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Multiplicities of a random sausage

by

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ABSTRACT. – Consider a particle that executes a transient random walk or a transient Lévy process on some group. Attach a set to the particle and trace out a sausage. Each point in the sausage that has been traced out over the inverval $[0, t]$ has an associated multiplicity–the amount of time in $[0, t]$ that the point has been covered by the moving set. Using potential theory, we investigate the asymptotics as $t \to \infty$ of the ensemble of multiplicities. Our results involve some interesting connections with the theory of Fredholm integral equations.

Key words: Random walk, Lévy process, sausage, capacity, integral equation, subadditive ergodic theory.

RÉSUMÉ. – On considère une particule qui se déplace selon une marche aléatoire transiente ou selon un processus de Lévy à valeurs dans un groupe. Cette particule entraîne avec elle un ensemble dont le déplacement crée une saucisse. A chaque point de la saucisse créée dans l'intervalle $[0, t]$ est associée une multiplicité : le temps pendant lequel ce point a été recouvert par l'ensemble mobile. En utilisant la théorie du potentiel, nous examinons les propriétés asymptotiques de l'ensemble des multiplicités lorsque $t \to \infty$. Nos résultats font intervenir des liens intéressants avec la théorie des équations intégrales de Fredholm.

1. INTRODUCTION AND STATEMENT OF RESULTS

We will be concerned primarily with certain questions regarding Lévy processes on locally compact groups in this paper, but in order to better

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understand the content of our results let us start by introducing the analogous questions for random walks. Suppose that $X = \{X_n, \mathbb{P}^x\}$ is a discrete time parameter, transient, right random walk on some countable, discrete group G. Write the group operation on G as $(x, y) \mapsto xy$ and let e denote the identity element. Note that G is not necessarily Abelian. Set $U = (U(x, y)) = \left(\sum_{n=0}^{\infty} \mathbb{P}^x \{X_n = y\}\right)$. Given a non-empty, finite set $A \subset G$, form the successive A-*sausages*

$$R_n = \bigcup_{k=0}^{n} \{X_k A^{-1}\}, \quad n = 0, 1, 2, \ldots$$

For each $x \in G$ let

$$N_n^x = |\{k : x \in X_k A^{-1}, \quad 0 \leqq k \leqq n\}|$$

be the *multiplicity* of the point x for R_n. Finally, let Π_n denote the empirical distribution of the number $\{N_n^x : x \in R_n\}$. That is, Π_n is the random probability measure on $\{1, 2, 3, \ldots\}$ given by

$$\Pi_n(\{j\}) = \frac{|\{x \in R_n : N_n^x = j\}|}{|R_n|}.$$

Typically, most of the mass of Π_n will be concentrated near 1 if and only if the walk X is, in some sense, "very transient" and the set A is small. We are interested in how X and A interact to determine the asymptotics of Π_n as $n \to \infty$.

Although the sequence of random probability measures $\{\Pi_n\}_{n=0}^{\infty}$ does not appear to have been considered in the literature, the associated sequence of random expectations

$$\sum_{j=1}^{\infty} j\, \Pi_n(\{j\}) = \frac{(n+1)\,|A|}{|R_n|}$$

has been considered by a number of authors. One can adapt the continuous time parameter arguments of Getoor (1965) and Spitzer in Kingman (1973) to show that $|R_n|/(n+1) \to C(A)$ as $n \to \infty$, where $C(A)$ is the capacity of the set A [*see* § 8.3 of Kemeny *et al.* (1976) for a discussion of capacities for Markov chains]. As a consequence of our continuous time parameter results, we obtain the following related asymptotics for $\{\Pi_n\}_{n=0}^{\infty}$. Let U_A denote the restriction of U to $A \times A$.

THEOREM 1.1. – *There exists a non-random probability measure* Π *such that* $\Pi_n \to \Pi$ *almost surely an* $n \to \infty$. *For each* $s \in [0, 1]$ *the equation*

$$sg(x) + (1-s)\, U_A\, g(x) = s, \quad x \in A, \tag{1.2}$$

has a unique solution, which we denote by g_s. The probability generating function of Π is given by

$$\sum_{j=1}^{\infty} s^j \,\Pi(\{j\}) = 1 - \frac{(1-s)\sum_{x \in \mathrm{A}} g_s(x)}{\mathrm{C}(\mathrm{A})}, \quad s \in [0, 1].$$

Let us now turn to the counterpart of this investigation for Lévy processes. With some abuse of notation, we will use similar notation for objects that are the analogues of ones above. When we need to distinguish which situation we are dealing with, we will use the terms *discrete or continuous.*

Let G be a separable, locally compact group, with the group operation written as $(x, y) \mapsto xy$ and identity element e. We do not suppose that G is Abelian. Write $\mathcal{G}$ for the Borel σ-field of G. Assume that G is unimodular; that is, the left and right Haar measures coincide when suitably normalised. Some classes of groups with this property are Abelian groups, discrete groups, semisimple Lie groups and connected nilpotent Lie groups. Let λ be a choice of a common Haar measure.

