_d$ which is not contained in $\{0\} \times \mathbf{O}(d) \times \mathbf{R}^d$, nor in a conjugate of $\mathbf{R} \times \mathbf{O}(d) \times \{0\}$. Then*

(i) *G is non-unimodular;*

(ii) *There is no Radon measure m on $\mathbf{R}^d$ such that $g \cdot m = m$ for all $g \in G$.*

Proof. – Let us first show that under the assumptions of the lemma, G has a simple structure. Let g_0 be an element of G such that $a(g_0) > 0$ and let

$$x = (0,\, \mathrm{Id},\, (\mathrm{Id} - e^{-a(g_0)}\, k(g_0))^{-1}\, b(g_0)).$$

By replacing G by its conjugate $x^{-1} G x$ and g_0 by $x^{-1} g_0 x$, we can and shall suppose without loss of generality that $b(g_0) = 0$. Since $\mathbf{O}(d)$ is a compact group, there exists a sequence (n_i) of integers such that $k(g_0)^{n_i}$ converges to Id as $i \to +\infty$. Then $g_0^{n_i} g g_0^{-n_i}$ converges to $(a(g),\, k(g),\, 0)$, for any $g \in G$. Since G is closed, $(a(g),\, k(g),\, 0) \in G$ and therefore, $(0,\, \mathrm{Id},\, b(g)) = g\,(a(g),\, k(g),\, 0)^{-1}$ is also in G. This proves that $G = G_1 \times G_2$, where G_1 is the closed subgroup of $\mathbf{R} \times \mathbf{O}(d)$ such that $G_1 \times \{0\} = G \cap \mathbf{R} \times \mathbf{O}(d) \times \{0\}$, and G_2 is the closed subgroup of $\mathbf{R}^d$ such that $\{(0,\, \mathrm{Id})\} \times G_2 = G \cap \{(0,\, \mathrm{Id})\} \times \mathbf{R}^d$. Since $e^{-a(g_0)}\, k(g_0)\, G_2 = G_2$, G_2 has no discrete component and is therefore isomorphic to $\mathbf{R}^p$, for some $p > 0$.

Let α_1 and α_2 be the Haar measures on G_1 and G_2. It is straightforward to check that the measure $\alpha_1 \otimes \alpha_2$ on G is right invariant but not left invariant under multiplication. Hence G is not unimodular.

Let us now suppose that there is a G-invariant Radon measure m on $\mathbf{R}^d$. Let H be the subspace orthogonal to G_2 in $\mathbf{R}^d$. We identify $\mathbf{R}^d$ with $H \times G_2$. Since m is invariant under G_2, we can write $m = m_1 \otimes m_2$, where m_1 is a Radon measure on H and m_2 is the Lebesgue measure on

G_2. Let $B_r = \{x \in H; \| x \| < r\}$ and $\varepsilon = e^{a(g_0)}$. Then, since $g_0^n \cdot m = m$, for any Borel set A in G_2,

$$\begin{aligned} m_1(B_r)\, m_2(A) &= \int_H \mathbf{1}_{B_r}(e^{-na(g_0)}\, k(g_0)^n\, x)\, dm_1(x) \\ &\quad \times \int_{G_2} \mathbf{1}_A(e^{-na(g_0)}\, k(g_0)^n\, y)\, dm_2(y) \\ &= \varepsilon^{np}\, m_1(B_{\varepsilon^n r})\, m_2(A). \end{aligned}$$

This leads to a contradiction as $n \to +\infty$.

