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Multiplicative chaos and random translation

by

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ABSTRACT. – Let $\mathbf{G} = \{ G_k \}_{k \geqq 1}$ be a standard Gaussian sequence and $\mathbf{Y} = \{ Y_k \}_{k \geqq 1}$ an independent non-negative random sequence which is also independent of $\mathbf{G}$. We shall analyse the conditions on $\mathbf{Y}$ for the equivalence (=mutual absolute continuity) of the measures $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ on $\mathbf{R}^\infty$ induced by $\mathbf{G}$ and $\mathbf{G}+\mathbf{Y}$, respectively. This problem implies a typical example of the multiplicative chaos. In particular we shall analyse in detail the case where Y_k's are two valued in view of the regularity of the multiplicative chaos and, as an application, give a negative answer to a conjecture of J.-P. Kahane on the regularity of the multiplicative chaos.

Key words : Multiplicative chaos, absolute continuity.

RÉSUMÉ. – Soit $\mathbf{G} = \{ G_k \}_{k \geqq 1}$ une suite de variables aléatoires gaussiennes et $\mathbf{Y} = \{ Y_k \}_{k \geqq 1}$ une suite de variables aléatoires non-négatives indépendantes qui soit aussi indépendante de $\mathbf{G}$. On donne les conditions sur $\mathbf{Y}$ pour l'équivalence des mesures $\mu_{\mathbf{G}}$ et $\mu_{\mathbf{G}+\mathbf{Y}}$ sur $\mathbf{R}^\infty$ induites par $\mathbf{G}$ et $\mathbf{G}+\mathbf{Y}$, respectivement. Ce problème fournit un exemple typique du chaos multiplicatif. En particulier, on analyse en détail le cas où Y_k sont deux-valuées au point de vue de la régularité du chaos multiplicatif et, comme application, on donne une réponse négative à une des conjectures de J.-P. Kahane sur la régularité du chaos multiplicatif.

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1. INTRODUCTION

Let T be a compact metric space and $\{X_k(t, \omega)\}_{t \in T} (k \geqq 1)$ an independent family of centered Gaussian processes on a probability space $(\Omega, \mathscr{F}, P)$. For every $k \in \mathbf{N}$ we assume that

$$p_k(s, t) = \mathbf{E}[X_k(s, \omega) X_k(t, \omega)] \geqq 0 \ (s, t \in T)$$

and $X_k(t, \omega)$ is $(\Sigma \otimes \mathscr{F})$-measurable, where Σ is the Borel field of T, and define

$$M_n(t, \omega) = \exp\left[\sum_1^n \left\{ X_k(t, \omega) - \frac{1}{2} p_k(t, t) \right\}\right]. \tag{1.1}$$

Then, for every fixed $t \in T$, $\{M_n(t, \omega)\}_{n \geqq 1}$ is naturally a positive martingale.

For every $\sigma \in \mathscr{M}(T)$, the collection of all finite measures on (T, Σ), define a random measure $M\sigma$ by

$$\int_T \phi(t)(M\sigma)(\omega, dt) = \lim_n \int_T M_n(t, \omega) \phi(t) \sigma(dt) \quad \text{a.s.},$$

for every continuous function ϕ on T. After Kahane [2] the above map M; $\sigma \in \mathscr{M}(T) \to M\sigma$ is called a *multiplicative chaos*, and $\sigma \in \mathscr{M}(T)$ is said to be M-*regular* or M-*singular* according as $\mathbf{E}[(M\sigma)(T)] = \sigma(T)$ or 0.

When T is the d-dimensional torus, σ is the Lebesgue measure and the covariance functions satisfy

$$\sum_k p_k(s, t) = \xi \log + \frac{1}{\|s-t\|} + O(1)$$

for some $\xi > 0$, Kahane [2] proved that σ is M-regular if $\xi < 2d$ and M-singular if $\xi \geqq 2d$. In other words σ is M-regular if and only if

$$\int_T \int_T \exp\left[\frac{1}{2} \sum_k p_k(s, t)\right] \sigma(ds) \sigma(dt) < \infty.$$

We should remark that this result implies the complete solution to the problem of absolute continuity of measures in the 2-space time dimensional Høegh-Krohn's model of quantum fields, which has been investigated by many authors (Høegh-Krohn [1], Kusuoka [7] and its references).

For $\sigma \in \mathscr{M}(T)$ and $u \geqq 0$ define

$$I(u; \sigma) = \int_T \int_T \exp\left[\frac{u}{2} \sum_k p_k(s, t)\right] \sigma(ds) \sigma(dt) \leqq \infty.$$

Then Kahane posed the following conjecture.

Conjecture (Kahane [4]). — Let M be a multiplicative chaos. Then $\sigma \in \mathcal{M}(T)$ is M-regular if and only if σ is expressed as a sum $\sum_n \sigma_n$ (convergence in total variation) of $\sigma_n \in \mathcal{M}(T)$ such that $I(1; \sigma_n) < \infty$.

On the other hand let $\mathbf{X} = \{X_k(\omega)\}_{k \geq 1}$ be an *i.i.d.* random sequence defined on $(\Omega, \mathcal{F}, P)$, $\sigma \in \mathcal{M}(T)$ a probability measure on (T, Σ) and $\mathbf{Y} = \{Y_k(t)\}_{k \geq 1}$ an independent random sequence defined on (T, Σ, σ). Then $\mathbf{X} + \mathbf{Y} = \{X_k(\omega) + Y_k(t)\}_{k \geq 1}$ is defined on the product probability space $(\Omega \times T, \mathcal{F} \otimes \Sigma, P \otimes \sigma)$ and $\mathbf{X}$ and $\mathbf{Y}$ are independent. The authors [6, 8, 9] investigated the problem of the equivalence of the probability measures $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ on $\mathbf{R}^\infty$ induced by $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y}$, respectively.

