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Symmetric reflected diffusions

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ABSTRACT. – Consider a Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$:

$$\mathscr{E}(f, f) = \frac{1}{2}\int_{D} \nabla f . a \nabla f p \, dx, f \in \mathscr{D}(\mathscr{E}),$$

$$\mathscr{D}(\mathscr{E}) = \{f \in L^2(D, p\,dx) \cap H^1_{loc}(D) : \mathscr{E}(f, f) < \infty\},$$

where D is a d-dimensional domain, a is a bounded, symmetric, locally elliptic, $d \times d$ matrix-valued function on D, p is a strictly positive probability density on D, and a, p are locally Lipschitz continuous on D. We investigate two methods for appoximating the stationary, symmetric Markov process X associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, which in the case of smooth non-degenerate data is a diffusion process with infinitesimal generator $\frac{1}{2p}\nabla . (ap\nabla)$ in D and conormal reflection at the boundary of D. The first (or exterior) approximation is a conventional penalty approximation by

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diffusions defined on all of $\mathbb{R}^d$. The second (or interior) approximation uses diffusions confined to D by singular drifts that tend to infinity at the boundary of D. The existence of such singular diffusions is established as a result of possible independent interest. For both approximation methods, the approximating sequences of processes are shown to be tight using a decomposition of Lyons and Zheng. The conditions under which one can identify any weak limit as a realization of X are most general for the interior approximation scheme and are satisfied for example if for any compact set $K \subset \mathbb{R}^d$, a is uniformly elliptic on $D \cap K$ and p is strictly bounded away from zero there. Finally, we show under further mild regularity and non-degeneracy conditions on a and p that if ∂D is locally of finite $(d-1)$-dimensional upper Minkowski content, then X is a semimartingale.

Key words : Dirichlet form, reflected diffusion, stationary, symmetric Markov process, penalty methods, singular drift, semimartingale, Minkowski content.

RÉSUMÉ. — Considérons une forme de Dirichlet $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$:

$$\mathscr{E}(f, f) = \frac{1}{2}\int_D \nabla f . a \nabla f p\, dx, f \in \mathscr{D}(\mathscr{E}),$$

$$\mathscr{D}(\mathscr{E}) = \{f \in L^2(D, p\, dx) \cap H^1_{loc}(D); \mathscr{E}(f, f) < \infty\},$$

où D est un domaine de $\mathbb{R}^d$, a est une fonction bornée de D dans l'ensemble des matrices $d \times d$ symétriques et définies positives, p est une densité de probabilité strictement positive sur D, a et p sont localement lipschitziennes sur D. Nous utilisons deux méthodes pour approcher le processus de Markov symétrique et stationnaire X associé à $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, qui dans le cas de données régulières est une diffusion de générateur infinitésimal $\frac{1}{2p}\nabla . (ap\nabla)$ dans D, avec réflexion conormale à la frontière de D. La première approximation est une approximation « extérieure »; elle utilise la méthode classique de pénalisation d'une diffusion dans tout $\mathbb{R}^d$. La seconde est une approximation « intérieure »; elle utilise des diffusions qui sont contraintes à rester dans D par une dérive singulière. Nous établissons l'existence de telles diffusions; il s'agit d'un résultat probablement intéresssant en soi. Dans les deux cas d'approximation, la tension est établie en utilisant une décomposition dûe à Lyons et Zheng. Les conditions sous lesquelles on peut identifier toutes les limites faibles comme des réalisations du processus X sont plus générales dans le cas de l'approximation intérieure, et sont satisfaites en particulier si pour tout compact K de $\mathbb{R}^d$, a est strictement elliptique et p uniformément positive sur $D \cap K$. Enfin on montre que sous des hypothèses assez faibles de régularité de a, p, ∂D et de non dégénérescence de a et p, X est une semimartingale.

