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Interacting random walk in a dynamical random environment I. Decay of correlations

by

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ABSTRACT. – We consider a random walk X_t, $t \in \mathbb{Z}_+$ and a dynamical random field $\xi_t(x)$, $x \in \mathbb{Z}^\nu$ $(t \in \mathbb{Z}_+)$ in mutual interaction with each other. The interaction is small, and the model is a perturbation of an unperturbed model in which walk and field evolve independently, the walk according to i.i.d. finite range jumps, and the field independently at each site $x \in \mathbb{Z}^\nu$, according to an ergodic Markov chain. Our main result in Part I concerns the asymptotics of temporal correlations of the random field, as seen in a fixed frame of reference. We prove that it has a "long time tail" falling off as an inverse power of t. In Part II we obtain results on temporal correlation in a frame of reference moving with the walk.

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RÉSUMÉ. – On considère un chemin aléatoire X_t, $t \in \mathbb{Z}_+$, et un milieu aléatoire dynamique $\xi_t(x)$, $x \in \mathbb{Z}^\nu\,(t \in \mathbb{Z}_+)$ en interaction mutuelle. L'interaction est petite, et le modèle est une perturbation d'un modèle imperturbé, dans lequel le chemin aléatoire et le milieu évoluent d'une façon indépendente pour chaque $x \in \mathbb{Z}^\nu$, selon une chaîne de Markov ergodique. Le résultat principal de la partie I regarde l'asymptotique des corrélations temporelles du milieu dans un repère fixé. On trouve que la décroissance dans le temps est donnée par une puissance négative de t. Dans la partie II on obtient des résultats sur les corrélations temporelles du milieu dans un repère qui se déplace avec la particule.

1. INTRODUCTION

The expression "random walk in random environment" can have several meanings. Most of the work is devoted to random motions in a fixed realization of the random environment. For this class of problems we refer the reader to the classical paper of Fisher [1], and to the paper [2] (and references therein) for recent rigorous results.

We are interested here in random walks in a dynamical environment. Except for the class of models that we consider in the present work, which were introduced in [4] and [5], there are, up to now, few results for random walks in random dynamical environments. An example is the result in [3].

The models that we consider here correspond to a particle performing a random walk on the lattice $\mathbb{Z}^\nu$ and interacting with a medium ("environment") consisting of a random field in evolution. The interaction is a mutual influence, *i. e.*, the random walk transition probabilities depend on the field and the evolution of the field depends on the position of the particle. The models are obtained as a modification of some "unperturbed" model, in which the particle and the environment evolve independently. This unperturbed model has, in addition, the following properties: the random walk makes jumps according to some homogeneous transition kernel with finite range, and the environment is an i.i.d. evolution of copies of some egordic Markov chain associated with the transitions at each site $x \in \mathbb{Z}^\nu$. The modification consists in adding a local interaction.

In the papers [4] and [5] we were mainly concerned with the proof of the central limit theorem for the displacement of the particle. Similar results obtained by other authors for models of random walks in dynamical environments are quoted in the reference lis ot [4].

In the present work we focus on the description of the influence of the walk on the environment, *i. e.*, on how the distribution of the random field is changed by the interaction with the particle. One can consider the problem in a fixed frame of reference (Part I), or in a "moving" one (Part II), *i. e.* in a reference frame that moves with the random walk. In the first case one would like to describe the relaxation of the system in time, as the random walk goes away from any finite region. In the second case one would like to know whether the system eventually reaches an invariant distribution for the environment around the position of the particle.

For a better understanding of the results and of methods used in the proofs we need a brief description of the model. Precise definitions can be found in Section 2.

As in [4] and [5], the particle position is given by a random walk $\{X_t : t \in \mathbb{Z}_+\}$, and the environments is described by a random field $\{\xi_t(x) : x \in \mathbb{Z}^\nu,\ t \in \mathbb{Z}_+\}$. At each site $x \in \mathbb{Z}^\nu$ the environment $\xi_t(x)$ takes values in some finite set S. The unperturbed model is described as follows. The particle performs an independent homogeneous random walk with transition probabilities $\{P_0(x, x+y) = P_0(y) : x, y \in \mathbb{Z}^\nu\}$. The evolution of the environment at different sites is independent and given by a Markov chain, with transition probabilities $\{q_0(s', s) : s', s \in S\}$, the same for all sites. We denote by Q_0 the corresponding stochastic matrix, and assume that the Markov chain defined by Q_0 is ergodic, with stationary measure π_0. Hence the unperturbed model admits a stationary measure $\Pi_0 = \pi_0^{\mathbb{Z}^\nu}$ for the process $\xi_t \equiv \{\xi_t(x) : x \in \mathbb{Z}^\nu\}$.

The "perturbed" or "interacting" model is a Markov process for which are assume conditional independence, *i. e.*, given the environment at time t, the transition probabilities at time t for the particle and for the environment at different sites are independent. We define the new transition probabilities by adding a term of order ϵ to the unperturbed ones, where ϵ is a small parameter:

$$\mathbb{P}(X_{t+1} = x + y | X_t = x,\ \xi_t = \bar{\xi}) = P_0(y) + \varepsilon\, c(y;\ \bar{\xi}(x)),$$

$$\mathbb{P}(\xi_{t+1}(z) = s | X_t = x,\ \xi_t = \bar{\xi}) = \begin{cases} q_0(\bar{\xi}(z), s) & z \neq x \\ q_0(\bar{\xi}(x), s) + \varepsilon\, \bar{q}(\bar{\xi}(x), s) & z = x. \end{cases}$$

This is the general form if the interaction is supposed to be localized at the site X_t where the particle is located at any given time t.

Our methods of proof are based on the analysis of the spectrum of the "transfer matrix" $\mathcal{T}$, the stochastic operator associated to the Markov process of the interacting model. $\mathcal{T}$ is considered as an operator acting on the Hilbert space $\hat{\mathcal{H}}$ of the square summable functions of ξ and z, with respect to the natural reference measure (the product of the counting measure in z and the measure Π_0). The main condition that we impose is that the perturbation is so small that the spectral gap, which at $\varepsilon = 0$ separates the leading invariant (with respect to $\mathcal{T}$) subspace ("leading" in the sense of the absolute values of the spectrum of $\mathcal{T}$) from the rest, does not vanish. The technical difficulties are connected with the construction of the leading invariant subspace for $\varepsilon > 0$, and with the analysis of the resolvent of $\mathcal{T}$.

Once the leading invariant subspace $\mathcal{H}_{le}$, is singled out, the restriction of $\mathcal{T}$ to $\mathcal{H}_{le}$, is reminiscent of the one-particle operator of Quantum Mechanics. In the case of the moving reference frame (Part II) and for models with two particles it reminds instead the two-particle operator. It is then not surprising that the constructions and the methods of analysis of the resolvent in our papers have close analogies in Quantum Mechanics (*see*, for example, [6] and [7]).

One may regret that the techniques that we use sometimes leave little room for probabilistic intuition. The main problem lies in the fact that from a probabilistic point of view we deal with very complicated objects. Particles interacting with an environment can be understood in physical terms as "dressed" or "quasi-"particles, as they are accompanied by a "cloud" of excitations of the medium. The cloud extends, as time goes to infinity, on the whole space $\mathbb{Z}^{\nu}$, although its intensity decreases exponentially with the distance. In our language, however, one can describe this complicated object in a rather simple way, by a "change of coordinates" in the space $\hat{\mathcal{H}}$, which changes to leading invariant subspace for $\varepsilon = 0$ into the new, or "perturbed" leading invariant subspace. We believe that methods of this kind, which "disentagle" the interaction, can be useful in the study of other types of many component stochastic models.

Similar ideas for "disentangling" the interaction are used in [4], though the setting and the techniques are rather different. On the other hand [5] is closely related to the present work and we will frequently refer to the results there.

For the present paper the main result concerns the case when the drift of the (perturbed) random walk is 0. We prove that the averages of local functions of the environment at time t tend to their equilibrium value (*i. e.*, with respect to the invariant measure Π_0) with a speed $\mathcal{O}\left(\frac{1}{t^{\nu/2}}\right)$, as $t \to \infty$. We also prove that the two-point correlation, between functions $f_1(\xi_0(x_1))$ and $f_2(\xi_t(x_2))$, behaves asymptotically as $\frac{C}{t^{(\nu/2)+1}}$, where C is a constant depending on f_1, f_2, x_1,x_2, on the initial position X_0 of the particle and on the initial distribution of the field Π. We further prove that, if the random walk has a nonzero drift, then the correlations decay exponentially fast in time. In the unperturbed reference model, of course, correlations are zero if $x_1 \neq x_2$ and decay exponentially in t if $x_1 = x_2$.

The results of the present paper provide a clear and simple example of a well kown phenomenon, namely that by adding a conserved component to a field with strong stochastic properties one forces a power-law decay of the temporal correlations (*see*, for example [8]).

The plan of the paper is as follows. In Section 2 we define the model and formulate the main results. In Section 3 we give the proofs. The Appendix A contains the proof of some technical facts, while in the Appendix B we study the constants appearing in the asymptotics of the correlations.

We observe in conclusion that our methods can allow to deal with models with a general local dependences, e. g., for which the functions c and $\bar{q}$ depend on the values $\xi_t(x+y)$ for $|y| < r$, where r is some positive "interaction range". One can also consider models with two particles, as in the paper [5], or more, which interact with the environment and with each other. Extensions in these directions will be the object of further work.

2. DEFINITION OF THE PROBLEM AND FORMULATION OF THE MAIN RESULTS

As in [4] and [5] the environment at the discrete time $t \in \mathbb{Z}_+$ is described by a random field $\xi_t \equiv \{\xi_t(x) : x \in \mathbb{Z}^\nu\}$, the single variables $\xi_t(x)$ taking values in a finite set S. We denote by $|S|$ the cardinality of S, and by $\Omega = S^{\mathbb{Z}^\nu}$ the state space of the random field ξ_t. We assign to Ω the topology of pointwise convergence. All measures are supposed to be Borel with respect to this topology. $X_t \in \mathbb{Z}^\nu$, $t \in \mathbb{Z}_+$ will denote the position of the particle performing the random walk.

2.1. Description of the model

Our model is, as in [4], [5], a Markov chain $\{(\xi_t, X_t), t \in \mathbb{Z}_+\}$, with state space $\hat{\Omega} = \Omega \times \mathbb{Z}^\nu$, and conditionally independent transition probabilities:

$$\begin{aligned} &\mathbb{P}(X_t = z, \xi_t \in A | X_{t-1} = x, \xi_{t-1} = \bar{\xi}) \\ &\quad = \mathbb{P}(X_t = z | X_{t-1} = x, \xi_{t-1} = \bar{\xi}) \\ &\qquad \times \mathbb{P}(\xi_t \in A | X_{t-1} = x, \xi_{t-1} = \bar{\xi}). \end{aligned} \tag{2.1}$$

Here $\bar{\xi} \in \Omega$ is a fixed configuration of the field $A \subset \Omega$ is an arbitrary measurable set of configurations of the environment. For the factors on the right-hand side of eq. (2.1) we make the following hypotheses.

2.1A. *Assumptions on the random walk transition probabilities*

I. The random walk transition probabilities are given by

$$\mathbb{P}(X_t = z | X_{t-1} = x, \xi_{t-1} = \bar{\xi}) = P_0(z - x) + \varepsilon\, c(z - x, \bar{\xi}(x)), \tag{2.2}$$

where P_0 is a probability distribution on $\mathbb{Z}^\nu$ (which defines the "unperturbed" random walk), and $c(\cdot, \cdot)$ is a function on $\mathbb{Z}^\nu \times S$ such that

$$\sum_{u \in \mathbb{Z}^\nu} c(u, s) = 0, \quad \text{for all } s \in S; \tag{2.3a}$$

$$1 \geqq P_0(u) + \varepsilon\, c(u, s) \geqq 0, \quad \text{for all } w \in \mathbb{Z}^\nu,\ s \in S. \tag{2.3b}$$

In what follows we will consider $c(\cdot, \cdot)$ to be fixed, and ϵ will be subject to the condition of being small enough. In view of the arbitrariness of the function $c(\cdot, \cdot)$ it is not restrictive to assume that $\varepsilon \geqq 0$.

Both P_0 and c are assumed to be of finite range: *i.e.*, there is some $D > 0$ such that

$$P_0(u) = c(u, s) = 0 \qquad \text{for } |u| > D, \qquad s \in S, \tag{2.3c}$$

where $|u|$ is defined as

$$|u| \equiv \sum_{i=1}^{\nu} |u^{(i)}|, \qquad u = (u^{(1)}, \cdots, u^{(\nu)}).$$

We assume furthermore that the characteristic function

$$\tilde{p}_0(\lambda) = \sum_{u \in \mathbb{Z}^\nu} P_0(u)\, e^{i(\lambda, u)}, \qquad \lambda = (\lambda_1, \cdots, \lambda_\nu) \in T^\nu \tag{2.3d}$$

define on the ν-dimensional torus T^ν, is subject to the following conditions:

II. $|\tilde{p}_0(\lambda)| < 1$, for all $\lambda \in T^\nu$, $\lambda \neq 0$.

III. The quadratic form $\sum_{ij} a_{i,j}^0 \lambda_i \lambda_j$ associated to the Taylor expansion

$$\log(\tilde{p}_0(\lambda)) = i \sum b_k^0 \lambda_k - \frac{1}{2} \sum a_{i,j}^0 \lambda_i \lambda_j + \cdots \tag{2.4}$$

in the neighborhood of $\lambda = 0$ is strictly positive for $\lambda \neq 0$.

2.1B. *Assumptions on the field transition probabilities*

IV. The distribution $\mathbb{P}(\xi_t \in \cdot | X_{t-1} = z, \xi_{t-1} = \bar{\xi})$ is a product measure for the random variables $\{\xi_t(x),\ x \in \mathbb{Z}^\nu\}$, each of which is distributed according to the law

$$\begin{aligned} &\mathbb{P}(\xi_t(z) = s | X_{t-1} = x, \xi_{t-1} = \bar{\xi}) \\ &\quad = \begin{cases} q_0(\bar{\xi}(z), s) & \text{if} \quad z \neq x \\ q_1(\bar{\xi}(x), s) & \text{if} \quad z = x, \end{cases} \end{aligned} \tag{2.5a}$$

where $q_0(s', s)$ and $q_1(s', s)\ (s', s \in S)$ are the matrix elements of two stochastic matrices, which we denote by Q_0 and Q_1, respectively. Q_0 defines the unperturbed evolution of the environment, and Q_1, is given by

$$q_1(s', s) = q_0(s', s) + \varepsilon\, \bar{q}(s', s). \tag{2.5b}$$

The condition that Q_1, is a stochastic matrix implies for the elements $\bar{q}(s', s)$ of $\bar{Q}$ that

$$\sum_s \bar{q}(s', s) = 0, \qquad s' \in S. \tag{2.5c}$$

V. The stochastic operator Q_0, acting on a function $f : S \to \mathbb{R}$ as

$$(Q_0 f)(s') = \sum_s q_0(s', s)\, f(s), \tag{2.6a}$$

can be diagonalized. We denote by $\{e_i(s),\ s \in S\}_{i=0}^{|S|-1}$ its eigenvectors (the normalization will be fixed below) and by $\{\mu_i : i = 0, \cdots, |S| - 1\}$ the corresponding eigenvalues. The labels are chosen in such a way that the eigenvalues are nonincreasing in absolute value: $|\mu_i| \geqq |\mu_{i+1}|$,

$i = 0, \cdots, |S| - 1$. (The general case, when Q_0 has Jordan cells, can also be treated, only formulas are more complicated.)