In particular let $\mathbf{G} = \{G_k(\omega)\}_{k \geq 1}$ be a standard Gaussian sequence on $(\Omega, \mathcal{F}, P)$, $\mathbf{Y} = \{Y_k(t)\}_{k \geq 1}$ an independent non-negative random sequence on (T, Σ, σ) and

$$M_n(t, \omega) = \exp\left[\sum_1^n \left\{ G_k(\omega) Y_k(t) - \frac{1}{2} Y_k(t)^2 \right\}\right]. \tag{1.2}$$

Then (1.2) defines a multiplicative chaos M for $X_k(t, \omega) = Y_k(t) G_k(\omega)$ and we have

$$I(u; \sigma) = \int_T \int_T \exp\left[\frac{u}{2} \sum_k Y_k(s) Y_k(t)\right] \sigma(ds) \sigma(dt).$$

On the other hand $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ are equivalent ($\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$) or singular ($\mu_{\mathbf{G}} \perp \mu_{\mathbf{G}+\mathbf{Y}}$) according as σ is M-regular or M-singular. Owed to the well known Kakutani dichotomy [5] we have either $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ or $\mu_{\mathbf{G}} \perp \mu_{\mathbf{G}+\mathbf{Y}}$. To characterize the equivalence of $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ is our first aim. In Section 2 we shall prove the following theorem.

THEOREM 1.

$$\sum_k \sigma(Y_k > \varepsilon) < \infty$$

and

$$\sum_k \mathbf{E}_\sigma [Y_k;\, Y_k \leqq \varepsilon]^2 < \infty \tag{1.3}$$

for some $\varepsilon > 0$ *imply* $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$.

Conversely $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ *implies*

$$\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$$

and (1.3) *for all* $\varepsilon > 0$.

As a cororally we obtain a positive answer to the conjecture in the case where $\sup_k Y_k \leqq L$ σ-*a.s.* for some $L \geqq 0$ (Proposition 1).

In particular we analyse, in Section 3, when $\mathbf{Y}=\{\varepsilon(a_k, p_k)\}_{k\geqq 1}$ is an independent random sequence with distributions

$$\sigma(\varepsilon(a_k, p_k)=a_k)=p_k, \qquad \sigma(\varepsilon(a_k, p_k)=0)=1-p_k, \tag{1.4}$$

where $a_k>0$ and $0<p_k<1$ for every $k\in\mathbf{N}$. Define

$$\alpha_k=\frac{1}{a_k^2}\log\frac{1+p_k}{p_k}.$$

Then, relating to the Kahane's conjecture, we shall prove:

THEOREM 2. — (*a*) *Assume* $\sup_k a_k<\infty$. *Then* $\mathrm{I}(u;\sigma)<\infty$ *for some* $u>0$ *implies* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$. *Conversely*, $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ *implies* $\mathrm{I}(u;\sigma)<\infty$ *for every* $u>0$. *Consequently we have* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ *if and only if* $\mathrm{I}(1;\sigma)<\infty$.

(*b*) *Assume* $\sup_k a_k=\infty$, *and define* $\underline{\alpha}=\liminf_{\{k;\, a_k>1\}}\alpha_k$ *and* $\bar{\alpha}=\limsup_{\{k;\, a_k>1\}}\alpha_k$.

(*b*-i) *Assume* $\underline{\alpha}>\frac{3}{2}$. *Then* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ *if and only if* $\mathrm{I}(2;\sigma)<\infty$.

(*b*-ii) *Assume* $\frac{1}{2}\leqq\underline{\alpha}\leqq\bar{\alpha}\leqq\frac{3}{2}$. *Then* $\mathrm{I}(2;\sigma)<\infty$ *implies* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$. *Conversely*, $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ *implies* $\mathrm{I}(1-\varepsilon;\sigma)<\infty$ *for every* $0<\varepsilon\leqq 1$.

(*b*-iii) *Assume* $\bar{\alpha}<\frac{1}{2}$. *Then* $\mathrm{I}(2\bar{\alpha}+\varepsilon;\sigma)<\infty$ *for some* $\varepsilon>0$ *implies* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$. *In particular* $\mathrm{I}(1;\sigma)<\infty$ *implies* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$. *Conversely* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ *implies* $I((2\underline{\alpha}-\varepsilon)_+;\sigma)<\infty$ *for every* $\varepsilon>0$, *where* a_+ *denotes* $\max(a, 0)$.

More precisely we shall analyse the case (*b*-ii). Define

$$\theta=\sup\{u\geqq 0;\ \mathrm{I}(u;\sigma)<\infty\}$$

and

$$\lambda(x)=2-\left(\frac{3}{2}-x\right)^2, \qquad x\geqq 0.$$

Then we shall prove:

THEOREM 3. — *Assume* $\sup_k a_k=\infty$ *and* $\frac{1}{2}\leqq\underline{\alpha}=\bar{\alpha}=\alpha\leqq\frac{3}{2}$. *Then* $\lambda(\alpha)<\theta$ *implies* $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$. *Conversely* $\lambda(\alpha)>\theta$ *implies* $\mu_{\mathbf{G}}\perp\mu_{\mathbf{G}+\mathbf{Y}}$.

As an application we shall give a negative answer to the Kahane's conjecture by giving examples in Section 4.

2. GENERAL CASE

Let $\mathbf{G}=\{G_k(\omega)\}_{k\geqq 1}$ be a standard Gaussian sequence defined on $(\Omega, \mathscr{F}, P)$, $\sigma\in\mathscr{M}(T)$ a probability measure and $\mathbf{Y}=\{Y_k(t)\}_{k\geqq 1}$ an independent non-negative random sequence defined on (T, Σ, σ). Define

$$g(x)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{2}\right], \qquad x\in\mathbf{R}$$

and

$$Z_k(x)=\int_T \exp\left[x\,Y_k(t)-\frac{1}{2}Y_k(t)^2\right]\sigma(dt)-1, \qquad x\in\mathbf{R},\ k\in\mathbf{N}. \quad (2.1)$$

Then the following theorem is our starting point.

THEOREM 4. [6, Theorem 2]. — *The next four statements are equivalent.*
(*a*) σ *is* M-*regular, where* M *is the multiplicative chaos defined by* (1.2).
(*b*) $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$.
(*c*) $\sum_k Z_k(G_k)$ *converges almost surely.*
(*d*) *For some, so that any*, $K\,(\geqq 1)$
(*d*-1) $\sum_k \mathbf{E}[Z_k(G_k);\ Z_k(G_k)>K]<\infty$

and

(*d*-2) $\sum_k \mathbf{E}[Z_k(G_k)^2;\ Z_k(G_k)\leqq K]<\infty$.

