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ENRIQUE D. ANDJEL
THOMAS S. MOUNTFORD
ROBERTO H. SCHONMANN

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Equivalence of exponential decay rates for bootstrap percolation like cellular automata

by

Enrique D. ANDJEL (1)

Université de Provence, Mathématiques,
case X, place Victor-Hugo, 13331 Marseille, France.

Thomas S. MOUNTFORD (2)

Mathematics Department, University of California,
Los Angeles, CA 90024, U.S.A.

and

Roberto H. SCHONMANN (3)

Mathematics Department, University of California,
Los Angeles, CA 90024, U.S.A.

ABSTRACT. – We consider a class of cellular automata on the state space $\{0, 1\}^{\mathbb{Z}^d}$, evolving in discrete time and starting from uniform product measures. In this class 1's do not change and the interaction is attractive, translation invariant and occurs among nearest neighbors (*see* [Sch1]). We prove the equivalence of the exponential decay rates related to (i) the probability of the origin being still vacant after a long time and to (ii) the probability that the core of a large finite block is not completely occupied by the dynamics restricted to this block. For the model in which 0's change to 1 when they have at least one neighboring 1 in each coordinate direction a further bound between exponential decay rates is also obtained, which in

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combination with results in [Mou] allows us to compute the exponential decay rate related to (i) for these models as being exactly $-2\log(1-p)$, where p is the initial density of 1's. In particular the exponent ν related to (i) is equal to 1 for these models. This improves a result in [And].

Key words: Cellular automata, bootstrap percolation, critical behavior, exponential rates.

RÉSUMÉ. – On considère une classe d'automates cellulaires dans l'espace $\{0, 1\}^{\mathbb{Z}^d}$, qui évoluent en temps discret et ont comme distribution initiale une mesure produit invariante par translations. Dans cette classe, l'état 1 est stable et l'interaction est monotone, invariante par translations et à plus proches voisins. On démontre l'équivalence des paramètres qui mesurent la décroissance exponentielle de (i) la probabilité que l'origine ne soit pas occupée après un long intervalle de temps et (ii) la probabilité que le noyau d'un grand hypercube ne soit pas entièrement occupé par la dynamique restreinte à cet hypercube. Pour les modèles dans lesquels un 0 devient un 1 s'il a au moins un 1 voisin dans chacune des directions parallèles aux axes de coordonnées, on démontre une seconde relation entre les paramètres de décroissance exponentielle, qui combinée aux résultats de [Mou] nous permet de conclure que pour ces modèles le paramètre lié à (i) est égal à $-2\log(1-p)$, où p est la densité initiale des 1. En particulier, l'exposant ν lié à (i) est égal à 1dans ces modèles. Ceci améliore un résultat de [And].

1. MODELS AND RESULTS

In this paper we continue to investigate the behavior of a class of cellular automata considered in [Sch1] and [Sch2], which generalizes in a natural direction the bootstrap percolation models. As in previous papers we are mostly concerned with the critical behavior of such models. The motivation for this projet comes from the increasing interest on cellular automata and especially from the surprising behavior of the bootstrap percolation and related models (*see* for instance the references to this paper and references therein; [Adl] is a recent review on bootstrap percolation from a physicist's point of view). A subset of this class of models had been introduced before in [AA], who called them "diffusion percolation".

The models considered in this paper are defined on the lattice $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers, and $d = 1, 2, \dots$ is the space dimensionality. The systems evolve in discrete time $t = 0, 1, 2, \dots$ To each element (site) of $\mathbb{Z}^d$,

x, we associate at each instant of time t, a random variable $\eta_t(x)$ which can assume the values 0 and 1. We say that the site x is empty (resp. occupied) at time t if $\eta_t(x) = 0$ (resp. 1). $\eta_t \in \{0, 1\}^{\mathbb{Z}^d}$ will represent the function that to $x \in \mathbb{Z}^d$ associates $\eta_t(x)$. Elements of $\{0, 1\}^{\mathbb{Z}^d}$ are called configurations. The system will be always started, at $t = 0$, from a translation invariant product random field, *i.e.*, the random variables $\eta_0(x)$, $x \in \mathbb{Z}^d$ are i.i.d. with $P(\eta_0(x) = 0) = 1 - p$, $P(\eta_0(x) = 1) = p$; $p \in [0, 1]$ is called the initial density. The system evolves then according to a deterministic rule satisfying the following conditions:

(i) $\eta_{t+1}(x) = 1$ if $\eta_t(x) = 1$ (1's are stable) or if η_t belongs to a certain set $\mathcal{C}_x$ (the sets $\mathcal{C}_x$, $x \in \mathbb{Z}^d$, specify the models).

