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PAUL DOUKHAN
PASCAL MASSART
EMMANUEL RIO

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The functional central limit theorem for strongly mixing processes

by

Paul DOUKHAN [1], **Pascal MASSART and Emmanuel RIO**

U.R.A.-C.N.R.S. D 0743,
Université de Paris-Sud,
Bât. n° 425 Mathématique,
91405 Orsay Cedex, France

ABSTRACT. – Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of $\mathbb{R}^d$-valued zero-mean random variables. Let $(\alpha_n)_{n>0}$ be the sequence of mixing coefficients. We define the *strong mixing function* α by $\alpha(t) = \alpha_{[t]}$ and we denote by Q the quantile function of $|X_0|$, which is the inverse function of $t \to \mathbb{P}(|X_0| > t)$. The main result of this paper is that the functional central limit theorem holds whenever the following condition is fulfilled:

$$(*) \qquad \int_0^1 \alpha^{-1}(t) [Q(t)]^2 \, dt < \infty$$

where f^{-1} denotes the inverse of the monotonic function f. Note that this condition is equivalent to the usual condition $\mathbb{E}(X_0^2) < \infty$ for m-dependent sequences. Moreover, for any $a > 1$, we construct a sequence $(X_i)_{i \in \mathbb{Z}}$ with strong mixing coefficients α_n of the order of n^{-a} such that the CLT does not hold as soon as condition $(*)$ is violated.

Key words : Central limit theorem, strongly mixing processes, Donsker-Prohorov invariance principle, stationary sequences.

RÉSUMÉ. – Soit $(X_i)_{i \in \mathbb{Z}}$ une suite stationnaire et fortement mélangeante de variables aléatoires réelles centrées, de suite de coefficients de mélange

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[1] And Département d'Économie, Université de Cergy-Pontoise, 53, boulevard du Port, 95011 Cergy-Pontoise, France.

$(\alpha_n)_{n>0}$. On définit la fonction de taux de mélange $\alpha(.)$ par $\alpha(t)=\alpha_{[t]}$ et on note Q la fonction de quantile de $|X_0|$. Sous la condition

$$(*) \qquad \int_0^1 \alpha^{-1}(t)[Q(t)]^2\,dt<\infty$$

on établit le théorème limite central fonctionnel pour la suite $(X_i)_{i\in\mathbb{Z}}$. Pour une suite m-dépendante, la condition $(*)$ est équivalente à la condition classique $\mathbb{E}(X_0^2)<+\infty$. De plus, pour tout $a>1$, on construit une suite $(X_i)_{i\in\mathbb{Z}}$ stationnaire dont les coefficients de mélange fort sont de l'ordre de n^{-a} telle que le théorème limite central n'est pas vérifié si la condition $(*)$ n'est pas satisfaite.

1. INTRODUCTION AND MAIN RESULTS

Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued zero-mean random variables with finite variance. As a measure of dependence we will use the strong mixing coefficients introduced by Rosenblatt (1956). For any two σ-algebras $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, let

$$\alpha(\mathscr{A}, \mathscr{B})=\sup_{(A, B)\in\mathscr{A}\times\mathscr{B}}|\mathbb{P}(A\cap B)-\mathbb{P}(A)\mathbb{P}(B)|=\sup_{(A, B)\in\mathscr{A}\times\mathscr{B}}|\mathrm{Cov}(\mathbf{1}_A, \mathbf{1}_B)|.$$

Since $(X_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence, the mixing coefficients α_n of the sequence $(X_i)_{i\in\mathbb{Z}}$ are defined by $\alpha_n=\alpha(\mathscr{F}_0, \mathscr{G}_n)$, where $\mathscr{F}_0=\sigma(X_i : i\leqq 0)$ and $\mathscr{G}_n=\sigma(X_i : i\geqq n)$. $(X_i)_{i\in\mathbb{Z}}$ is called a strongly mixing sequence if $\lim_{n\to+\infty}\alpha_n=0$. Examples of such sequences may be found in Davydov (1973), Bradley (1986) and Doukhan (1991).

Let $S_n=\sum_{i=1}^n X_i$. If the distribution of $S_n/\sqrt{n}$ is weakly convergent to a (possibly degenerate) normal distribution, then $(X_i)_{i\in\mathbb{Z}}$ is said to satisfy the central limit theorem (CLT). Let define the process $\{Z_n(t) : t\in[0, 1]\}$ by

$$(1.0) \qquad \sqrt{n}\,Z_n(t)=\sum_{i=1}^{[nt]} X_i,$$

square brackets designing the integer part, as usual. For each ω, $Z_n(.)$ is an element of the Skorohod space D([0, 1]) of all functions on [0, 1] which have left-hand limits and are continuous from the right. It is equipped with the Skorohod topology (*cf.* Billingsley, 1968, sect. 14). Let W denote the standard Brownian motion on [0, 1]. If the distribution of $Z_n(.)$ converges weakly in D([0, 1]) to σW for some nonnegative σ, $(X_i)_{i \in \mathbb{Z}}$ is said to satisfy the functional central limit theorem.

For stationary strongly mixing sequences, the CLT and the functional CLT may fail to hold when only the variance of the r.v.'s is assumed to be finite (*cf.* Davydov, 1973). So, the aim of this paper is to provide a sharp condition on the tail function $t \to \mathbb{P}(|X_0| > t)$ and on the mixing coefficients implying the CLT and the functional CLT. By sharp condition, we mean that, given a sequence of mixing coefficients and a tail function violating this condition, one can construct a strictly stationary sequence $(X_i)_{i \in \mathbb{Z}}$ with corresponding tail function and mixing coefficients for which the CLT does not hold. Let us recall what is known on this topic at this time.

As far as we know, all the results in the field are of the following type: assume that for some adequate function ϕ, $\phi(X_0^2)$ is integrable and that the mixing coefficients satisfy some summability condition (depending of course on ϕ), then the CLT holds.

The first result of this type is due to Ibragimov (1962) who takes $\phi(x) = x^r$ with $r > 1$ and gives the summability condition

$$\sum_{n>0} \alpha_n^{1-1/r} < +\infty.$$

The functional CLT was studied by Davydov (1968): he obtained the summability condition $\sum_{n>0} \alpha_n^{1/2-1/2r} < +\infty$. Next, Oodaira and Yoshihara (1972) obtained the functional CLT under Ibragimov's condition.