0. INTRODUCTION

In this paper we consider the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$:

$$\mathscr{E}(f, f) = \frac{1}{2}\int_D \nabla f . a \nabla f \, d\mu, \; f \in \mathscr{D}(\mathscr{E}), \tag{1}$$

$$\mathscr{D}(\mathscr{E}) = \{ f \in L^2(D, \mu) \cap H^1_{loc}(D) : \mathscr{E}(f, f) < \infty \}, \tag{2}$$

where D is a domain in $\mathbb{R}^d$, a is a symmetric bounded, locally elliptic, $d \times d$ matrix-valued function on D, $d\mu = p dx$ and p is a strictly positive probability density on D. To ensure that this is a Dirichlet form (*see* Theorem 2.1), we impose the mild regularity assumption that a and p are locally Lipschitz in D [*see* (A1)-(A2) of section 1].

The theory of Fukushima [9] associates to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ a *stationary* symmetric Markov process X with strongly continuous semigroup on $L^2(D, \mu)$. The infinitesimal generator of this semigroup is an extension of the partial differential operator

$$L = \frac{1}{2p} \nabla . (ap \nabla) \quad \text{on } C_c^\infty(D).$$

We paraphrase this by saying that X behaves in D like a diffusion with diffusion coefficient a and drift $\frac{1}{2p} \nabla . (ap)$. Here and throughout this paper we use the term diffusion loosely to mean a Markov process that on the interior of its state space has an infinitesimal generator that is an elliptic partial differential operator. When we need the more restrictive notion of a continuous strong Markov process, we shall use precise words to this effect. Indeed, for there to be a continuous strong Markov process with paths in $\bar{D}$ associated to $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, one needs to impose additional assumptions on the data (D, a, p). In particular, if D is bounded and has smooth boundary, and a, p can be extended off D to smooth functions defined on $\mathbb{R}^d$ such that a is uniformly elliptic and $p > 0$, then there are many ways to define a continuous strong Markov process in $\bar{D}$ with associated Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. For instance, when "smooth" is interpreted to mean C^2-smooth, the work of Lions and Sznitman [17], Theorem 4.4, on solutions of stochastic differential equations with reflecting boundary conditions, guarantees that given a d-dimensional Brownian motion W and initial point $x \in \bar{D}$ and letting $\sigma = \sqrt{a}$ denote the positive definite, symmetric, square root of a, $b = \frac{1}{2p} \nabla . (ap)$, and n denote the inward unit normal to ∂D, there is a unique solution (X^x, L^x) of (3)-(5) that is adapted to W:

$$X_t^x = x + \int_0^t \sigma(X_s^x)\, dW_s + \int_0^t b(X_s^x)\, ds + \int_0^t (an)(X_s^x)\, dL_s^x,\ t \geqq 0, \tag{3}$$

(4) L^x is a continuous, one-dimensional, non-decreasing process,

$$L_t^x = \int_0^t 1_{\partial D}(X_s^x)\, dL_s^x, \qquad t \geqq 0. \tag{5}$$

The boundary behavior of X^x is captured by the last integral in (3) and (4)-(5), which indicate that X is reflected at the boundary of D in the *conormal* direction an. The process L^x is called the local time of X^x on ∂D. By Itô's formula, the process X^x generates a solution starting from x of the submartingale problem used by Stroock and Varadhan [26] to characterize diffusions with smooth boundary conditions. Indeed, the uniqueness of solutions of that problem [or of (3)-(5)] guarantees that $\{X^x, x \in \bar{D}\}$ has the strong Markov property. The process obtained by randomizing the initial condition of this process so that it has the stationary law μ on D can be verified by integration by parts to be a realization of the stationary symmetric Markov process associated with the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. An extension of the above claims to unbounded D can be achieved by imposing suitable growth conditions on a and p.

When ∂D is not smooth or a or p may degenerate at the boundary of D, little is known about representations of the form (3)-(5) for the Markov (or strong Markov) process associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. However, by analogy with the smooth case, we shall refer to such processes as symmetric reflected diffusion processes.

In the special case where $a \equiv I$, D is of finite Lebesgue measure and p is constant, X is called (normally) reflected Brownian motion in D and more is known about this process. In particular, Bass and Hsu [2], [1], have shown that when D is a bounded Lipschitz domain, there is a continuous strong Markov process on $\bar{D}$ associated to $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and relative to the filtration generated by X one has the semimartingale decomposition:

$$X_t = X_0 + W_t + \int_0^t n(X_s)\, dL_s, \qquad t \geqq 0, \tag{6}$$

where W is a d-dimensional Brownian motion martingale, L is a continuous, adapted, non-decreasing process that increases only when X is on ∂D, and n is the inward unit normal vector field defined a.e. relative to surface measure on ∂D. Since L does not charge the set of times for which X is at points where n does not exist, (6) is well defined.