VI. There is a nonzero mass gap in the spectrum of Q_0:

$$1 = \mu_0 > |\mu_1|.$$

Condition VI implies that the Markov chain with space state S and stochastic operator Q_0 is ergodic. We denote by π_0 the (unique) stationary measure. We normalize e_0 by taking $e_0(s) = 1$, for all $s \in S$. The eigenvectors $\{e_i : i > 0\}$ of the matrix Q_0 are orthogonal to the eigenvector π_0 of the transposed matrix:

$$\sum_s e_i(s)\, \pi_0(s) = 0, \qquad i > 0. \tag{2.6b}$$

Consider the Hilbert space $l_2(S, \pi_0)$ and the operator Q_0^* (adjoint to Q_0) acting according to the formula

$$(Q_0^* f)(s) = \sum_{s'} q_0(s', s) \frac{\pi_0(s')}{\pi_0(s)} f(s'), \qquad s \in S. \tag{2.6c}$$

Its eigenvalues are

$$\mu_0^* = 1,\ \mu_1^* = \bar{\mu}_1, \cdots, \mu_{|S|-1}^* = \bar{\mu}_{|S|-1},$$

and we denote by $e_i^*(s)$ the corresponding eigenvectors. As above we can take $e_0^*(s) \equiv 1$ and the eigenvectors $\{e_i^*\}$ are bi-orthogonal to the eigenvectors $\{e_i\}$:

$$(e_i, e_j^*) = 0 \qquad \text{if} \quad i \neq j. \tag{2.6d}$$

We normalize the eigenvectors in such a way that

$$(e_i, e_j^*) = \delta_{ij}, \qquad \|e_i\| = \|e_i^*\|, \qquad i = 0, \cdots, |S| - 1. \tag{2.6e}$$

As in [5] we will need the coefficients of the following expansions:

$$\left.\begin{aligned} c(u, s) &= \sum_{j=0}^{|S|-1} c_j(u)\, e_j(s), \\ \sum_s \bar{q}(s', s)\, e_j(s) &= \sum_{r=0}^{|S|-1} q_r^j\, e_r(s'), \\ e_j(s)\, e_k(s) &= \sum_{r=0}^{|S|-1} b_{j,k}^r\, e_r(s). \end{aligned}\right\} \tag{2.6f}$$

Some properties of the coefficients are given in Appendix A.

2.1C. *Further conditions on the transition probabilities*

VII. The function $c(\cdot, \cdot)$ in eq. (2.2) is such that

$$c_0(u) = \sum_s c(u, s)\, \pi_0(s) = 0, \qquad u \in \mathbb{Z}^\nu. \tag{2.7}$$

This condition is not restrictive, and can be considered as some kind of normalization: if it is not satisfied, then replace $P_0(x)$ by $P_0(x) + \sum_s c(x, s)\, \pi_0(s)$. It means that the average (over the measure π_0) of the perturbed random walk reproduces the unperturbed one.

We now come to the important spectral condition.

VIII. The following inequality holds

$$\min_{\lambda \in T^\nu} |\tilde{p}_0(\lambda)| > |\mu_1|. \tag{2.8}$$

Condition VIII states that there is a finite spectral gap for $\varepsilon = 0$.

2.2. Notation

We fix the initial position of the particle at time $t = 0$ to be $X_0 = x_0$, and assign an initial distribution Π to the random field ξ_0. Let $f(\xi, z)$, $(\xi, z) \in \hat{\Omega}$, be a bounded function, measurable with respect to the variable ξ. Its conditional average under the condition $(\bar{\xi}, x) \in \hat{\Omega}$, *i. e.*, the average with respect to the transition probability (2.1), is denoted by

$$(\mathcal{T} f)(\bar{\xi}, x) \equiv \langle f | \bar{\xi}, x \rangle, \tag{2.9}$$

where $\mathcal{T}$ is the stochastic operator of the Markov process, which, by analogy with Statistical Physics (understanding time as an additional space dimension) is sometimes called "transfer matrix".

The measure $\Pi_0 \equiv \pi_0^{\mathbb{Z}^\nu}$ is clearly invariant under the unperturbed evolution of the field. We set $\mathcal{H} = L_2(\Omega, \Pi_0)$, and denote by $\hat{\mathcal{H}} = L_2(\Omega, \Pi_0) \otimes l_2(\mathbb{Z}^\nu)$ the Hilbert space with scalar product

$$(f, g)_{\hat{\mathcal{H}}} = \sum_{z \in \mathbb{Z}^\nu} \int_\Omega f(\xi, z)\, \overline{g(\xi, z)}\, d\Pi_0 \tag{2.10}$$

We denote by $\mathcal{P}$, or $\mathcal{P}_{\Pi, x_0}$, the distribution on the space $\hat{\Omega}^{\mathbb{Z}_+}$ of the trajectories $\{ (\xi_t(y), X_t) : y \in \mathbb{Z}^\nu, t \in \mathbb{Z}_+ \}$, which arises by the stochastic evolution from the initial distribution $\Pi \times \delta_{x, x_0}$ on $\hat{\Omega}$. Averaging with respect to $\mathcal{P}$ will be denoted simply by $\langle \cdot \rangle$.

We shall consider correlations of the type

$$\langle f_1(\xi_t(x_1)), f_2(\xi_0(x_2)) \rangle \equiv \langle f_1(\xi_t(x_1)) f_2(\xi_0(x_2)) \rangle - \langle f_1(\xi_t(x_1)) \rangle \langle f_2(\xi_0(x_2)) \rangle, \quad (2.11)$$

where $x_1, x_2 \in \mathbb{Z}^\nu$ and f_1, f_2 are functions on S. Averages and correlations with respect to any other measure ν will be denoted by $\langle \cdot \rangle_\nu$ and $\langle \cdot, \cdot \rangle_\nu$, respectively

2.3. Main results

In [5] we proved, for ϵ small enough, the local limit theorem for the displacement of the particle with respect to its initial position, which implies in particular that, as $t \to \infty$, we have the asymptotic expansion

$$\langle X_t - X_0 \rangle = bt + o(t).$$

Here the vector $b \in \mathbb{R}^\nu$ is the "drift", and its components are given by

$$b_k = i \frac{\partial \tilde{p}(\lambda)}{\partial \lambda_k}|_{\lambda=0} \quad k = 1, \cdots \nu, \quad (2.12)$$

where $\tilde{p}(\lambda)$ is an analytic function such that $\tilde{p}(\lambda)|_{\varepsilon=0} = \overline{\tilde{p}_0(\lambda)}$ (*see* Lemma 3.1 below).

As is to be expected, the time asymptotics of the correlations depends on whether $b = 0$ or $b \neq 0$. For $b \neq 0$ we have the following theorem.

THEOREM 2.1. – *If conditions I-VIII above are satisfied and $b \neq 0$, then*

$$|\langle f_1(\xi_t(x_1)), f_2(\xi_0(x_2)) \rangle| < C\,\theta^t,$$

where $\theta \in (0, 1)$ is a constant depending on the parameters of the model, and C is a constant depending on the functions f_1, f_2, on the sites x_1, x_2 and on the initial data x_0, Π.

If $b = 0$ then one should expect a power-law decay. For technical reasons we need the following more restrictive assumption of symmetry for the random walk.

IX. (Symmetry condition on the random walk.)

$$\mathbb{P}(X_t = z, \xi_t \in A | X_{t-1} = x, \xi_{t-1} = \eta) = \mathbb{P}(X_t = -z, \xi_t \in \mathcal{V} A | X_{t-1} = -x, \xi_{t-1} = \mathcal{V}\eta) \quad (2.13)$$

where $\mathcal{V}$ is the space reflection on Ω: $(\mathcal{V}\,\xi)(x) = \xi(-x)$. From (2.1), (2.2) one can see that condition (2.13) is satisfied for all ϵ small enough if and only if

$$P_0(u) = P_0(-u), \qquad c(u, s) = c(-u, s), \qquad u \in \mathbb{Z}^\nu, \; s \in S.$$

Condition IX implies that the drift is zero. Other consequences are stated in Lemma 3.8 below.

THEOREM 2.2. – *If conditions I-IX above are satisfied, then*

$$\langle f_1(\xi_t(x_1)),\, f_2(\xi_0(x_2)) \rangle = \frac{C}{t^{(\nu/2)+1}}(1 + o(1)), \qquad t \to \infty, \tag{2.14}$$

with

$$C = \sum_{j_1 \neq 0,\, j_2 \neq 0} C_{j_1, j_2},\, c_{j_1}^{(1)}\, c_{j_2}^{(2)}.$$

Here $c_j^{(k)}$ are the coefficients in the expansion

$$f_k(s) = \sum_{j=0}^{|S|-1} c_j^{(k)}\, e_j(s) \qquad k = 1, 2,$$

and the constants C_{j_1, j_2}, are given, to leading order in ε, by the expression

$$\begin{aligned} C_{j_1, j_2} = \varepsilon^2 \frac{(2\pi)^{1/2}}{2(Det\, A)^{1/2}} \frac{q_0^{j_1}}{1 - \mu_{j_1}} \sum_{u \in \mathbb{Z}^\nu} \sum_{j,x} c_j(u) \sum_{k_1, k_2} b_{k_1, k_2}\, u_{k_1}\, u_{k_2} \\ \times \int \frac{1}{(1 - \mu_j\, \tilde{p}_0(\lambda))} e^{-i(\lambda, x)}\, d\lambda \\ \times \langle e_j(\xi_0(x - x_0)),\, e_{j_2}(\xi_0(x_2)) \rangle_\Pi + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{2.15}$$

where $A = \{a_{i,j}\}$ is a positive definite matrix, $B = \{b_{i,j}\} = A^{-1}$, and the coefficients q_j^k and $c_j(\,\cdot\,)$ are defined by eq. (2.6 f).

The matrix A has a complicated expression and will be explicitely given in Section 3.

It is worthwhile to formulate a further result, which follows easily from the proof of Theorem 2.2, and describes the "relaxation" of the environment to the unperturbed equilibrium state Π_0.

A function $F(\xi)$, $\xi \in \Omega$ is called a local functional of the field, if it depends only on the subconfiguration $\xi(x)$, $x \in B$, for some finite set $B \subset \mathbb{Z}^\nu$. The following theorem holds.

THEOREM 2.3. – *If conditions I-IX above are satisfied and F is a local functional of the field, then for any $\bar{\xi}$ and x_0*

$$\begin{aligned}\langle F(\xi_t)|X_0 = x_0, \xi_0 = \bar{\xi}\rangle \\ = C_F^{(0)} + \frac{C_F^{(1)}}{t^{\nu/2}} + \frac{C_F^{(2)}(\bar{\xi}, x_0)}{t^{(\nu/2)+1}} + o\left(\frac{1}{t^{(\nu/2)+1}}\right), \qquad t \to \infty, \quad (2.16)\end{aligned}$$

where $C_F^0 = \langle F\rangle_{\Pi_0}$, and the term $o\left(\frac{1}{t^{(\nu/2)+1}}\right)$ is uniformly small in $\bar{\xi}$.

The explicit expression of the coefficients $C_F^{(i)}$ $i = 1, 2$ will be given at the end of Section 3.

Remark 2.1. – The fact that the constant $C_F^{(1)}$ does not depend on the starting point of the random walk x_0 is not surprising, since the asymptotic expansion (2.16) does not hold uniformly in x_0, *i.e.*, the residual term $o\left(\frac{1}{t^{(\nu/2)+1}}\right)$ is not uniformly small in x_0.

3. PROOFS

We report here for convenience some of the results of [5], and deduce some consequences which are needed in the proofs.

3.1. Notations

In what follows the expression "const" may denote any positive absolute constant, *i.e.*, any constant independent of the variables and of ϵ.

In [5] we introduced a basis $\{\Psi_{\Gamma, z} : \Gamma \in \mathfrak{M}, z \in \mathbb{Z}^\nu\}$ in $\hat{\mathcal{H}}$, with

$$\Psi_{\Gamma, z}(\xi, x) = \Psi_\Gamma(\xi)\,\delta_{x, z}, \qquad \Psi_\Gamma(\xi) = \prod_{y \in \mathbb{Z}^\nu} e_{\gamma(y)}(\xi(y)). \qquad (3.1a)$$

Here $\Gamma = \{\gamma(y) : y \in \mathbb{Z}^\nu\}$ is a multi-index, *i.e.*, a function γ on $\mathbb{Z}^\nu$ with values in $\{0, 1, \cdots, |S| - 1\}$ and finite support $\operatorname{supp}\Gamma = \{y \in \mathbb{Z}^\nu : \gamma(y) > 0\}$. $\mathfrak{M}$ denotes the collection of all such multi-indices. The functions $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$ are a basis in $\mathcal{H} = L^2(\Omega, \Pi_0)$.

We use the special notation $\bar{0}$ for the multi-index that takes the value 0 everywhere, *i.e.*, $\Gamma = \bar{0}$ means $\gamma(x) = 0$ for all $x \in \mathbb{Z}^\nu$. $|A|$ will denote the cardinality of a set $A \subset \mathbb{Z}^\nu$.