Since a polynomial moment condition is not well adapted to exponential mixing rates, Herrndorf (1985) introduces more flexible moment assumptions. Let $\mathscr{F}$ denote the set of convex and increasing differentiable functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and $\lim_{+\infty} x^{-1} \phi(x) = \infty$. Assume that ϕ belongs to $\mathscr{F}$, then Herrndorf gives the summability condition:

$$\sum_{n>0} \alpha_n \phi^{-1}(1/\alpha_n) < +\infty, \tag{1.1}$$

where ϕ^{-1} denote the inverse function of ϕ. Actually, Herrndorf proves a functional CLT for nonstationary sequences under an additional condition on the variances of the partial sums of the r.v.'s X_i, and this condition is ensured by (1.1) for stationary sequences via Theorem 2 of Bulinskii and Doukhan (1987).

All these results rely on covariance inequalities (such as Davydov's) which hold under moment assumptions. Our approach to improve on previous results is to use a new covariance inequality recently established by Rio (1992), which introduces an explicit dependence between the mixing coefficients and the tail function. We first need to introduce some notations that we shall use all along this paper.

Notations. — If (u_n) is a nonincreasing sequence of nonnegative real numbers, we denote by $u(.)$ the rate function defined by $u(t)=u_{[t]}$. For any nonincreasing function f, let f^{-1} denote the inverse function of f,

$$f^{-1}(u)=\inf\{t: f(t)\leqq u\}.$$

For any random variable X with distribution function F, we denote indefferently by Q_X or Q_F the quantile function which is the inverse of the tail function $t \to \mathbb{P}(|X|>t)$.

LEMMA 1 [Rio, 1992]. — *Let* X *and* Y *be two real-valued r.v.'s with finite variance. Let* $\alpha=\alpha(\sigma(X), \sigma(Y))$. *Let* $Q_X(u)=\inf\{t: \mathbb{P}(|X|>t)\leqq u\}$ *denote the quantile function of* $|X|$. Then $Q_X Q_Y$ *is integrable on* [0, 1], *and*

$$|\operatorname{Cov}(X, Y)|\leqq 2\int_0^{2\alpha} Q_X(u)\, Q_Y(u)\, du.$$

By Theorem 1.2 in Rio (1992), the following result holds.

PROPOSITION 1 [Rio, 1992]. — *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a strictly stationary and strongly mixing sequence of real-valued random variables satisfying*

$$M=\int_0^1 \alpha^{-1}(u)\,[Q_{X_0}(u)]^2\, du<+\infty.$$

Then,

$$\sum_{t\in\mathbb{Z}} |\operatorname{Cov}(X_0, X_t)|\leqq 8\,M,$$

and denoting by σ^2 *the sum of the series* $\sum_{t\in\mathbb{Z}} \operatorname{Cov}(X_0, X_t)$, *we have:*

$$\operatorname{Var} S_n \leqq 8\,M\,n \qquad \textit{and} \qquad \lim_{n\to+\infty} n^{-1}\operatorname{Var} S_n=\sigma^2.$$

Now the question raises whether the above integral condition (which ensures the convergence of the variance of the normalized partial sums) is sufficient to imply the CLT. In fact the answer is positive as shown by the following theorem.

THEOREM 1. — *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a strictly stationary and strongly mixing sequence of real-valued centered random variables satisfying*

$$\int_0^1 \alpha^{-1}(u)\,[Q_{X_0}(u)]^2\, du<+\infty. \qquad (1.2)$$

Then the series $\sum_{t \in \mathbb{Z}} \operatorname{Cov}(X_0, X_t)$ *is absolutely convergent, and* Z_n *converges in distribution to* σW *in the Skorohod space* $D([0, 1])$, *where* σ *is the nonnegative real defined by* $\sigma^2 = \sum_{t \in \mathbb{Z}} \operatorname{Cov}(X_0, X_t)$.

Remark. — This result still holds for $\mathbb{R}^d$-valued random variables. Then $|X_0|$ denotes the norm of X_0 in $\mathbb{R}^d$, and the multivariate process Z_n converges in distribution to a multivariate Wiener measure with some covariance function $(s, t) \to \sum_{i \in \mathbb{Z}} \mathbb{E}((X_0 . s)(X_i . t)) = \Gamma(s, t)$, where $a . b$ denotes the canonical inner product on $\mathbb{R}^d$.

APPLICATIONS. — 1. *Bounded random variables.* — If X_0 is a bounded r.v., Q is uniformly bounded over [0, 1], and condition (1.2) is equivalent to the condition $\sum_{n>0} \alpha_n < +\infty$. So, in that case, the CLT obtained by Ibragimov and Linnik (1971) (Theorems 18.5.3 and 18.5.4) and the functional CLT of Hall and Heyde (1980) (Corollary 5.1) are recovered.

2. *Conditions on the tail function.* — Let ϕ be some element of $\mathscr{F}$. Assume that there exists some positive constant C_ϕ such that the distribution of X_0 fulfills:

$$\mathbb{P}(X_0^2 > u) \leqq 1/\phi(u/C_\phi).$$

If, for some $r > 1$, $x \to x^{-r} \phi(x)$ is nondecreasing, Theorem 1 shows that the functional CLT holds whenever the summability condition (1.1) is satisfied. Hence the functional CLT is ensured by a weaker condition on the tail of X_0 than Herrndorf's moment condition $\mathbb{E}(\phi(X_0^2)) < +\infty$.

3. *Moment conditions.* — Assume that $\mathbb{E}(\phi(X_0^2)) < +\infty$ for some $\phi \in \mathscr{F}$. This condition is equivalent to the following moment condition on Q:

$$\int_0^1 \phi([Q(u)]^2)\, du < +\infty.$$

Note that, if U is uniformly distributed over [0, 1], Q(U) has the distribution of $|X_0|$.

So, if $\phi^*(y) = \sup_{x>0} [xy - \phi(x)]$ denotes the dual function of ϕ, Young's inequality ensures that (1.2) holds if

$$\int_0^1 \phi^*(\alpha^{-1}(u))\, du < +\infty. \tag{1.3}$$

An elementary calculation shows that (1.3) holds if

$$\sum_{n>0} (\phi')^{-1}(n)\, \alpha_n < +\infty. \tag{1.4}$$

Some calculations show that this summability condition is weaker than (1.1) [one can use the convexity of ϕ and the monotonicity of the sequence $(\alpha_n)_{n>0}$ to prove that $(\phi')^{-1}(n) \leqq \phi^{-1}(1/\alpha_n)$ if n is large enough; *cf.* Rio, 1992]. In particular, when $\phi(x) = x^r$ for some $r > 1$, (1.4) holds if and only if the series $\sum_{k>0} k^{1/(r-1)} \alpha_k$ is convergent, which improves on (1.1). For example, when $\alpha_n = O(n^{-r/(r-1)} (\log n)^{-\theta})$ for some $\theta > 0$, (1.4) needs $\theta > 1$, while (1.1) needs $\theta > r/(r-1)$, which shows that condition (1.2) is weaker than (1.1).