For an arbitrary domain D of finite Lebesgue measure, Williams and Zheng [29] have shown that one can approximate the stationary reflected Brownian motion X in D by a sequence of stationary diffusions with drifts that tend to infinity at ∂D in such a way as to keep the sample paths of these diffusions in D. We refer to this as an interior approximation. It further follows from the work of [29] that X is a semimartingale if the boundary of D is locally of finite $(d-1)$-dimensional upper Minkowski content [*see* (A8) for the definition] and in this case an averaged version of the decomposition (6) holds (*see* Theorem 6.1 below).

In this paper we generalize the results of [29] to symmetric reflected diffusions associated with the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and we compare the interior approximation with a conventional penalty (or exterior) approximation method. We stress here that we only consider the *stationary* Markov process associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and we do not address the question of when it has an associated continuous strong Markov process on $\bar{D}$. The latter is an interesting open problem. (For the case where a and p do not degenerate at the boundary of D, which includes the case of reflected Brownian motion, this question has recently been addressed by Chen [6], and some sufficient [6] and necessary [28] conditions for such a strong Markov process to have a certain semimartingale decomposition have been given. Since the submission of our paper, further progress has been made on this question in the non-degenerate case by Chen, Fitzsimmons and Williams [7].)

The structure of our paper is as follows.

Section 1 describes some notation and assumptions. In section 2 we verify that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is indeed a Dirichlet form under these assumptions and characterize it as the maximal one associated with self-adjoint, non-negative definite, Markovian extensions of the operator $(L, C_c^\infty(D))$. This generalizes Lemma 2.3.4 of [9] from the Brownian motion case to the diffusion case treated here. In sections 3 and 4 we develop exterior (penalization) and interior approximations, respectively, to X. The arguments in these two sections have the following common skeleton:

(i) define an approximating sequence of stationary, symmetric diffusions, and identify the associated Dirichlet forms,

(ii) establish tightness of this sequence of processes using the decomposition of Lyons and Zheng [18] and a uniform bound on the diffusion coefficients of these processes,

(iii) verify there is strong convergence in $L^2(\mu)$ of the semigroups on L^∞ functions whenever there is weak convergence and conclude that a weak limit of the approximating sequence is a stationary, symmetric Markov process with strongly continuous semigroup on $L^2(D, \mu)$ and that its infinitesimal generator is an extension of $(L, C_c^\infty(D))$,

(iv) under suitable conditions, identify the Dirichlet form associated with any weak limit as $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, and conclude that the approximating sequence converges weakly to X.

The main differences between sections 3 and 4 lie in steps (i) and (iv). The definition of the approximating sequence of processes is significantly more difficult in section 4 than in section 3. Indeed, the penalty approximation of section 3 uses a sequence of ordinary diffusions defined on $\mathbb{R}^d$ with drifts that outside of D point back toward D and grow in magnitude to $+\infty$ as one proceeds along the sequence. On the other hand, the interior approximation of section 4 employs a sequence of diffusions confined to D by singular drifts that for each diffusion in the sequence tend to $+\infty$ in magnitude as the boundary ∂D is approached. The proof that such diffusions exist and do not exit D is deferred to section 5. The result stated there may be of independent interest since it is related to work of such authors as Carlen [4], Zheng [30], Norris [21], Fukushima [10], Cattiaux and Léonard [5], and references cited therein on diffusions with singular drift. The extra complication in step (i) for the interior approximation is rewarded in step (iv) with more general conditions under which one can identify the limit process. In section 3 we need to assume that ∂D is of zero d-dimensional Lebesgue measure and that there is a set of functions that is dense in the Hilbert space associated with the form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and which can be extended to a set of functions defined on $\mathbb{R}^d$ that is dense in each of the domains of the Dirichlet forms for the approximating penalized processes. This *extension property* usually requires some additional conditions on (D, a, p) (see Remark 3.10). On the other hand, in section 4 a sufficient condition for the identification is that for each compact set K in $\mathbb{R}^d$, on $K \cap D$, a is uniformly elliptic and p is strictly bounded away from zero, conditions which require no direct assumptions on D.