Suppose that $\mathrm{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{X}_t, \theta_t, \mathbb{P}^x)$ is a right Lévy process on G. That is, X is a conservative, G-valued Hunt process with the property that $\mathbb{P}^e \{\mathrm{X}_t \in \mathrm{B}\} = \mathbb{P}^x \{\mathrm{X}_t \in x\,\mathrm{B}\}$ for all $t \geqq 0$, $\mathrm{B} \in \mathcal{G}$ and $x \in \mathrm{G}$. Let $\{\mathrm{U}^\alpha\}_{\alpha \geqq 0}$ denote the family of potential operators of X. Assume that compact sets are transient for X. This is equivalent to requiring that $\mathrm{U}^0(e, \mathrm{K}) < \infty$ for all compact sets $\mathrm{K} \subset \mathrm{G}$ [*see* Getoor (1980) for a thorough discussion of various equivalent definitions of transience].

Given a set $\mathrm{A} \in \mathcal{G}$ with a compact closure and $\lambda(\mathrm{A}) > 0$, set

$$\mathrm{R}_t = \bigcup_{s \in [0, t]} \{\mathrm{X}_s\, \mathrm{A}^{-1}\}, \quad t \geqq 0.$$

For each $x \in \mathrm{G}$ put

$$\mathrm{N}_t^x = m\{s : x \in \mathrm{X}_s\, \mathrm{A}^{-1},\ 0 \leqq s \leqq t\},$$

where m is Lebesgue measure on $\mathbb{R}$; and define a random measure Π_t on $\mathbb{R}_+$ by

$$\Pi_t(\mathrm{B}) = \frac{\lambda\{x \in \mathrm{R}_t : \mathrm{N}_t^x \in \mathrm{B}\}}{\lambda(\mathrm{R}_t)}.$$

In order to obtain an analogue of Theorem 1.1 in this setting, we need to impose some regularity conditions on X. Specifically, we assume that for each $\alpha \geqq 0$ and each $x \in \mathrm{G}$ the measure $\mathrm{B} \mapsto \mathrm{U}^\alpha(x, \mathrm{B})$ is absolutely continuous with respect to λ. Using arguments almost identical to those in Hawkes (1979), one can show that this property is equivalent to the property that each operator U^{α}, $\alpha > 0$, is strong Feller (that is, maps bounded measurable functions into bounded continuous functions) or to the property that the α-excessive functions are all lower semicontinuous for each $\alpha \geqq 0$. In this case there exist Borel functions $u^{\alpha} : G \to [0, \infty]$ such that

$$\int_B u^{\alpha}(x^{-1} y) \lambda(dy) = U^{\alpha}(x, B).$$

Let $C(A)$ denote the capacity of A (*see* § 2 and the start of § 3 for a discussion of the relevant potential theory). Write U_A for the operator that maps the set of bounded Borel functions on A into itself by $U_A f(x) = \int_A u^0(x^{-1} y) f(y) \lambda(dy)$, $x \in A$.

THEOREM 1.3. – *There exists a non-random probability measure* Π *such that* $\Pi_t \to \Pi$ *almost surely as* $t \to \infty$. *For each* $\beta \in [0, \infty[$ *the equation*

$$g(x) + \beta U_A g(x) = 1, \quad x \in A, \tag{1.4}$$

has a bounded, nonnegative solution. This solution is unique and we denote it by g_{β}. *The Laplace transform of* Π *is given by*

$$\int_0^{\infty} e^{-\beta u} \Pi(du) = 1 - \frac{\beta}{C(A)} \int_A g_{\beta} \, d\lambda, \quad \beta \in [0, \infty[.$$

Equation (1.4) is a Fredholm integral equation of the second type [*see*, for example, Ch IV of Riesz and Sz.-Nagy (1990)]. In special situations, it is possible to use the theory that has been developed for such equations to analyse the structure of the solutions to (1.4). In particular, we can apply the spectral theory of self-adjoint operators on Hilbert space when the process X is symmetric; that is when $u^{\alpha}(z) = u^{\alpha}(z^{-1})$ for $\alpha \geqq 0$ and $z \in G$. All the operator theory we use may be found in Chs XIII and IX of Riesz and Sz.-Nagy (1990).

Write $\langle \cdot, \cdot \rangle_A$ for the inner product in the complex Hilbert space $L^2(A, \lambda)$. Let $\mathcal{D}$ denote the set of functions $f \in L^2(A, \lambda)$ such that

$$\int_A u^0(x^{-1} y) f(y) \lambda(dy)$$

exists for λ-almost all $x \in A$ and

$$x \mapsto \int_A u^0(x^{-1} y) f(y) \lambda(dy)$$

belongs to $L^2(A, \lambda)$. Observe that $\mathcal{D}$ contains all the bounded functions, and hence is dense in $L^2(A, \lambda)$. It is not hard to check that if we extend our definition of U_A and regard it as a (possibly) unbounded operator on $L^2(A, \lambda)$ with domain $\mathcal{D}$, then U_A is closed. Moreover, U_A is self-adjoint and nonnegative definite. There therefore exists a spectral family of projections $\{E_\gamma\}_{\gamma \geqq 0}$ such that U_A has the spectral representation

$$U_A = \int_{0-}^{\infty} \gamma\, dE_\gamma.$$

COROLLARY 1.5. – *Suppose that* X *is symmetric. Suppose also that* A *is open. In the above notation,* Π *has the mixture distribution*

$$\Pi = \int_{0+}^{\infty} \Gamma_\gamma\, p(d\gamma);$$

where Γ_γ *is the exponential distribution with mean* γ, *and*

$$p(d\gamma) = \frac{\gamma^{-1}\, d\langle E_\gamma 1, 1\rangle}{\displaystyle\int_{0+}^{\infty} \eta^{-1}\, d\langle E_\eta 1, 1\rangle}, \qquad \gamma > 0.$$

In particular, Π *is infinitely divisible.*

The plan of the rest of the paper is a follows. In § 2 we give a brief overview of the general Markov potential theory we use. In § 3 we prove Theorem 1.3 and give a sketch of the proof of Theorem 1.1. We prove Corollary 1.5 in § 4. Finally, in § 5 we consider the case of Brownian motion with drift on $\mathbb{R}$. When A is an interval, it is possible to solve (1.4) in closed form. Again using the theory of Fredholm integral equations, we also obtain the rather unexpected result that Π is an infinite convolution of exponential distributions in this case.