Proof of Theorem 1. — For any $\varepsilon>0$ decompose $Z_k(G_k)$ into

$$\begin{aligned}Z_k(G_k)&=\mathbf{E}_\sigma\left[\exp\left[G_k Y_k-\frac{1}{2}Y_k^2\right]-1;\ Y_k>\varepsilon\right]\\&\quad+\mathbf{E}_\sigma\left[\exp\left[G_k Y_k-\frac{1}{2}Y_k^2\right]-1;\ Y_k\leqq\varepsilon\right]\\&=V_\varepsilon(G_k)+W_\varepsilon(G_k),\end{aligned}$$

where $\mathbf{E}_\sigma$ denotes the expectation with respect to σ. Then, from the same arguments as in [6, Lemma 1, Theorem 4] and [8, Theorem 3.2 (B)], $\sum_k \sigma(Y_k>\varepsilon)<\infty$ implies the almost sure absolute convergence of $\sum_k V_\varepsilon(G_k)$, and (1.3) implies the L^2-convergence, therefore the almost sure convergence, of $\sum_k W_\varepsilon(G_k)$. Thus we obtain the sufficiency.

Conversely assume the almost sure convergence of $\sum_k Z_k(G_k)$. Then, since $Y_k\geqq 0$, $Z_k(x)$ is increasing and continuous in x, and

$Z_k(1) < \exp\left[\frac{1}{2}\right]$, Theorem 4 ($d$-2) implies

$$\infty > \sum_k \mathbf{E}\left[Z_k(G_k)^2;\ Z_k(G_k) \leqq \exp\left[\frac{1}{2}\right]\right]$$
$$\geqq \sum_k \int_0^1 Z_k(x)^2 g(x)\,dx \geqq \int_0^1 g(x)\,dx \sum_k Z_k(0)^2.$$

Therefore we have $\sum_k Z_k(0)^2 = \sum_k \mathbf{E}_\sigma\left[1 - \exp\left[-\frac{1}{2}Y_k^2\right]\right]^2 < \infty$, which implies $\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$ for every $\varepsilon > 0$.

Next we shall prove (1.3). For any $\varepsilon > 0$ we have $Z_k(x) < \exp\left[\frac{9}{2}\varepsilon^2\right]$ for every $x \in [0\ 3\varepsilon)$. Then Theorem 4 (d-2) implies

$$\infty > \sum_k \mathbf{E}\left[Z_k(G_k)^2;\ Z_k(G_k) \leqq \exp\left[\frac{9}{2}\varepsilon^2\right]\right]$$
$$\geqq \sum_k \int_{2\varepsilon}^{3\varepsilon} Z_k(x)^2 g(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}} \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma\left[\exp\left[-\frac{(x-Y_k)^2}{2}\right] - \exp\left[-\frac{x^2}{2}\right]\right]^2 \exp\left[\frac{1}{2}x^2\right] dx.$$

Since $\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$, we have

$$\infty > \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma\left[\exp\left[-\frac{(x-Y_k)^2}{2}\right] - \exp\left[-\frac{x^2}{2}\right];\ Y_k \leqq \varepsilon\right]^2 \exp\left[\frac{1}{2}x^2\right] dx$$
$$= \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma\left[\left[\int_0^1 (x - tY_k)\exp\left[-\frac{(x-tY_k)^2}{2}\right] dt\, Y_k;\ Y_k \leqq \varepsilon\right]^2 \exp\left[\frac{1}{2}x^2\right] dx$$
$$\geqq \varepsilon^2 \exp[-9\varepsilon^2] \int_{2\varepsilon}^{3\varepsilon} \exp\left[\frac{1}{2}x^2\right] dx \sum_k \mathbf{E}_\sigma[Y_k;\ Y_k \leqq \varepsilon]^2,$$

which completes the proof of Theorem 1. □

PROPOSITION 1. — (*a*) *Assume* $\sup_k Y_k < \infty$, σ-a.s.. *Then* $I(1;\sigma) < \infty$ *implies* $\mu_G \sim \mu_{G+Y}$.

(*b*) *Assume* $\sup_k Y_k < L$, σ-a.s. *for some* $L > 0$. *Then* $\mu_G \sim \mu_{G+Y}$ *if and only if* $I(1;\sigma) < \infty$.

Proof. – Since $\{Y_k(t)\}_{k \geqq 1}$ is an independent random sequence, we have

$$I(1; \sigma) = \prod_k \int_T \int_T \exp\left[\frac{1}{2} Y_k(t) Y_k(s)\right] \sigma(ds)\, \sigma(dt),$$

and thus we have $I(1; \sigma) < \infty$ if and only if

$$\sum_k \left\{ \int_T \int_T \exp\left[\frac{1}{2} Y_k(t) Y_k(s)\right] \sigma(ds)\, \sigma(dt) - 1 \right\}$$
$$= \frac{1}{2} \sum_k \int_T Y_k(t)\, \sigma(dt) \int_T Y_k(s)\, \sigma(ds) \int_0^1 \exp\left[\frac{1}{2} Y_k(t) Y_k(s)\, x\right] dx < \infty.$$

(*a*) Assume $\sup_k Y_k < \infty$, σ-a.s. and $I(1; \sigma) < \infty$. Then we have $\sum_k \sigma(Y_k > L) < \infty$ for some $L > 0$ so that $\sum_k \mathbf{E}_\sigma [Y_k : Y_k \leqq L]^2 < \infty$ implies the almost sure convergence of $\sum_k Z_k(G_k)$ by Theorem 1. In fact, since $Y_k \geqq 0$, $I(1; \sigma) < \infty$ implies

$$\infty > \sum_k \int_T Y_k(t)\, \sigma(dt) \int_T Y_k(s)\, \sigma(ds) \int_0^1$$
$$\times \exp\left[\frac{1}{2} Y_k(t) Y_k(s)\, x\right] dx \geqq \sum_k \mathbf{E}_\sigma [Y_k : Y_k \leqq L]^2.$$

(*b*) Assume $\sup_k Y_k < L$, σ-a.s. for some $L > 0$. Then we obtain "*if*" part by (*a*). Conversely assume the almost sure convergence of $\sum_k Z_k(G_k)$. Then, by Theorem 1, we have $\sum_k \mathbf{E}_\sigma [Y_k]^2 = \sum_k \mathbf{E}_\sigma [Y_k : Y_k \leqq L]^2 < \infty$, so that

$$\sum_k \int_T Y_k(t)\, \sigma(dt) \int_T Y_k(s)\, \sigma(ds) \int_0^1 \exp\left[\frac{1}{2} Y_k(t) Y_k(s)\, x\right] dx$$
$$\leqq \exp\left[\frac{1}{2} L^2\right] \sum_k \mathbf{E}_\sigma [Y_k]^2 < \infty,$$

which proves (*b*). □

In Section 4 (4.3) we shall give an example that $\sup_k Y_k < \infty$, σ-a.s. and $\mu_G \sim \mu_{G+Y}$ do not imply $I(1; \sigma) < \infty$.