In section 6 we address the question of when X is a semimartingale. For this we impose slightly stronger regularity assumptions on a and p, formulated precisely in (A1″)-(A2″) of section 6. Using the tightness criterion for semimartingales of Meyer and Zheng [19], we are able to verify in a similar manner to that in [29] that X is a semimartingale if the boundary of D is locally of finite $(d-1)$-dimensional upper Minkowski content and to obtain some properties of its semimartingale decomposition (*see* Theorem 6.1). Whilst this manuscript was in preparation, we received a preprint of the work of Chen [6] on sufficient conditions for symmetric reflected diffusions in bounded domains to be semimartingales. Under slightly weaker regularity conditions on a and p than ours in section 6, Chen shows that X is a semimartingale when the boundary of D is of finite $(d-1)$-dimensional lower Minkowski content. Since his method of approximation is different from ours, we feel our result may still be of interest. We would like to thank Z. Q. Chen for sending us his preprint and for telling us about the paper [10] of Fukushima.

1. NOTATIONS AND ASSUMPTIONS

For a vector $x \in \mathbb{R}^d$ and set $F \subset \mathbb{R}^d$, $|x|$ will denote the Euclidean norm of $x: |x| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$ and $d(x, F) = \inf_{y \in F} |x-y|$. For two sets F and G in $\mathbb{R}^d (d \geqq 1)$ we write $F \subset\subset G$ if $F \subset \bar{F} \subset G$ and the closure $\bar{F}$ of F is *compact*. For a domain $D \subset \mathbb{R}^d$, the σ-field of Borel subsets of D will be denoted by $\mathscr{B}(D)$, and for a σ-finite measure ν on $(D, \mathscr{B}(D))$ and $p \in [1, \infty]$, we let $L^p(D, \nu)$ denote the space of Borel measurable functions $f: D \to \mathbb{R}$ that are in L^p with respect to the measure ν, and we let $\|\cdot\|_{L^p(D, \nu)}$ denote the L^p-norm on $L^p(D, \nu)$. If $D = \mathbb{R}^d$, we write $L^p(\nu)$ for $L^p(\mathbb{R}^d, \nu)$. If ν is Lebesgue measure on D, we simply write $L^p(D)$ for $L^p(D, \nu)$. For the inner product on $L^2(D, \nu)$, we shall use $(.,.)_{L^2(D, \nu)}$, *i.e.*, $(f, g)_{L^2(D, \nu)} = \int_D fg\, d\nu$ for all $f, g \in L^2(D, \nu)$. When there is no ambiguity as to the measure involved, we shall simply write $(.,.)$ for $(.,.)_{L^2(D, \nu)}$.

For each non-negative integer n, we let $W^{n, p}(D)$ denote the Sobolev space of functions $f \in L^p(D)$ that have all distributional derivatives up to and including those of order n in $L^p(D)$. The norm on $W^{n, p}(D)$ is taken to be

$$\|f\|_{n, p} = \sum_{0 \leqq |\alpha| \leqq n} \|D^\alpha f\|_{L^p(D)} \qquad \text{for} \quad 1 \leqq p \leqq \infty,$$

where $D^\alpha f$ denotes the α-th derivative of f for any multi-index α, and in particular, $D^0 f = f$. The local spaces $W^{n, p}_{loc}(D)$ consist of functions $f: D \to \mathbb{R}$ that belong to $W^{n, p}(D')$ for all domains $D' \subset\subset D$. We note that for $n \geqq 1$, $W^{n, \infty}_{loc}(D)$ is the same as $C^{n-1, 1}(D)$, the space of functions which have derivatives up to and including those of order $(n-1)$, and which together with those derivatives are locally Lipschitz continuous on D. We use the usual notation that $H^n(D) = W^{n, 2}(D)$ and $H^n_{loc}(D) = W^{n, 2}_{loc}(D)$.