2. AN OVERVIEW OF SOME POTENTIAL THEORY

In this section we will recall some notation and ideas from Markov potential theory. The reader is referred to Getoor (1984) for the details.

Let E be a locally compact, separable space. Suppose that $Y = (\Sigma, \mathcal{H}, \mathcal{H}_t, Y_t, \psi_t, \mathbb{Q}^y)$ (respectively, $\hat{Y} = (\hat{\Sigma}, \hat{\mathcal{H}}, \hat{\mathcal{H}}_t, \hat{Y}_t, \hat{\psi}_t, \hat{\mathbb{Q}}^y)$) is a conservative Hunt process on E. Let $\{V^\alpha\}_{\alpha \geqq 0}$ (respectively, $\{\hat{V}^\alpha\}_{\alpha \geqq 0}$) denote the family of potential operators of Y (respectively, $\hat{Y}$).

Suppose that Y and $\hat{Y}$ are in strong duality with respect to some σ-finite measure μ. That is, $V^\alpha(y, \cdot) \ll \mu$ and $\hat{V}^\alpha(y, \cdot) \ll \mu$ for all $y \in E$ and $\alpha \geqq 0$, and

$$\int (V^\alpha f) \cdot g\, d\mu = \int f \cdot (\hat{V}^\alpha g)\, d\mu$$

for all $\alpha \geqq 0$ and all nonnegative, Borel f and g.

Suppose that the α-excessive functions for Y and the α-excessive functions for $\hat{Y}$ (that is, the α-co-excessive functions) are lower semi continuous for all $\alpha \geqq 0$.

Finally, assume that compact sets are transient for Y (respectively, $\hat{Y}$). This is equivalent to requiring that $y \mapsto V^0(y, K)$ (respectively, $y \mapsto \hat{V}^0(y, K)$) is bounded for all compact sets $K \subset E$. As a consequence of the above, for each $y \in E$ and each $\alpha \geqq 0$ there is a unique α-co-excessive function $v^\alpha(y, \cdot)$ such that

$$V^\alpha(y, dz) = v^\alpha(y, z)\,\mu(dz).$$

Functions $\hat{v}^\alpha(y, z)$ are defined similarly for $\hat{V}^\alpha$ and we have $\hat{v}^\alpha(y, z) = v^\alpha(z, y)$.

Let $B \subset E$ be a Borel set with compact closure. Set

$$T_B = \inf\{t > 0 : Y_t \in B\}$$

and

$$\phi_B(y) = \mathbb{Q}^y\{T_B < \infty\}, \quad y \in E.$$

Define $\hat{T}_B$ and $\hat{\phi}_B$ similarly in terms of $\hat{Y}$. There is a unique finite measure π_B such that

$$\phi_B(y) = \int v^0(y, z)\,\pi_B(dz),$$

for all $y \in E$. A measure $\hat{\pi}_B$ is determined similarly by $\hat{\phi}_B$. The measures π_B and $\hat{\pi}_B$ are called, respectively, the capacitary and co-capacitary measures of B. The quantities $C(B) = \pi_B(E)$ and $\hat{C}(B) = \hat{\pi}_B(E)$ are called, respectively, the capacity and co-capacity of B.

We will require the following two facts. Firstly, the measure π_B is concentrated on $B \cup B^{cr}$, where B^{cr} is the set of co-regular points for B, and an analogous result holds for $\hat{\pi}_B$. Secondly,

$$C(B) = \hat{C}(B). \tag{2.1}$$

3. PROOFS OF THEOREM 1.1 AND THEOREM 1.3

Let us begin with the proof of the continuous result Theorem 1.3. The proof of the discrete result Theorem 1.1 is outlined briefly at the end of the section.

We first recall the construction of a strong dual for X with respect to λ. Let u^α be as in § 1. Set

$$\hat{u}^\alpha(z) = u^\alpha(z^{-1}), \quad z \in \mathrm{G},$$

so that

$$\hat{u}^\alpha(x^{-1} y) = u^\alpha(y^{-1} x). \tag{3.1}$$

Define a kernel $\hat{\mathrm{U}}^\alpha$ on G by

$$\hat{\mathrm{U}}^\alpha(x, \mathrm{B}) = \int_{\mathrm{B}} \hat{u}^\alpha(x^{-1} y)\, \lambda(dy).$$

Observe that

$$\begin{aligned}\hat{\mathrm{U}}^\alpha f(x) &= \int u^\alpha(y^{-1} x) f(y)\, \lambda(dy)\\ &= \int u^\alpha((xy)^{-1} x) f(xy)\, \lambda(dy)\\ &= \int u^\alpha(y^{-1}) f(xy)\, \lambda(dy)\\ &= \int u^\alpha(y) f(xy^{-1})\, \lambda(dy),\end{aligned} \tag{3.2}$$

where the last equality follows the fact that the transformation $y \mapsto y^{-1}$ is measure-preserving for a unimodular group. In particular, the definition of $\hat{\mathrm{U}}^\alpha$ is independent of the version of u^α that we use and $\hat{\mathrm{U}}^\alpha(x, G) = \int u^\alpha(y)\, \lambda(dy) = 1/\alpha$. Note from (3.1) that

$$\int (\mathrm{U}^\alpha f) \cdot g\, d\lambda = \int f \cdot (\hat{\mathrm{U}}^\alpha g)\, d\lambda. \tag{3.3}$$

It is straightforward to use (3.2) and the resolvent equation for $\{\mathrm{U}^\alpha\}_{\alpha>0}$ to show that the family $\{\hat{\mathrm{U}}^\alpha\}_{\alpha>0}$ also obeys the resolvent equation.