4. *Exponential mixing rates.* – Assume that the mixing coefficients satisfy $\alpha_k = O(a^k)$ for some a in $]0, 1[$. Then there exists some $s > 0$ such that (1.3) holds with $\phi^*(x) = \exp(sx) - sx - 1$. Since $\phi = (\phi^*)^*$, (1.2) holds if

$$\mathbb{E}(X_0^2 \log^+ |X_0|) < +\infty. \tag{1.5}$$

Now, when $\phi(x) \sim x \log^+ x$ as $x \to +\infty$, the summability condition (1.1) holds iff

$$\sum_{n>0} [\log(1/\alpha_n)]^{-1} < +\infty. \tag{1.6}$$

If the mixing rate is truly exponential, that is for some positive $c \log(1/\alpha_k) \geqq ck$ for any large enough k, then (1.6) does not hold. Hence Theorem 1 still improves on Herrndorf's result. ■

We now show that Theorem 1 is sharp for power-type mixing rates.

THEOREM 2. – *Let $a > 1$ be given and* F *be any continuous distribution function of a zero-mean real-valued random variable such that:*

$$(a) \qquad \int_0^1 u^{-1/a} [Q_F(u)]^2 \, du = +\infty.$$

Then there exist a stationary Markov chain $(Z_i)_{i \in \mathbb{Z}}$ of random variables with d.f. F *such that:*

(i) $0 < \liminf_{n \to +\infty} n^a \alpha_n \leqq \limsup_{n \to +\infty} n^a \alpha_n < \infty$,

(ii) $n^{-1/2} \sum_{i=1}^{n} Z_i$ *does not converge in distribution to a Gaussian random variable.*

The organization of the paper is the following: in section 2, we prove the CLT for stationary sequences satisfying (1.2). In section 3, we prove the tightness of Z_n and we derive the functional CLT. Next, in section 4, we prove Theorem 2 and give some additional results concerning the optimality of condition (1.2).

2. A NEW CLT FOR THE PARTIAL SUMS OF A STRONGLY MIXING SEQUENCE

In this, section, starting from the covariance inequalities of Rio (1992), we prove a central limit theorem for stationary and strongly mixing sequences $(X_i)_{i\in\mathbb{Z}}$ satisfying condition (1.2). The proof of this CLT is based on the following theorem of Hamm and Heyde (1980) (Theorem 5.2) which relies on a deep result of Gordin (1969).

THEOREM 3 [Hall and Heyde], 1980. – *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a stationary and ergodic sequence with* $\mathbb{E}(X_0)=0$ *and* $\mathbb{E}(X_0^2)<\infty$. *Let* $\mathscr{F}_0=\sigma(X_i:i\leqq 0)$. *If* $\sum_{k>0}\mathbb{E}(X_k\,\mathbb{E}(X_n\,|\,\mathscr{F}_0))$ *converges for each* $n>0$ *and*

$$\lim_{n\to+\infty}\sum_{k\geqq K}|\mathbb{E}(X_k\,\mathbb{E}(X_n\,|\,\mathscr{F}_0))|=0$$

uniformly in $K\geqq 1$, *then* $\lim_{\infty} n^{-1}\,\mathbb{E}(S_n^2)=\sigma^2$ *for some nonnegative read* σ. *If* $\sigma>0$, *then* $S_n/\sigma\sqrt{n}$ *converges in distribution to the standard normal law.*

We now prove the CLT under condition (1.2).

THEOREM 4. – *Let* $(X_i)_{i\in\mathbb{Z}}$ *be strictly stationary sequence of real-valued centered r.v.'s satisfying* (1.2). *Then:*

(i) *the séries* $\sum_{i\in\mathbb{Z}}\mathbb{E}(X_0X_i)$ *converges to a nonnegative number* σ^2 *and* $n^{-1}\operatorname{Var}S_n$ *converges to* σ^2.

(ii) *if* $\sigma>0$, $S_n/\sqrt{n}$ converges in distribution to the normal distribution $N(0,\sigma^2)$.

Proof. – By Proposition 1, (i) holds. In order to apply Theorem 3, we note that, on the one hand $(X_i)_{i\in\mathbb{Z}}$ is an ergodic sequence because it is a strongly mixing stationary sequence and on the other hand that the other assumptions of Theorem 3 are implied by

$$\lim_{n\to+\infty}\sum_{n>0}|\mathbb{E}(X_k\,\mathbb{E}(X_n\,|\,\mathscr{F}_0))|=0. \tag{2.1}$$

Our program to prove (ii) is now to ensure that (2.1) holds by means of Lemma 1. To this aim, we need to control the tail of $\mathbb{E}(X_n\,|\,\mathscr{F}_0)$. This is precisely what is done in the following claims.

CLAIM 1. – *For any nonincreasing positive function* $Q(.)$ *satisfying* (1.2) *there exists some nonincreasing function* Q_*, *still satisfying* (1.2), *and such that:*

$$Q_*(t)\geqq Q(t) \quad \textit{and} \quad t\to t[Q_*(t)]^{3/2} \textit{ is nondecreasing.} \tag{2.2}$$

Proof. – Let $R(t)=t[Q(t)]^{3/2}$ and $R_*(t)=\sup_{s\leq t} R(s)$. The monotonicity properties of the above functions imply that

$$R_*(t)\leqq R_*(t/2) \vee t[Q(t/2)]^{3/2}. \tag{2.3}$$

Let $Q_*(t)=(R_*(t)/t)^{2/3}$. Clearly, Q_* satisfies (2.2). It remains to prove that Q_* still satisfies condition (1.2). Now, it follows from (2.3) that

$$Q_*(t)\leqq Q(t/2) \vee 2^{-2/3} Q_*(t/2). \tag{2.4}$$

Assume now that $Q(.)$ is uniformly bounded over]0, 1]. Both (2.4) and the monotonicity of α^{-1} and Q imply, using the change of variable $u=t/2$, that:

$$\int_0^1 \alpha^{-1}(t)[Q_*(t)]^2\,dt \leqq 2^{-1/3}\int_0^{1/2} \alpha^{-1}(u)[Q_*(u)]^2\,du+2\int_0^{1/2} \alpha^{-1}(u)[Q(u)]^2\,du.$$

Hence

$$\int_0^{1/2} \alpha^{-1}(t)[Q_*(t)]^2\,dt\leqq 10\int_0^{1/2} \alpha^{-1}(u)[Q(u)]^2\,du, \tag{2.5}$$

therefore establishing Claim 1 for Q_* in the case to uniformly bounded functions. The corresponding result for unbounded quantile functions follows from (2.5) applied to $Q_A(t)=Q(t)\wedge A$ and from the fact that $Q_*=\lim_{A\nearrow\infty}\uparrow(Q_A)_*$ combined with Beppo-Levi Lemma. ■

From now on, let $Q=Q_{X_0}$ and let Q_* satisfy the properties of Claim 1.