The norm of a vector $f \in \mathcal{H}$ can be written in terms of the components $\{f_\Gamma\}$ of the expansion $f = \sum_\Gamma f_\Gamma \Psi_\Gamma$ as

$$\left.\begin{aligned} \|f\|^2 &= (f, f) = \sum_{\substack{\Gamma, \Gamma' \\ \operatorname{supp}\Gamma = \operatorname{supp}\Gamma'}} a_{\Gamma, \Gamma'}\, f_\Gamma \overline{f_{\Gamma'}}, \\ a_{\Gamma, \Gamma'} &= \prod_{x \in \operatorname{supp}\Gamma} (e_{\gamma(x)}, e_{\gamma'(x)}). \end{aligned}\right\} \tag{3.1b}$$

The group $\{\mathcal{U}_v : v \in \mathbb{Z}^\nu\}$ of all shifts on $\mathbb{Z}^\nu$ is a unitary group of operators which acts on $\hat{\mathcal{H}}$ as follows

$$\mathcal{U}_v f(\xi, x) = f(\xi + v, x + v), \qquad f \in \hat{\Omega},$$

where $(\xi + v)(x) = \xi(x - v)$ is the space shift of the field ξ. The space $\hat{\mathcal{H}}$ decomposes into a direct integral

$$\hat{\mathcal{H}} = \int_{T^\nu} d\lambda\, \mathcal{H}_\lambda \tag{3.2}$$

where $d\lambda$ is the normalized Haar measure on the ν–dimensional torus T^ν. $\mathcal{H}_\lambda$ is the space of all generalized functions of the type

$$f = \sum_{\Gamma, z} f_{\Gamma - z}\, e^{-i(\lambda, z)}\, \Psi_{\Gamma, z},$$

such that the components $\{f_\Gamma\}$ make the norm (3.1 b) finite. Such functions are eigenfunctions of $\mathcal{U}_v$ with eigenvalue $e^{-i(\lambda, v)}$, $v \in \mathbb{Z}^\nu$. The representation (3.2) means that we can write any vector $f = \sum_{\Gamma, z} f_{\Gamma, z} \Psi_{\Gamma, z} \in \hat{\mathcal{H}}$ as

$$f = \sum_{\Gamma, z} \int d\lambda\, \tilde{f}_{\Gamma - z}()\, e^{-i(\lambda, z)}\, \Psi_{\Gamma, z}, \tag{3.3a}$$

where $\{\hat{f}_\Gamma(\lambda) : \Gamma \in \mathfrak{M}\}$ are the components of the Fourier transform

$$\tilde{f}_\Gamma(\lambda) = \sum_{z \in \mathbb{Z}^\nu} f_{\Gamma + z, z}\, e^{i(\lambda, z)}, \tag{3.3b}$$

and $\Gamma + z$, $z \in \mathbb{Z}^\nu$ is the translation of the multi-index Γ, *i.e.*, $\tilde{\Gamma} = \Gamma + z$ is defined by the relation $\tilde{\gamma}(x) = \gamma(x - z)$.

We recall briefly some facts on the Fourier transform, referring to Section 3 of [5] for details. Let $l_2(\mathfrak{M})$ denote the space of the sequences $\{f_\Gamma : \Gamma \in \mathfrak{M}\}$, with norm defined by eq. (3.1 b). The Fourier transform

(3.3 b) defines a unitary application of the space $\hat{\mathcal{H}}$ onto the Hilbert space $l_2(\mathfrak{M}) \times L_2(T^\nu, d\lambda)$ of the functions $\tilde{f} = \{ \tilde{f}_\Gamma(\lambda) : \Gamma \in \mathfrak{M}, \lambda \in T^\nu \}$ with norm

$$\|\hat{f}\|^2 = \int d\lambda \|\hat{f}(\lambda)\|^2_{l_2(\mathfrak{M})}.$$

Since the transfer matrix $\mathcal{T}$ defined by (2.9) commutes with the shifts $\{\mathcal{U}_v : v \in \mathbb{Z}^\nu\}$, the representation (3.2) generates a corresponding representation for $\mathcal{T}$ as

$$\mathcal{T} = \int_{T^\nu} d\lambda\, \mathcal{T}(\lambda),$$

and the operator $\mathcal{T}(\lambda)$ acts on $\mathcal{H}_\lambda$. Under the Fourier transform the operator $\mathcal{T}$ is transformed in the operator $\tilde{\mathcal{T}}$, acting on $l_2(\mathfrak{M}) \times L_2(T^\nu, d\lambda)$ as

$$(\tilde{\mathcal{T}}\tilde{f})_\Gamma(\lambda) = \sum_{\Gamma'} \hat{R}_{\Gamma,\Gamma'}(\lambda)\, \tilde{f}_{\Gamma'}(\lambda).$$

Here

$$\hat{R}_{\Gamma,\Gamma'}(\lambda) = \sum_{u \in \mathbb{Z}^\nu} \mathcal{T}_{\Gamma',0,\Gamma+u,u}\, e^{i(\lambda,u)}, \tag{3.4a}$$

and the matrix elements of $\mathcal{T}$ are defined by the position

$$\mathcal{T}\Psi_{\Gamma,z} = \sum_{\Gamma',z'} \mathcal{T}_{\Gamma,z,\Gamma',z'}\Psi_{\Gamma',z'}. \tag{3.4b}$$

The explicit expression of the matrix elements of $\mathcal{T}$ is given in the Appendix A (A.1 a). From (A.1) follows

$$\hat{R}_{\bar{0},\bar{0}}(\lambda) = \overline{\tilde{p}_0(\lambda)}. \tag{3.4c}$$

For $M > 0$ we consider the subspace $\mathcal{H}_M \subset \mathcal{H}$ of the vectors f for which

$$\|f\|_M = \sum_{\Gamma \in \mathfrak{M}} M^{|\operatorname{supp}\Gamma|}|f_\Gamma| < \infty. \tag{3.5a}$$

$\mathcal{H}_M$ with the norm $\|\cdot\|_M$ is a Banach space. If

$$M > M^* \equiv \max\{\max_{i,s}|e_i(s)|, 2\},$$

as we suppose from now on, then the obvious upper bound

$$|\Psi_\Gamma(\xi)| \leqq (M^*)^{|\operatorname{supp}\Gamma|} \tag{3.5b}$$

leads to the inequality

$$\|f\|_{\mathcal{H}} \leqq \|f\|_{\mathcal{H}_M}. \tag{3.5c}$$

$\mathcal{H}_M$ is obviously dens in $\mathcal{H}$.

3.2. Preliminary results: invariant subspaces

The following result is an obvious consequence of Lemmas 3.5 and 4.2 of [5].

LEMMA 3.1. – *Assume I-VIII. Then $\mathcal{T}(\lambda)$ has, for all $\lambda \in T^\nu$, a unique eigenvector $\chi(\lambda) \in \mathcal{H}_M$ with eigenvalue $\tilde{p}(\lambda)$, such that $\tilde{p}(\lambda)|_{\varepsilon=0} = \overline{\tilde{p}_0(\lambda)}$. Furthermore $\tilde{p}(\lambda)$ and $\chi(\lambda)$ can be extended to analytic functions in some complex neighborhood*

$$W_d = \{(\lambda_1, \cdots, \lambda_\nu) : |\mathrm{Im}\,\lambda_i| < d,\ i = 1, \cdots, \nu\}, \qquad d > 0$$

of the torus T^ν.

Moreover, $\chi(\lambda)$ has the following properties:

(i) $\chi_\Gamma(0) = 0$ *for* $\Gamma \neq \bar{0}$;

(ii) *if $\chi(\lambda)$ is normalized by setting $\chi_{\bar{0}}(\lambda) = 1$, then one can find a constant $q \in (0, 1)$, depending only on the range D and on $|\mu_1|$, such that for any positive M and ϵ small enough the following estimates hold for all $\lambda \in T^\nu$*

$$\sum_{\Gamma \neq \bar{0}} |\chi_\Gamma(\lambda)| M^{|\mathrm{supp}\,\Gamma|}\ const\,\varepsilon \tag{3.6a}$$

$$|\chi_\Gamma(\lambda)| < const\ \varepsilon\, \frac{q^{d_{\mathrm{supp}\,\Gamma \cup \{0\}}}}{M^{|\mathrm{supp}\,\Gamma|}}, \qquad \Gamma \neq \bar{0}, \tag{3.6b}$$

where d_B, $B \subset \mathbb{Z}^\nu$, denotes the minimal length of the connecd graphs connecting all points of the set B.

Proof. – The proof follows the same lines as for Lemmas 3.5 and 4.2 of [5]. We will explain here only the main points, referring to [5] for the details.

The invariance condition for the vector χ leads to the following equation

$$\left.\begin{aligned} \chi_\Gamma(\lambda) = \frac{1}{\hat{R}_{\bar{0},\bar{0}}(\lambda)} [\sum_{\Gamma' \neq \bar{0}} \hat{R}_{\Gamma,\Gamma'}(\lambda)\, \chi_{\Gamma'}(\lambda) - \chi_\Gamma(\lambda) \\ \times \sum_{\Gamma' \neq \bar{0}} \hat{R}_{\bar{0},\Gamma'}(\lambda)\, \chi_{\Gamma'}(\lambda) + \hat{R}_{\Gamma',\bar{0}}(\lambda)], \\ \Gamma \neq \bar{0}, \qquad \chi_{\bar{0}}(\lambda) \equiv 1, \end{aligned}\right\} \tag{3.6c}$$

where the coefficients $\hat{R}_{\Gamma,\Gamma'}(\lambda)$ are defined by (3.4 a, b, c). Setting $\delta = \min_{\lambda \in T^\nu} |\tilde{p}_0(\lambda)| - |\mu_1|$, by condition VIII we have $\delta > 0$, and one can find $d > 0$ so small that

$$|\tilde{p}_0(\lambda)| > \left(|\mu_1| + \frac{\delta}{2}\right) e^{dD}, \qquad \lambda \in W_d.$$

Consider the Banach space $\mathcal{L}_q$ of the vectors $\{\chi_\Gamma : \Gamma \neq \bar{0}\}$, analytic in the neighborhood W_d with the norm

$$\|\chi\| \equiv \sup_{\Gamma \neq \bar{0}} \sup_{\lambda \in W_d} |\chi_\Gamma(\lambda)| q^{-d_{\operatorname{supp}\Gamma \cup \{0\}}} M^{|\operatorname{supp}\Gamma|}.$$

The right-hand side of eq. (3.6 c) defines a map $B : \mathcal{L}_q \to \mathcal{L}_q$. Existence and uniqueness of χ will be established if we show that B is a contraction in $\mathcal{L}_q$ in some sphere $\|\chi\| < \kappa$ for κ small enough.

By the formulas (3.4 a, b, c) and (A.1 a) of Appendix A, we see that

$$\hat{R}_{\Gamma,\Gamma'}(\lambda) = \hat{R}^0_{\Gamma,\Gamma'}(\lambda) + \varepsilon \Delta \hat{R}_{\Gamma,\Gamma'}(\lambda), \qquad \hat{R}^0_{\Gamma,\Gamma'}(\lambda) = \hat{R}_{\Gamma,\Gamma'}(\lambda)|_{\varepsilon=0}.$$

$\hat{R}^0_{\Gamma,\Gamma'}(\lambda)$ is easily computed, and we find

$$\begin{aligned}
&\left| \frac{1}{\hat{R}_{\bar{0},\bar{0}}(\lambda)} \sum_{\Gamma' \neq \bar{0}} \hat{R}^0_{\Gamma,\Gamma'}(\lambda)\chi_{\Gamma'}(\lambda) \right| \\
&= \left| \frac{\mu_\Gamma}{\hat{R}_{\bar{0},\bar{0}}(\lambda)} \sum_{u \in \mathbb{Z}^\nu} \chi_{\Gamma - u} P_0(u) e^{-i(\lambda, u)} \right| \\
&\leqq \frac{\|\chi\| e^{dD}}{M^{|\operatorname{supp}\Gamma|}} \left| \frac{\mu_\Gamma}{\tilde{p}_0(\lambda)} \right| \sum_{u \in \mathbb{Z}^\nu} P_0(u) q^{d_{\operatorname{supp}\Gamma \cup \{u\}}} \\
&\leqq \|\chi\| \frac{|\mu_1|}{|\mu_1| + \delta/2} q^{-D} q^{d_{\operatorname{supp}\Gamma \cup \{0\}}}.
\end{aligned}$$

Here we have used the inequality

$$|\chi_{\Gamma-u}(\lambda)| \leqq \|\chi\| \frac{q^{d_{\operatorname{supp}\{\Gamma - u\} \cup \{0\}}}}{M^{|\operatorname{supp}\Gamma|}} = \|\chi\| \frac{q^{d_{\operatorname{supp}\Gamma \cup \{0\}}}}{M^{|\operatorname{supp}\Gamma|}},$$

and the relation

$$d_{\operatorname{supp}\Gamma \cup \{0\}} \leqq d_{\operatorname{supp}\Gamma \cup \{u\}} + |u| \leqq d_{\operatorname{supp}\Gamma \cup \{u\}} + D.$$

Choosing q in such a way that

$$\left(\frac{|\mu_1|}{|\mu_1| + (\delta/2)}\right)^{1/D} < q < 1$$

we see that the $\mathcal{L}_q$ norm of the term $\dfrac{1}{\hat{R}_{\bar{0},\bar{0}}(\lambda)} \sum_{\Gamma'} \hat{R}^0_{\Gamma,\Gamma'}(\lambda)\chi_{\Gamma'}(\lambda)$ is bounded by $\beta\|\chi\|$, with $\beta \in (0, 1)$.

In the same way one can prove that the term containing $\Delta \hat{R}_{\Gamma, \Gamma'}$ has $\mathcal{L}_q$ norm less than const $\varepsilon \|\chi\|$, and that the quadratic (in χ) term of (3.6 c) has norm bounded by const $\varepsilon \|\chi\|^2$. The last term, that does not depend on χ, is explicitely computed in the Appendix B [eq. (B.3)], and has also norm of order ε.

Hence the map B is a contraction for small ε, in some small sphere. The estimates (3.6 a, b) are an easy consequence. As for the function $\tilde{p}(\lambda)$, its expression [5] is given by

$$\tilde{p}(\lambda) = \hat{R}_{\bar{0}, \bar{0}}(\lambda) + \sum_{\Gamma \neq \bar{0}} \hat{R}_{\bar{0}, \Gamma}(\lambda) \chi_\Gamma(\lambda). \tag{3.6d}$$

Its properties follow once again from the expression of the coefficients $\hat{R}_{\Gamma, \Gamma'}(\lambda)$. The lemma is proved. ■

From now on $\chi(\lambda)$ will denote the eigenvector of $\mathcal{T}(\lambda)$ of Lemma 3.1, with the normalization described in the lemma.

Remark 3.1. – The expansion in ε of χ_Γ, for small ε, is given by Proposition B.2 of Appendix B.

$\chi(\lambda)$ identifies an invariant (with respect to $\mathcal{T}$) subspace, as is shown by the following result.