The family $\{\hat{\mathrm{U}}^\alpha\}_{\alpha>0}$ is therefore a resolvent in the sense of Definition VI.1.1 of Blumenthal and Getoor (1968). Following Remark VI.1.14 in Blumenthal and Getoor (1968), it is easy to check a Feller condition and conclude that $\{\hat{\mathrm{U}}^\alpha\}_{\alpha\geqq 0}$ is the family of potential operators of a conservative Hunt process, which we will call $\hat{\mathrm{X}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathrm{X}}_t, \hat{\theta}_t, \hat{\mathbb{P}}^x)$. The process $\hat{\mathrm{X}}$ is also a right Lévy process for which, by (3.2), compact sets are transient. By (3.3), X and $\hat{\mathrm{X}}$ are in strong duality with respect to λ. Note that λ is an invariant measure for both X and $\hat{\mathrm{X}}$.

Having disposed of these preliminaries, let us check that Π_t converges weakly to some non-random probability measure Π almost surely as $t \to \infty$. Define a random finite measure M_t on $\mathbb{R}_+$ by

$$M_t(B) = \lambda\{x \in R_t : N_t^x \in B\}.$$

For each $s \geqq 0$, the process $t \mapsto M_t([0, s])$ satisfies the conditions of Kingman's subadditive ergodic theorem [*see* Kingman (1973)]; and hence $t^{-1} M_t([0, s])$ converges almost surely and in L^1 as $t \to \infty$. By Kolmogorov's zero-one law, this limit is a constant. As

$$\int s\, M_t(ds) = \int_{R_t} N_t^x\, \lambda(dx) = t\, \lambda(A),$$

it follows that the family $\{t^{-1} M_t\}_{t>0}$ is tight almost surely, and hence there is a non-random finite measure M such that $t^{-1} M_t$ converges weakly to M almost surely as $t \to \infty$. Moreover,

$$\lim_{t\to\infty} \mathbb{P}^e\left[t^{-1} \int f(s)\, M_t(ds)\right] = \int f(s)\, M(ds)$$

for all bounded, continuous functions f.

From a similar argument, we also have

$$M(\mathbb{R}_+) = \lim_{t\to\infty} t^{-1} M_t(\mathbb{R}_+) = \lim_{t\to\infty} t^{-1} \lambda(R_t) = \lim_{t\to\infty} t^{-1}\, \mathbb{P}^e[\lambda(R_t)].$$

Now

$$\begin{aligned}
\mathbb{P}^e[\lambda(R_t)] &= \int \mathbb{P}^e\{\exists\, 0 \leqq s \leqq t : x \in X_s A^{-1}\}\, \lambda(dx) \\
&= \int \mathbb{P}^e\{\exists\, 0 \leqq s \leqq t : X_s \in A\}\, \lambda(dx) \\
&= \mathbb{P}^\lambda\{\exists\, 0 \leqq s \leqq t : X_s \in A\};
\end{aligned}$$

and so, by Theorem 1 of Getoor (1965),

$$M(\mathbb{R}_+) = C(A) > 0.$$

Thus there is a non-random probability measure Π such that Π_t converges weakly to Π almost surely at $t \to \infty$.

We will now calculate $\int e^{-\beta u}\, M(du)$ for some fixed $\beta > 0$. Set

$$D_t = m\{0 \leqq s \leqq t : X_s \in A\} = \int_0^t 1_A(X_s)\, ds.$$

Observe that

$$\mathbb{P}^e \left[\mathrm{M}_t (\mathbb{R}_+) - \int e^{-\beta u} \, \mathrm{M}_t (du) \right]$$

$$\begin{aligned}
&= \mathbb{P}^e \left[\int (1 - \exp(-\beta m \{s : x \in \mathrm{X}_s \, \mathrm{A}^{-1}, \, 0 \leqq s \leqq t\})) \, \lambda (dx) \right] \\
&= \int \mathbb{P}^{x^{-1}} [1 - \exp(-\beta \, \mathrm{D}_t)] \, \lambda (dx) \\
&= \mathbb{P}^\lambda [1 - \exp(-\beta \, \mathrm{D}_t)] \\
&= \mathbb{P}^\lambda \left[\int_0^t \frac{d}{ds} \exp(-\beta \, \mathrm{D}_{t-s} \circ \theta_s) \, ds \right] \\
&= \mathbb{P}^\lambda \left[\beta \int_0^t \exp(-\beta \, \mathrm{D}_{t-s} \circ \theta_s) \, 1_\mathrm{A} (\mathrm{X}_s) \, ds \right] \\
&= \beta \, \mathbb{P}^\lambda \left[\int_0^t \mathbb{P}^{\mathrm{X}_s} [\exp(-\beta \, \mathrm{D}_{t-s}) \, 1_\mathrm{A} (\mathrm{X}_s) \, ds \right] \\
&= \beta \int_0^t \int_\mathrm{A} \mathbb{P}^x [\exp(-\beta \, \mathrm{D}_{t-s})] \, \lambda (dx) \, ds \\
&= \beta \int_0^t \int_\mathrm{A} \mathbb{P}^x [\exp(-\beta \, \mathrm{D}_{t-s})] \, \lambda (dx) \, ds.
\end{aligned}$$