Proof of Theorem 2.3. – Let G_μ be the smallest closed subgroup of $\mathbf{S}_d$ that carries μ. Under the hypotheses of the theorem, it is easy to see that G_μ has the properties of Lemma 2.10. In particular it is non-unidomodular. Let U be the potential kernel of the random walk (X_n) of law μ, defined by

$$U(g, D) = \sum_{n=0}^{+\infty} \mathbf{P}(X_n\, g \in D), \qquad g \in \mathbf{S}_d,\ D \text{ Borel set in } \mathbf{S}_d.$$

Since μ is adapted to the non-unimodular group G_μ, (X_n) is a transient random walk, which implies that for any compact set D in $\mathbf{S}_d$, $U(g, D)$ is a bounded function of $g \in \mathbf{S}_d$ (*see*, e.g., Guivarc'h *et al.* [16], Theorem 51, Prop. 31). By Proposition 2.9, for any compact set C in $\mathbf{R}^d$,

$$m(C) = \int U((a, \mathrm{Id}, b), (0, 1] \times \mathbf{O}(d) \times C)\, d\lambda(a, b),$$

thus $m(C)$ is finite. It then follows from Lemma 2.8 that m is a μ-invariant Radon measure. If $\mu\{g \in \mathbf{S}_d;\ g \cdot m = m\} = 1$, then $g \cdot m = m$ for all $g \in G_\mu$, which is impossible by Lemma 2.10.

Remark. – There are other natural μ-invariant measures on $\mathbf{R}^d$. For instance, the formula

$$\tilde{m}(C) = \mathbf{E}_\nu \left[\sum_{n=0}^{\tau-1} \mathbf{1}_C(B_n)\right],$$

also defines a μ-invariant positive measure. One can show that $\tilde{m}$ is a Radon measure (by making use of the fact that m is Radon). Actually, we conjecture that all the μ-invariant Radon measures are proportional.

3. THE GENERAL CASE

We shall first prove Theorem 1.4 by a serie of reductions to the group of affine similarities $\mathbf{S}_d$.

LEMMA 3.1. – *Let μ be an adapted centered probability measure on a compactly generated group G with a moment of order α. Let $\phi : G \to H$ be a continuous homomorphism from G to the l.c. group H and let K be the closure of $\phi(G)$. Then the image $\phi(\mu)$ of μ is adapted to K, is centered and has a moment of order α.*

Proof. – Let V be a compact set in G such that $G = \bigcup_{n \geq 0} V^n$. Let W be a symmetric open set in K, relatively compact, that contains $\phi(V)$. Then $\bigcup_{n \geq 0} W^n$ is an open subgroup in K that contains $\phi(G)$. Since an open subgroup is closed, $\bigcup_{n \geq 0} W^n = K$. This shows in particular that K is compactly generated. Since $\phi(V^n)$ is contained in W^n, we see that $\delta_W(\phi(g)) \leq \delta_V(g)$ for any $g \in G$. Therefore, $\int \delta_W(h)^\alpha \, d\phi(\mu)(h) \leq \int \delta_V(g)^\alpha \, d\mu(g)$. If χ is an additive character of K, then $\chi \circ \phi$ is an additive character of G and thus, $\int \chi \, d\phi(\mu) = \int \chi \circ \phi \, d\mu = 0$, when μ is centered.

We shall need the following well known lemma (*see*, e.g. [16]):

LEMMA 3.2. – *A random walk on a finite group with an adapted law is an irreducible Markov chain.*

Let H be a normal closed subgroup of a locally compact group G of finite index. To each probability measure μ on G, we associate a probability measure μ_H on H, called the induced measure, by the following recipe: let S_n, $n \in \mathbf{N}$, be the right random walk of law μ starting from the unit element. By definition

$$S_n = Y_1 Y_2 \cdots Y_n$$

where $\{Y_n, \ n \geq 1\}$ is a sequence of independent random elements of G with the distribution μ. Let $T = \inf\{n \geq 1; \ S_n \in H\}$. The induced measure μ_H is defined as the distribution of S_T. If $\pi : G \to G/H$ is the canonical projection, then $\pi(S_n)$ is a random walk on the finite group G/H. The state $\pi(H)$ is recurrent for this Markov chain. Since T is the first return time of $\pi(S_n)$ to this state, T is finite almost surely and has an exponential moment.