CLAIM 2. – *Let* $X_n^0=\mathbb{E}(X_n\mid\mathcal{F}_0)$. *For any* $t\in]0, 1]$, $Q_{X_n^0}(t)\leqq 2Q_*(t/2)$.

Proof. – On the one hand, the convexity of the function $h_x(u)=2/x(|u|-x/2)^+$ together with Jensen inequality conditionally to $\mathcal{F}_0$ and the elementary inequality $h_x(u)\geqq 1$ whenever $|u|>x$ show that

$$\mathbf{1}_{(|X_n^0|>x)}\leqq h_x(X_n^0)\leqq \mathbb{E}(h_x(X_n)\mid\mathcal{F}_0). \tag{2.6}$$

On the other hand, the stationarity of $(X_i)_{i\in\mathbb{Z}}$ and an integration by parts give

$$\mathbb{E}(h_x(X_n))=\mathbb{E}(h_x(X_0))=\frac{2}{x}\int_{x/2}^{\infty} Q^{-1}(x)\,dx,$$

which, combined with (2.6), provides:

$$\mathbb{P}(|X_n^0|>x)\leqq \frac{2}{x}\int_{x/2}^{\infty} Q^{-1}(x)\,dx.$$

Both the above inequality and (2.2) imply then that

$$\mathbb{P}(|X_n^0|>x)\leqq\sqrt{(x/2)}\,Q_*^{-1}(x/2)\int_{x/2}^{\infty}u^{-3/2}\,du\leqq 2\,Q_*^{-1}(x/2),$$

therefore completing the proof. ■

Proof of Theorem 4. — By Lemma 1 and claim 2,

$$|\mathbb{E}(X_k X_n^0)|\leqq 8\int_0^{\alpha_k}[Q_*(t)]^2\,dt. \tag{2.7}$$

Since Q_* has the properties of claim 1, it follows that the series $\sum_{k>0}|\mathbb{E}(X_k X_n^0)|$ is uniformly convergent w.r.t. n. Hence the proof of Theorem 4 will be achieved if we prove that, for any $k>0$, $\lim_{n\to+\infty}\mathbb{E}(X_k X_n^0)=0$. Since $\mathbb{E}(X_k X_n^0)=\mathbb{E}(X_n X_k^0)$, by (2.7),

$$|\mathbb{E}(X_k X_n^0)|\leqq 8\int_0^{\alpha_n}[Q_*(t)]^2\,dt,$$

therefore completing the proof of (2.1). Hence Theorem 4 holds. ■

3. ON THE DONSKER-PROHOROV INVARIANCE PRINCIPLE

In this section, we derive the functional CLT from Theorem 4.

Proof of Theorem 1. — Assume first that $\sigma>0$. Theorem 4 ensures that the sequence $(S_n^2/n)_{n>0}$ is uniformly integrable, via Theorem 5.4 in Billingsley (1968). Set $S_n^*=\sup_{k\in[1,n]}|S_k|$. By Theorem 1.4 in Peligrad (1985), it is enough to prove that, for any positive ε, there exists $\lambda>0$ and an integer n_0 such that, for any $n>n_0$,

$$\mathbb{P}(S_n^*\geqq\lambda\sqrt{n})\leqq\frac{\varepsilon}{\lambda^2}. \tag{3.1}$$

Second, if $\sigma=0$, the convergence in distribution of Z_n to the null random process follows fom (3.2) and Theorem 4.

Proof of 3.1. — Clearly, there is no loss of generality in assuming that $\mathbb{E}(|X_0|)\leqq 1$, which we shall do throughout. Since $(S_n^2/n)_{n>0}$ is uniformly integrable, by Theorem 22 in Dellacherie and Meyer (1975), there exists an increasing convex function G such that

$$\lim_{x\to\infty}\frac{G(x)}{x}=\infty\quad\text{and}\quad\mathbb{E}(G(S_n^2/n))\leqq 1. \tag{3.2}$$

Since $(X_i)_{i \in \mathbb{Z}}$ is a stationary sequence satisfying (1.2), the sequence $(Y_i)_{i \in \mathbb{Z}}$ defined by $Y_i = |X_i| - \mathbb{E}(|X_i|)$ still satisfies Theorem 4. Let $T_n = \sum_{i=1}^{n} Y_i$. So, we may (modifying G) assume that

$$\mathbb{E}(G(T_n^2/n)) \leqq 1/2. \tag{3.3}$$

Hence, the tightness of Z_n is a consequence of the following proposition.

PROPOSITION 2. – *Let $(X_i)_{i \in \mathbb{Z}}$ be stationary sequence satisfying $\lim_{n \to +\infty} n \alpha_n = 0$. Assume furthermore that $(X_i)_{i \in \mathbb{Z}}$ satisfies the uniform integrability conditions* (3.3) *and* (3.4). *Then $(X_i)_{i \in \mathbb{Z}}$ fulfills the tightness criterion* (3.2).

Proof. – Set $p = p(n) = [\sqrt{n}]$, $q = q(n) = n/p$ and

$$U_i = \int_{q(i-1)}^{qi} X_{[s+1]} \, ds, \qquad V_i = \int_{q(i-1)}^{qi} |X_{[s+1]}| \, ds.$$

Let A_n be the following event: there exists some $i \in [1, p]$ such that $V_i \geqq 2\sqrt{n}$. Since $(X_i)_{i \in \mathbb{Z}}$ is a stationary sequence and $\mathbb{E}(V_i) \leqq \sqrt{n}$, (3.4) yields

$$\mathbb{P}(A_n) \leqq p/G(p). \tag{3.4}$$

Hence, $\lim_{n \to \infty} \mathbb{P}(A_n) = 0$. Let $W_j = U_1 + \ldots + U_j$ and $W_p^* = \max_{j \leqq p} |W_j|$. Clearly

$$S_n^* \leqq W_p^* + \sup_{i \leqq p} (V_i/2). \tag{3.5}$$

It remains to control the r.v. W_p^*. Here, we adapt the proof of Kolmogorov's inequality for i.i.d. r.v.'s to the mixing case. Set $E_j = \{W_j^* \geqq 2(1+\lambda)\sqrt{n},\ W_{j-1}^* < 2(1+\lambda)\sqrt{n}\}$. Then,

$$\mathbb{P}(A_n^c \cap \{W_p^* \geqq 2(1+\lambda)\sqrt{n}\}) \leqq \mathbb{P}(|W_p| \geqq \lambda\sqrt{n}) + \sum_{j=1}^{p} \mathbb{P}(E_j \cap \{|W_p - W_{j+1}| \geqq \lambda\sqrt{n}\}). \tag{3.6}$$