LEMMA 3.2. – *One can find a subspace $\hat{\mathcal{H}}_1 \subset \hat{\mathcal{H}}$ invariant with respect to $\mathcal{T}$ and a basis $\{\psi_u : u \in \mathbb{Z}^\nu\}$ in $\hat{\mathcal{H}}_1$ of the form*

$$\psi_u(\xi, x) = \sum_{\Gamma, z} h^u_{\Gamma, z} \Psi_{\Gamma, z}(\xi, x), \tag{3.7a}$$

where the coefficients $h^u_{\Gamma, z}$ are given by

$$h^u_{\Gamma, z} = \int_{T^\nu} e^{i(\lambda, u-z)} \chi_{\Gamma-z}(\lambda)\, d\lambda. \tag{3.7b}$$

Proof. – Let $g(\lambda)$ be an analytic function of λ on T^ν. Consider the vector $f^{(g)} \in \hat{\mathcal{H}}$ with components

$$f^{(g)}_{\Gamma, z} = \int d\lambda\, g(\lambda) \chi_{\Gamma-z}(\lambda) e^{-i(\lambda, z)}$$

Clearly the space $\mathcal{G}$ of all functions $f^{(g)}$, for g analytic, is left invariant by $\mathcal{T}$. By expanding in Fourier series $g(\lambda) = \sum_u g_u e^{i(\lambda, u)}$, it is easily seen that $\mathcal{G}$ is spanned by the functions $\{\psi_u : u \in \mathbb{Z}^\nu\}$, with the coefficients given by (3.7 b). $\hat{\mathcal{H}}_1$ is then obtained by taking the closure of $\mathcal{G}$ in the norm of $\hat{\mathcal{H}}$.

Lemma 3.2 is proved. ■

We need some bounds on the coefficients $h^u_{\Gamma,z}$, which are provided by the following result.

LEMMA 3.3. – *There is a constant $\hat{q} \in (0, 1)$ such that, for M as in Lemma 3.1,*

$$|h^u_{\Gamma,z}| < const\, \varepsilon \frac{\hat{q}^{d_{\operatorname{supp}\Gamma \cup \{z\} \cup \{u\}}}}{M^{|\operatorname{supp}\Gamma|}}, \qquad \Gamma \neq \bar{0}, \quad z \in \mathbb{Z}^\nu. \tag{3.8}$$

Proof. – The proof follows from the expression (3.7 b) and Lemma 3.1 (*see* [5], Lemmas 4.1, 4.2). ■

It is easy to see, by Lemma 3.1, making use of formulas (3.3 a) and (3.3 b) for the Fourier transform and its inverse, that the action of $\mathcal{T}$ on the basis $\{\psi_u\}$ is given by

$$\mathcal{T}\psi_u = \sum_z r(u - z)\,\psi_z \tag{3.9a}$$

where

$$r(u) = \int d\lambda\, \tilde{p}(\lambda)\, e^{-i(\lambda, u)}. \tag{3.9b}$$

The decomposition of the space $\hat{\mathcal{H}}$ in invariant subspaces is stated by the following lemma.

LEMMA 3.4. – (i) *The space $\hat{\mathcal{H}}$ can be decomposed into two invariant subspaces*

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 + \bar{\mathcal{H}}_1.$$

(ii) *in the subspace $\bar{\mathcal{H}}_1$ on can find a basis $\{\psi_{\Gamma,u} : \Gamma \neq \bar{0},\ u \in \mathbb{Z}^\nu\}$ for which the inequality*

$$|\psi_{\Gamma,u}(\xi, x)| \leqslant const\, \bar{\theta}^{|u-x|}\, M^{|\operatorname{supp}\Gamma|}, \tag{3.10}$$

holds for some $\bar{\theta} \in (0, 1)$, any $M > M^*$, and ε small enough.

Proof. – Following [5], Section 3, one can introduce in $\mathcal{H}$ a new basis $\{\Psi^*_\Gamma : \Gamma \in \mathfrak{M}\}$, bi-orthogonal to the basis $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$, and a corresponding basis $\{\Psi^*_{\Gamma,z} : \Gamma \in \mathfrak{M},\ z \in \mathbb{Z}^\nu\}$ in $\hat{\mathcal{H}}$ of the form

$$\Psi^*_{\Gamma,z}(\xi, x) = \prod_{y \in \mathbb{Z}^\nu} e^*_{\gamma(y)}(\xi(y))\,\delta_{x,z},$$

where e^*_j are the eigenvectors of the matrix Q^*_0, adjoint to Q_0 [with respect to the scalar product in $l_2(S, \pi_0)$]. The basis $\{\Psi^*_{\Gamma,z}\}$ is bi-orthogonal to the basis $\{\Psi_{\Gamma,z}\}$.

If $\mathcal{T}^*$ denotes the operator adjoint to $\mathcal{T}$ [with respect to the scalar product (2.10)], one finds, reasoning exactly as above, a new subspace $\hat{\mathcal{H}}_1^*$, invariant with respect to $\mathcal{T}^*$, and a basis $\{\psi_u^* : u \in \mathbb{Z}^\nu\}$ in $\hat{\mathcal{H}}_1^*$, given by

$$\psi_u^* = \sum h_{\Gamma, z}^{*u} \Psi_{\Gamma, z}^*, \tag{3.11}$$

where $h_{\Gamma, z}^{*u}$ is given by eq. (3.7 b), in which $\chi(\lambda)$ is replaced by the eigenfunction $\chi^*(\lambda)$ of $\mathcal{T}^*(\lambda)$ (*see* [5], Section 3). $\chi^*(\lambda)$ is normalized by setting $\chi_{\bar{0}}^*(\lambda) = 1$ for all $\lambda \in T^\nu$, and satisfies the same inequalities (3.6 a, b) as χ.

By the bi-orthogonality property it is easy to see that the functions

$$\psi_{\Gamma, u} = \Psi_{\Gamma, u} - \sum_z \overline{h_{\Gamma, u}^{*z}}\, \Psi_{\bar{0}, z} \tag{3.12}$$

are orthogonal to the functions of $\hat{\mathcal{H}}_1^*$. They satisfy the estimate (3.10), as one can see by observing that the quantities $h_{\Gamma, z}^{*, u}$ satisfy the same estimate (3.8) as $h_{\Gamma, z}^u$. [This follows from the validity of the estimate (3.7 b) for χ^*.]

Consider the set of functions $\{\psi_{\Gamma, z} : \Gamma \in \mathfrak{M}, z \in \mathbb{Z}^\nu\}$, where the functions $\psi_{\bar{0}, u} = \psi_u$ are given by eq. (3.7 a). If we show that this system is a basis in $\hat{\mathcal{H}}$ it follows that $\bar{\mathcal{H}}_1$ is identified with the span of $\{\psi_{\Gamma, z} : \Gamma \neq \bar{0}\, z \in \mathbb{Z}^\nu\}$, which completes the proof of the lemma. One can also see that $\bar{\mathcal{H}}_1 = (\hat{\mathcal{H}}_1^*)^\perp$.

By (3.7 a) and (3.12) we can write

$$\psi_{\Gamma, z} = (\mathcal{E} + \mathcal{C})\, \Psi_{\Gamma, z}, \tag{3.13a}$$

where $\mathcal{E}$ denotes the identity matrix, and the matrix elements of the operator $\mathcal{C}$, defined by the position

$$\mathcal{C}\, \Psi_{\Gamma, z} = \sum_{\Gamma', z'} \mathcal{C}_{\Gamma, z, \Gamma', z'}\, \Psi_{\Gamma', z'}, \tag{3.13b}$$

are easily computed, to find that $\mathcal{C}_{\Gamma, z, \Gamma', z'} = 0$ if $\Gamma = \Gamma' = \bar{0}$, or $\Gamma \neq \bar{0}$, $\Gamma' \neq \bar{0}$, and

$$\mathcal{C}_{\Gamma, z, \bar{0}, z'} = -\overline{h_{\Gamma, z}^{*, z'}}, \qquad \mathcal{C}_{\bar{0}, z, \Gamma, z'} = h_{\Gamma, z'}^z, \qquad \Gamma \neq \bar{0}. \tag{3.14}$$

Let $f \in \hat{\mathcal{H}}$ and $f' = \mathcal{C} f$. The relation between the Fourier transforms $\tilde{f}(\lambda)$ and $\tilde{f}'(\lambda)$ [computed by means of formulas (3.3 a, b) and (3.7 b)] is given by the formulas

$$\tilde{f}'_{\bar{0}}(\lambda) = -\sum_{\Gamma \neq \bar{0}} \tilde{f}_\Gamma(\lambda)\, \overline{\chi_\Gamma^*(\lambda)}$$

$$\tilde{f}'_\Gamma(\lambda) = \chi_\Gamma(\lambda)\, \tilde{f}_{\bar{0}}(\lambda), \qquad \Gamma \neq \bar{0}.$$

If now $g = (\mathcal{E} + \mathcal{C}) f$ some simple algebra shows that

$$\tilde{f}_{\bar{0}}(\lambda) = \frac{\tilde{g}_{\bar{0}}(\lambda) + \sum_{\Gamma' \neq \bar{0}} \overline{\chi^*_{\Gamma'}(\lambda)}\, \tilde{g}_{\Gamma'}(\lambda)}{1 + \sum_{\Gamma' \neq \bar{0}} \overline{\chi^*_{\Gamma'}(\lambda)}\, \chi_{\Gamma'}(\lambda)}$$

$$\tilde{f}_{\Gamma}(\lambda) = \tilde{g}_{\Gamma}(\lambda) + \chi_{\Gamma}(\lambda) \frac{\tilde{g}_{\bar{0}}(\lambda) - \sum_{\Gamma' \neq \bar{0}} \overline{\chi^*_{\Gamma'}(\lambda)}\, \tilde{g}_{\Gamma'}(\lambda)}{1 + \sum_{\Gamma' \neq \bar{0}} \overline{\chi^*_{\Gamma'}(\lambda)}\, \chi_{\Gamma'}(\lambda)}, \qquad \Gamma \neq \bar{0}.$$

Observe that by the estimate (3.6 b) and the analogous one for χ^* it follows that $\left| \sum_{\Gamma' \neq \bar{0}} \chi_{\Gamma'}(\lambda) \overline{\chi^*_{\Gamma'}(\lambda)} \right| < \text{const}\, \varepsilon^2$. Hence for ϵ small enough $\mathcal{E} + \mathcal{C}$ is invertible, and the set $\{ \psi_{\Gamma, z} : \Gamma \in \mathfrak{M},\ z \in \mathbb{Z}^\nu \}$ is a basis in $\hat{\mathcal{H}}$. The lemma is proved. ■

Remark 3.2. – One can prove, exactly in the same way, that the decomposition

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}^*_1 + \bar{\mathcal{H}}^*_1$$

holds, where the subspaces $\hat{\mathcal{H}}^*_1$ and $\bar{\mathcal{H}}^*_1$ are invariant with respect to $\mathcal{T}^*$. Moreover $\hat{\mathcal{H}}^*_1$ is spanned by the basis $\{ \psi^*_u \}$, given by formula (3.11), and $\bar{\mathcal{H}}^*_1$ is spanned by the basis

$$\psi^*_{\Gamma,u} = \Psi^*_{\Gamma, u} - \sum_z \overline{h^z_{\Gamma, u}}\, \Psi^*_{\bar{0}, z}, \qquad \Gamma \neq \bar{0},\ u \in \mathbb{Z}^\nu. \tag{3.15}$$

3.3. Projections on the invariant subspaces

We now want to separate the contributions to the correlation that come from the invariant subspaces $\hat{\mathcal{H}}_1$ and $\bar{\mathcal{H}}_1$.

The functions f_1, f_2 can be expanded in the basis $\{ e_j(s) \}_{j=0, \cdots, |S|-1}$ of the eigenvectors of the matrix Q_0

$$f_k(s) = \sum_{j \neq 0} c^{(k)}_j e_j(s) + c^{(k)}_0 e_0(s), \qquad k = 1, 2.$$

Recalling that $e_0(s) \equiv 1$, the correlation (2.11) becomes

$$\sum_{j_1 \neq 0,\, j_2 \neq 0} c^{(1)}_{j_1} c^{(2)}_{j_2} \langle e_{j_1}(\xi_t(x_1)),\ e_{j_2}(\xi_0(x_2)) \rangle,$$

and it is enough to study correlations of the type

$$\langle e_{j_1}(\xi_t(x_1)),\ e_{j_2}(\xi_0(x_2))\rangle, \qquad j_1, j_2 = 1, \cdots, |S| - 1. \tag{3.16}$$

By the definition (3.1 a) of the functions $\Psi_{\Gamma, z}$ we have for $j \neq 0$

$$e_j(\xi_t(y)) = \sum_{z \in \mathbb{Z}^\nu} \Psi_{\Gamma_j^{(y)}, z}(\xi_t, x), \qquad y \in \mathbb{Z}^\nu$$

where $\Gamma_j^{(y)}$ denotes the multi-index Γ such that $\operatorname{supp}\Gamma = \{y\}$ and $\gamma(y) = j$. Setting $F_i(\xi, x) = e_{j_i}(\xi(x_i))$, $i = 1, 2$ (note that the functions F_i, $i = 1, 2$ are constant in x), we find

$$\begin{aligned} \langle e_{j_1}(\xi_t(x_1))\rangle &= \int_\Omega (\mathcal{T}^t F_1)(\xi, x_0)\, d\Pi(\xi) \\ &= \sum_{z \in \mathbb{Z}^\nu} \int_\Omega (\mathcal{T}^t \Psi_{\Gamma_{j_1}^{(x_1)}, z})(\xi, x_0)\, d\Pi(\xi), \end{aligned} \tag{3.17a}$$

$$\begin{aligned} &\langle e_{j_1}(\xi_t(x_1))\, e_{j_2}(\xi_0(x_2))\rangle \\ &\quad = \int_\Omega (\mathcal{T}^t F_1)(\xi, x_0)\, F_2(\xi, x_0)\, d\Pi(\xi) \\ &\quad = \sum_{z \in \mathbb{Z}^\nu} \int_\Omega (\mathcal{T}^t \Psi_{\Gamma_{j_1}^{(x_1)}, z})(\xi, x_0)\, \Psi_{\Gamma_{j_2}^{(x_2)}}(\xi)\, d\Pi(\xi). \end{aligned} \tag{3.17b}$$

The convergence of the series on the right-hand side of equations (3.17 a, b) is established by Lemmas 3.5 and 3.7 below.