PROPOSITION 3.3. – *Consider a probability measure μ on a compactly generated group G, that is adapted, centered and with a third moment. Let μ_H be the induced measure on a normal subgroup of finite index H of G. Then μ_H is adapted, centered and has a third moment on H.*

Proof. – The proof of this proposition is given in three independent parts.

Part 1. – We first show that μ_H is centered. Let $s : G/H \to G$ be a section of G/H, which means that $\pi \circ s = \mathrm{id}$. Let χ be an additive character of H. For any $(g, x) \in G \times G/H$, $s(x)\, g s(x\pi(g))^{-1}$ is in H. We set $A(g, x) = \chi(s(x)\, g s(x\pi(g))^{-1})$. It is straightforward to check that, for all $g_1, g_2 \in G$ and $x \in G/H$,

$$A(g_1 g_2, x) = A(g_1, x) + A(g_2, x\pi(g_1)).$$

(A is actually the cocycle associated with the representation induced by χ). Let ν be the uniform probability measure on the finite set G/H and let $\xi(g) = \int A(g, x)\, d\nu(x)$. It follows from the relation above that $\xi(g_1 g_2) = \xi(g_1) + \xi(g_2)$. Therefore, ξ is an additive character of G, and since μ is centered,

$$\int\int A(g, x)\, d\nu(x)\, d\mu(g) = \int \xi(g)\, d\mu(g) = 0.$$

We can suppose that $s(\dot{e}) = e$, where e and $\dot{e}$ are the unit elements of G and G/H. Then,

$$\chi(S_T) = A(S_T, \dot{e}) = A(Y_1 Y_2 \cdots Y_T, \dot{e}) = \sum_{k=0}^{T-1} A(Y_{k+1}, \pi(Y_1 \cdots Y_k)).$$

Therefore, using the fact that $\{T > k\}$ is independent of Y_{k+1},

$$\begin{aligned}
\mathbf{E}(\chi(S_T)) &= \sum_{k=0}^{+\infty} \mathbf{E}(A(Y_{k+1}, \pi(S_k))\, \mathbf{1}_{\{T>k\}}) \\
&= \sum_{k=0}^{+\infty} \mathbf{E}\left(\int A(g, \pi(S_k))\, \mathbf{1}_{\{T>k\}}\, d\mu(g)\right) \\
&= \mathbf{E}\left(\sum_{k=0}^{T-1} \int A(g, \pi(S_k))\, d\mu(g)\right).
\end{aligned}$$

The Markov chain $\pi(S_n)$ on the finite space G/H is irreducible by Lemma 3.2, and its unique invariant distribution is ν. Since T is the return time at $\dot{e}$ of this chain, we know that for any function $f : G/H \to \mathbf{R}^+$,

$$\mathbf{E}\left(\sum_{k=0}^{T-1} f(\pi(S_k))\right) = \mathbf{E}(T) \int f(x)\, d\nu(x).$$

Hence,

$$\mathbf{E}(\chi(S_T)) = \mathbf{E}(T) \int\int A(g, x)\, d\mu(g)\, d\nu(x) = 0.$$

This proves that distribution μ_H of S_T is centered.

Part 2. – Let us now show that μ_H is adapted. We consider the semigroup $S = \bigcup_{n\geq 0} (\operatorname{Supp} \mu)^n$, where $\operatorname{Supp} \mu$ is the support of μ. Let H_1 be the subgroup of H generated by $S \cap H$. Let us prove, by induction on $n \in \mathbf{N}$, that

$$(SS^{-1})^n \cap H \subset H_1.$$

This is clear when $n = 0$. We shall use the fact that the order of each element of G/H divides $r = \operatorname{Card}(G/H)$, and therefore that $g^r \in H$ for any $g \in G$. Let $h \in (SS^{-1})^{n+1} \cap H$. We can write $h = g_1 g_2^{-1} g$, where $g_1, g_2 \in S$ and $g \in (SS^{-1})^n$. Then

$$h = g_1^r (g_2 g_1^{r-1})^{-1} g = g_1^r (g_2 g_1^{r-1})^{-r} (g_2 g_1^{r-1})^{r-1} g.$$