We let $C^n(D)$ denote the set of functions $f: D \to \mathbb{R}$ that are n-times continuously differentiable on D, $n = 0, 1, 2, \ldots$ We shall write $C(D)$ in place of $C^0(D)$. The set $C^\infty(D)$ will denote those functions that are infinitely differentiable on D. For $n = 0, 1, 2, \ldots, \infty$, $C^n_c(D)$ will denote those functions in $C^n(D)$ that have compact support in D and $C_b^n(D)$ will denote those functions in $C^n(D)$ which together with their partial derivatives up to and including those of order n are bounded on D. Again, for $n=0$, the superscript n will be suppressed.

Occasionally, for a vector or matrix valued function f defined on D or $\mathbb{R}^d$, we shall write for example, $f \in W^{n,p}(D)$ to mean that each component of the vector or matrix valued function belongs to $W^{n,p}(D)$.

For $k \geqq 1$, let $C([0, 1], \mathbb{R}^k)$ denote the space of continuous $\mathbb{R}^k$-valued functions defined on $[0, 1]$, and, unless indicated otherwise, consider $C([0, 1], \mathbb{R}^k)$ to be endowed with the topology of uniform convergence.

Throughout this paper d will be a fixed positive integer and D will be a fixed but arbitrary domain in $\mathbb{R}^d$. In sections 2-4, functions $a: D \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $p: D \to \mathbb{R}$ will be assumed given such that they satisfy

(A1) $a \equiv (a_{ij}) \in W_{loc}^{1,\infty}(D)$ is symmetric, bounded and locally elliptic on D, in the sense that for each compact set $K \subset D$, there is $\lambda_K > 0$ such that

$$\xi' a(x) \xi \equiv \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geqq \lambda_K |\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and $x \in K$; and

(A2) $p \in W_{loc}^{1,\infty}(D)$, $p > 0$ on D, and $\int_D p\,dx = 1$.

In particular, note that since $W_{loc}^{1,\infty}(D) \subset C(D)$, for any compact set $K \subset D$ there are constants c_K and C_K such that $0 < c_K \leqq p(x) \leqq C_K < \infty$ for all $x \in K$. We regard (D, a, p) as the data for the stationary symmetric reflected diffusion that we wish to construct. Such a process should live in $\bar{D}$, have infinitesimal generator on functions $f \in C_c^\infty(D)$ given by

$$Lf = \frac{1}{2p} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij} p \frac{\partial f}{\partial x_j}\right),$$

and have associated Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ given by (1)-(2). We shall let $b = \frac{1}{2p} \nabla . (ap)$, *i.e., for* $j = 1, \ldots, d$,

$$b_j = \frac{1}{2p} \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(a_{ij} p).$$

2. THE DIRICHLET FORM

The aim of this section is to study the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ defined by (1)-(2). We shall closely follow the terminology of Fukushima [9] and freely use the results proved there. We shall first prove that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is indeed a Dirichlet form, and then identify the associated unbounded operator on $L^2(D, \mu)$, which will be the infinitesimal generator of the process to be constructed in the next section. Note that the results of this section will be applied later not only to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ under study, but also to the approximating forms which will be introduced in the next two sections.

Recall that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called a Dirichlet form on $L^2(D, \mu)$ if it is a symmetric, closed and Markovian bilinear form on $\mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E})$, where $\mathscr{D}(\mathscr{E})$ – called the domain of $\mathscr{E}$ – is a dense linear subspace of $L^2(D, \mu)$. In this section, $(.,.)$ denotes the inner product in $L^2(D, \mu)$, where as before $d\mu = p dx$ on D.

THEOREM 2.1. – *$(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, as defined by* (1)-(2), *is a Dirichlet form.*

Proof. – The symmetry of $\mathscr{E}$ follows readily from that of the matrix (a_{ij}). The Markovian property of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ follows from the fact that for $\varphi_\varepsilon \in C_b^1(\mathbb{R})$ as in [9], page 5, $\nabla \varphi_\varepsilon(f) = \varphi'_\varepsilon(f) \nabla f$ where $|\varphi'_\varepsilon| \leqq 1$.