3. TWO-VALUED CASE

In this section we consider the case $\mathbf{Y}=\{\varepsilon(a_k, p_k)\}$. By definition (1.4) and (2.1) we have

$$Z_k(x)=p_k\left(\exp\left[a_k x-\frac{1}{2}a_k^2\right]-1\right),\qquad k\in\mathbf{N},\ x\in\mathbf{R}.$$

By Theorem 4, M-regularity of σ, the equivalence of $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ and the almost sure convergence of $\sum_k Z_k(G_k)$ are equivalent. Relating to the conjecture, we shall characterize them in terms of

$$I(u;\sigma)=\prod_k\left\{1+p_k^2\left(\exp\left[\frac{u}{2}a_k^2\right]-1\right)\right\}.$$

First we shall prove the following.

PROPOSITION 2. – *(a)* $\sum_k p_k<\infty$ *implies the almost sure absolute convergence of* $\sum_k Z_k(G_k)$.

(b) $I(2;\sigma)<\infty$ *implies the almost sure convergence of* $\sum_k Z_k(G_k)$.

Proof. – (a) Assume $\Sigma_k p_k<\infty$. Then

$$\mathbf{E}\left[\sum_k|Z_k(G_k)|\right]=\sum_k p_k\int_{-\infty}^{\infty}\left|\exp\left[a_k x-\frac{1}{2}a_k^2\right]-1\right|g(x)\,dx\leqq 2\sum_k p_k<\infty,$$

which proves *(a)*.

(b) Assume $I(2;\sigma)<\infty$. Since $\{Z_k(G_k)\}_{k\geqq 1}$ is a sequence of independent random variables with mean 0, $\sum_k \mathbf{E}[Z_k(G_k)^2]<\infty$ implies the L^2-convergence, consequently the almost sure convergence, of $\sum_k Z_k(G_k)$. In fact we have

$$\sum_k \mathbf{E}[Z_k(G_k)^2]=\sum_k p_k^2\int_{-\infty}^{\infty}\left(\exp\left[a_k x-\frac{1}{2}a_k^2\right]-1\right)^2 g(x)\,dx$$
$$=\sum_k p_k^2(\exp[a_k^2]-1)<\infty.\qquad\square$$

Decompose $\mathbf{N}$ into

$$\mathbf{N}=\{k\geqq 1;\ a_k\leqq 1\}\cup\{k\geqq 1;\ a_k>1\}=\mathcal{N}_1\cup\mathcal{N}_2.$$

Remark 1. – We have $\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty$ if and only if

$$\sum_{k \in \mathcal{N}_1} p_k^2 \left(\exp\left[\frac{u}{2} a_k^2 \right] - 1 \right) < \infty$$

for some, so that any, $u > 0$.

The next lemma is immediately derived from Proposition 2 and Remark 1.

LEMMA 1. – *(a)* $\sum_{k \in \mathcal{N}_2} p_k < \infty$ *implies the almost sure convergence of* $\sum_{k \in \mathcal{N}_2} Z_k(G_k)$.

(b) $\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty$ *implies the almost sure convergence of* $\sum_{k \in \mathcal{N}_1} Z_k(G_k)$.

The following lemma plays a central role in our discussion.

LEMMA 2. – $\sum_k Z_k(G_k)$ *converges almost surely if and only if*

$$\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty, \tag{3.1}$$

$$\sum_{k \in \mathcal{N}_2} p_k^2 < \infty, \tag{3.2}$$

$$\sum_{k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x)\, dx < \infty \tag{3.3}$$

and

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x)\, dx < \infty. \tag{3.4}$$

Proof. – Assume the almost sure convergence of $\sum_k Z_k(G_k)$. Then, by Theorem 1, we have

$$\infty > \sum_k \mathbf{E}_\sigma [\varepsilon(a_k, p_k) : \varepsilon(a_k, p_k) \leqq 1]^2 = \sum_{k \in \mathcal{N}_1} p_k^2 a_k^2,$$

$$\infty > \sum_k \sigma(\varepsilon(a_k, p_k) > 1)^2 = \sum_{k \in \mathcal{N}_2} p_k^2,$$

which proves (3.1) and (3.2).

Since $Z_k(x)$ is strictly increasing and $Z_k\left(a_k\left(\alpha_k+\frac{1}{2}\right)\right)=1$, we have by Theorem 4 ($d$-1) and ($d$-2)

$$\infty>\sum_k \mathbf{E}[Z_k(G_k);\ Z_k(G_k)>1]=\sum_{k\in\mathcal{N}_2} p_k \int_{a_k(\alpha_k+(1/2))}^{\infty} \times\left(\exp\left[a_k x-\frac{1}{2}a_k^2\right]-1\right)g(x)\,dx$$

$$=\sum_{k\in\mathcal{N}_2} p_k \int_{a_k(\alpha_k-(1/2))}^{a_k(\alpha_k+(1/2))} g(x)\,dx,$$

which proves (3.3), and

$$\infty>\sum_k \mathbf{E}[Z_k(G_k)^2;\ Z_k(G_k)\leqq 1]=\sum_{k\in\mathcal{N}_2} p_k^2 \int_{-\infty}^{a_k(\alpha_k+(1/2))} \times\left(\exp\left[a_k x-\frac{1}{2}a_k^2\right]-1\right)^2 g(x)\,dx$$

$$=\sum_{k\in\mathcal{N}_2} p_k^2\left\{\exp[a_k^2]\int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx-2 \times\int_{a_k((1/2)-\alpha_k)}^{\infty} g(x)\,dx+\int_{-a_k((1/2)+\alpha_k)}^{\infty} g(x)\,dx\right\}$$

$$\geqq\sum_{k\in\mathcal{N}_2} p_k^2\left\{\exp[a_k^2]\int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx-2\right\},$$

thus, by (3.2), this proves (3.4).