Since g_1^r and $(g_2 g_1^{r-1})^r$ are in $H \cap S$, we see that $g_1^r (g_2 g_1^{r-1})^{-r} \in H_1$. This implies, in particular, that $(g_2 g_1^{r-1})^{r-1} g \in H$. On the other hand, $(g_2 g_1^{r-1})^{r-1} g \in S (SS^{-1})^n = (SS^{-1})^n$. Thus $(g_2 g_1^{r-1})^{r-1} g$ is in $(SS^{-1})^n \cap H$, hence in H_1 by the induction hypothesis. This proves that $h \in H_1$.

The cofinite subgroup H is open in G and $\bigcup_{n\geq 0} (SS^{-1})^n$ is dense in G since μ is adapted. Hence H_1 is dense in H. Now let H_2 be the closed subgroup of H generated by the support of μ_H. If $x \in S \cap H$, then for some $n \in \mathbf{N}$, $x \in \operatorname{supp} \mu^n \cap H$. Therefore, for any neighborhood V of x in H, $\mathbf{P}(S_n \in V) \neq 0$. If $T_0 = 0$ and $T_{n+1} = \inf \{k > T_n,\ S_k \in H\}$, then $\{S_{T_n},\ n \in \mathbf{N}\}$ is the random walk of law μ_H. We see that $\sum_{k=1}^{+\infty} \mathbf{P}(S_{T_k} \in V) \neq 0$, hence $x \in H_2$. Thus H_1 is contained in H_2. Since H_1 is dense in H and H_2 is closed, $H_2 = H$.

Part 3. – Let us show now that μ_H has a third moment. Let V be a compact set in G with e in its interior, such that $G = \bigcup_{n\geq 0} V^n$. There exists a generating compact neighborhood W of e in H, and α, $\beta > 0$, such that

$$\delta_W(h) \leq \alpha\, \delta_V(h) + \beta, \qquad h \in H,$$

(*see* Kaimanovitch [20]). The function δ_V is subadditive, thus

$$\int \delta_V^3\, d\mu_H = \mathbf{E}(\delta_V(S_T)^3) \leq \mathbf{E}\left[\left(\sum_{k=1}^{T} \delta_V(Y_k)\right)^3\right].$$

Let us remark that $\sum_{k=1}^{n} \delta_V(Y_k)$ is an usual random walk on $\mathbf{R}$ with a third moment, and that T is a stopping time with an exponential moment. It is classical that such a stopped sum has a third moment (*see*, e.g., Gut [17]). Therefore $\int \delta_V^3 \, d\mu_H < +\infty$.

LEMMA 3.4. – *Let μ be an adapted probability measure on a locally compact group G. We suppose that there is a normal subgroup H of finite index of G such that the induced measure μ_H has non-constant positive continuous harmonic functions. Then μ also has non-constant positive continuous harmonic functions.*

Proof. – Let h be a non-constant positive continuous μ_H-harmonic function on H. We shall use a classical construction. We consider the right random walk $(\Omega, (S_n), \mathbf{P}_g, g \in G)$ of law μ. Let $f(g) = \mathbf{E}_g(h(S_T))$, $g \in G$. Since h is μ_H-harmonic, $f(g) = h(g)$ when $g \in H$. Let θ be the shift on Ω. We have by the Markov property, for any $x \in G$,

$$\begin{aligned}\int f(xg)\, d\mu(g) &= \mathbf{E}_x[f(S_1)] = \mathbf{E}_x[\mathbf{1}_{\{S_1 \in H\}} f(S_1) + \mathbf{1}_{\{S_1 \notin H\}}(h(S_T))] \\ &= \mathbf{E}_x[\mathbf{1}_{\{S_1 \in H\}} h(S_1)] \\ &\quad + \mathbf{E}_x[\mathbf{1}_{\{S_1 \notin H\}} \mathbf{E}_x(h(S_T \circ \theta)/\sigma(S_0, S_1))] \\ &= \mathbf{E}_x[\mathbf{1}_{\{S_1 \in H\}} h(S_T)] + \mathbf{E}_x[\mathbf{1}_{\{S_1 \notin H\}} h(S_T)] = f(x)\end{aligned}$$