Hence, if we set $h(x) = \mathbb{P}^x [\exp(-\beta \, \mathrm{D}_\infty)]$, then we have

$$\lim_{t \to \infty} t^{-1} \, \mathbb{P}^e \left[\mathrm{M}_t (\mathbb{R}_+) - \int e^{-\beta u} \, \mathrm{M}_t (du) \right] = \beta \int_\mathrm{A} h(x) \, \lambda (dx).$$

Therefore,

$$\int e^{-\beta u} \, \mathrm{M} (du) = \mathrm{C} (\mathrm{A}) - \beta \int_\mathrm{A} h(x) \, \lambda (dx)$$

and

$$\int e^{-\beta u} \, \Pi (du) = 1 - \frac{\beta}{\mathrm{C} (\mathrm{A})} \int_\mathrm{A} h(x) \, \lambda (dx).$$

Except for the question of uniqueness, the proof will be completed if we can show that the restriction of h to A satisfies (1.4). To this end, define an operator $\tilde{\mathrm{U}}^0$ by

$$\tilde{\mathrm{U}}^0 f(x) = \mathbb{P}^x \left[\int_0^\infty \exp(-\beta \, \mathrm{D}_t) \, f(\mathrm{X}_t) \, dt \right]$$

for bounded Borel functions f. From the resolvent form of the Feynman-Kac formula [*see*, for example, § III.39 of Williams (1979)] we have

$$\mathrm{U}^0 f = \tilde{\mathrm{U}}^0 f + \mathrm{U}^0 (\beta 1_{\mathrm{A}} \tilde{\mathrm{U}}^0 f).$$

Thus

$$\begin{aligned} 1 - h(x) &= -\mathbb{P}^x \left[\int_0^\infty \frac{d}{dt} \exp(-\beta \mathrm{D}_t)\, dt \right] \\ &= \mathbb{P}^x \left[\int_0^\infty \exp(-\beta \mathrm{D}_t)\, \beta 1_{\mathrm{A}}(\mathrm{X}_t)\, dt \right] \\ &= \beta \tilde{\mathrm{U}}^0 1_{\mathrm{A}}(x) \\ &= \beta \{ \mathrm{U}^0 1_{\mathrm{A}}(x) - \mathrm{U}^0 (\beta 1_{\mathrm{A}} \tilde{\mathrm{U}}^0 1_{\mathrm{A}})(x) \} \\ &= \beta \{ \mathrm{U}^0 1_{\mathrm{A}}(x) - \mathrm{U}^0 (1_{\mathrm{A}} (1 - h))(x) \} \\ &= \beta \mathrm{U}^0 (1_{\mathrm{A}} h)(x), \end{aligned}$$

as required.

Finally, let us show that (1.4) has a unique bounded, nonnegative solution. Suppose that g and g' are two such solutions. Let f be the function on G that is equal to $g - g'$ on A and is zero elsewhere. As $f + \beta \mathrm{U}^0 f = 0$ on A, we conclude that on the set $\{f^- > 0\} \subset \mathrm{A}$ we have

$$\mathrm{U}^0 f^+ = \mathrm{U}^0 f^- - \beta^{-1} f = \mathrm{U}^0 f^- + \beta^{-1} f^- \geqq \mathrm{U}^0 f^-.$$

From the domination principle [*see*, for example, § XII.28 of Dellacherie and Meyer (1988)], we see that $\mathrm{U}^0 f^+ \geqq \mathrm{U}^0 f^-$ everywhere. Similarly, $\mathrm{U}^0 f^- \geqq \mathrm{U}^0 f^+$ everywhere, and hence $\mathrm{U}^0 f^+ \geqq \mathrm{U}^0 f^-$. Thus $f = -\beta \mathrm{U}^0 f = 0$ and $g = g'$, as required. □

COROLLARY 3.4. – *The probability measure* Π *is the distribution of* D_∞ *under* $\mathbb{P}^{\hat{\rho}_{\mathrm{A}}}$, *where* $\hat{\rho}_{\mathrm{A}} = \hat{\pi}_{\mathrm{A}} / \hat{\pi}_{\mathrm{A}}(\mathrm{G}) = \hat{\pi}_{\mathrm{A}} / \mathrm{C}(\mathrm{A})$.