Now, both the mixing property and the uniform and the uniform integrability property (3.3) yield

$$\mathbb{P}(E_j \cap \{|W_p - W_{j+1}| \geqq \lambda\sqrt{n}\}) \leqq \alpha_{[q]-1} + \frac{\mathbb{P}(E_j)}{G(\lambda^2)}. \tag{3.7}$$

In the same way, $\mathbb{P}(|W_p| \geqq \lambda\sqrt{n}) \leqq 1/G(\lambda^2)$. Since the events E_j are disjoint, both (3.4), (3.5), (3.6) and (3.7) imply:

$$\mathbb{P}(S_n^* \geqq (3+2\lambda)\sqrt{n}) \leqq \frac{p(n)}{G(p(n))} + p(n)\,\alpha_{[q(n)]-1} + \frac{2}{G(\lambda^2)}. \tag{3.8}$$

Now, $\lim_{n \to \infty} p(n)\alpha_{[q(n)]-1}=0$, therefore completing the proof of Proposition 2. ■

Since condition (1.2) implies $\lim_{n \to +\infty} n\alpha_n=0$, Theorem 1 holds.

4. ON THE OPTIMALITY OF THE CENTRAL LIMIT THEOREM

In this section, we construct strictly stationary and strongly mixing sequences of r.v.'s such that Theorem 4 does not hold as soon as condition (1.2) is violated. The general way to obtain these sequences is the following. We construct a stationary and β-mixing sequence $(U_i)_{i \in \mathbb{Z}}$ of real-valued r.v.'s. with uniform distribution over [0, 1]. Then, by means of the so-called quantile transformation, we will obtain stationary and β-mixing sequences of real-valued r.v.'s with arbitrary distribution function F. Let us recall the definition of the β-mixing coefficients of $(X_i)_{i \in \mathbb{Z}}$ (Rozanov and Volkonskii, 1959). Given two σ-fields $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, the β-mixing coefficient $\beta(\mathscr{A}, \mathscr{B})$ between $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$\beta(\mathscr{A}, \mathscr{B})=\frac{1}{2}\sup\{\sum_{i \in \mathrm{I}}\sum_{j \in \mathrm{J}}|\mathbb{P}(A_i \cap B_j)-\mathbb{P}(A_i)\mathbb{P}(B_j)|\}$$

where the supremum is taken over finite partitions $(A_i)_{i \in \mathrm{I}}$ and $(B_j)_{j \in \mathrm{J}}$ respectively $\mathscr{A}$ and $\mathscr{B}$ measurable [notice that $\beta(\mathscr{A}, \mathscr{B}) \leqq 1$]. The mixing coefficients β_n of the sequence $(X_i)_{i \in \mathbb{Z}}$ are defined by $\beta_n=\beta(\mathscr{F}_0, \mathscr{G}_n)$. $(X_i)_{i \in \mathbb{Z}}$ is called a β-mixing sequence if $\lim_{n \to +\infty} \beta_n=0$.

It follows from Theorem 4 and from the well known inequality $2\alpha_n \leqq \beta_n$ that the central limit theorem holds whenever the following integral condition holds:

$$\int_0^1 \beta^{-1}(u)[Q_F(u)]^2\,du < +\infty. \tag{4.1}$$

Here, we obtain some kind of converse to Theorem 4 for polynomial rates of decay of the mixing coefficients. Furthermore, our counterexample works for β-mixing sequences as well.

THEOREM 5. — *For any* $a>1$, *there exists a stationary Markov chain* $(U_i)_{i \in \mathbb{Z}}$ *of r.v.'s with uniform distribution over* [0, 1] *and sequence of*

β-mixing coefficients $(\beta_n)_{n \geq 0}$, *such that:*

(i) $0 < \liminf_{n \to +\infty} n^a \beta_n \leq \limsup_{n \to +\infty} n^a \beta_n < \infty$.

(ii) *For any measurable and integrable function* $f:]0, 1] \to \mathbb{R}$ *satisfying*

$$(a) \qquad \int_0^1 u^{-1/a} [f(u)]^2 \, du = +\infty,$$

$n^{-1/2} \sum_{i=1}^{n} [f(U_i) - \mathbb{E}(f(U_i))]$ *does not converge in distribution to a Gaussian r.v.*

We derive from Theorems 4 and 5 the following corollary, yielding Theorem 2.

COROLLARY 1. — *Let* $a > 1$ *be given and* F *be any continuous distribution function of a zero-mean real-valued random variable such that:*

$$(a) \qquad \int_0^1 u^{-1/a} [Q_F(u)]^2 \, du = +\infty.$$

Then there exists a stationary Markov chain $(Z_i)_{i \in \mathbb{Z}}$ *of r.v.'s with d.f.* F *such that:*

(i) $0 < \liminf_{n \to +\infty} 2 . n^a \alpha_n \leq \limsup_{n \to +\infty} n^a \beta_n < \infty$. Here $(\alpha_n)_{n \geq 0}$ *and* $(\beta_n)_{n \geq 0}$ *denote the sequences of strong mixing and* β*-mixing coefficients of* $(Z_i)_{i \in \mathbb{Z}}$.

(ii) $n^{-1/2} \sum_{i=1}^{n} Z_i$ does not converge in distribution to a Gaussian random variable.

Furthermore, $(Z_i)_{i \in \mathbb{Z}}$ does not satisfy the Kolmogorov LIL:

PROPOSITION 3. — *For any continuous d.f.* F *of a zero-mean r.v. satisfying* (*a*) *of Corollary* 1, *there exists a stationary Markov chain* $(Z_i)_{i \in \mathbb{Z}}$ *of r.v.'s with d.f.* F, *satisfying* (i) *of Corollary* 4.1 *and such that, we have, setting* $S_n = \sum_{i=1}^{n} Z_i$,

$$\limsup_{n \to +\infty} \frac{|S_n|}{\sqrt{n \log \log n}} = +\infty \text{ a. s.}$$

Proof of Theorem 5. — The sequence $(U_i)_{i \in \mathbb{Z}}$ will be defined from a strictly stationary Markov chain $(X_i)_{i \in \mathbb{Z}}$.