since $S_T \circ \theta = S_T$ when $S_1 \notin H$. This proves that f is a non-constant positive harmonic function. Let us show that f is locally integrable with respect to a Haar measure on G. We consider the measure $\nu = \sum_{n=0}^{+\infty} \mu^n / 2^{n+1}$.

Since f is harmonic, for all $x \in G$, $f(x) = \int f(xg)\, d\nu(g)$. The function f is equal to h on H, thus it is locally integrable on H. Let V be a relatively compact open set in H, and m_H be a right Haar measure on H. Then for all $y \in H$,

$$+\infty > \int_V f(xy)\, dm_H(x) = \int_V \left\{ \int_G f(xyg)\, d\nu(g) \right\} dm_H(x),$$

therefore $\int_V f(xyg)\, dm_H(x) =< +\infty$, for $m_H \otimes \nu$-almost all (y, g). Since μ is adapted on G, the support of the image of ν on G/H is equal to G/H (*see* lemma 3.2). Thus there exists a subset $\{g_i,\ i = 1, \cdots, r\}$ of G such that $G = \bigcup_{i=1}^{r} H\, g_i$ and such that

$$\int_{Vy^{-1}} f(xg_i)\, dm_H(x) = \int_V f(xyg_i)\, dm_H(x) < +\infty,$$

for m_H-almost all $y \in H$. Since $m = \sum_{i=1}^{r} m_H \star \varepsilon_{g_i}$ is a right Haar measure on G, this implies that f is locally m-integrable. Finally, let ϕ be a positive continuous function on G with compact support, then $\phi \star f$ is a positive continuous harmonic function.

We shall now establish some algebraic preliminary results. Let us introduce the following definitions:

DEFINITION 3.5. – *The class* $\mathbf{C}$ *is the class of the topological groups* G *which admit a continuous homomorphism* ϕ *in a group of affine similarities* $\mathbf{S}_d$ *such that the closure of* $\phi(G)$ *in* $\mathbf{S}_d$ *is of exponential growth.*

DEFINITION 3.6. – *We consider a group* G *acting on a finite-dimensional real vector space* V *by linear transformations* $\rho(g)$, $g \in G$. *We say that this action is of type* $\mathbf{S}$ *is there exists a compact subgroup* K *of* $Gl(V)$ *such that for all* $g \in G$, $\rho(g)$ *can be written as*

$$\rho(g) = \Delta(g)\, k(g)$$

where $\Delta(g) \in \mathbf{R}_+^*$, $k(g) \in K$. *We call* Δ *the associated multiplicative character.*

If $\alpha = \dim(V)$, then $\Delta(g) = |\det \rho(g)|^{1/\alpha}$, therefore Δ is indeed a multiplicative real character of G, *i.e.* a continuous homomorphism from G into the multiplicative group $\mathbf{R}_+^*$.

LEMMA 3.7. – *Let* G *be a subgroup of* $Gl(d, \mathbf{R})$ *which is not in the class* $\mathbf{C}$. *If there is a* G*-invariant linear subspace* V *of* $\mathbf{R}^d$ *such that the actions of* G *on* V *and on* $\mathbf{R}^d/V$ *are of type* $\mathbf{S}$ *with different multiplicative characters, then* V *has a* G*-invariant supplementary subspace.*

Proof. – Without loss of generality, we can suppose that V is the linear subspace of $\mathbf{R}^d$ spanned by the first $n = \dim(V)$ vectors of the canonical basis, for some $n < d$.