(ii) $\eta_{t+1}(x) = 0$ if $\eta_t(x) = 0$ and η_t does not belong to the set $\mathcal{C}_x$.

In this paper the sets $\mathcal{C}_x$ will always obey several restrictions:

a) Translation invariance. – We define $\theta_x \eta$ by $(\theta_x \eta)(y) = \eta(y - x)$ and we assume that $\mathcal{C}_x = \{\eta : \theta_{-x} \eta \in \mathcal{C}_0\}$. In particular the set $\mathcal{C}_0 =: \mathcal{C}$ specifies the model.

b) Nearest neighbor interaction. – We define $\mathcal{N}_x = \{y \in \mathbb{Z}^d : \|x - y\| = 1\}$, where $\|\cdot\|$ is the $l_1 - norm$ on $\mathbb{Z}^d$ ($\|x\| = |x_1| + \cdots + |x_d|$). We assume that if $\eta \in \mathcal{C}_x$ and $\eta(y) = \eta'(y)$ for every $y \in \mathcal{N}_x$, then $\eta' \in \mathcal{C}_x$. Informally, each site is influenced only by its nearest neighbors at each step of the evolution.

c) Attractiveness. – We define on $\{0, 1\}^{\mathbb{Z}^d}$ the partial order given by $\eta \leq \eta'$ if $\eta(x) \leq \eta'(x)$ for every $x \in \mathbb{Z}^d$. We assume that if $\eta \in \mathcal{C}_x$ and $\eta \leq \eta'$, then $\eta' \in \mathcal{C}_x$. Informally, the more 1's we have at time t, the more 1's we will have at time $t + 1$.

The set $\mathcal{C}$ may be specified by a set $\mathcal{D}$ of subsets of $\mathcal{N} := \mathcal{N}_0$ via

$$\mathcal{D} = \{A \subset \mathcal{N} : \eta(x) = 1 \text{ for all } x \in A \Rightarrow \eta \in \mathcal{C}\}.$$

Observe that, by attractiveness, if $A \in \mathcal{D}$ and $A \subset B$, then $B \in \mathcal{D}$.

In order to give some examples we define the elements of $\mathbb{Z}^d$ $e_1 = (1, 0, 0, \ldots, 0), \ldots, e_d = (0, 0, 0, \ldots, 1)$ and denote by $|A|$ the cardinality of the set A.

Examples:

1) *Bootstrap percolation.* – Take $l \in \{0, \ldots, 2d\}$ and set

$$\mathcal{D} = \{A \subset \mathcal{N} : |A| \geq l\}.$$

A 0 becomes a 1 if at least l of its neighbors are 1's.

2) *The basic model.* – This is the particular case of bootstrap percolation with $l = d$.

3) *The modified basic model*

$$\mathcal{D} = \{ A \subset \mathcal{N} : A \cap \{ -e_i, +e_i \} \neq \emptyset \text{ for } i = 1, \ldots, d \},$$

In this model a 0 becomes a 1 if in each one of the d coordinate directions it has at least one neighbor which is a 1.

4) *Oriented models.* – Take $(a_1, \ldots, a_d) \in \{ -1, +1 \}^d$. For each one these 2^d choices we have one of the oriented models defined by

$$\mathcal{D} = \{ A \subset \mathcal{N} : \{ a_1 e_1, a_2 e_2, \ldots, a_d e_d \} \subset A \}.$$

In case $a_i = +1$, for $i = 1, \ldots, d$, we call the model the *basic oriented model.*

Given two models defined respectively by $\mathcal{D}_1$ and $\mathcal{D}_2$, we say that the latter dominates the former if $\mathcal{D}_1 \subset \mathcal{D}_2$. Informally, if a 0 becomes a 1 in the former, the same occurs in the latter. The following statements are clearly true. The bootstrap percolation model with $l = l_1$, dominates the one with $l = l_2$ if $l_1 \leq l_2$. The basic model dominates the modified basic model and this one dominates all the oriented models.