By Lemma 3.4 we have

$$\left.\begin{aligned} &\Psi_{\Gamma_{j_1}^{(x_1)}, z} = \Psi^{(1)}_{\Gamma_{j_1}^{(x_1)}, z} + \bar{\Psi}^{(1)}_{\Gamma_{j_1}^{(x_1)}, z}, \\ &\Psi^{(1)}_{\Gamma_{j_1}^{(x_1)}, z} \in \hat{\mathcal{H}}_1, \qquad \bar{\Psi}^{(1)}_{\Gamma_{j_1}^{(x_1)}, z} \in \bar{\mathcal{H}}_1 \end{aligned}\right\} \tag{3.18}$$

The expansion of $\Psi^{(1)}_{\Gamma_{j_1}^{(x_1)}, z} \in \hat{\mathcal{H}}_1$ in the basis $\{\psi_u\}$ can be written as

$$\Psi^{(1)}_{\Gamma_{j_1}^{(x_1)}, z} = \sum_u C^u_{j_1, x_1, z}\, \psi_u.$$

Since the vectors ψ_v^* are orthogonal to the subspace $\bar{\mathcal{H}}_1$, we have the following equation for the coefficients $C^u_{j_1, x_1, z}$

$$\left.\begin{aligned} (\Psi_{\Gamma_{j_1}^{(x_1)}, z}, \psi_v^*) = (\Psi^{(1)}_{\Gamma_{j_1}^{(x_1)}, z}, \psi_v^*) = \sum_u C^u_{j_1, x_1, z}(\psi_u, \psi_v^*), \\ v \in \mathbb{Z}^\nu \end{aligned}\right\} \tag{3.19}$$

where $(\cdot,\ \cdot)$ denotes as usual the scalar product in $\hat{\mathcal{H}}$. Eq. (3.11) gives

$$(\Psi_{\Gamma_{j_1}^{(x_1)},\, z_0},\ \psi_v^*) = \overline{h^{*v}_{\Gamma_{j_1}^{(x_1)},\, z_0}} \tag{3.20}$$

and

$$(\psi_u,\ \psi_v^*) = \sum_{\Gamma,\, y} h^u_{\Gamma,\, y}\, \overline{h^{*v}_{\Gamma,\, y}} = \int e^{i\,(\lambda,\, u-v)} \Big(\sum_{\Gamma} \chi_\Gamma(\lambda)\, \overline{\chi_\Gamma^*(\lambda)}\Big)\, d\lambda. \tag{3.21}$$

By the estimate (3.6 a, b), and similar ones for χ^*, the series under the integration sign converges, and we can write

$$\sum_{\Gamma} \chi_\Gamma(\lambda)\, \overline{\chi_\Gamma^*(\lambda)} = 1 + \varepsilon^2\, \kappa(\lambda), \tag{3.22}$$

where $\kappa(\lambda)$ may depend on ε, but is uniformly bounded in ε (and has a limit as $\varepsilon \to 0$). Furthermore it is an analytic function in the complex neighborhood W_d of the torus T^ν, as a consequence of the analiticity of the functions $\chi_\Gamma(\lambda)$ and $\chi_\Gamma^*(\lambda)$ (Lemma 3.1). Hence if ε is small $1 + \varepsilon^2\, \kappa(\lambda)$ is bounded away from 0, and the function $(1 + \varepsilon^2\, \kappa(\lambda))^{-1}$ is analytic. From (3.19-22) we find

$$C^u_{j_1,\, x_1,\, z} = \int_{T^\nu} \frac{\overline{\chi^*_{\Gamma_{j_1}^{(x_1)} - z}(\lambda)}}{1 + \varepsilon^2\, \kappa(\lambda)}\, e^{i\,(\lambda,\, z-u)}\, d\lambda. \tag{3.23}$$

By applying $\mathcal{T}^t$ to both sides of the decomposition (3.18) we find

$$(\mathcal{T}^t\, \Psi_{\Gamma_{j_1}^{(x_1)},\, z})(\xi,\ x_0) = (\mathcal{T}^t\, \Psi^{(1)}_{\Gamma_{j_1}^{(x_1)},\, z})(\xi,\ x_0) + (\mathcal{T}^t\, \bar{\Psi}^{(1)}_{\Gamma_{j_1}^{(x_1)},\, z})(\xi,\ x_0). \tag{3.24}$$

We first study the projection in $\hat{\mathcal{H}}_1$.

LEMMA 3.5. – *The series*

$$\sum_z (\mathcal{T}^t\, \Psi^{(1)}_{\Gamma_{j_1}^{(x_1)},\, z})(\xi,\ x_0) \tag{3.25}$$

converges absolutely, uniformly in ξ, x_0, *and for any* $\Gamma \in \mathfrak{M}$ *we have*

$$\sum_z \langle\, (\mathcal{T}^t\, \Psi^{(1)}_{\Gamma_{j_1}^{(x_1)},\, z})(\cdot,\ x_0),\ \Psi_\Gamma(\,\cdot\,)\,\rangle = \int g_\Gamma(\lambda)\, (\tilde{p}(\lambda))^t\, d\lambda, \tag{3.26a}$$

where the function

$$g_\Gamma(\lambda) = \frac{\sum_z \overline{\chi^*_{\Gamma_{j_1}^{(x_1)} - z}(\lambda)}\, e^{i\,(\lambda,\, z)}}{1 + \varepsilon^2\, \kappa(\lambda)} \times \sum_{\Gamma'} e^{-i\,(\lambda,\, x_0)}\, \chi_{\Gamma' - x_0}(\lambda)\, \langle\, \Psi_{\Gamma'},\ \Psi_\Gamma\,\rangle_\Pi \tag{3.26b}$$

is analytic in W_d.

Proof. – By Lemmas 3.1, 3.2 and equation (3.9 a) we get

$$(\mathcal{T}^t \Psi^{(1)}_{\Gamma^{(x_1)}_{j_1}, z})(\xi, x_0) = \sum_{u, v} C^u_{j_1, x_1, z} r_t(u - v) \psi_v(\xi, x_0)$$
$$= \sum_{u, v, \Gamma'} C^u_{j_1, x_1, z} r_t(u - v) h^v_{\Gamma', x_0} \Psi_{\Gamma'}(\xi), \quad (3.27)$$

where

$$r_t(u) = \int e^{i(u, \lambda)} (\tilde{p}(\lambda))^t \, d\lambda. \quad (3.28)$$

From expression (3.23), and the analyticity of χ^*, by shifting the integration path to complex values of λ, it is easily seen that for d small enough

$$\sum_{z, u} |C^u_{j_1, x_1, z}| \leqq \text{const} \sum_z \sup_{\lambda \in W_d} \frac{|\chi^*_{\Gamma^{(x_1)}_{j_1} - z}(\lambda)|}{|1 + \varepsilon^2 \kappa(\lambda)|}.$$

By formulas (3.27), (3.23), and (3.28), using some simple algebra, it is easy to see that eq. (3.26 a) holds with the function g_Γ is given by eq. (3.26 b). The analyticity of g_Γ follows from (3.26 b), and the analogous estimate for χ^*, and the fact that χ^*, χ, and κ are analytic functions. ■

3.4. Estimate of the projection on $\bar{\mathcal{H}}_1$

The proofs of Theorems 2.1 and 2.2 are based on formula (3.26 a). We first need to show that the second term in (3.24) gives an exponentially (in t) vanishing contribution to the correlations (3.16).

For $f \in \bar{\mathcal{H}}_1$ we write the expansion

$$f = \sum_{\Gamma, u} f_{\Gamma, u} \psi_{\Gamma, u}.$$

Let $A \subset \mathbb{Z}^\nu$ be a finite subset. We denote by $d(A, u)$ the distance of a point $u \in \mathbb{Z}^\nu$ from the set A and fix $\tilde{M} > M^*$. Consider the subspace $\mathcal{H}_{A, \hat{q}} \subset \bar{\mathcal{H}}_1$ of the vectors $f \in \bar{\mathcal{H}}_1$ such that the norm

$$\|f\|_{A, \hat{q}} = \sup_u \sum_\Gamma |f_{\Gamma, u}| \tilde{M}^{|\operatorname{supp} \Gamma|} \hat{q}^{-d(A, u)} \quad (3.29)$$

is finite. Clearly $\mathcal{H}_{A, \hat{q}}$ with the norm (3.29) is a Banach space. The following assertion holds.

LEMMA 3.6. – *On can find $\hat{q} \in (0, 1)$ such that for ϵ small enough and any finite $A \subset \mathbb{Z}^\nu$ the space $\mathcal{H}_{A, \hat{q}}$ is invariant with respect to the restriction $\mathcal{T}|_{\bar{\mathcal{H}}_1}$ and*

$$\|\mathcal{T}|_{\bar{\mathcal{H}}_1}\|_{\mathcal{H}_{A, \hat{q}}} \leqq \theta,$$

where $\theta \in (0, 1)$ is some constant independent of A.

Proof. – The proof is analogous to the proof of Lemma 4.8 of [5] and is deferred to the Appendix A. ■

The estimate of the second term in (3.24) is given by the next lemma.

LEMMA 3.7. – *The following estimate holds, for some constants θ, $\bar{q} \in (0, 1)$*

$$|(\mathcal{T}^t \, \bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z}) (\xi, x_0)| < const \; \theta^t \, \bar{q}^{|z-x_0|}. \tag{3.30}$$

Proof. – We prove first that $\bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0} \in \mathcal{H}_{A, q}$, for $A = \{ z_0 \}$. To do this we shall first construct a basis $\{ \hat{\psi}^*_{\Gamma, u} \, : \, \Gamma \neq \bar{0}, \; u \in \mathbb{Z}^\nu \}$ in $\bar{\mathcal{H}}^*_1$, bi-orthogonal to the basis $\{ \psi_{\Gamma, u} \}$ in $\bar{\mathcal{H}}_1$. We have

$$(\psi_{\Gamma, z}, \, \psi^*_{\Gamma', z'}) = \delta_{\Gamma, \Gamma'} \, \delta_{z, z'} + \sum_u \overline{h^{*u}_{\Gamma, z}} \, h^u_{\Gamma', z'} \equiv (\mathcal{E} + \mathcal{D})_{\Gamma, z, \Gamma', z'}.$$

Setting

$$\hat{\psi}^*_{\Gamma, z} = \psi^*_{\Gamma, z} + \sum_{\Gamma', z'} \mathcal{B}_{\Gamma, z, \Gamma', z'} \, \psi^*_{\Gamma' z'} = \sum_{\Gamma', z'} (\mathcal{E} + \mathcal{B})_{\Gamma, z, \Gamma', z'} \, \psi^*_{\Gamma' z'}, \tag{3.31}$$

where $\mathcal{E}$ denotes the identity matrix, some simple algebra shows that $\mathcal{B}$ has to be a solution of the equation

$$(\mathcal{E} + \mathcal{D}) \, (\mathcal{E} + \mathcal{B}^*) = \mathcal{E}.$$

It is convenient to consider the matrices $\mathcal{D}$ and $\mathcal{B}^*$ as operators acting on the Banach space $l_{A, \hat{q}}$ on the double sequences $f = \{ f_{\Gamma, z} \, : \, \Gamma \neq \bar{0}, \; z \in \mathbb{Z}^\nu \}$, with the norm given by (3.29). Using the explicit expression for the matrix elements of $\mathcal{D}$ and the estimate (3.6b) we have for $A = \{ z_0 \}$

$$\|\mathcal{D}f\|_{l_{A,\hat{q}}} \leqq \text{const}\, \varepsilon^2 \sup_z \sum_{\substack{z', z'' \\ \Gamma, \Gamma'}} \frac{q^{d_{\operatorname{supp}\Gamma \cup \{ z \} \cup \{ z'' \}} + d_{\operatorname{supp}\Gamma' \cup \{ z' \} \cup \{ z'' \}}}}{M^{|\operatorname{supp}\Gamma| + |\operatorname{supp}\Gamma'|}}$$

$$\times \; |f_{\Gamma', z'}| M^{|\operatorname{supp}\Gamma|} \hat{q}^{-|z_0 - z|}$$

$$\leqq \text{const}\, \varepsilon^2 \|f\|_{l_{A,\hat{q}}} \sup_z \sum_{\Gamma, z', z''} q^{|z' - z''| + d_{\operatorname{supp}\Gamma \cup \{ z \} \cup \{ z'' \}}}$$

$$\hat{q}^{-|z' - z|} \left(\frac{\tilde{M}}{M} \right)^{|\operatorname{supp}\Gamma|}.$$

Here we have used the triangular inequality and the obvious relation $d_{\operatorname{supp}\Gamma' \cup \{ z' \} \cup \{ z'' \}} \geqq |z' - z''|$. Taking $z' = z + v$, $z'' = z + w$, $\hat{q} \in (q, 1)$, the right-hand side is bounded from above by the quantity

$$\text{const}\, \varepsilon^2 \|f\|_{A,\hat{q}} \sum_{\Gamma, v, w} r^{d_{\operatorname{supp}\Gamma \cup \{ v \} \cup \{ w \} \cup \{ 0 \}}} \left(\frac{\tilde{M}}{M} \right)^{|\operatorname{supp}\Gamma|}, \tag{3.32}$$

with $r = \dfrac{q}{\hat{q}}$. Here we have used the inequality $d_{\operatorname{supp}\Gamma \cup \{ 0 \} \cup \{ w \}} + |v - w| \geqq d_{\operatorname{supp}\Gamma \cup \{ v \} \cup \{ w \} \cup \{ 0 \}}$. We can take M and $\tilde{M}$ in such a way that

$$M^* < \tilde{M} < \frac{e\, M}{\sum_{x \in \mathbb{Z}^\nu} r^{|x|}}.$$

The proof that this choice of $\tilde{M}$ implies the convergence of the series (3.32) is complicated and is deferred to the Appendix A.

Hence for the operators on $l_{A,\hat{q}}$ defined by the matrix $\mathcal{D}$ and its adjoint $\mathcal{D}^*$ we have $\|\mathcal{D}\| < \text{const}\, \varepsilon^2$, $\|\mathcal{D}^*\| < \text{const}\, \varepsilon^2$, so that the operator

$$\mathcal{B} = -\mathcal{D}^* (\mathcal{E} + \mathcal{D}^*)^{-1}$$

and its adjoint $\mathcal{B}^*$ exist as bounded operators and have norms of order $\mathcal{O}(\varepsilon^2)$.

Next we write the expansion

$$\bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0} = \sum_{\Gamma, u} C^{(z_0)}_{\Gamma, u} \psi_{\Gamma, u}.$$

The coefficients, using the explicit expressions (3.15) and (3.31) for $\psi^*_{\Gamma, u}$ and $\hat{\psi}^*_{\Gamma, u}$, are given by

$$C^{(z_0)}_{\Gamma, u} = (\Psi_{\Gamma^{(x_1)}_{j_1}, z_0}, \hat{\psi}^*_{\Gamma, u}) = \delta_{\Gamma^{(x_1)}_{j_1}, \Gamma} \delta_{z_0, u} + \mathcal{B}^*_{\Gamma^{(x_1)}_{j_1}, z_0, \Gamma, u}.$$

Taking $A = \{ z_0 \}$, and recalling that the operator norm of $\mathcal{B}^*$ in $l_{A,\hat{q}}$ is finite, we see that $\bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0} \in \mathcal{H}_{A,\hat{q}}$.