Then we can write each $g \in G$ as $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix}$, where for instance g_1 is a $n \times n$ matrix. Let W be the vector space of $n \times (d-n)$ matrices. We set $\rho(g)\, M = g_1\, M\, g_3^{-1}$, for any $M \in W$. This defines an action of type $\mathbf{S}$ of G on W with the non-trivial character $\Delta = \Delta_1/\Delta_2$, where Δ_1 and Δ_2 are the characters of the action on V and on $\mathbf{R}^d/V$, respectively. We identify W with the vector space $\mathbf{R}^m$, where $m = n\,(d-n)$, equipped with a scalar product for which the maps $\rho\,(g)/\Delta\,(g)$ are in the orthogonal group $\mathbf{O}\,(m)$. One checks easily that the formula

$$\xi\,(g) = (-\log\,\Delta\,(g),\ \rho\,(g)/\Delta\,(g),\ g_2\,g_3^{-1})$$

defines an homomorphism ξ from G to the group of affine similarities $\mathbf{S}_m$. We have supposed that G is not in the class $\mathbf{C}$. Thus the closure of $\xi\,(G)$ is not of exponential growth, and in particular it is unimodular. Since the character Δ is not trivial, it follows from Lemma 2.10 that $\xi\,(G)$ is contained in a conjugate of $\mathbf{R} \times \mathbf{O}\,(m) \times \{0\}$. This yields that there is $M \in W$ such that

$$\rho\,(g)\,M + g_2\,g_3^{-1} = M, \quad \text{for all } g \in G.$$

Then the linear subspace $\left\{\begin{pmatrix} M\,x \\ x \end{pmatrix},\ x \in \mathbf{R}^{d-n}\right\}$ of $\mathbf{R}^d$ is G-invariant and supplementary to V.

LEMMA 3.8. – *Let G be a closed subgroup of $Gl\,(d,\,\mathbf{R})$ which is not in the class $\mathbf{C}$. We suppose that there is a finite sequence $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{R}^d$ of G-invariant linear subspaces of $\mathbf{R}^d$ such that the action of G on each V_i/V_{i-1} is of type $\mathbf{S}$. Then G has a polynomial growth.*

Proof. – We shall say that a closed subgroup G of $Gl\,(d,\,\mathbf{R})$ is of type $\mathbf{S}^*$ is there is a finite sequence $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{R}^d$ of G-invariant linear subspaces of $\mathbf{R}^d$ such that the action of G on each V_i/V_{i-1} is of type $\mathbf{S}$ and all the associated characters are equal. Such a subgroup is contained in the direct product of $\mathbf{R}_+^*$ with a compact extension of a nilpotent (upper triangular) group. It has therefore a polynomial growth.

Let us prove by induction on d that for any subgroup G of $Gl\,(d,\,\mathbf{R})$ satisfying the assumptions of the lemma, there exists a direct sum decomposition

$$\mathbf{R}^d = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

such that, for each $i = 1, \cdots, k$, the linear subspace W_i is G-invariant and the action on G on W_i is of type $\mathbf{S}^*$. This will imply that G is isomorphic to the direct product of subgroups of $Gl\,(d,\,\mathbf{R})$ of type $\mathbf{S}^*$. Hence G has a polynomial growth. Let us suppose that the induction hypothesis holds true until $d-1$. Let L be a proper G-invariant subspace of $\mathbf{R}^d$ containing V_{n-1} of maximal dimension. Then the action on $\mathbf{R}^d/L$ is irreducible and of type $\mathbf{S}$. Let Δ be its multiplicative character. By the induction hypothesis, we can write

$$L = W_1 \oplus W_2 \oplus \cdots \oplus W_p,$$

where, for each $i = 1, \cdots, p$, the subspace W_i is G-invariant and the action of G on W_i is of type $\mathbf{S}^*$. If all the characters of these actions are equal to Δ then G is of type $\mathbf{S}^*$ and the conclusion of the lemma holds. Let us suppose that one of these characters, for instance the character Δ_1 associated with the action on $\mathbf{W}_1$, is different from Δ. Owing to the induction hypothesis, it suffices to prove that W_1 has a supplementary G-invariant subspace.