Once we have specified the dimension d, the set $\mathcal{C}$ and the initial density p, we denote by $P_p(\cdot)$ the probability measure corresponding to the process $(\eta_t)_{t \geq 0}$.

Define

$$T = \inf \{ t \geq 0 : \eta_t(0) = 1 \},$$

$$\gamma_T(p) = \sup \{ \gamma \geq 0 : \text{there exists } C < \infty \text{ such that } P_p(T > t) \leq C e^{-\gamma t} \text{ for all } t > 0 \},$$

and

$$\pi_c = \inf \{ p \in [0, 1] : \gamma_T(p) > 0 \}.$$

In Section V of [Sch1] it was proven that when $\pi_c < 1$, then

$$\gamma_T(\pi_c) = 0$$

[when $\pi_c = 1$ then $\gamma_T(\pi_c) = +\infty$] and that for the modified basic model (and hence for the bootstrap percolation models with $l \leq d$) $\pi_c = 0$. On the other hand for the oriented models $0 < \pi_c < 1$, and for bootstrap percolation models with $l > d$, $\pi_c = 1$. When $\pi_c < 1$, the critical exponent ν is defined by

$$\nu = \lim_{p \searrow \pi_c} \frac{\log \gamma_T(p)}{\log(p - \pi_c)},$$

provided that the limit exists.

Given $\Gamma \subset \mathbb{Z}^d$ we define the dynamics restricted to Γ, $(\eta_t^\Gamma : t = 0, 1, \ldots)$ by changing at $t = 0$ the state of each site $x \in \Gamma^c$ to 0 and also modifying the evolution in Γ^c by keeping this condition forever [*i.e.* for $x \in \Gamma^c$ $\eta_t^\Gamma(x) = 0$ for all t and for $x \in \Gamma$, $\eta_0^\Gamma(x) = \eta_0(x)$ and $\eta_{t+1}^\Gamma(x)$ is obtained from $\eta_t^\Gamma(x)$ by the same rules used for $(\eta_t : t = 0, 1, \ldots)$]. Given $\Gamma_2 \subset \Gamma_1$ we say that "Γ_2 is Γ_1-spanned (by the initial configuration η_0)" if

$$\text{for all } x \in \Gamma_2, \quad \lim_{t \to \infty} \eta_t^{\Gamma_1}(x) = 1.$$

If Γ is Γ-spanned, we say as in [AL] that "Γ is internally spanned".

Consider the cubes

$$Q_N = \{x \in \mathbb{Z}^d : |x_i| \leq N \text{ for } i = 1, \ldots, d\}.$$

Define

$$S(N, p) = P_p(Q_N \text{ is } Q_{2N}\text{-spanned}),$$

and

$$\gamma_S(p) = \sup\{\gamma \geq 0 : \text{there exists } C < \infty \text{ such that } 1 - S(N, p) \leq C e^{-\gamma N} \text{ for all } N > 0\}.$$

From arguments in Section V of [Sch1] it follows that $\gamma_S(p) > 0 \Leftrightarrow \gamma_T(p) > 0$. We will strengthen this result by showing

THEOREM 1. – *For all the models in the class defined above and all* $0 \leq p \leq 1$

$$\frac{1}{2(d+1)} \gamma_S(p) \leq \gamma_T(p) \leq \gamma_S(p).$$

Define a critical exponent related to $\gamma_S(p)$ by

$$\nu_S = \lim_{p \searrow \pi_c} \frac{\log \gamma_S(p)}{\log(p - \pi_c)},$$

provided that the limit exists. Theorem 1 implies that

$$\nu = \nu_S$$

in the sense that either both exist and are identical or both do not exist.