Proof. – Fix $\beta > 0$. As in the proof of the theorem, set

$$h(x) = \mathbb{P}^x [\exp(-\beta \mathrm{D}_\infty)].$$

From the calculations in the theorem we have,

$$\begin{aligned}\int_0^\infty e^{-\beta_u}\,\Pi(du) &= 1 - \frac{\beta}{C(A)}\int_A g_\beta\,d\lambda \\ &= 1 - \frac{\beta}{C(A)}\int_G 1_A\,h\,d\lambda \\ &= 1 - \frac{\beta}{C(A)}\int_G (\hat{U}^0\,\hat{\pi}_A)\cdot(1_A\,h)\,d\lambda \\ &= 1 - \frac{\beta}{C(A)}\int_G U^0\,(1_A\,h)\;d\hat{\pi}_A \\ &= \int_G \{1 - \beta\,U^0\,(1_A\,h)\}\,d\hat{\rho}_A \\ &= \int_G h\,d\hat{\rho}_A \\ &= \int_G \mathbb{P}^x\,[\exp(-\beta\,D_\infty)]\,d\hat{\rho}_A.\end{aligned}$$

The third equality follows from the fact that $\{x \in A : \hat{U}^0\,\hat{\pi}_A\,(x) < 1\} \subset A\backslash A^{cr}$ is semipolar and hence has zero Haar measure [*cf.* Proposition II.3.3 of Blumenthal and Getoor (1968)]. The fourth equality follows from duality. □

Let us now turn to a brief sketch of the proof of Theorem 1.1. We can associate the discrete time parameter process $\{X_n\}_{n=0}^\infty$ with a continuous time parameter process $\{\bar{X}_t\}_{t\geqq 0}$ by the usual device of introducing exponential holding times with expectation 1. That is, if $(P(x, y)) = (\mathbb{P}^x\{X_1 = y\})$ is the transition matrix of $\{X_n\}_{n=0}^\infty$, then $\{\bar{X}_t\}_{t\geqq 0}$ has semigroup given by

$$\mathbb{P}^x\{\bar{X}_t = y\} = \sum_{k=0}^{\infty}(e^{-t}\,t^k/k\,!)\,P^k\,(x, y).$$

It is easy to check that conditions of Theorem 1.3 apply to $\{\bar{X}_t\}_{t\geqq 0}$. Moreover, the capacity operator for $\bar{X}$ (with λ taken to be counting measure) coincides with the usual Markov chain definition of the capacity operator for X with counting measure as the reference measure [*see* § 8.3 of Kemeny *et al.* (1976)].

Combining a strong law of large numbers argument with the conclusion of Theorem 1.3 applied to $\bar{X}$ shows that Π_n converges almost surely as

$n \to \infty$ to a non-random measure Π with

$$\sum_{j=1}^{\infty} \left(\frac{1}{1+\beta}\right)^j \Pi(\{j\}) = 1 - \frac{\beta}{\mathrm{C}(\mathrm{A})} \sum_{x \in \mathrm{A}} \bar{g}_\beta(x), \quad \beta \in [0, \infty[,$$

where $\bar{g}_\beta$ is the unique solution of

$$\bar{g}(x) + \beta\, \mathrm{U}_\mathrm{A}\, \bar{g}(x) = 1, \quad x \in \mathrm{A}.$$

Setting $\beta = (1-s)/s$, we obtain the claim of Theorem 1.1.

4. SYMMETRIC PROCESSES — PROOF OF COROLLARY 1.5

Recalling the discussion prior to the statement of the corollary, we have the spectral representation

$$\mathrm{U}_\mathrm{A} = \int_{0-}^{\infty} \gamma\, d\mathrm{E}_\gamma.$$

Thus the bounded operator $(\mathrm{I} + \beta\, \mathrm{U}_\mathrm{A})^{-1}$ is well-defined for all $\beta \geqq 0$, and we have

$$(\mathrm{I} + \beta\, \mathrm{U}_\mathrm{A})^{-1} = \int_{0-}^{\infty} (1 + \beta_\gamma)^{-1}\, d\mathrm{E}_\gamma.$$

Applying Theorem 1.3, we find that

$$\int e^{-\beta_u}\, \Pi(du) = 1 - \frac{\beta}{\mathrm{C}(\mathrm{A})} \int_{0-}^{\infty} (1 + \beta\gamma)^{-1}\, d\langle \mathrm{E}_\gamma 1, 1\rangle. \tag{4.1}$$

Recall from § 2 that, for a general Borel set B, $\hat{\pi}_\mathrm{B}$ is concentrated on the set $\mathrm{B} \cup \mathrm{B}^r$, where B^r is the set of regular points of B. As A is open, we see from Corollary 3.4 that Π does not have an atom at 0. Thus

$$\lim_{\beta \to \infty} \int e^{-\beta u}\, \Pi(du) = 0$$

and

$$\mathrm{C}(\mathrm{A}) = \int_{0-}^{\infty} \gamma^{-1}\, d\langle \mathrm{E}_\gamma 1, 1\rangle.$$

In particular, $\langle \mathrm{E}_0 1, 1\rangle = 0$. Substituting this value for $\mathrm{C}(\mathrm{A})$ into (4.1) shows that

$$\int e^{-\beta u}\, \Pi(du) = \int_{0+}^{\infty} (1 + \beta\gamma)^{-1}\, p(d\gamma),$$

and this is equivalent to the first claim of the corollary.

Finally, it follows from Goldie (1967) or Steutel (1967) that all mixtures of exponential distritbutions are infinitely divisible. □

Remarks. – (i) If we remove the assumption that A is open, then it is no longer necessarily true that Π will be a mixture of nondegenerate exponential random variables. It is clear from the proof that if the other conditions of the corollary are still in place then, in general, Π may be a mixture of nondegenerate exponential random variables and a point mass at 0. For example, one can take X to Brownian motion in $\mathbb{R}^3$ and $A = A_1 \cup A_2$, where

$$A_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$$

and

$$A_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\}.$$

The co-capacitary measure of A (which is, of course, also the capacitary measure) will assign all of its mass to the set A_2 and it is clear that if $x \in A_2$ then

$$\mathbb{P}^x \{D_\infty = 0\} > 0$$

and hence, from Corollary 3.4, Π has an atom at 0. In this case, it will not be possible to determine Π by considering the spectral structure of U_A as an operator on $L^2(A, \lambda)$, as this operator is isometrically equivalent to the operator U_{A_1} on $L^2(A_1, \lambda)$.