Let V be a linear subspace of $\mathbf{R}^d$ such that $\mathbf{R}^d = L \oplus V$. We consider the quotient action π of G on $W_1 \oplus V = \mathbf{R}^d/(W_2 \oplus \cdots \oplus W_p)$. Let E be a proper subspace of minimal dimension of W_1 invariant under this action. By minimality, this action on E is of type $\mathbf{S}$. Applying the induction hypothesis to the quotient action on $(W_1 \oplus V)/E$, we see that there exist two subspaces U_1 and U_2 of $W_1 \oplus V$ such that $W_1 \oplus V = E \oplus U_1 \oplus U_2$,

$$\pi(G)\, U_1 \subset E \oplus U_1, \qquad \pi(G)\, U_2 \subset E \oplus U_2,$$

and the quotient action of $\pi(G)$ on $(E \oplus U_2)/E$ is of type $\mathbf{S}$, with character Δ. We remark that $\pi(G)(E \oplus U_2) = E \oplus U_2$ and that the character of the type $\mathbf{S}$ action of $\pi(G)$ on E is Δ_1. Thus we can apply the preceding lemma to conclude that E has a supplementary subspace F in $E \oplus U_2$, invariant under the action of $\pi(G)$. Then $F \oplus W_2 \oplus W_3 \oplus \cdots \oplus W_p$ is a G-invariant subspace, supplementary to W_1. This proves the induction hypothesis.

PROPOSITION 3.9. – *Let G be a closed amenable subgroup of $Gl(d, \mathbf{R})$ of exponential growth. Then there is a normal subgroup D of G of finite index in the class* $\mathbf{C}$.

Proof. – Since G is amenable, it has a normal subgroup D of finite index, for which there is a finite sequence $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{R}^d$ of D-invariant linear subspaces of $\mathbf{R}^d$ such that the action of D on each V_i/V_{i-1} is of type $\mathbf{S}$ (cf. Guivarc'h [13]). This group D has an exponential growth. Thus it follows from Lemma 3.8 that it is in the class $\mathbf{C}$.

LEMMA 3.10. – *Let T be a closed subgroup of exponential growth of a connected Lie group L. There exist an integer $d > 0$ and a continuous homomorphism $\psi : L \to Gl(d, \mathbf{R})$ such that the closure of $\psi(T)$ has an exponential growth.*

Proof. – We consider the image $Ad(L)$ of L in the adjoint representation. Since $\mathrm{Ad}(L)$ is a connected subgroup of a linear group, there exists some integer $d > 0$ and a one-to-one continuous homomorphism $\phi : Ad(L) \to Gl(d, \mathbf{R})$ such that $\phi(Ad(L))$ is closed (*see* Bourbaki [8], Ex. III.9.42). Let $\psi = \phi \circ Ad$. Then $C = \operatorname{Ker} \psi$ is central and $\psi(L)$ is closed. Let G be the closure of $\psi(T)$, G_0 be its connected component of the identity and let $H = \psi^{-1}(G)$. Let us suppose that G is not of exponential growth. This is equivalent to suppose that the weights of the adjoint representation of G_0 are of absolute value one, and that the discrete group G/G_0 is a finite extension of a nilpotent group (*see* Guivarc'h [13], Losert [23]). Let H_0 be the connected component of the identity of H. Then $G_0 = \psi(H_0)$ and G_0 is isomorphic to $H_0/H_0 \cap C$. Since C is central, the weights of the adjoint representation of H_0 are weights of the adjoint representation of G_0, and therefore are of absolute value one. On the other hand, $(H/H_0)/(H_0 C/H_0) = H/H_0 C = G/G_0$ is a finite extension of a nilpotent group, and $H_0 C/H_0$ is central in H/H_0. This immediately implies that H/H_0 is itself a finite extension of a nilpotent group, because a central extension of a nilpotent group is nilpotent. Thus H has a polynomial growth. This is impossible, since T is a closed subgroup of H with an exponential growth.