To relate this fact to the results in [And] and [Mou] we define also

$$R(N, p) = P_p (Q_N \text{ is internally spanned}),$$

and

$$\gamma_R(p) = \sup \{ \gamma \geq 0 \;:\; \text{there exists } C < \infty \text{ such that } 1 - R(N, p) \leq C e^{-\gamma N} \text{ for all } N > 0 \}.$$

Remark. – The cube Q_N has side $2N + 1$. For this reason $\gamma_R(p) = 2\bar{\gamma}(p)$, where $\bar{\gamma}(p)$ is the corresponding exponential decay rate considered in [And].

Clearly, by attractiveness, $R(N, p) \leq S(N, p)$ and hence $\gamma_R(p) \leq \gamma_S(p)$. But observe that for models for which the orientation is relevant, we may have values of p [possibly every $p \in (0, 1)$] for which $\gamma_R(p) = 0 < \gamma_S(p)$. For this reason, when one considers the whole class of models above, $S(N, p)$ is in general more useful than $R(N, p)$. On the other hand for the modified basic model in $d = 2$ (for which $\pi_c = 0$), [And] proved that for each fixed $p > 0$, $R(N, p)$ converges to 1 as $N \to \infty$ so fast that

$$\bar{\nu} := \lim_{p \searrow 0} \frac{\log \gamma_R(p)}{\log p} = 1,$$

and [Mou] later proved the much stronger result which states that in every dimension for the same modified basic model

$$\gamma_R(p) = -2 \log (1 - p).$$

Using this result and the simple observation that

$$1 - S(N, p) \geq P_p (\eta_0(x) = 0 \text{ for each } x \in \{-2N, \ldots, 2N\} \times \{0\}) = (1 - p)^{4N+1},$$

on obtains from Theorem 1, and the already observed inequality $\gamma_R(p) \leq \gamma_S(p)$, the following consequence.

COROLLARY 1. – *For the modified basic model in every dimension*

$$\nu = \bar{\nu} = \nu_S = 1.$$

Previously the best result on ν for this model had been obtained in [And], who proved that in dimension 2 if ν exists then $1 \leq \nu \leq 2$. Motivated by the explicit computation of $\gamma_R(p)$ for the modified basic model in [Mou], we will strengthen also Theorem 1 for this model and show

THEOREM 2. – *For the modified basic model in any dimension and every* $0 \leq p \leq 1$

$$\gamma_T(p) \geq \gamma_R(p).$$

This inequality together with the trivial bound ($\lfloor \cdot \rfloor$ denotes integer part)

$$P_p(T > t) \geq P_p(\eta_0(x) = 0 \text{ for all } x \in [-t, t] \times \{0\}) = (1-p)^{2\lfloor t \rfloor + 1}$$

implies that

COROLLARY 2. – *For the modified basic model in every dimension*

$$\gamma_T(p) = -2 \log(1-p).$$

Our results for the modified basic model say that besides of the fact that the probability that the origin is still vacant at time t goes to 0 exponentially fast with t, also the corresponding rate of exponential decay is not too small even when p is small, since it behaves asymptotically as $2p$ when $p \to 0$. This should be contrasted with an important result in [AL], according to which in $d = 2$ there are $0 < C_1 < C_2 < \infty$ such that for every $\varepsilon > 0$

$$\exp(C_1/p) < \inf\{t : P_p(T > t) < \varepsilon\} < (\exp C_2/p),$$

so that for small p the system needs an extremely large time for the origin to be likely to become occupied. Our results on the other hand (and for this matter also the weaker result $\nu \leq 2$ in [And] in the $d = 2$ case) say that once the time is so long that the origin is very likely to be occupied, waiting even longer makes this probability go to 1 relatively fast.