Conversely (3.1) implies the almost sure convergence of $\sum_{k\in\mathcal{N}_1} Z_k(G_k)$ by Lemma 1 (b). On the other hand, by Theorem 4, (3.2), (3.3) and (3.4) implies the almost sure convergence of $\sum_{k\in\mathcal{N}_2} Z_k(G_k)$, which completes the proof. □

Remark 2. — We have proved Lemma 2 for a decomposition of $\mathbf{N}$ according as $a_k\leqq 1$ or $a_k>1$. But it is not difficult to show that Lemma 2 is true for any decomposition of $\mathbf{N}$ according as $a_k\leqq\varepsilon$ or $a_k>\varepsilon$, where ε is an arbitrary positive number.

Remark 3. — Since the series (3.1)$\sim$(3.4) are of positive terms, for any decomposition $\mathbf{N}=\bigcup_{1}^{n}\mathcal{K}_j$, $\Sigma_k Z_k(G_k)$ converges almost surely if and only if $\sum_{k\in\mathcal{K}_j} Z_k(G_k)$, $j=1, 2, \ldots, n$, separately converge almost surely.

PROPOSITION 3. — *(a) If* $\alpha_k > -\frac{1}{a_k}+\frac{3}{2}$ *for every* $k\in\mathcal{N}_2$, *then* $\sum_k Z_k(G_k)$ *converges almost surely if and only if* $I(2;\sigma)<\infty$.

(b) If $\alpha_k \leqq \frac{1}{2}$ *for every* $k\in\mathcal{N}_2$, *then* $\sum_k Z_k(G_k)$ *converges almost surely if and only if* (3.1) *and* $\sum_{k\in\mathcal{N}_2} p_k<\infty$.

Proof. — (*a*) $I(2;\sigma)<\infty$ implies the almost sure convergence of $\sum_k Z_k(G_k)$ by Proposition 2(*b*).

Conversely assume $\alpha_k > -\frac{1}{a_k}+\frac{3}{2}$ for every $k\in\mathcal{N}_2$ and the almost sure convergence of $\sum_k Z_k(G_k)$. Then we have by Lemma 2

$$\infty > \sum_{k\in\mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx \geqq \int_1^{\infty} g(x)\,dx \sum_{k\in\mathcal{N}_2} p_k^2 \exp[a_k^2].$$

On the other hand we have by Lemma 2(3.1) and Remark 1

$$\sum_{k\in\mathcal{N}_1} p_k^2(\exp[a_k^2]-1)<\infty.$$

Therefore we have $\sum_k p_k^2(\exp[a_k^2]-1)<\infty$ and, consequently, $I(2;\sigma)<\infty$.

(*b*) By Lemma 1, (3.1) and $\sum_{k\in\mathcal{N}_2} p_k<\infty$ imply the almost sure convergence of $\sum_k Z_k(G_k)$.

Conversely assume the almost sure convergence of $\sum_k Z_k(G_k)$. Since $\alpha_k-\frac{1}{2}\leqq 0$ and $a_k\left(\alpha_k+\frac{1}{2}\right)\geqq\frac{1}{2}$ for every $k\in\mathcal{N}_2$, we have, by Lemma 2

$$\infty > \sum_{k\in\mathcal{N}_2} p_k \int_{a_k(\alpha_k-(1/2))}^{a_k(\alpha_k+(1/2))} g(x)\,dx \geqq \int_0^{1/2} g(x)\,dx \sum_{k\in\mathcal{N}_2} p_k,$$

consequently $\sum_{k\in\mathcal{N}_2} p_k<\infty$. Then Lemma 2 completes the proof of (*b*). □

Proof of theorem 2. – (*a*) is proved by Proposition 1 (*b*).

(*b*) Assume $\sup_k a_k = \infty$.

(*b*-i) Assume $\underline{\alpha} > \frac{3}{2}$. Then $\alpha_k > \frac{3}{2}$ for large $k \in \mathrm{N}$, so that Proposition 3 (*a*) and Theorem 4 prove (*b*-i).

(*b*-ii) Assume $\frac{1}{2} \leqq \underline{\alpha} \leqq \bar{\alpha} \leqq \frac{3}{2}$.

$\mathrm{I}(2; \sigma) < \infty$ implies $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ by Proposition 2 (*b*) and Theorem 4.

Conversely assume $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ and fix any $0 < \varepsilon \leqq 1$. Then, by Theorem 4, $\sum_k \mathrm{Z}_k(\mathrm{G}_k)$ converges almost surely and we have (3.2) by Lemma 2. Choose $\tau > 0$ such that $\tau < \sqrt{1+\varepsilon} - 1$ and also choose $k_0 \in \mathcal{N}_2$ such that $k \geqq k_0$, $k \in \mathcal{N}_2$ implies $\alpha_k \geqq \frac{1}{2} - \tau$. Then we have by Theorem 4 and Lemma 2

$$\infty > \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx$$

$$\geqq \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{(1+\tau)a_k}^{\sqrt{1+\varepsilon}\,a_k} g(x)\,dx$$

$$\geqq \frac{\sqrt{1+\varepsilon} - (1+\tau)}{\sqrt{2\pi}} \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1-\varepsilon}{2} a_k^2\right],$$

which proves (*b*-ii).