Definition of the Markov Chain. — Let μ be an atomless probability distribution on [0, 1] and $T:]0, 1] \to [0, 1[$ be a measurable function. The conditional distribution $\Pi(x, .)$ of X_{n+1}, given $(X_n = x)$, is defined by

$$\Pi(x, .) = \Pi(\delta_x, .) = T(x) \delta_x + (1 - T(x)) \mu.$$

Assume furthermore the function T to fulfill the integral condition

$$(4.2)\qquad C(\mu)=\int_0^1 (1-T(x))^{-1}\,\mu(dx)<+\infty.$$

Then the nonnegative measure ν defined by

$$C(\mu)\,\nu=(1-T(x))^{-1}\,\mu$$

is an invariant probability. By Kolmogorov's extension theorem, there exists a strictly stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ with transition probabilities $\Pi(x,.)$ and stationary law ν. If the d.f. F_ν of ν is continuous, setting $U_i=F_\nu(X_i)$, we obtain a stationary Markov chain of r.v.'s with uniform distribution over [0, 1].

Mixing properties of the Markov chain. — Let $\mathcal{F}_0=\sigma(X_i:i\leqq 0)$ and $\mathcal{G}_n=\sigma(X_i:i\geqq n)$. Let $\beta_n=\beta(\mathcal{F}_0,\mathcal{G}_n)$ denote the coefficient of β-mixing between $\mathcal{F}_0$ and $\mathcal{F}_n$. Let F be any continuous distribution function. If F_ν is continuous, β_n also is the coefficient of β-mixing of order n of the chain $(Y_i)_{i\in\mathbb{Z}}$ defined by $Y_i=F^{-1}(F_\nu(X_i))$.

For any probability law m on [0, 1], for any measurable function $f:[0,1]\to\mathbb{R}^+$, we set:

$$\mathbb{E}_m(f)=\int_0^1 f(x)\,m(dx).$$

The following result provides a characterization of those distributions μ such that the Markov chain $(Y_i)_{i\in\mathbb{Z}}$ does not satisfy (4.1).

LEMMA 2. — *For any positive integer n,*

$$\mathbb{E}_\nu(T^n)\leqq\beta_n\leqq 3\,\mathbb{E}_\nu(T^{[n/2]}).$$

Comment. — Let $\beta_{\nu,n}=\mathbb{E}_\nu(T^n)$. Lemma 2 ensures that, for any u in [0, 1],

$$(4.3)\qquad \beta_\nu^{-1}(u)\leqq\beta^{-1}(u)\leqq 3\,\beta_\nu^{-1}(u/3).$$

If follows that the Markov chain $(Y_i)_{i\in\mathbb{Z}}$ satisfies (4.1) if and only if

$$\int_0^1 \beta_\nu^{-1}(u)\,[Q_F(u)]^2\,du<+\infty.$$

Proof of Lemma 2. — Since $(X_i)_{i\in\mathbb{Z}}$ is a stationary Markov chain, by Proposition 1 of Davydov (1973), $\beta_n=\beta(\mathcal{F}_0,\mathcal{G}_n)=\beta(\sigma(X_0),\sigma(X_n))$. So, if $P_{0,n}$ denotes the bivariate distribution of (X_0, X_n),

$$(4.4)\qquad 2\beta_n=\|P_{0,n}-(\nu\otimes\nu)\|=\int_0^1\|\Pi^n(x,.)-\nu\|\,\nu(dx),$$

where $\|m\|$ denotes the variation norm of the measure m. Let $\tau=\inf\{i>0:X_i\neq X_{i-1}\}$. Since $(X_i)_{i\in\mathbb{Z}}$ is a Markov chain, the strong

Markov property ensures that, starting from $(X_0=x)$, for any $k\in]0, n]$, the conditional distribution of X_n given $(\tau=k)$ is $\Pi^{n-k}(\mu, .)$. Let $\mathbb{P}_x$ denote the probability of the chain, starting from $(X_0=x)$:

$$\|\Pi^n(x, .)-\nu\| \leqq 2\,\mathbb{P}_x(\tau>n)+\sum_{k=1}^{n} \mathbb{P}_x(\tau=k)\,\|\Pi^{n-k}(\mu, .)-\nu\|.$$

Clearly, $\mathbb{P}_x(\tau>k)=[T(x)]^k$. Hence, taking the $l^1(\nu)$-norm in (4.4), we obtain:

$$(4.5)\qquad 2\beta_n \leqq 2\,\mathbb{E}_\nu(T^n)+\sum_{k=1}^{n} \mathbb{E}_\nu(T^{k-1}(1-T))\,\|\Pi^{n-k}(\mu, .)-\nu\|.$$

The main tool is then the majorization of the total variations of the signed measures $\Pi^{n-k}(\mu, .)-\nu$. First, we give a polynomial expansion of the measure $\Pi^n(\mu, .)$. Let m be any signed measure on [0, 1]. The linear operator $\Pi: m\to\Pi(m, .)$ has an unique extension to the set signed measures, which we still denote by Π, and

$$\Pi(m, .)=T\,m+\left[\int_0^1 (1-T(x)(m(dx)\right]\mu$$

We now prove that there exists an unique sequence $(a_k)_{k\geqq 0}$ of numbers such that

$$(4.6)\qquad \Pi^n(\mu, .)-\nu=\sum_{k=0}^{n} a_k\,T^{n-k}\mu-T^n\nu.$$

Proof. – We prove (4.6) by induction on n. First the assertion holds if $n=0$. Secondly, if we assume that the assertion holds for any $l<n$, then, at the step n,

$$\Pi^n(\mu, .)-\nu=\Pi(\Pi^{n-1}(\mu, .)-\nu)=T(\Pi^{n-1}(\mu, .)-\nu)+a_n\,\mu,$$

where $a_n=\int_0^1 (1-T(x))(\Pi^{n-1}(\mu, .)-\nu)(dx)$. Now, by the induction hypothesis,

$$T(\Pi^{n-1}(\mu, .)-\nu)=\sum_{k=0}^{n-1} a_k\,T^{n-k}\mu-T^n\nu,$$

which implies the existence of the sequence $(a_n)_{n\geqq 0}$. Since $\Pi_n(\mu, .)$ is a probability measure, the sequence $(a_n)_{n\geqq 0}$ has to satisfy the equations

$$(4.7)\qquad \sum_{k=0}^{n} a_k\,\mathbb{E}_\mu(T^{n-k})=\mathbb{E}_\nu(T^n),$$

for any nonnegative integer n. Hence the coefficients a_n are uniquely defined.

Now, recall that we have in view to bound $\|\Pi^n(\mu, .)-\nu\|$. This bound will be derived from the following main claim.

CLAIM 3. — *For any nonnegative integer n,* $a_n \geqq 0$.