(ii) It is possible to prove an analogue of Corollary 1.5 in the discrete case. Here Π will be a finite mixture of geometric random variables. We leave the details to the reader.

5. BROWNIAN MOTION WITH DRIFT

Suppose for this section that X is Brownian motion on $\mathbb{R}$ with unit positive drift. Thus

$$\mathbb{P}^x \{X_t \in dy\} = (2\pi t)^{-1/2} \exp(-(y - x - t)^2/(2t))\, \lambda(dy),$$

where λ is Lebesgue measure on $\mathbb{R}$, and

$$u^0(z) = \exp(z - |z|).$$

The strong dual of X with respect to λ is, of course, just Brownian motion with unit negative drift. Consider $A = [0, 1[$. In this case it is possible to calculate the Laplace transform of Π explicitly.

Note that $\hat{\pi}_{[0,\,1[}$ is the unit point mass at 0 and so $C([0,\,1[) = 1$. Thus, from Theorem 1.3,

$$\int e^{-\beta u}\,\Pi(du) = 1 - \beta \int_0^1 e^{-x} f(x)\,dx, \tag{5.1}$$

where f is the unique solution of the equation

$$f(x) + \beta \int_0^1 e^{-|y-x|} f(y)\,dy = e^x, \quad x \in [0,\,1[. \tag{5.2}$$

Observe that if

$$h(x) = (a+1)\,e^{ax} + (a-1)\,e^{-ax}$$

for some constant a, then

$$\begin{aligned} h(x) &+ \frac{a^2-1}{2} \int_0^1 e^{-|y-x|}\,h(y)\,dy \\ &= \left[2a + \frac{(a+1)^2}{2}\,(e^{a-1}-1) - \frac{(a-1)^2}{2}\,(e^{-(a+1)}-1)\right] e^x, \end{aligned}$$

$$x \in [0,\,1[.$$

From this we can conclude that

$$\int e^{-\beta u}\,\Pi(du) = e \Big/ \left[\left(\frac{b^2+1}{2b}\right)\sinh(b) + \cosh(b)\right],$$

where $b = (2\beta+1)^{1/2}$.

Recall from Corollary 3.4 that Π is just the distribution of the total amount of time that X starting at 0 spends in $[0,\,1[$. While the above Laplace transform does not seem to appear in the literature, J. Pitman has pointed out that it may be derived from the remark (6.3) and the formula (2.*k*) in Pitman and Yor (1982).

It follows from excursion theory considerations that Π is infinitely divisible. The rest of this section is devoted to showing that Π may, in fact, be written as a convolution of exponential distributions.

Let J be the bounded operator on $L^2([0,\,1[,\,\lambda)$ given by

$$J f(x) = \int_0^1 e^{-|y-x|} f(y)\,dy.$$

Note that J is a self-adjoint and compact. Moreover, since

$$e^{-|z|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\zeta z}}{1+\zeta^2}\,d\zeta, \tag{5.3}$$

J is also positive definite. From § 94 of Riesz and Sz.-Nagy (1990) we see that $(\mathrm{I}+\beta\,\mathrm{J})$ has a bounded inverse for every $\beta \geqq 0$, where I is the identity operator.

For $n = 1, 2, \ldots$, let J_n be the operator on $\mathrm{L}^2([0, 1[, \lambda)$ defined by

$$\mathrm{J}_n f(x) = \int_0^1 \mathrm{K}_n(x, y) f(y)\, dy,$$

where

$$\mathrm{K}_n(x, y) = \exp\left(-\left|\frac{k}{n} - \frac{j}{n}\right|\right), \quad \frac{j-1}{n} \leqq x < \frac{j}{n},$$
$$\frac{k-1}{n} \leqq y < \frac{k}{n}, \quad 1 \leqq j, k \leqq n.$$

Observe that $\mathrm{J}_n \to \mathrm{J}$ in norm as $n \to \infty$, and hence $(\mathrm{I}+\beta\,\mathrm{J}_n)$ has a bounded inverse for n sufficiently large. Moreover, $(\mathrm{I}+\beta\,\mathrm{J}_n)^{-1}$ converges in norm to $(\mathrm{I}+\beta\,\mathrm{J})^{-1}$. Set

$$e_n(x) = \exp\left(\frac{j}{n}\right), \quad \frac{j-1}{n} \leqq x < \frac{j}{n}, \quad 1 \leqq j \leqq n.$$

Then $e_n \to \exp(\cdot)$ in $\mathrm{L}^2([0, 1[, \lambda)$ as $n \to \infty$. Thus $(\mathrm{I}+\beta\,\mathrm{J}_n)^{-1} e_n \to (\mathrm{I}+\beta\,\mathrm{J})^{-1} \exp(\cdot)$ and, by (5.1) and (5.2),

$$\int e^{-\beta u}\, \Pi(du) = \lim_{n\to\infty} \left[1 - \beta \int_0^1 e^{-x}\, (\mathrm{I}+\beta\,\mathrm{J}_n)^{-1} e_n(x))\, dx\right].$$

Write I_n for the $n \times n$ identity matrix. Define an $n \times n$ matrix $\tilde{\mathrm{K}}_n$ by

$$\tilde{\mathrm{K}}_n(j, k) = \exp\left(-\left|\frac{k}{n} - \frac{j}{n}\right|\right), \quad 1 \leqq j, k \leqq n;$$

and define n-vectors $\tilde{d}_n$ and $\tilde{e}_n$ by

$$\tilde{d}_n(j) = \exp\left(-\frac{j}{n}\right), \quad 1 \leqq j \leqq n,$$

and

$$\tilde{e}_n(j) = \exp\left(\frac{j}{n}\right), \quad 1 \leqq j \leqq n.$$

It follows from (5.3) that $\tilde{\mathrm{K}}_n$ is a positive definite, symmetric matrix for all n.