Returning to the full class of models to which Theorem 1 applies, we observe that still another natural probability to look at, as discussed in Section 5 of [AL], is

$$M(N, p) = P_p(\eta_t^{Q_N}(0) = 1 \text{ for some } t).$$

Define

$$\gamma_M(p) = \sup\{\gamma \geq 0 : \text{there exists } C < \infty \text{ such that } 1 - M(N, p) \leq C e^{-\gamma N} \text{ for all } N > 0\}.$$

And

$$\nu_M(p) := \lim_{p \searrow \pi_c} \frac{\log \gamma_M(p)}{\log(p - \pi_c)},$$

provided that the limit exists. From attractiveness translation invariance and the observation that "effects travel with maximum speed 1", since the interaction is among nearest neighbors, one obtains as in relation (III.5) and (V.5) in [Sch1] the following

PROPOSITION 1. – *For all the models above and all* $0 \leq p \leq 1$,

$$1 - M(2N, p) \leq 1 - S(N, p) \leq (2N + 1)^d (1 - M(N, p)).$$

And as a consequence $\gamma_M(p) \leq \gamma_S(p) \leq 2\gamma_M(p)$ and $\nu_M = \nu_S$.

2. PROOF OF THEOREM 1

The second inequality follows easily from relation (V.5) in [Sch1] (obtained from the observation that "effects travel with maximum speed 1", since the interaction is among nearest neighbors): If $\gamma_T(p) = 0$ there is nothing to prove, otherwise, given $0 < \gamma_1 < \gamma_2 < \gamma_T(p)$ there are C_1, $C_2 < \infty$ such that

$$\begin{aligned} 1 - S(N, p) &\leq |Q_N| \cdot P_p(T > N) \\ &\leq (2N + 1)^d C_2 e^{-\gamma_2 N} \\ &\leq C_1 e^{-\gamma_1 N}. \end{aligned}$$

So

$$\gamma_T(p) \leq \gamma_S(p).$$

To prove the other inequality, observe first that if the model does not dominate any oriented model, then by Proposition V.1 in [Sch1] $\pi_c = 1$ and $\gamma_T(p) = 0$ for all $p \in [0, 1)$, while $\gamma_T(1) = +\infty$. On the other hand, by the same simple argument which proves this proposition (a cube of 0's is never eaten),

$$1 - S(N, p) \geq P_p(\eta_0(x) = 0 \text{ for all } x \in Q_1) = (1-p)^{3^d},$$

so that $\gamma_S(p) = \gamma_T(p)$ for all p in this case.

Suppose from this moment on that the model dominates an oriented model, the basic oriented model say. Consider the following cubes, where $N \in \mathbb{Z}_+$, $k \in \mathbb{Z}^d$

$$\begin{aligned} Q_N &= \{x \in \mathbb{Z}^d : |x_i| \leq N, i = 1, \ldots, d\}, \\ Q_{N,k} &= \{x \in \mathbb{Z}^d : x - (2N+1)k \in Q_N\}, \\ \bar{Q}_{N,k} &= \{x \in \mathbb{Z}^d : x - (2N+1)k \in Q_{2N}\}. \end{aligned}$$

Observe that the $Q_{N,k}$ have side $2N+1$ and for $k \in \mathbb{Z}^d$ form a partition of $\mathbb{Z}^d$. The $\bar{Q}_{N,k}$ have sides $4N+1$, and intersect each other when the k's are close. But if $\|k' - k''\| \geq d+1$ (where $\|\cdot\|$ is the l_1-norm), then

$$\bar{Q}_{N,k'} \cap \bar{Q}_{N,k''} = \varnothing.$$

For each k, $Q_{N,k}$ and $\bar{Q}_{N,k}$ are centered on the same point $(2N+1)k$.

In what follows take a fixed N (to be chosen later). Say that the block $\bar{Q}_{N,k}$ is "good" if $Q_{N,k}$ is $\bar{Q}_{N,k}$-spanned by the initial configuration. Think of $k \in \mathbb{Z}^d$ as sites of a renormalized lattice. For $k \in \mathbb{Z}^d$, let $\beta(k)$ be the indicator function of the event $\bar{Q}_{N,k}$ is good. The random field $\beta\{(k) : k \in \mathbb{Z}^d\}$ clearly has a finite range of dependency (which is d in the l_1-norm). It is translation invariant and for all k