We can now apply Lemma 3.6, which implies

$$\|\mathcal{T}^t \bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0}\|_{\mathcal{H}_{\{z_0\},\hat{q}}} \leqq \operatorname{const} \theta^t. \tag{3.33}$$

In order to get the estimate (3.30) we write the expansion

$$\mathcal{T}^t \bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0} = \sum_{\Gamma, u} f^{z_0, t}_{\Gamma, u} \psi_{\Gamma, u}.$$

With the help of (3.10), (3.33), we get for some $\tilde{M} > M^*$

$$\begin{aligned} |\mathcal{T}^t \bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z_0}(\xi, x_0)| &\leqq \sum_{\Gamma, u} |f^{z_0, t}_{\Gamma, u}| \tilde{M}^{|\operatorname{supp}\Gamma|} \hat{q}^{|u - x_0|} \\ &< \operatorname{const} \theta^t \sum_u \hat{q}^{|u - z_0| + |u - x_0|} < \operatorname{const} \theta^t \bar{q}^{|z_0 - x_0|}, \end{aligned}$$

where we have assumed that $\hat{q}$ is larger than the constant $\bar{\theta}$ appearing in the estimate (3.10), and have set $\bar{q} = \sqrt{\hat{q}}$. Lemma 3.7 is proved. ■

Remark 3.3. – From Lemma 3.7 it follows immediately that

$$|\langle \sum_z (\mathcal{T}^t \bar{\Psi}^{(1)}_{\Gamma^{(x_1)}_{j_1}, z})(\cdot, x_0), \Psi_{\Gamma^{(x_2)}_{j_2}}(\cdot)\rangle_\Pi| \leqq \operatorname{const} \theta^t. \tag{3.34}$$

3.5. Proof of Theorem 2.1

In order to prove Theorem 2.1 we have to find the asymptotics of the first term in (3.24).

In formula (3.26 a) we can deform the integration contour in a complex neighborhood of the point $\lambda = 0$ in such a way that along the deformed contour the inequality $|\tilde{p}(\lambda)| < 1$ holds. In fact we can write the power series expansion of $\tilde{p}(\lambda)$ in a neighborhood of $\lambda = 0$ as

$$\tilde{p}(\lambda) = 1 - i(b, \lambda) - \frac{1}{2}(A\lambda, \lambda) + \mathcal{O}(\lambda^3), \tag{3.35}$$

where $b \in \mathbb{R}^\nu$ is the drift, which in [5] is shown to be given by eq. (2.12), where $\tilde{p}$ is given by eq. (3.6 d). By assumption III, $(A\lambda, \lambda)$ is a positive quadratic form for ε small enough (*see* Section 3 in [5]). Setting $\lambda = \tau + i\,\sigma$ we get

$$|\tilde{p}\,(\tau + i\sigma)|^2 = 1 - 2\,(b,\, \sigma) + (A\,\sigma,\, \sigma) - (A\,\tau,\, \tau) + (b,\, \tau)^2 + (b,\, \sigma)^2 + \mathcal{O}\,((\tau^2 + \sigma^2)^{3/2}). \quad (3.36)$$

Since we know that $|\tilde{p}\,(\lambda)| < 1$ for $\tau \neq 0$, $\sigma = 0$ (*see* Lemma 3.5 of [5]), we have

$$(A\,\tau,\, \tau) > (b,\, \tau)^2.$$

On the other hand, setting $\sigma = \alpha\, b$ for $\alpha > 0$, we get

$$-2\,(b,\, \sigma) + (A\,\sigma,\, \sigma) = -2\,\alpha\,(b,\, b) + \alpha^2\,(A\,b,\, b) < 0 \quad (3.37)$$

for $\alpha < \alpha_0$ sufficiently small. We now take a sufficiently small sphere $|\tau| < \kappa_1$ on the torus T^ν in such a way that outside of this sphere $|\tilde{p}\,(\lambda)| < \tilde{\theta} < 1$. We then deform the interior of this sphere to a "contour"

$$\Sigma = \{\, \lambda = \lambda\,(\tau) = i\,\alpha\,(\tau)\, b + \tau,\ |\tau| < \kappa_1 \,\}$$

where the function $\alpha\,(\tau)$ is chosen in such a way that $\alpha_0 > \alpha\,(\tau) \geqq 0$ and $\alpha\,(\tau) = 0$ on the boundary of the sphere $|\tau| = \kappa_1$. By (3.36), (3.37), we get

$$|\tilde{p}\,(\lambda)| < \bar{\theta} < 1, \qquad \lambda \in \Sigma. \quad (3.38)$$

Since the contour Σ is within the region of analiticity of the functions $\tilde{p}\,(\lambda)$ and $g\,(\lambda)$, the integration region in the integral (3.26 a) can be changed to

$$\{\, \lambda = \tau + i\,\sigma \,:\, |\tau| \geqq \kappa_1,\ \sigma = 0 \,\} \cup \Sigma$$

The assertion of Theorem 2.1 follows from Remark 3.3, which gives the decay in t of the second term of (3.24), and from relations (3.26 a), (3.38), which given the analyticity of $g_{\Gamma_{j_2}^{(x_2)}}$, lead to the result.

3.6. Proof of Theorem 2.2

We first prove the following lemma:

LEMMA 3.8. – *For the symmetric random walk $\tilde{p}\,(\lambda)$ is an even real function of λ.*

Proof. – Consider the following antilinear transformation (reflection) $\mathcal{R}$ acting on the functions on $\hat{\Omega}$

$$(\mathcal{R}\, f)(\xi,\, x) = \overline{f\,(\mathcal{V}\,\xi,\, -x)}.$$

The symmetry condition on the random walk implies that $\mathcal{T}$ commutes with $\mathcal{R}$, and moreover

$$\mathcal{R}\mathcal{U}_s = \mathcal{U}_{-s}\mathcal{R}, \qquad s \in \mathbb{Z}^\nu,$$

where $\mathcal{U}_s$ is the translation operator. It follows that the space $\mathcal{H}_\lambda$, which is an eigenspace of the group $\{\mathcal{U}_s\}$ with eigenvalue $e^{i(s,\lambda)}$ [*see* (3.2)], is transformed by $\mathcal{R}$ into itself. Moreover, as $\mathcal{R}$ and $\mathcal{T}$ commute, the vector $\mathcal{R}\chi(\lambda) \in \mathcal{H}_\lambda$ is an eigenvector of $\mathcal{T}(\lambda)$ with eigenvalue $\overline{\tilde{p}(\lambda)}$. It was proved in [5] that the eigenvector of $\mathcal{T}(\lambda)$ with eigenvalue of maximum absolute value is unique, hence $\mathcal{R}\chi(\lambda) = \beta\chi(\lambda)$, where β is a constant. This implies $\overline{\tilde{p}(\lambda)} = \tilde{p}(\lambda)$.

If we now consider the linear transformation $\tilde{\mathcal{R}}$

$$(\tilde{\mathcal{R}}) f(\xi, x) = f(\mathcal{V}\xi, -x),$$

then by observing again that $\tilde{\mathcal{R}}$ commutes with $\mathcal{T}$, we can repeat the above arguments to find that

$$\tilde{\mathcal{R}}\mathcal{H}_\lambda = \mathcal{H}_{-\lambda}, \qquad \tilde{\mathcal{R}}\chi(\lambda) = \text{const}\,\chi(-\lambda),$$

i. e., $\tilde{p}(\lambda) = \tilde{p}(-\lambda)$. The lemma is proved. ■

To accomplish the proof of Theorem 2.2 we need the asymptotic expansion of the integral (3.26 a) for $\Gamma = \Gamma_{j_2}^{(x_2)}$. Note that by formula (3.28), and the fact that $\chi_\Gamma(0) = \delta_{\bar{0},\Gamma}$ (Lemma 3.1) we have $g_{\Gamma_{j_2}^{(x_2)}}(0) = 0$. In the asymptotic expansion of the integral (3.26 a) one has to take into account that $\lambda = 0$ is the absolute maximum of the function $\tilde{p}(\lambda)$, $\lambda \in T^\nu$. The proof is technically complicated and is deferred to the Appendix B. ■

3.7. Proof of Theorem 2.3

The first step is the extension of the results of Lemma 3.7 and Remark 3.3 to any function $\Psi_{\Gamma,z}$, $\Gamma \neq \bar{0}$, of the basis. Namely inequality (3.30) holds if $\Gamma_{j_1}^{(x_1)}$ is replaced by any $\Gamma \neq \bar{0}$, $\bar{\Psi}_\Gamma^{(1)}$ being the projection of Ψ_Γ on $\bar{\mathcal{H}}_1$. Similarly inequality (3.34) holds if we replace $\Gamma_{j_1}^{(x_1)}$ by $\Gamma \neq \bar{0}$, and $\Gamma_{j_2}^{(x_2)}$ by any Γ'. The proof of these facts does not differ from the previous proofs and we omit it.

As a consequence we have the asymptotics

$$(\mathcal{T}^t\,\Psi_\Gamma)(\xi, x_0) = \sum_z (\mathcal{T}^t\,\Psi_{\Gamma,z}^{(1)})(\xi, x_0) + o(\theta^t), \qquad \Gamma \neq \bar{0}. \tag{3.39}$$

as $t \to \infty$. Here, $\theta \in (0, 1)$ and $\Psi^{(1)}_{\Gamma, z}$ is the projection of $\Psi_{\Gamma, z}$ on $\hat{\mathcal{H}}_1$. Furthermore, by the same procedure as in Lemma 3.5, it is easy to see that the first term on the right of eq. (3.39) is equal to

$$\int_{T^\nu} g_\Gamma(\lambda;\, \xi,\, x_0)\, (\tilde{p}(\lambda))^t\, d\lambda$$

where

$$g_\Gamma(\lambda;\, \xi,\, x_0) = \sum_{\Gamma'} \Psi_{\Gamma'}(\xi) \sum_z \frac{\overline{\chi^*_{\Gamma - z}(\lambda)\, e^{i(\lambda,\, z - x_0)}}}{1 + \varepsilon^2\, \kappa(\lambda)}\, \chi_{\Gamma' - x_0}(\lambda).$$

In analogy with the proof of Lemma 3.5 one can see that $g_\Gamma(\cdot;\, \xi,\, x_0)$ is analytic in W_d for d small enough, and its derivatives in W_d are bounded uniformly in ξ. We are in a situation in which the classical Laplace method for the asymptotic (in t) expansion applies (*see* [9], Chap. 8). We obtain

$$(\mathcal{T}^t\, \Psi_\Gamma)(\xi,\, x_0) = \frac{C^{(1)}_\Gamma}{t^{\nu/2}} + \frac{C^{(2)}_\Gamma(\xi,\, x_0)}{t^{(\nu/2)+1}} + o\left(\frac{1}{t^{(\nu/2)+1}}\right), \tag{3.40a}$$

where

$$C^{(1)}_\Gamma = (2\,\pi)^{\nu/2}\, (\mathrm{Det} A)^{-1/2} \sum_z \overline{\chi^*_{\Gamma - z}(0)}, \tag{3.40b}$$

A is again the matrix in eq. (3.35), and the function $C^{(2)}_\Gamma(\xi,\, x_0)$ turns out to be given by the formula

$$\begin{aligned} C^{(2)}_\Gamma&(\xi,\, x_0) \\ &= (2\,\pi)^{\nu/2}\, (\mathrm{Det}\, A)^{-1/2} \left\{ \frac{1}{2} \sum_{(k_1,\, k_2)} b_{k_1,\, k_2} \frac{\partial^2 g_\Gamma}{\partial\lambda_{k_1}\, \partial\lambda_{k_2}} \Big|_{\lambda=0} \right. \\ &\quad + \frac{1}{4!}\, g_\Gamma \Big|_{\lambda=0} \sum_{(k_1, k_2,\, k_3,\, k_4)} a_{k_1, k_2,\, k_3,\, k_4} \\ &\quad \left. \times\, (b_{k_1, k_2}\, b_{k_3,\, k_4} + b_{k_1,\, k_3}\, b_{k_2,\, k_4} + b_{k_1,\, k_4}\, b_{k_2,\, k_3}) \right\}. \end{aligned} \tag{3.40c}$$

In the first sum the indices run over all the ordered pairs $(k_1,\, k_2)$, and in the second sum over all ordered quadruples $(k_1,\, k_2,\, k_3,\, k_4)$, $k_i = 1, \cdots, \nu$, $i = 1, 2$. $b_{i,\, j}$ are the matrix elements of $B = A^{-1}$, and

$$a_{k_1,\, k_2,\, k_3,\, k_4} = \frac{\partial^4 \log \tilde{p}(\lambda)}{\partial\lambda_{k_1}\, \partial\lambda_{k_2}\, \partial\lambda_{k_3}\, \partial\lambda_{k_4}} \Big|_{\lambda=0}.$$

The assertion of Theorem 2.3 follows easily from relations (3.40 a-c), and the quantities $C_F^{(i)}$ in formula (2.16) take the form

$$C_F^{(i)} = \sum_{\Gamma} C_\Gamma^{(i)} F_\Gamma \qquad i = 1, 2$$

where the coefficients $\{ F_\Gamma \}$ refer to the basis $\{ \Psi_\Gamma : \Gamma \in \mathfrak{M} \}$.

CONCLUDING REMARKS

Theorem 2.1 could be made more precise, with some more work, by finding out the exact asymptotics, which, under some general assumptions, should be of the type

$$\frac{\text{const}}{t^\gamma} e^{-\alpha t}, \qquad \alpha > 0, \gamma > 0,$$

where α and γ depend on the parameters of the model. In general one should have $\gamma = \frac{\nu}{2}$ or $\gamma = \nu$ according to whether the "one-particle" or the "two-particle" branch of the spectrum dominates the asymptotics. This problem requires a more detailed investigation, and we plan to come back to it in a future paper.

Condition VIII is certainly strong, but it has the advantage of making proofs much simpler. One could replace it by a weaker one, requiring inequality (2.8) to hold only in some neighborhood of $\lambda = 0$, and using ideas similar to the ones introduced at the end of Section 3 of [5]. We hope to come back to this problem as well.

There is a simple intuitive (or "physical") explanation of the results of Theorems 2.2 and 2.3, which can be formulated as follows. By the local limit theorem for the position of the random walk (*see* [4] and [5]), the particle starting at the point, say, $x_0 = 0$, in the absence of drift, falls at some large time t on any fixed point $x \in \mathbb{Z}^\nu$, at a finite distance from the origin, with probability $\frac{\text{const}}{t^{\nu/2}}$. Since the "information" on the field is carried by the particle performing the random walk, one can say that the "fraction of information" on the value of the field $\xi_0(0)$ which gets to the point x at the moment t falls off as $\frac{\text{const}}{t^{\nu/2}}$. Actually the correlation tends to zero faster, as cancellations occur in the computation, due to the fact that the coefficient $C_\Gamma^{(1)}$ in front of $\frac{1}{t^{\nu/2}}$. in (3.30) does not depend on x_0 and ξ.