Proof. — By equation (4.7),

$$a_n + \sum_{k=0}^{n-1} a_k \mathbb{E}_\mu(T^{1+(n-1-k)}) = \mathbb{E}_\upsilon(T^n). \tag{4.8}$$

By the convexity of $l \to \mathbb{E}_\mu(T^l)$,

$$\mathbb{E}_\mu(T^{1+(n-1-k)}) \leqq \mathbb{E}_\mu(T^{n-1-k}) \frac{\mathbb{E}_\mu(T^n)}{\mathbb{E}_\mu(T^{n-1})}.$$

By induction, it follows that

$$a_n + \frac{\mathbb{E}_\nu(T^{n-1})\,\mathbb{E}_\mu(T^n)}{\mathbb{E}_\mu(T^{n-1})} \geqq \mathbb{E}_\nu(T^n). \tag{4.9}$$

Hence,

$$a_n \mathbb{E}_\mu(T^{n-1}) \geqq \mathbb{E}_\nu(T^n)\,\mathbb{E}_\mu(T^{n-1}) - \mathbb{E}_\mu(T^n)\,\mathbb{E}_\nu(T^{n-1}). \tag{4.10}$$

The elementary equality

$$C(\mu)\,\mathbb{E}_\nu(T^n) = \sum_{k \geqq n} \mathbb{E}_\mu(T^k)$$

implies that

$$C(\mu)\,\mathbb{E}_\mu(T^{n-1})\,a_n \geqq \sum_{k \geqq n} [\mathbb{E}_\mu(T^k)\,\mathbb{E}_\mu(T^{n-1}) - \mathbb{E}_\mu(T^n)\,\mathbb{E}_\mu(T^{k-1})] \geqq 0, \tag{4.11}$$

therefore establishing Claim 3. ■

Equation (4.6) and Claim 3 imply now that

$$\|\Pi^n(\mu, .)-\nu\| \leqq 2\,\mathbb{E}_\nu(T^n). \tag{4.12}$$

Both (4.4), (4.5) and a few calculation show that

$$\beta_n \leqq 3\,\mathbb{E}_\nu(T^{[n/2]}). \tag{4.13}$$

Now $\mathbb{P}_x(\tau > n) = \mathbb{P}_x(X_n = x) = (T(x))^n$, which together with (4.3) ensures that $2\beta_n \geqq 2\,\mathbb{E}_\nu(T^n)$. Hence Lemma 4.1. holds. ■

Applications to the lower bounds for the CLT. — Throughout, let $T(x) = 1 - x$. In order to obtain power rates of decay for the mixing coefficients of the sequence $(X_i)_{i \in \mathbb{Z}}$, we define the probability measures μ and ν by

$$\frac{d\nu}{d\lambda} = \mathbf{1}_{[0,1]}(x).\,a x^{a-1} \tag{4.14}$$

for some positive a. With the above choice of ν

$$\frac{d\mu}{d\lambda} = \mathbf{1}_{[0,1]}(x).\,(1+a)\,x^a.$$

Let k be a positive integer. Clearly,

$$\mathbb{E}_\nu(\mathrm{T}^k)=k^{-a}\int_0^k (1-x/k)^k\, ax^{a-1}\,dx. \tag{4.15}$$

Hence

$$\lim_{k\to+\infty} k^a\,\mathbb{E}_\nu(\mathrm{T}^k)=a\,\Gamma(a).$$

Here Γ denotes the Γ-function. Since $\mathrm{F}_\nu(x)=x^a$, setting $\mathrm{U}_i=\mathrm{X}_i^a$, we obtain a stationary Markov chain of uniformly distributed r.v.'s. with the same mixing coefficients. So, both (4.16) and Lemma 2 imply (i) of Theorem 5.

Clearly, there is no loss of generality in assuming that $\mathbb{E}(f(\mathrm{U}_i))=0$, which we shall do throughout the sequel. In order to prove that the sequence $(f(\mathrm{U}_i))_{i\in\mathbb{Z}}$ does not satisfy the CLT if (a) of Theorem 5 holds, we now prove that some compound sums of the sequence $(f(\mathrm{U}_i)_{i\in\mathbb{Z}}$ are partial sums of i.i.d. random variables.

Let $(\mathrm{T}_k)_{k\geqq 0}$ be the increasing sequence of positive stopping times defined by $\mathrm{T}_0=\tau$, where τ is defined just before (4.4), and $\mathrm{T}_k=\inf\{i>\mathrm{T}_{k-1}:\mathrm{X}_i\neq\mathrm{X}_{i-1}\}$ for any $k>0$. Set $\tau_k=\mathrm{T}_{k+1}-\mathrm{T}_k$. Clearly $\mathrm{X}_{\mathrm{T}_k}$ has the distribution μ. Hence, by the strong Markov property, the r.v.'s $(\tau_k)_{k\geqq 0}$ are i.i.d. with tail function $\mathbb{P}(\tau_k>n)=\mathbb{E}_\mu(\mathrm{T}^n)$. Clearly,

$$\sum_{i=1}^{\mathrm{T}_n-1} f(\mathrm{U}_i)=\sum_{i=1}^{\tau-1} f(\mathrm{U}_i)+\sum_{k=0}^{n-1}\tau_k\, f(\mathrm{U}_{\mathrm{T}_k}). \tag{4.17}$$

We now prove that

$$\sum_{i=1}^{\mathrm{T}_n-1} f(\mathrm{U}_i)-\sum_{i=1}^{[n\,\mathbb{E}(\tau_1)]} f(\mathrm{U}_i)=o_{\mathrm{P}}(\sqrt{n}) \tag{4.18}$$

Proof of 4.18. – Clearly, the bivariate r.v.'s $(\mathrm{X}_{\mathrm{T}_k},\tau_k)_{k>0}$ are i.i.d. Let $\zeta_k=\mathrm{X}_{\mathrm{T}_k}$. The r.v.s ζ_k are i.i.d. with common distribution μ, and

$$\mathbb{P}(\tau_k>n\,|\,\zeta_k=\zeta)=(1-\zeta)^n. \tag{4.19}$$

Since $a>1$, by (4.19), $\mathbb{E}(\tau_1^2)<\infty$, which implies that $(\mathrm{T}_n-n\,\mathbb{E}(\tau_1))/\sqrt{n}$ is weakly convergent a normal distribution. Hence, for any $\varepsilon>0$, there exist $\mathrm{A}>0$ such that

$$\liminf_{n\to+\infty}\mathbb{P}(n\,\mathbb{E}(\tau_1)\in[\mathrm{T}_{[n-\mathrm{A}\sqrt{n}]},\mathrm{T}_{[n+\mathrm{A}\sqrt{n}]}])\geqq 1-\varepsilon. \tag{4.20}$$