From equation (A.2.3k) of Mardia *et al.* (1979) we have

$$\begin{aligned}
&1-\beta\int_0^1 e^{-x}\left((\mathrm{I}+\beta\,\mathrm{J}_n)^{-1}\,e_n(x)\right)dx\\
&\quad=1-\beta\,n^{-1}\,\tilde{d}'_n\,(I_n+\beta\,n^{-1}\,\tilde{\mathrm{K}}_n)^{-1}\,\tilde{e}_n\\
&\quad=1+\tilde{d}'_n\,(-\beta^{-1}\,n\,\mathrm{I}_n-\tilde{\mathrm{K}}_n)^{-1}\,\tilde{e}_n\\
&\quad=\det\,[-\beta^{-1}\,n\,\mathrm{I}_n-\tilde{\mathrm{K}}_n+\tilde{e}_n\,\tilde{d}'_n]/\det\,[-\beta^{-1}\,n\,\mathrm{I}_n-\tilde{\mathrm{K}}_n].
\end{aligned}$$

Now $-\beta^{-1}\,n\,\mathrm{I}_n-\tilde{\mathrm{K}}_n+\tilde{e}_n\,\tilde{d}'_n$ is a lower triangular matrix which has $-\beta^{-1}\,n$ in each of the diagonal positions, and so the last quantity above is

$$\begin{aligned}
&(-\beta^{-1}\,n)^n/\det\,[-\beta^{-1}\,n\,\mathrm{I}_n-\tilde{\mathrm{K}}_n]\\
&\quad=(\det\,[\mathrm{I}_n+\beta\,n^{-1}\,\tilde{\mathrm{K}}_n])^{-1}=\prod_{k=1}^{n}(1+\beta\,\mu_k^n)^{-1},
\end{aligned}$$

where $\mu_1^n \geqq \mu_2^n \geqq \ldots \geqq \mu_n^n > 0$ are the eigenvalues of the matrix $n^{-1}\,\tilde{\mathrm{K}}_n$ repeated according to their multiplicity. Note that the eigenvalues of the matrix $n^{-1}\,\tilde{\mathrm{K}}_n$ are also the nonzero eigenvalues of the operator J_n. As $\mathrm{J}_n \to \mathrm{J}$ in norm, we see from § 95 of Riesz and Sz.-Nagy (1990) that the eigenvalues of J_n converge to those of J, and hence

$$\int e^{-\beta_u}\,\Pi\,(du)=\prod_{k=1}^{\infty}(1+\beta\mu_k)^{-1},$$

where $\mu_1 \geqq \mu_2 \geqq \ldots > 0$ are the eigenvalues of J repeated according to their multiplicity. Thus

$$\Pi=\Gamma_1\star\Gamma_2\star\ldots,$$

where Γ_k is the exponential distribution with mean μ_k.

Remarks. – (i) As it stands, the result we have just obtained is something of an analytic curiosity. It would be of interest to know if there is an argument that derives this decomposition of Π from a suitable decomposition of the sample paths of X.

(ii) Given our explicit closed form for the Laplace transform of Π, it is possible to obtain the infinite product development from the Hadamard factorisation formula. This latter method has the advantage that it easily identifies the sequence $\{\mu_k\}$ as the sequence $\{2/(1+y_k^2)\}$, where $y_1 \leqq y_2 \leqq \ldots$ are the positive solutions of the equation $\tan\,(y)=2\,y/(y^2-1)$. The merit of the above method is that it can be applied to obtain an analogous decomposition in the case of other sets for which we may be unable to explicitly solve the counterpart of (5.2).

(iii) D. Aldous has remarked to us that, at least on a heuristic level, the fact that Π is a convolution of exponential distributions is to be expected, given results in the literature of birth and death processes. It is shown by a purely analytic argument in Keilson (1971) that if Y is a birth and death process on $\{0, 1, 2, \ldots\}$ with 0 as a reflecting boundary, then the first passage time from 0 to any state n is a convolution of exponential distributions. Suppose now that Z is a birth and death process on $\{\ldots, -1, 0, 1, 2, \ldots\}$ such that $Z_t \rightarrow +\infty$ almost surely as $t \rightarrow \infty$. By an obvious device, it is possible to construct a birth and death process on $\{0, 1, 2, \ldots\}$ with 0 as a reflecting boundary such that the first passage time of this process to $n + 1$ starting at 0 has the same distribution as the total amount of time that Z spends in $\{0, 1, \ldots, n\}$ starting at 0. The latter random variable therefore has a distribution that is a convolution of exponential distributions. Our result can thus be expected to follow from approximating Brownian motion with drift by a suitable sequence of birth and death processes. Moreover, similar reasoning should apply to obtain a similar decomposition for the distribution of the total occupation time of an interval for a wide class of one-dimensional diffusions. We note that the analogue of Keilson's decomposition for the first passage time of a one-dimensional diffusion is fully explored in Kent (1980).

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