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APPENDIX A

Proof of Lemma 3.6. – The coefficients in the expansion (3.4 b), computed in [5], are given by

$$\begin{aligned}\mathcal{T}_{\Gamma, z, \Gamma', z'} = \mu_\Gamma\, \delta_{\Gamma, \Gamma'}\, P_0 (z - z') + \varepsilon \frac{\mu_\Gamma}{\mu_{\gamma (z')}} \\ \times L(\gamma (z'), \gamma' (z'); z - z')\, \delta_{A_0 (z') \Gamma, A_0 (z') \Gamma'}\end{aligned} \tag{A.1a}$$

Here $A_j (x) \Gamma$, $j = 0, \cdots, |S| - 1$, $x \in \mathbb{Z}^\nu$, denotes the multi-index which is obtained from Γ by replacing its value at x by j, and

$$\begin{aligned}L (j, j'; u) = q^j_{j'}\, P_0 (u) + \mu_j \sum_{l=0}^{|S|-1} b^{j'}_{j, l}\, c_l (u) \\ + \varepsilon \sum_{l, k=0}^{|S|-1} q^j_l\, b^{j'}_{k, l}\, c_k (u).\end{aligned} \tag{A.1b}$$

The coefficients are given by (2.6 f), and the following properties, which are easily derived

$$\left.\begin{aligned}c_0 (u) = 0, \qquad c_j (u) = 0, \qquad |u| > D, \\ q^0_k = 0, \qquad b^j_{m, k} = b^j_{k, m}, \qquad b^j_{0, m} = \delta_{j, m}\end{aligned}\right\} \tag{A.1c}$$

It follows in particular that

$$\left.\begin{aligned}L (0, 0; u) = 0, \qquad L (0, m; u) = c_m (u), \\ \sum_{u \in \mathbb{Z}^\nu} L (0, m; u) = 0.\end{aligned}\right\} \tag{A.1d}$$

We introduce, as in [5] the subspaces $\hat{\mathcal{H}}_1^0$, spanned by the vectors $\{ \Psi_{\bar{0}, u} : u \in \mathbb{Z}^\nu \}$, and $\bar{\mathcal{H}}_1^0$, spanned by the vectors $\{ \Psi_{\Gamma, u} : \Gamma \neq \bar{0}\ u \in \mathbb{Z}^\nu \}$. The operator $\mathcal{C}$ defined by formulas (3.13 a, b) can be written as an operator matrix

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}^{00} & \mathcal{C}^{01} \\ \mathcal{C}^{10} & \mathcal{C}^{11} \end{pmatrix} \tag{A.2}$$

where $\mathcal{C}^{00} : \hat{\mathcal{H}}_1^0 \to \hat{\mathcal{H}}_1^0$, $\mathcal{C}^{01} : \bar{\mathcal{H}}_1^0 \to \hat{\mathcal{H}}_1^0$, $\mathcal{C}^{10} : \hat{\mathcal{H}}_1^0 \to \bar{\mathcal{H}}_1^0$, and $\mathcal{C}^{11} : \bar{\mathcal{H}}_1^0 \to \bar{\mathcal{H}}_1^0$. We have $\mathcal{C}^{00} = 0$ and $\mathcal{C}^{11} = 0$, and, setting for convenience $\mathcal{C}^{10} = \mathcal{S}$, $\mathcal{C}^{01} = \mathcal{S}^*$,

$$\mathcal{S}_{\bar{0}, z, \Gamma, z'} = h^z_{\Gamma, z'}, \qquad \mathcal{S}^*_{\Gamma, z, \bar{0}, z'} = -\overline{h^{*z'}_{\Gamma, z}},\ \Gamma \neq \bar{0}, \qquad z, z' \in \mathbb{Z}^\nu.$$

The spaces $\hat{\mathcal{H}}_1$ and $\bar{\mathcal{H}}_1$ are written as

$$\hat{\mathcal{H}}_1 = \{ f + \mathcal{S} f : f \in \hat{\mathcal{H}}_1^0 \}, \qquad \bar{\mathcal{H}}_1 = \{ f + \mathcal{S}^* g : g \in \bar{\mathcal{H}}_1^0 \}.$$

Writing $\mathcal{T}$ as an operator matrix, in the same way as $\mathcal{C}$ in eq. (A.2), the invariance under $\mathcal{T}$ of $\hat{\mathcal{H}}_1$ and $\bar{\mathcal{H}}_1$ is easily seen to be equivalent to the following conditions

$$\left.\begin{aligned} \mathcal{S}\mathcal{T}^{00} + \mathcal{S}\mathcal{T}^{01}\mathcal{S} &= \mathcal{T}^{11}\mathcal{S} + \mathcal{T}^{10}, \\ \mathcal{S}^*\mathcal{T}^{11} + \mathcal{S}^*\mathcal{T}^{10}\mathcal{S}^* &= \mathcal{T}^{00}\mathcal{S}^* + \mathcal{T}^{01}. \end{aligned}\right\} \tag{A.3}$$

Eq. (A.3) implies that if $g \in \bar{\mathcal{H}}_1$ is written in the basis $\{ \psi_{\Gamma, z} : \Gamma \neq \bar{0}, z \in \mathbb{Z}^\nu \}$ as $g = \sum_{\Gamma \neq \bar{0}, z} g_{\Gamma, z} \psi_{\Gamma, z}$, then

$$\mathcal{T}|_{\bar{\mathcal{H}}_1} g = \sum_{\substack{\Gamma \neq \bar{0}, z \\ \Gamma' \neq \bar{0}, z'}} (\mathcal{T}^{11} + \mathcal{T}^{10}\mathcal{S}^*)_{\Gamma, z, \Gamma', z'}\, g_{\Gamma, z}\, \psi_{\Gamma', z'}.$$

Hence the operator $\mathcal{T}|_{\hat{\mathcal{H}}_1}$ is represented in the basis $\{ \psi_{\Gamma, z} \}$ by the matrix

$$\tilde{\mathcal{T}}_{\Gamma, z, \Gamma', z'} = \mathcal{T}_{\Gamma, z, \Gamma', z'} - \sum_u \overline{h^{*u}_{\Gamma, z}}\, \mathcal{T}_{\bar{0}, u, \Gamma', z'}.$$

Consider now the norm (3.29) for g as before. The estimate of $\|\mathcal{T} g\|_{A, \hat{q}}$ can be reduced to the separate estimate of three terms. The first one is bounded from above as follows

$$\begin{aligned} &|\mu_1| \sup_u \sum_{\Gamma, z} |g_{\Gamma, z}| P_0(u - z)\, \hat{q}^{-d(A, u)}\, M^{|\mathrm{supp}\,\Gamma|} \\ &\leqq |\mu_1| \sup_u \sum_{\Gamma, z} |g_{\Gamma, z}| P_0(u - z)\, \hat{q}^{-d(A, z)}\, M^{|\mathrm{supp}\,\Gamma|}\, \hat{q}^{-|u - z|} \\ &\leqslant |\mu_1|\, \|g\|_{A, \hat{q}}\, \hat{q}^{-D}. \end{aligned}$$

Choosing $\hat{q}$ by setting $\log \hat{q} = \frac{\eta}{D} \log |\mu_1|$, with $\eta \in (0, 1)$ the right-hand side is bounded by $|\mu_1|^{1-\eta} \|g\|_{A, \hat{q}}$. The second term, corresponding to the second term on the right of eq. (A.1 b), taking into account that $||\operatorname{supp} \Gamma| - |\operatorname{supp} \Gamma'|| \leqq 1$, and that for fixed Γ', u Γ can take only $|\mathcal{S}|$ values, is estimated by

$$\begin{aligned} &\operatorname{const} \varepsilon M \sup_u \sum_{\Gamma, \Gamma', z} |L(\gamma'(u), \gamma(u); u - z)| \, |g_{\Gamma', z}| M^{|\operatorname{supp} \Gamma'|} \hat{q}^{-d(A, u)} \\ &\quad \leqq \operatorname{constant} \varepsilon M |\mathcal{S}| \hat{q}^{-D} \|g\|_{A, \hat{q}}. \end{aligned}$$

The third term corresponds to the second term on the right in equation (A.3) and, using the estimate (3.8) for h^*, is estimated by

$$\begin{aligned} &\operatorname{const} \varepsilon^2 M \sup_u \sum_{\Gamma, \Gamma', z, v} q^{d_{\operatorname{supp} \Gamma' \cup z \cup v}} |L(0, \gamma(u); u - v)| |g_{\Gamma', z'}| \hat{q}^{-d(A, u)} \\ &\quad \leqq \operatorname{const} \varepsilon^2 M |\mathcal{S}| \sup_u \sum_{\Gamma', z, v} q^{|z-v|} 1_{\{|u-v| < D\}} |g_{\Gamma', z}| \\ &\qquad \times M^{|\operatorname{supp} \Gamma'|} |\hat{q}^{-d(A, z)} \hat{q}^{-|z-u|} \\ &\quad \leqq \operatorname{const} \varepsilon^2 M |\mathcal{S}| \, \|g\|_{A, \hat{q}} \sup_u \sum_{z, v} q^{|z-v|} 1_{\{|u-v| < D\}} \hat{q}^{-|z-u|}, \end{aligned} \tag{A.4}$$

where we used the inequality $d_{\operatorname{supp} \Gamma' \cup z \cup v} > |z - v|$. Taking $\hat{q} \in (q, 1)$, and setting $q = \beta \hat{q}$ we find that the last expression is bounded by

$$\operatorname{const} \varepsilon^2 M |\mathcal{S}| \, \|g\|_{A, \hat{q}} \hat{q}^{-D} D^\nu \sum_u \beta^{|\mu|}$$

and the series converges since $\beta \in (0, 1)$. Putting all together we find

$$\|\mathcal{T}|_{\mathcal{H}_1} g\|_{A, \hat{q}} \leqq |\mu_1|^{(1-\eta)} (1 + \operatorname{constant} \varepsilon) \|g\|_{A, \hat{q}},$$

which proves the Lemma.

Proof of the convergence of the series in (3.32). – We shall prove for simplicity the convergence of the series

$$\sum_\Gamma q^{d_{\operatorname{supp} \Gamma \cup \{0\}}} \beta^{|\operatorname{supp} \Gamma|}$$

for $q \in (0, 1)$ and $\beta < (C e)^{-1}$, with $C = \sum_{x \in \mathbb{Z}^\nu} q^{|x|}$. This implies the convergence of the series in (3.32).

We have

$$\sum_{|\operatorname{supp}\Gamma|=n} q^{d_{\operatorname{supp}\Gamma\cup\{0\}}} \leqq \sum_{|\operatorname{supp}\Gamma|=n} \sum_{T_n} \prod_{(x,y)\in T_n} q^{|x-y|} \tag{A.5}$$

where $\sum_{T_n}$ denotes the sum over all tree-graphs with vertices at the points of the set $\operatorname{supp}\Gamma \cup \{0\}$. If 0 coincides with some point of $\operatorname{supp}\Gamma$ it is convenient to consider it as distinct, so that the trees T_n have always $n+1$ vertices, two of them may be at distance 0. Passing from summation over $\operatorname{supp}\Gamma$ to summation over the ordered sets of points $\{x_1, x_2, \cdots, x_n\}$, and changing the summation order, we find that the right side of (A.5) is bounded by

$$\sum_{\mathcal{T}_n} \frac{1}{n!} \sum_{x_1,\cdots,x_n} \prod_{(i,j)\in\mathcal{T}_n} q^{|x_i-x_j|},$$

where the sum is over all the abstract trees $\mathcal{T}_n$ with $n+1$ vertices. The sum over $x_1, \cdots, x_n$ is estimated by C^n, and, as the number of abstract trees with $n+1$ vertices is estimated by $(n+1)^n$, we get that the series is majorized by the series

$$\sum_{n=0}^{\infty} \frac{(C\beta)^n}{n!} (n+1)^n,$$

which converges if $C\beta e < 1$.

APPENDIX B

We assume Conditions I-IX, which imply that the functions $c_j(\lambda)$, $j = 1, \cdots, |S| - 1$ and $\tilde{p}_0(\lambda)$ are even in λ.

We first prove a result which is interesting by itself on the behavior of the function $\{\chi_\Gamma(\lambda) : \Gamma \neq \bar{0}\}$ solution of eq. (3.6c) for small ε.

PROPOSITION B.1. – *As $\varepsilon \to 0$ the following asymptotics holds.*

$$\chi_\Gamma(\lambda) = \varepsilon \sum_{j=1}^{|S|-1} \tilde{c}_j(\lambda) \sum_{x\in\mathbb{Z}^\nu} \delta_{\Gamma,\Gamma_j^{(x)}} \times \int \frac{e^{-i(\mu,x)}}{\tilde{p}_0(\lambda) - \mu_j \tilde{p}_0(\mu-\lambda)} d\mu + \varepsilon^2 \rho_\Gamma(\lambda), \tag{B.1}$$

where $\tilde{c}_j(\lambda) = \sum_u c_j(u)\, e^{i(\lambda, u)}$. *Moreover the function* $\{\rho_\Gamma : \Gamma \neq \bar{0}\}$ *is uniformly bounded in the norm of* $\mathcal{L}_q$.

Proof. – We recall that $\Gamma_j^{(x)}$ denotes the multi-index with $\operatorname{supp}\Gamma = \{x\}$ and $\gamma(x) = j$.

As in the proof of Lemma 3.1, we consider equation (3.6 c) as an equation in the space $\mathcal{L}_q$, and we rewrite it in the form

$$((1 - \hat{\mathcal{R}}(\lambda))\,\chi(\lambda))_\Gamma = h_\Gamma(\lambda) + g_\Gamma(\lambda) \qquad \Gamma \neq \bar{0}, \tag{B.2}$$

with

$$\left.\begin{aligned} h_\Gamma(\lambda) &= \frac{\hat{R}_{\Gamma,\bar{0}}(\lambda)}{\tilde{p}_0(\lambda)} = \varepsilon\delta_{\Gamma_0,\bar{0}}\frac{\tilde{c}_{\gamma(0)}(\lambda)}{\tilde{p}_0(\lambda)} \\ g_\Gamma(\lambda) &= -\frac{\chi_\Gamma(\lambda)}{\tilde{p}_0(\lambda)} \sum_{\Gamma'\neq\bar{0}} \hat{R}_{\bar{0},\Gamma'}(\lambda)\,\chi_{\Gamma'}(\lambda). \end{aligned}\right\} \tag{B.3}$$

Here $\hat{\mathcal{R}}(\lambda)$ is the operator with matrix elements $(\tilde{p}_0(\lambda)^{-1}\,\hat{R}_{\Gamma,\Gamma'}(\lambda)$, and we have used relations (3.4 a, b), (A.1 a). Γ_0 denotes the multi-index obtained by deleting the point 0 from $\operatorname{supp}\Gamma$. We can write

$$\hat{\mathcal{R}}(\lambda) = \hat{\mathcal{R}}^{(0)}(\lambda) + \varepsilon\,\Delta\hat{\mathcal{R}}(\lambda), \qquad \hat{\mathcal{R}}^{(0)}(\lambda) = \hat{\mathcal{R}}(\lambda)|_{\varepsilon=0},$$

and $\hat{\mathcal{R}}^{(0)}(\lambda)$ is identified by eq. (A.1 a). The Neumann series (of operators on $\mathcal{L}_q$)

$$\begin{aligned} (1 - \hat{\mathcal{R}}(\lambda))^{-1} &= (1 - \hat{\mathcal{R}}^{(0)}(\lambda) - \varepsilon\,\Delta\hat{\mathcal{R}}(\lambda))^{-1} \\ &= (1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1} + (1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1} \\ &\quad \times \sum_{k=1}^{\infty} [(1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1}\,\varepsilon\,\Delta\hat{\mathcal{R}}(\lambda)]^k \end{aligned} \tag{B.4}$$

converges, since, by reasoning as in the proof of Lemma 3.1, one can see that the norm of the operartor $\hat{\mathcal{R}}(\lambda)$ is less than $\dfrac{|\mu_1|}{|\mu_1| + \delta/2} + \mathcal{O}(\varepsilon)$. Hence the operator $1 - \hat{\mathcal{R}}(\lambda)$ is invertible.