(*b*-iii) Assume $\bar{\alpha} < \frac{1}{2}$. Then $\alpha_k < \frac{1}{2}$ for large $k \in \mathbf{N}$, thus we have $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ if and only if (3.1) and $\sum_{k \in \mathcal{N}_2} p_k < \infty$ by Proposition 3 (*b*) and Theorem 4. On the other hand, for any $\varepsilon > 0$, we may choose $k_0 \in \mathcal{N}_2$, by definition, such that $k \geqq k_0$, $k \in \mathcal{N}_2$ implies

$$\left(\underline{\alpha} - \frac{1}{2}\varepsilon\right)_+ \leqq \frac{1}{a_k^2} \log \frac{1+p_k}{p_k} \leqq \bar{\alpha} + \frac{1}{2}\varepsilon.$$

It is easy to check

$$\frac{1}{2} p_k^2 \exp\left[\left(\underline{\alpha} - \frac{1}{2}\varepsilon\right)_+ a_k^2\right] \leqq p_k \leqq p_k^2 \exp\left[\left(\bar{\alpha} + \frac{1}{2}\varepsilon\right) a_k^2\right]$$

for $k \geqq k_0$, $k \in \mathcal{N}_2$, which proves (*b*-iii). □

For the proof of Theorem 3 we shall give the next lemma. Define

$$\lambda'(x, y)=2x-\left(y-\frac{1}{2}\right)^2, \qquad x, y \geqq 0,$$

and note that $\lambda'(x, x)=\lambda(x)$ for every $x \geqq 0$.

LEMMA 3. – *Assume* $\limsup_k a_k > 1$, (3.1), (3.2) and $\frac{1}{2} < \alpha_k \leqq -\frac{1}{a_k}+\frac{3}{2}$ *for every* $k \in \mathcal{N}_2$.

(*a*) (i) *If* $\lambda'(\bar{\alpha}, \underline{\alpha}) < \theta$, *then we have* (3.3).

(ii) *If* $\lambda'(\underline{\alpha}, \bar{\alpha}) > \theta$, *then* (3.3) *does not hold.*

(*b*) (i) *If* $\lambda(\bar{\alpha}) < \theta$, *then we have* (3.4).

(ii) *If* $\lambda(\underline{\alpha}) > \theta$, *then* (3.4) *does not hold.*

Proof. – Before proving the lemma we shall remark that $\frac{1}{2} \leqq \underline{\alpha} \leqq \bar{\alpha} \leqq \frac{3}{2}$, $0 \leqq \lambda'(\underline{\alpha}, \bar{\alpha}) \leqq \lambda'(\bar{\alpha}, \underline{\alpha}) \leqq 2$ and $1 \leqq \lambda(\underline{\alpha}) \leqq \lambda(\bar{\alpha}) \leqq 2$, and that (3.2) implies $\lim_{k \in \mathcal{N}_2} p_k = 0$, consequently,

$$\underline{\alpha}=\liminf_{k \in \mathcal{N}_2} \alpha_k=\liminf_{k \in \mathcal{N}_2} \frac{1}{a_k^2}(-\log p_k).$$

Therefore, for any $0<\delta<\frac{1}{2}$, we may choose $k(\delta) \in \mathcal{N}_2$ such that

$$\frac{1}{\bar{\alpha}+\delta} \log \frac{1+p_k}{p_k} \leqq a_k^2 \leqq \frac{1}{\underline{\alpha}-\delta}(-\log p_k) \tag{3.5}$$

for every $k \geqq k(\delta)$, $k \in \mathcal{N}_2$.

On the other hand (3.1) implies

$$\sum_{k \in \mathcal{N}_1} p_k^2\left(\exp\left[\frac{u}{2} a_k^2\right]-1\right)<\infty$$

for every $u>0$ and we have by (3.2)

$$\theta=\sup\left\{u \geqq 0 ;\ \sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right]<\infty\right\}. \tag{3.6}$$

(a-i) Assume $\lambda'(\bar{\alpha}, \underline{\alpha})<\theta$ and fix any u, $u_0 \geqq 0$ such that $\lambda'(\bar{\alpha}, \underline{\alpha})<u_0<u<\theta$. Then, by (3.6), we have $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right]<\infty$. According as $\underline{\alpha}>\frac{1}{2}$ or $\underline{\alpha}=\frac{1}{2}$, choose $0<\delta<\frac{1}{2}$ such that $\lambda'(\bar{\alpha}+\delta, \underline{\alpha}-\delta) \leqq u_0$ and $\underline{\alpha} \geqq \frac{1}{2}+\delta$ or $\lambda'\left(\bar{\alpha}+\delta, \frac{1}{2}\right) \leqq u_0$. Then we have

$$\frac{(\underline{\alpha}-\delta-(1/2))_+^2+u_0}{2(\bar{\alpha}+\delta)}-1 \geqq 0,$$

$$a_k\left(\alpha_k-\frac{1}{2}\right)=\left(\frac{1}{a_k^2} \log \frac{1+p_k}{p_k}-\frac{1}{2}\right) a_k \geqq\left(\underline{\alpha}-\delta-\frac{1}{2}\right)_+ a_k$$

and

$$\frac{1}{2} a_k^2 \geqq \frac{1}{2(\bar{\alpha}+\delta)}(-\log p_k)$$

for every $k \geqq k(\delta)$, $k \in \mathcal{N}_2$. Consequently we have

$$\begin{aligned}
& \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k-(1/2))}^{a_k(\alpha_k+(1/2))} g(x)\, dx \\
\leqq & \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} a_k p_k g\left(a_k\left(\alpha_k-\frac{1}{2}\right)\right) \\
\leqq & \frac{1}{\sqrt{2 \pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} a_k p_k \exp\left[-\frac{a_k^2}{2}\left(\underline{\alpha}-\delta-\frac{1}{2}\right)_+^2\right] \\
\leqq & \frac{1}{\sqrt{2 \pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} a_k p_k^2 \exp\left[\frac{u_0}{2} a_k^2\right] p_k^{u_0+(\underline{\alpha}-\delta-(1/2))_+^2/2(\bar{\alpha}+\delta)-1} \\
\leqq & \frac{1}{\sqrt{2 \pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] a_k \exp\left[-\frac{u-u_0}{2} a_k^2\right]<\infty,
\end{aligned}$$

which proves (a-i).