Now, by (4.19),

$$\mathbb{E}\,|\tau_k f(\mathrm{U}_{\mathrm{T}_k})|=\int_0^1 |f(\zeta^a)|\,\alpha\zeta^{a-1}\,d\zeta<\infty$$

and

$$\mathbb{E}(\tau_k f(U_{T_k}))=0.$$

Hence the law of large numbers applied to the sequence of i.i.d. integrable random variables $(\tau_k f(U_{T_k}))_{k>0}$ ensures that

$$n^{-1/2} \sup_{m \in [n-A\sqrt{n},\, n+A\sqrt{n}]} \left| \sum_{k=1}^{n} \mathbb{E}(\tau_k f(U_{T_k})) - \sum_{k=1}^{m} \mathbb{E}(\tau_k f(U_{T_k})) \right| \to 0 \text{ in probability}$$

as n tends to infinity. Since the above random variable majorizes the random variable $n^{-1/2} \left| \sum_{i=1}^{T_n - 1} f(U_i) - \sum_{i=1}^{[n\,\mathbb{E}(\tau_1)]} f(U_i) \right|$ on the event $(n\,\mathbb{E}(\tau_1) \in [T_{[n-A\sqrt{n}]}, T_{[n+A\sqrt{n}]}])$ via (4.17), both (4.20) and the above inequality imply (4.18). ■

It follows from (4.18) that, if

$$\Delta_n = n^{-1/2} \sum_{k=0}^{n-1} \tau_k f(U_{T_k})$$

does not converges to a normal distribution, the same holds for $n^{-1/2} \sum_{i=1}^{n} f(U_i)$. Now, by the converse to the usual central limit theorem (*cf.* Feller, 1950), Δ_n converges to a normal distribution if and only if $\mathbb{E}(\tau_k^2 f^2(U_{T_k})) < \infty$. By (4.20), it holds iff

$$\mathbb{E}(\zeta_1^{-2}[f(\zeta_1^a)]^2) = (1+a)\int_0^1 \zeta^{a-2}[f(\zeta^a)]^2\, d\zeta < \infty.$$

Hence, using the change of variable $u=\zeta^a$, we get Theorem 5.

Proof of Corollary 1. – The r.v.'s Z_i are defined from the corresponding r.v.'s U_i via a quantile transformation. So, $(Z_i)_{i\in\mathbb{Z}}$ and $(U_i)_{i\in\mathbb{Z}}$ have the same mixing structure. We now prove that the sequence $(\alpha_n)_{n\geq 0}$ of strong mixing coefficients of the sequence $(U_i)_{i\in\mathbb{Z}}$ satisfies the left-hand side inequality

$$\liminf_{n\to+\infty} n^a \alpha_n > 0. \tag{4.22}$$

Proof of (4.22). – For any continuous d.f. F, the sequences $(F^{-1}(U_i))_{i\in\mathbb{Z}}$ and $(U_i)_{i\in\mathbb{Z}}$ have the same strong mixing coefficients. Hence, by (ii) of Theorem 5 and Theorem 4, for any continuous and symmetric d.f. F such that $\int_0^{1/2} u^{-1/a}[F^{-1}(u)]^2\,du=\infty$, we have:

$$\int_0^{1/2} \alpha^{-1}(u)[F^{-1}(u)]^2\,du=\infty,$$

where α denotes the strong mixing rate function of the sequence $(U_i)_{i\in\mathbb{Z}}$. Using a density argument (one can find a sequence (F_n) of continuous and symmetric d.f. such that $G=\lim_{n\to+\infty}\uparrow[F_n^{-1}]^2)$ we get that for any nonincreasing function $G:[0, 1/2]\to\mathbb{R}^+$ such that $\int_0^{1/2} u^{-1/a}G(u)\,du=\infty$, we have:

$$\int_0^{1/2} \alpha^{-1}(u)\,G(u)\,du=\infty.$$

Suppose that $(\alpha_n)_{n\geqq 0}$ does not satisfy (4.22). Then there exists some increasing sequence $(n_k)_{k>0}$ of positive integers such that, for any $k>0$,

$$\alpha_{n_{k+1}}\leqq\frac{\alpha_{n_k}}{2}\quad\text{and}\quad n_k^a\,\alpha_{n_k}\leqq 2^{-k}. \tag{4.23}$$

For any $u\in]\alpha_{n_{k+1}}, \alpha_{n_k}]$, set $G(u)=n_k^{a-1}$. If $u>\alpha_{n_1}$, let $G(u)=0$. By (4.23), on one hand,

$$\int_0^{1/2} \alpha^{-1}(u)\,G(u)\,du\leqq\frac{a}{a-1}\sum_{k>0}\alpha_{n_k}^{1-1/a}\,n_k^{a-1}<+\infty,$$

and, on the other hand,

$$\int_0^{1/2} u^{-1/a}G(u)\,du\geqq\frac{a}{a-1}\sum_{n>0}(\alpha_{n_k}^{1-1/a}-\alpha_{n_{k+1}}^{1-1/a})\,n_k^{a-1}=+\infty,$$

which implies (4.22).

Notice now that the integral $\int_0^1 u^{-1/a}[Q_F(u)]^2\,du$ is convergent if and only if both the two above integrals $\int_0^{1/2} u^{-1/a}[F^{-1}(u)]^2\,du$ and $\int_0^{1/2} u^{-1/a}[F^{-1}(1-u)]^2\,du$ are convergent. So, setting $Z_i=F^{-1}(U_i)$ iff $\int_{k=1}^{n} u^{-1/a}[F^{-1}(u)]^2\,du=\infty$, and $Z_i=F^{-1}(1-U_i)$ otherwise, and applying Theorem 5, we get Corollary 1.

Proof of Proposition 3. – By definition of T_n,

$$S_{T_n - 1} = \sum_{i=1}^{\tau - 1} Z_i + \sum_{k=0}^{n-1} \tau_k Z_{T_k}.$$

Since the r.v.'s $\tau_k Z_{T_k}$ are i.i.d. with $\mathbb{E}(|\tau_k Z_{T_k}|^2) = +\infty$, Strassen's converse to the law of the iterated logarithm (1966) shows that:

$$\limsup_{n \to +\infty} \frac{|S_{T_n - 1}|}{\sqrt{n \log \log n}} = +\infty \text{ a. s.} \tag{4.24}$$

Proposition 3 then follows from (4.24) and from the fact that $\lim_{+\infty} T_n/n = \mathbb{E}(\tau_1)$ a. s.

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