$$P_p(\beta(k) = 1) = S(N, p).$$

Now we have to recall some notions related to oriented site percolation. We say that $(x^{(1)}, \ldots, x^{(n)})$ is an oriented path in $\mathbb{Z}^d$ if $x^{(i)} \in \mathbb{Z}^d$, $i = 1, \ldots, n$ and $n = 1$ or $x^{(i+1)} - x^{(i)} \in \{e_1, \ldots, e_d\}$, $i = 1, \ldots, n-1$. Given a random field $(\alpha(x) : x \in \mathbb{Z}^d)$, where $\alpha(x) \in \{0, 1\}$ we say that the oriented vacant cluster of the origin w.r.t. α is the random set

$$\begin{aligned} C_\alpha^v = \{ y \in \mathbb{Z}^d :\ & \text{there is an oriented path } (x^{(1)}, \ldots, x^{(n)}) \text{ such that} \\ & 0 = x^{(1)}, y = x^{(n)} \text{ and } \alpha(x^{(i)}) = 0 \text{ for } i = 1, \ldots, n\} \end{aligned}$$

(If $\alpha(0) = 1$ then $C_\alpha^v = \varnothing$.) The range of C_α^v is defined as 0 if $C_\alpha^v = \varnothing$ and otherwise as

$$\begin{aligned} A_\alpha^v &= \sup\left\{ 1 + \sum_{i=1}^{d} |y_i| \; : \; y \in C_\alpha^v \right\} \\ &= 1 + \sup\{ \|y\| \; : \; y \in C_\alpha^v \} \end{aligned}$$

Given now $s > 0$, consider the block construction above, with $N = \lfloor s^\delta \rfloor$, where δ is arbitrary except for the restrictions $0 < \delta < 1/d$. Given $\varepsilon > 0$, let F_ε be the event that for the corresponding random field β on the normalized lattice we have

$$A_\beta^v > s(1 - 2\varepsilon)/(2N + 1).$$

At time $(4N + 1)^d$ ($\leq 5^d s^{\delta d}$ for large s) all the $Q_{N,k}$ which correspond to a k with $\beta(k) = 1$ will be completely occupied, because the dynamics restricted to a finite set Γ has to reach a final configuration in a time $\leq |\Gamma|$. Consider the vacant cluster of the origin on the originl (not the normalized) lattice w.r.t. the configuration $\eta_{(4N+1)^d} =: \zeta$. It is clear that

$$C_\zeta^v \subset \bigcup_{k \in C_\beta^v} Q_{N,k}.$$

From the facts that $\|(2N+1)k\| = (2N+1) \cdot \|k\|$ and $\sup\{ \|y - (2N + 1)k\| \; : \; y \in Q_{N,k} \} = dN$, it follows from the triangular inequality that

$$A_\zeta^v \leq A_\beta^v \cdot (2N + 1) + dN.$$

So for large s on $(F_\varepsilon)^c$, $A_\zeta^v \leq s(1 - 2\varepsilon) + dN \leq s(1 - \varepsilon)$. Compare the model we are considering with one that has the same dynamics up to time $(4N + 1)^d$ and afterwards evolves as the basic oriented model. Since we are supposing that our model dominates the basic oriented model, it follows that for arbitrary t

$$P_p(T > 5^d s^{d\delta} + t) \leq P_p(A_\zeta^v > t).$$

(For a formal argument *see* Proposition IV of [Sch1].) Therefore for large s

$$P_p(T > 5^d s^{d\delta} + s(1 - \varepsilon)) \leq P_p(F_\varepsilon).$$

Set $m = \lfloor s(1-2\varepsilon)/(2N+1)\rfloor$. if F_ε occurs, then in the renormalized lattice there is an oriented path $(k^{(1)}, \dots, k^{(m)})$, with $k^{(1)} = 0$ and $\beta(k^{(i)}) = 0$, $i = 1, \dots, m$. The random variables $\beta(k^{(1)})$, $\beta(k^{(1+(d+1))})$, $\beta(k^{(1+2(d+1))})$, $\dots$, $\beta(k^{(1+\lfloor(m-1)/(d+1)\rfloor\cdot(d+1))})$ are mutually independent and since there are d^{m-1} oriented paths starting at a given site and crossing $m-1$ other sites, it follows that for large s

$$\begin{aligned} P_p(F_\varepsilon) &\le d^{m-1}(P(\beta(0)=0))^{\lfloor(m-1)/(d+1)\rfloor} \\ &\le d^{s^{1-\delta}}(1-S(\lfloor s^\delta\rfloor, p))^{(1-3\varepsilon)\, s^{1-\delta}/2\,(d+1)}. \end{aligned}$$