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. – We consider a probability measure μ on the group G, that is centered, adapted and with a moment of order 3. Let R be the closure of $\phi(G)$ in L. Since L is almost connected, it has a compact normal subgroup K such that $\tilde{L} = L/K$ is a Lie group with a finite number of connected components (*see* Montgomery-Zippin [27]). Let $\pi : L \to \tilde{L}$ be the canonical projection. By Lemma 3.1 the image $\tilde{\mu} = (\pi \circ \phi)(\mu)$ is adapted to $\tilde{R} = \pi(R)$, is centered and has a moment of order 3. Moreover $\tilde{R}$ is a closed subgroup of $\tilde{L}$ with exponential growth. If $f : \tilde{R} \to \mathbf{R}^+$ is $\tilde{\mu}$-harmonic, then $f \circ \pi \circ \phi$ is μ-harmonic. Thus, it suffices to prove that the probability measure $\tilde{\mu}$ on R has non-constant continuous positive harmonic functions. Let $\tilde{L}_0$ be the connected component of the identity in $\tilde{L}$ and let $T = \tilde{R} \cap \tilde{L}_0$. There is a natural injection from $\tilde{R}/T$ into $\tilde{L}/\tilde{L}_0$, hence $\tilde{R}/T$ is finite. By Lemma 3.4, it suffices to show that the induced measure $\tilde{\mu}_T$ on T has non-constant continuous positive harmonic functions. As T is a closed subgroup of exponential growth of $\tilde{L}_0$, there exists an homomorphism ψ from $\tilde{L}_0$ to $Gl(d, \mathbf{R})$ such that the closure H of $\psi(T)$ is of exponential growth (*see* Lemma 3.10). The image $\eta = \psi(\tilde{\mu}_T)$ is centered, has a moment of order 3, and is adapted to H (by Proposition 3.3 and Lemma 3.1). Once again, it suffices to prove that η has non-constant continuous positive harmonic functions.

If H is is non-amenable, then there are positive (and bounded) η-harmonic functions by Theorem 1.1. When H is amenable, it has a normal subgroup D of finite index in the class **C**, by Proposition 3.9. In other words, there is a continuous homomorphism ξ from D to an affine similarity group $\mathbf{S}_m$ such that the closure $\tilde{D}$ of $\xi(D)$ is of exponential growth. Let η_D be the induced measure by η on D and let $\tilde{\eta}$ be the image of η_D by ξ. Then $\tilde{\eta}$, considered as a measure on $\tilde{D}$, is centered, adapted and with a moment of order 3 and it suffices to see that it has non-constant continuous positive harmonic functions. This follows from Theorem 2.3.

Proof of Theorem 1.6. – It follows from the discretization procedure of Lyons and Sullivan [24], Ancona [2], Ballmann and Ledrappier [7], that there exists a probability measure μ on Γ such that any positive μ-harmonic function on Γ can be extended to a positive harmonic function on M. The support of μ is Γ. The fact that the covering is co-compact implies that μ has an exponential moment (*see* Ancona [2], p. 67 or Kaimanovitch [21]), and μ can be chosen symmetric (*see* Ballmann and Ledrappier [7]) and thus centered. If M is of exponential growth, then Γ is also of exponential growth (*see* Milnor [25]), thus the existence of positive harmonic functions follows from Theorem 1.4. When M is not of exponential growth then it is proved in Guivarc'h [15] that the positive harmonic functions are constant (notice that Γ is of polynomial growth by [13]).

Under the hypotheses of Theorem 1.6, the bounded harmonic functions are constant if and only if there is no spectral gap. Indeed, it is shown in Brooks [9] that there is a spectral gap if and only if Γ is non-amenable. In this case there exist non-constant bounded harmonic functions on Γ, and thus on M. If Γ is amenable, then it is a finite extension of a polycyclic group (*see* Guivarc'h [13]), and the Liouville property of M in that setting is proved in Ancona [2] and Kaimanovitch [21].

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