(a-ii) Assume $\lambda'(\underline{\alpha}, \bar{\alpha})>\theta$ and fix any $u \geqq 0$ such that $\lambda'(\underline{\alpha}, \bar{\alpha})>u>\theta$. Then, by (3.6), we have $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right]=\infty$. Choose $0<\delta<\frac{1}{2}$ such that $\lambda'(\underline{\alpha}-\delta, \bar{\alpha}+2\delta) \geqq u$. Then we have

$$\frac{(\bar{\alpha}+2\delta-(1/2))^2+u}{2(\underline{\alpha}-\delta)}-1 \leqq 0,$$

so that

$$\sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x)\,dx$$

$$\geqq \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k - (1/2)) + \delta a_k} g(x)\,dx$$

$$\geqq \delta \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k a_k g\left(a_k\left(\alpha_k - \frac{1}{2}\right) + \delta a_k\right)$$

$$\geqq \frac{\delta}{\sqrt{2\pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k \exp\left[-\frac{a_k^2}{2}\left\{\bar{\alpha} + 2\delta - \frac{1}{2}\right\}^2\right]$$

$$\geqq \frac{\delta}{\sqrt{2\pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] = \infty,$$

which proves (a-ii).

(b-i) Assume $\lambda(\bar{\alpha}) < \theta$.

If $\bar{\alpha} = \frac{3}{2}$, then $\sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] < \infty$ by $\lambda\left(\frac{3}{2}\right) = 2$ and (3.6), thus we obtain the conclusion.

Next we assume $\bar{\alpha} < \frac{3}{2}$ and fix any $u > 0$ such that $\lambda(\bar{\alpha}) < u < \theta$. Then, by (3.6), $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] < \infty$. Choose $0 < \delta < \frac{1}{2}$ such that

$$\bar{\alpha} + \delta < \frac{3}{2} \quad \text{and} \quad u \geqq \lambda(\bar{\alpha} + \delta) = 2 - \left(\frac{3}{2} - (\bar{\alpha} + \delta)\right)^2.$$

Then from (3.5) we have

$$a_k\left(\frac{3}{2} - \alpha_k\right) \geqq \left(\frac{3}{2} - (\bar{\alpha} + \delta)\right) a_k > 0$$

for every $k \geqq k(\delta)$, $k \in \mathcal{N}_2$. Thus

$$\sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x)\,dx$$

$$\leqq \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} \frac{p_k^2 \exp[a_k^2]}{a_k(3/2 - \alpha_k)} g\left(a_k\left(\frac{3}{2} - \alpha_k\right)\right)$$

$$\leqq \frac{1}{\sqrt{2\pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \exp\left[-\frac{(3/2 - (\bar{\alpha} + \delta))^2}{2} a_k^2\right]$$

$$\leqq \frac{1}{\sqrt{2\pi}} \sum_{k \geqq k(\delta),\, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] < \infty,$$

which proves (b-i).

(b-ii) Assume $\lambda(\underline{\alpha})>\theta$ and fix any $u\geqq 0$ such that $\lambda(\underline{\alpha})>u>\theta$. Then, by (3.6), we have $\sum_{k\in\mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2}a_k^2\right]=\infty$. Choose $0<\delta<\frac{1}{2}$ such that $\lambda(\underline{\alpha}-2\delta)\geqq u$. Then we have from (3.5)

$$0<a_k\left(\frac{3}{2}-\alpha_k\right)\leqq\left(\frac{3}{2}-(\underline{\alpha}-\delta)\right)a_k$$

for every $k\geqq k(\delta)$, $k\in\mathcal{N}_2$, and

$$\sum_{k\geqq k(\delta),\, k\in\mathcal{N}_2} p_k^2 \exp[a_k^2]\int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx$$

$$\geqq \sum_{k\geqq k(\delta),\, k\in\mathcal{N}_2} p_k^2 \exp[a_k^2]\int_{((3/2)-(\underline{\alpha}-\delta))\,a_k}^{((3/2)-(\underline{\alpha}-\delta))\,a_k+\delta\, a_k} g(x)\,dx$$

$$\geqq \frac{\delta}{\sqrt{2\pi}}\sum_{k\geqq k(\delta),\, k\in\mathcal{N}_2} p_k^2 \exp[a_k^2]\exp\left[-\frac{((3/2)-(\underline{\alpha}-2\delta))^2}{2}a_k^2\right]$$

$$\geqq \frac{\delta}{\sqrt{2\pi}}\sum_{k\geqq k(\delta),\, k\in\mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2}a_k^2\right]=\infty,$$

which proves (b-ii). □

Proof of theorem 3. – First we consider the case $\alpha=\frac{3}{2}$. $\lambda\left(\frac{3}{2}\right)=2<\theta$ implies $I(2;\sigma)<\infty$, thus we have $\mu_{\mathbf{G}}\sim\mu_{\mathbf{G}+\mathbf{Y}}$ by Proposition 2(b) and Theorem 4.

Conversely $\lambda\left(\frac{3}{2}\right)>\theta$ implies $I(2-2\tau;\sigma)=\infty$ for some $0<\tau<1$, thus we have

$$\sum_k p_k^2(\exp[(1-\tau)\,a_k^2]-1)=\infty.$$

Without loss of generality, by Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), and consequently we have

$$\sum_{k\in\mathcal{N}_2} p_k^2 \exp[(1-\tau)\,a_k^2]=\infty.$$

Choose $\delta>0$ such that $\delta^2<2\tau$ and $k_0\in\mathcal{N}_2$ such that $k\geqq k_0$, $k\in\mathcal{N}_2$ implies $\alpha_k+\delta\geqq\frac{3}{2}$. Then we have

$$\sum_{k\geqq k_0,\, k\in\mathcal{N}_2} p_k^2 \exp[a_k^2]\int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx$$

$$\geqq \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{\delta a_k}^{\sqrt{2\tau}\, a_k} g(x)\, dx$$

$$\geqq \frac{\sqrt{2\tau}-\delta}{\sqrt{2\pi}} \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp[(1-\tau)\, a_k^2] = \infty,$$

which implies $\mu_{\mathbf{G}} \perp \mu_{\mathbf{G}+\mathbf{Y}}$ by Lemma 2 and Theorem 4.