Suppose that $\gamma_S(p) > 0$ since otherwise there is nothing to be proven. Given now $0 < \gamma < \gamma_S(p)$ there exists C_1, $C_2 < \infty$ such that

$$1 - S(\lfloor s^\delta\rfloor, p) \le C_1\, e^{-\gamma\lfloor s^\delta\rfloor} \le C_2\, e^{-\gamma s^\delta}.$$

From the last three displayed inequalities and the fact that $\delta < 1/d$ it follows that for large s

$$P_p(T > s) \le d^{s^{1-\delta}}\,(C_2 \vee 1)^{s^{1-\delta}} \exp(-\gamma\, s\,(1-3\varepsilon)/2\,(d+1)).$$

Now since $\delta > 0$, there exists $C_3 < \infty$ such that

$$P_p(T > s) \le C_3 \exp(-\gamma\, s\,(1-4\varepsilon)/2\,(d+1)).$$

Therefore $\gamma_T(p) \ge ((1-5\varepsilon)/2\,(d+1))\,\gamma_S(p)$. And since $\varepsilon > 0$ is arbitrary

$$\gamma_T(p) \ge \frac{1}{2\,(d+1)}\,\gamma_S(p).$$

3. PROOF OF THEOREM 2

The proof of Theorem 2 is an adaptation of that of Theorem 1. For $k \in \mathbb{Z}^d$ let $\varphi(k)$ be the indicator function of the event $\{\, Q_{N,k}$ is internally spanned$\}$. The random variables $\{\varphi(k) : k \in \mathbb{Z}^d\}$ are clearly mutually independent and identically distributed, with

$$P_p(\varphi(k) = 1) = R(N, p).$$

Given a random field $\{\alpha(k) : k \in \mathbb{Z}^d\}$, define $\alpha'(k) = \alpha(-k)$ for each $k \in \mathbb{Z}^d$, and

$$A'^v_\alpha = A^v_{\alpha'}.$$

A'^v_α can be thought of as the range of the vacant oriented cluster of the origin w.r.t. the random field α, when the orientation is the opposite of the one considered before.

Choose as before $N = \lfloor s^\delta \rfloor$ with $0 < \delta < 1/d$ and define $\xi := \eta_{(2N+1)^d}$. By time $(2N+1)^d$ ($\leq 3^d s^{\delta d}$ for large s) all the $Q_{N,k}$ which correspond to a k with $\varphi(k) = 1$ will be completely occupied. Since the modified basic model dominates all the oriented models, it follows from Proposition IV.1 of [Sch1] that for arbitrary $t > 0$

$$P_p(T > 3^d s^{\delta d} + t) \leq P_p(A^v_\xi > t,\ A'^v_\xi > t).$$

But observe that for $t > 0$

$$\begin{aligned} &P_p(A^v_\xi > t,\ A'^v_\xi > t)\\ &\quad \leq P_p(A^v_\varphi > (t - dN)/(2N+1),\ A'^v_\varphi > (t - dN)/(2N+1))\\ &\quad = (P_p(\varphi(0) = 0))^{-1} \cdot P_p(A^v_\varphi > (t - dN)/(2N+1))\\ &\qquad \times P(A'^v_\varphi > (t - dN)/(2N+1)). \end{aligned}$$

The factor $(P_p(\varphi(0) = 0))^{-1}$ is irrelevant for the rest of the argument [where t will be chosen as $s(1 - \varepsilon)$] since it is of order $\exp(-\gamma_R s^{\delta d})$ and $\delta < 1/d$.

From this point on, one can proceed as in the proof of Theorem 1, but with obvious simplifications due to the independence of the $\{\varphi(k)\}$. The presence of the two terms above (those with A^v_φ and A'^v_φ) gives raise to an extra factor 2, crucial to make the inequality between $\gamma_T(p)$ and $\gamma_R(p)$ sharp.

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