Substituting (B.4) in (B.2) we get

$$\begin{aligned} \chi_\Gamma(\lambda) = &\sum_{\Gamma'\neq\bar{0}} (1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1}_{\Gamma,\Gamma'}\,(h_{\Gamma'}(\lambda) + g_{\Gamma'}(\lambda)) \\ &+ \sum_{\Gamma'\neq\bar{0}} \Big[(1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1} \sum_{k=1}^{\infty} \varepsilon^k \end{aligned}$$

$$\times \left[(1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1} \Delta \hat{\mathcal{R}}(\lambda)]^k\right]_{\Gamma, \Gamma'}$$
$$\times (h_{\Gamma'}(\lambda) + g_{\Gamma'}(\lambda)) \tag{B.5}$$

Taking into account the estimate (3.6 b), we see that the only contribution of order ε in (B.5) is given by the term

$$\varphi_\Gamma(\lambda) = \sum_{\Gamma' \neq \bar{0}} (1 - \hat{\mathcal{R}}^{(0)}(\lambda))^{-1}_{\Gamma, \Gamma'} h_{\Gamma'}(\lambda),$$

the rest being at least $\mathcal{O}(\varepsilon^2)$ in the norm of $\mathcal{L}_q$. By the above expression for h_Γ, one sees that φ_Γ is different from zero only if $|\operatorname{supp}\Gamma| = 1$, *i. e.*, if $\Gamma = \Gamma_j^{(x)}$ for some $x \in \mathbb{Z}^\nu$, $j = 1, \cdots, |S| - 1$. Assuming that $\Gamma = \Gamma_j^{(x)}$, taking into account that $\hat{R}^0_{\Gamma, \Gamma'}(\lambda) = \mu_j \sum_{u \in \mathbb{Z}^\nu} \delta_{\Gamma, \Gamma' + u} P_0(u) e^{i(\lambda, u)}$, we get

$$\varphi_\Gamma(\lambda) = \mu_j \sum_u \frac{P_0(u)}{\tilde{p}_0(\lambda)} e^{i(\lambda, u)} \varphi_{\Gamma+u}(\lambda) + \varepsilon \delta_{x,0} \frac{\tilde{c}_j(\lambda)}{\tilde{p}_0(\lambda)}.$$

Using the Fourier transform in the x variable we get

$$\varphi_\Gamma(\lambda) = \varepsilon \tilde{c}_j(\lambda) \int e^{-i(\mu, x)} \frac{1}{\tilde{p}_0(\lambda) - \mu_j \tilde{p}_0(\mu - \lambda)} d\mu. \tag{B.6}$$

This concludes the proof of Proposition B.1. ■

We now turn to the asymptotics as $t \to \infty$ of the integral on the right-hand side of eq. in (3.26 a), which we shall denote by $I(t)$. By applying the classical Laplace method (*see* [9], ch. 8) we find

$$I(t) = \frac{1}{2} \frac{1}{t^{(\nu/2)+1}} (2\pi)^{\nu/2} \frac{1}{(\operatorname{Det} A)^{(1/2)}}$$
$$\times \left[\sum_{k_1, k_2} b_{k_1, k_2} \frac{\partial^2}{\partial \lambda_{k_1} \partial \lambda_{k_2}} g_\Gamma \Big|_{\lambda=0}\right] (1 + o(1)) \tag{B.7}$$

where $A \equiv \{a_{i,j}\}$ is the matrix given by (3.35) $B \equiv \{b_{i,j}\} = A^{-1}$, and the function $g_\Gamma(\lambda)$ is given by eq. (3.26 b). We are interested in the expansion (B.7) for $\Gamma = \Gamma_{j_2}^{(x_2)}$. We shall write g, instead of $g_{\Gamma_{j_2}^{(x_2)}}$, for brevity. The expansion in ϵ of the leading term in right-hand side of eq. (B.7) is given by the following proposition.

PROPOSITION B.2.

$$\frac{\partial^2}{\partial\lambda_{k_1}\,\partial\lambda_{k_2}}\,g\,\Big|_{\lambda=0}=\varepsilon^2\sum_{z_0}\int_{T^\nu}e^{-i\,(\omega',\,x_1-z_0)}\,\frac{\hat{\alpha}\,(\omega')}{1-\mu_{j_1}\,\tilde{p}_0\,(\omega')}\,d\omega'$$

$$\times\sum_x\sum_{j=1}^{|S|-1}\int_{T^\nu}e^{-i\,(\omega,\,x)}\,\frac{\sum_u u_{k_1}\,u_{k_2}\,c_j\,(u)}{1-\mu_j\,\tilde{p}_0\,(\omega)}$$

$$\times\,d\omega\,\langle\,\Psi_{\Gamma_j^{(x)}-x_0},\,\Psi_{\Gamma_{j_2}^{(x_2)}}\,\rangle_\Pi+\varepsilon^3\,\bar{g},\qquad\text{(B.8a)}$$

where

$$\hat{\alpha}\,(\omega)=\sum_u\,[q_0^{j_1}\,P_0\,(u)+\sum_j\,c_j\,(u)\,b_{j_1,\,j}^0\,\mu_{j_1}]\,e^{i(\omega,\,u)},\qquad\text{(B.8b)}$$

and $\bar{g}$ *is finite as* $\varepsilon\to 0$.

Proof. – We recall that the quantities q_j^k, $c_j\,(u)$ and $b_{i,\,j}^k$ are defined by eq. (2.6 f), and $\tilde{p}_0$ is given by eq. (2.3 d). For the proof of Proposition B.1 we need the asymptotics as $\varepsilon\to 0$ of the various terms appearing in the expression for the second derivatives of g. The proof is based on the following two lemmas.

LEMMA B.1

$$\frac{\partial^2}{\partial\lambda_{k_1}\,\partial\lambda_{k_2}}\,g\,\Big|_{\lambda=0}=\sum_{z_0}\overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}\,(0)\,\frac{\partial^2}{\partial\lambda_{k_1}\,\partial\lambda_{k_2}}$$

$$\times\sum_\Gamma\,\chi_{\Gamma-x_0}\,(\lambda)\,\langle\,\Psi_\Gamma,\,\Psi_{\Gamma_{j_2}^{(x_2)}}\,\rangle_\Pi|_{\lambda=0}.$$

LEMMA B.2. – *We have*

$$\overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}\,(0)=-\varepsilon\int_{T^\nu}e^{-i\,(\omega,\,x_1-z_0)}\,\frac{\hat{\alpha}\,(\omega)}{1-\mu_{j_1}\,\tilde{p}_0\,(\omega)}\,d\omega+\varepsilon^2\,g\,(z_0)$$

where $\hat{\alpha}\,(\omega)$ *is defined in eq.* (B.8 b) *and* $\sum_{z_0}|g(z_0)|<\infty$.

We postpone the proof of the lemmas and complete the proof of Proposition B.2.

By Proposition B.1 the derivatives of $\rho_\Gamma\,(\lambda)$ are uniformly bounded in W_d, since the functions $\rho_\Gamma\,(\lambda)$ are analytic in W_d, and the $\mathcal{L}_q$ norm of ρ_Γ

is bounded as $\varepsilon \to 0$. Hence $\Gamma = \Gamma_j^{(x)}$ we have

$$\frac{\partial^2}{\partial \lambda_{k_1}\, \partial \lambda_{k_2}}\, \chi_\Gamma(\lambda)\Big|_{\lambda=0} = -\varepsilon \int_{T^\nu} e^{-i(\omega,x)} \frac{\sum_u u_{k_1}\, u_{k_2}\, c_j(u)}{1 - \mu_j\, \tilde{p}_0(\omega)}\, d\omega + \varepsilon^2\, \phi(x), \tag{B.9}$$

with $\sum_x |\phi(x)| < \infty$. Proposition B.2 now follows from Lemmas B.1 and B.2. We actually need to take the sum on z_0, and we get as the dominant term of the expression (B.8 a)

$$\frac{q_0^{j_1}}{1-\mu_{j_1}} \sum_x \sum_{j=1}^{|S|-1} \int_{T^\nu} e^{-i(\omega,x)} \frac{\sum_u u_{k_1}\, u_{k_2}\, c_j(u)}{1-\mu_j\, \tilde{p}_0(\omega)} \times \langle \Psi_{\Gamma_j^{(x)}-x_0}, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi\, d\omega. \quad \blacksquare$$

We now pass to the proof of the lemmas.

Proof of Lemma B.1. – The proof is a straightforward calculation, namely

$$\begin{aligned}
\frac{\partial^2}{\partial \lambda_{k_1}\, \partial \lambda_{k_2}}\, g\Big|_{\lambda=0} = \sum_{z_0} \Bigg\{ & \frac{\partial^2}{\partial \lambda_{k_1}\, \partial \lambda_{k_2}} \\
& \times \left[\frac{e^{-i(\lambda, x_1 - z_0)}}{(\chi(\lambda),\, \chi^*(\lambda))_{l_2(\mathfrak{M})}}\, \overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}(\lambda) \right] \Bigg|_{\lambda=0} \\
& \times \sum_\Gamma \chi_{\Gamma - x_0}(0)\, \langle \Psi_\Gamma, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi \\
& + \frac{\partial}{\partial \lambda_{k_1}} \left[\frac{e^{-i(\lambda, x_1 - z_0)}}{(\chi(\lambda),\, \chi^*(\lambda))_{l_2(\mathfrak{M})}}\, \overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}(\lambda) \right] \Bigg|_{\lambda=0} \\
& \times \frac{\partial}{\partial \lambda_{k_2}} \sum_\Gamma \chi_{\Gamma - x_0}(\lambda)\, \langle \Psi_\Gamma, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi \big|_{\lambda=0} \\
& + \frac{\partial}{\partial \lambda_{k_2}} \left[\frac{e^{-i(\lambda, x_1 - z_0)}}{(\chi(\lambda),\, \chi^*(\lambda))_{l_2(\mathfrak{M})}}\, \overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}(\lambda) \right] \Bigg|_{\lambda=0} \\
& \times \frac{\partial}{\partial \lambda_{k_1}} \sum_\Gamma \chi_{\Gamma - x_0}(\lambda)\, \langle \Psi_\Gamma, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi \Big|_{\lambda=0} \\
& + \overline{\chi^*_{\Gamma_{j_1}^{(x_1)}-z_0}}(0)\, \frac{\partial^2}{\partial \lambda_{k_1}\, \partial \lambda_{k_2}} \sum_\Gamma \chi_{\Gamma - x_0}(\lambda) \\
& \times \langle \Psi_\Gamma, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi \Big|_{\lambda=0} \Bigg\}.
\end{aligned} \tag{B.10}$$

The first term on the r.h.s. of (B.10) is zero, as $\chi_\Gamma(0) = 0$ for $\Gamma \neq \bar{0}$. For $\Gamma = \bar{0}$ we have $\Psi_{\bar{0}} \equiv 1$ and obviously

$$\langle 1, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_\Pi = 0. \tag{B.11}$$

For the second and third term in (B.10) we need to compute $\frac{\partial}{\partial \lambda_k} \chi_\Gamma(\lambda) \Big|_{\lambda=0}$.

Obviously $\frac{\partial}{\partial \lambda_k} \chi_{\bar{0}}(\lambda) = 0$ for all $\lambda \in T^\nu$, and if $\Gamma = \bar{0}$ then a straighforward calculation, which makes use of the symmetry of the function $c(\cdot, \cdot)$, gives

$$\frac{\partial}{\partial \lambda_k} \chi_\Gamma(\lambda) \Big|_{\lambda=0} = \sum_{\Gamma' \neq \bar{0}} \frac{\hat{R}_{\Gamma,\Gamma'}(\lambda)}{\tilde{p}_0(\lambda)} \frac{\partial}{\partial \lambda_k} \chi_{\Gamma'}(\lambda) \Big|_{\lambda=0}, \qquad \Gamma \neq \bar{0}. \tag{B.12}$$

(B.12) is a linear equation for the quantities $\frac{\partial}{\partial \lambda_k} \chi_\Gamma(\lambda) \Big|_{\lambda=0}$, $\Gamma \neq \bar{0}$, which can be considered, for example on a subspace of $\mathcal{H}_M$. Since the operator with matrix elements $\{\hat{R}_{\Gamma,\Gamma'} : \Gamma, \Gamma' \neq \bar{0}\}$ is bounded in this space, as it follows from formulas (A.1 a,b), and (3.6 c), with a norm $|\mu_1| + \mathcal{O}(\varepsilon)$, eq. (B.12) has, as a unique solution, the $\frac{\partial}{\partial \lambda_k} \chi_\Gamma(\lambda) \Big|_{\lambda=0} = 0$. Lemma B.1 is proved. ■

Proof of Lemma B.2. – The proof could be based on a small ϵ expansion for χ_Γ^* similar to the one established in Proposition B.1 for χ_Γ. To be short, observe that the following equations holds, as a consequence of the invariance of χ^* under $\mathcal{T}^*$ [5],

$$\overline{\chi^*_{\Gamma_{j_1}^{(x_1)} - z_0}}(0) = \sum_{\Gamma' \neq 0} \hat{S}_{\Gamma_{j_1}^{(x_1)} - z_0, \Gamma'}(0) \overline{\chi^*_{\Gamma'}}(0) - \hat{S}_{\Gamma, \bar{0}}(0)$$

where

$$\hat{S}_{\Gamma,\Gamma'}(\lambda) = \overline{\hat{R}_{\Gamma',\Gamma}(\lambda)}$$

Lemma B.2 is then proved by solving this equation with the methods used above. ■

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