Next we consider the case $\frac{1}{2} < \alpha < \frac{3}{2}$. Choose $0 < \eta < 1$ and $k_0 \in \mathcal{N}_2$ such that

$$\frac{1}{2} < \alpha_k < -\eta + \frac{3}{2} \tag{3.7}$$

for every $k \geqq k_0$, $k \in \mathcal{N}_2$. $\lambda(\alpha) < \theta$ implies (3.1) and (3.2) since $\lambda(\alpha) > 1$ for $\frac{1}{2} < \alpha < \frac{3}{2}$. On the other hand, by (3.2) and (3.7), we have $\lim_{k \in \mathcal{N}_2} p_k = 0$, so that $\lim_{k \in \mathcal{N}_2} a_k = \infty$. Therefore we may choose $k_1 \geqq k_0$, $k_1 \in \mathcal{N}_2$ such that

$$\frac{1}{2} < \alpha_k < -\frac{1}{a_k} + \frac{3}{2}$$

for every $k \geqq k_1$. Thus we have $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ by Lemma 3, Lemma 2 and Theorem 4.

Conversely assume $\lambda(\alpha) > \theta$. By Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), consequently we have $\mu_{\mathbf{G}} \perp \mu_{\mathbf{G}+\mathbf{Y}}$ by the same argument from above.

Finally we consider the case $\alpha = \frac{1}{2}$. $\lambda\left(\frac{1}{2}\right) = 1 < \theta$ implies $I(1+\tau;\, \sigma) < \infty$ for some $0 < \tau < 1$, then we have (3.1), (3.2) and

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1+\tau}{2} a_k^2\right] < \infty.$$

Choose $k_0 \in \mathcal{N}_2$ such that $\alpha_k < \frac{1+\tau}{2}$ for every $k \geqq k_0$, $k \in \mathcal{N}_2$. Then we have

$$\sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k \leqq \sum_{k \geqq k_0,\, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1+\tau}{2} a_k^2\right] < \infty,$$

so that $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ by Lemma 1 and Theorem 4.

Conversely $\lambda\left(\frac{1}{2}\right)>\theta$ implies $I(1-\tau;\sigma)=\infty$ for some $0<\tau<1$, so that

$$\sum_k p_k^2\left(\exp\left[\frac{1-\tau}{2}a_k^2\right]-1\right)=\infty.$$

By Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), and consequently we have

$$\sum_{k\in\mathcal{N}_2} p_k^2\exp\left[\frac{1-\tau}{2}a_k^2\right]=\infty.$$

Choose $\delta>0$ such that $\delta<\sqrt{1+\tau}-1$ and $k_0\in\mathcal{N}_2$ such that $k\geqq k_0$, $k\in\mathcal{N}_2$ implies $\alpha_k+\delta\geqq\frac{1}{2}$. Then we have

$$\sum_{k\geqq k_0,\,k\in\mathcal{N}_2} p_k^2\exp[a_k^2]\int_{a_k((3/2)-\alpha_k)}^{\infty} g(x)\,dx$$
$$\geqq \sum_{k\geqq k_0,\,k\in\mathcal{N}_2} p_k^2\exp[a_k^2]\int_{(1+\delta)a_k}^{\sqrt{1+\tau}\,a_k} g(x)\,dx$$
$$\geqq \frac{\sqrt{1+\tau}-(1+\delta)}{\sqrt{2\pi}}\sum_{k\geqq k_0,\,k\in\mathcal{N}_2} p_k^2\exp\left[\frac{1-\tau}{2}a_k^2\right]=\infty,$$

which implies $\mu_{\mathbf{G}}\perp\mu_{\mathbf{G}+\mathbf{Y}}$ by Lemma 2 and Theorem 4. □

4. EXAMPLES

In this section we shall give negative answers to Kahane's conjecture. For the two-valued sequence $\mathbf{Y}=\{\varepsilon(a_k,p_k)\}_{k\geqq 1}$ on (T,Σ,σ), define

$$a_k=\sqrt{\beta\log(k^\gamma+1)},\qquad p_k=k^{-\gamma},\qquad k\in\mathbf{N},$$

where β is positive constant and $\gamma>\frac{1}{2}$. Then we have

$$\alpha_k=\frac{1}{a_k^2}\log\frac{1+p_k}{p_k}=\frac{1}{\beta},\qquad k\in\mathbf{N},$$

and $\alpha=\lim_k\alpha_k=\frac{1}{\beta}$. Moreover we have

$$\sum_1^n p_k^2\exp\left[\frac{u}{2}a_k^2\right]=\sum_1^n k^{-2\gamma}(k^\gamma+1)^{(u/2)\beta}=O\left(\sum_1^n k^{-2\gamma+(u/2)\beta\gamma}\right),\qquad n\to\infty,$$

for every $u>0$, so that $\theta=\frac{2(2\gamma-1)}{\beta\gamma}$ and $I(\theta;\sigma)=\infty$. The multiplicative chaos M defined in (1.2) is given by

$$M_n(t,\omega)=\exp\left[\sum_1^n\left\{G_k(\omega)\,\varepsilon(a_k,p_k)(t)-\frac{1}{2}\varepsilon(a_k,p_k)(t)^2\right\}\right].$$

(4.1) $\beta=\frac{1}{2}$ and $\gamma=\frac{2}{3}$.

In this case $I(1;\sigma)<\infty$ but σ is not M-regular. In fact we have $\sup_k a_k=\infty$, $\alpha=2>\frac{3}{2}$ and $\theta=2$, so that $I(2;\sigma)=\infty$, $I(1;\sigma)<\infty$ and σ is not M-regular by Theorem 2(b-i) and Theorem 4.

(4.2) $\beta=\frac{5}{6}$ and $\gamma=\frac{2}{3}$.

In this case also $I(1;\sigma)<\infty$ but σ is not M-regular. In fact we have $\sup_k a_k=\infty$, $\frac{1}{2}<\alpha=\frac{6}{5}<\frac{3}{2}$, $\theta=\frac{6}{5}>1$ and $\lambda(\alpha)=\frac{191}{100}>\theta$, so that $I(1;\sigma)<\infty$ and σ is not M-regular by Theorem 3 and Theorem 4.

(4.3) $\beta=3$ and $\gamma=2$.

In this case $I(1;\sigma)=\infty$ but σ is M-regular. In fact we have

$$\sum_k p_k=\sum_k k^{-2}<\infty,$$

so that $\sup_k \varepsilon(a_k,p_k)<\infty$ σ-a.s. and σ is M-regular by Proposition 1(a) and Theorem 4.

On the other hand $\theta=1$, hence $I(1;\sigma)=\infty$.

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