To show that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is closed, suppose that $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathscr{D}(\mathscr{E})$ such that $\{f_n\}$ is Cauchy relative to the norm $\|.\|_{\mathscr{D}(\varepsilon)}$ given by

$$\|f\|_{\mathscr{D}(\varepsilon)}^2 = (f, f) + \mathscr{E}(f, f), f \in \mathscr{D}(\mathscr{E}).$$

We first note that there exists $f \in L^2(D, \mu)$ such that $f_n \to f$ in $L^2(D, \mu)$. Let $\{D_k\}_{k=1}^\infty$ be an increasing sequence of open subsets of D such that

(i) $D_k \subset\subset D$, $k = 1, 2, \ldots$,

(ii) $\bigcup_k D_k = D$.

By the local ellipticity of a and since p is bounded away from zero on D_k, for any $k \geqq 1$, $\exists c_k > 0$ such that

$$\|g\|_{H^1(D_k)} \leqq c_k \|g\|_{\mathscr{D}(\mathscr{E})}, g \in \mathscr{D}(\mathscr{E}).$$

Hence for each $k \geqq 1$,

$$f_n|_{D_k} \to f|_{D_k} \quad \text{in } H^1(D_k) \tag{7}$$

and so $f \in H_{loc}^1(D)$. Moreover $\nabla f_n \to \nabla f$ in μ-measure. Hence from Fatou's Lemma

$$\mathscr{E}(f, f) \leqq \liminf_{n \to \infty} \mathscr{E}(f_n, f_n) < \infty$$

and $f \in \mathscr{D}(\mathscr{E})$. To show that $\mathscr{E}(f - f_n, f - f_n) \to 0$ as $n \to \infty$, note that

$$\mathscr{E}(f - f_n, f - f_n) = \int_{D_k} \nabla(f - f_n) . a \nabla(f - f_n) \, d\mu + \int_{D \setminus D_k} \nabla(f - f_n) . a \nabla(f - f_n) \, d\mu.$$

Since (7) holds and $f \in \mathscr{D}(\mathscr{E})$, it suffices to show that $\forall \varepsilon > 0$, $\exists k(\varepsilon)$ and $N(\varepsilon) \in \mathbb{N}$ such that $\forall n \geqq N(\varepsilon)$,

$$\int_{D \setminus D_{k(\varepsilon)}} \nabla f_n . a \nabla f_n \, d\mu \leqq \varepsilon. \tag{8}$$

This follows from the Cauchy property of $\{ f_n \}$ in $\mathscr{D}(\mathscr{E})$ and the fact that $\nabla f_N \in L^2(D, \mu)$ for each N. □

It follows from Theorem 1.3.1 and 1.4.1 in [9] that there exists a unique negative semi-definite self-adjoint operator A on $L^2(D, \mu)$, which generates a strongly continuous symmetric Markovian (3) semigroup $\{ P_t, t \geqq 0 \}$, and satisfies:

$$\mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{-A})$$
$$\mathscr{E}(f, g) = (\sqrt{-A} f, \sqrt{-A} g), f, g \in \mathscr{D}(\mathscr{E}).$$

Note that the following properties hold:

$$\mathscr{D}(A) \subset \mathscr{D}(\mathscr{E})$$
$$\mathscr{E}(f, g) = (-A f, g), f \in \mathscr{D}(A), g \in \mathscr{D}(\mathscr{E}).$$

We shall use the following definition of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in terms of the semi-group $\{ P_t, t \geqq 0 \}$:

$$\mathscr{D}(\mathscr{E}) = \left\{ f \in L^2(D, \mu) : \lim_{t \downarrow 0} \uparrow \frac{1}{t}(f, f - P_t f) < \infty \right\}$$

$$\mathscr{E}(f, g) = \lim_{t \downarrow 0} \frac{1}{t}(f - P_t f, g), f, g \in \mathscr{D}(\mathscr{E}).$$

Finally we associate to A the resolvent $\{ G_\alpha \triangleq (\alpha I - A)^{-1}, \alpha > 0 \}$, which satisfies

$$G_\alpha(L^2(D, \mu)) \subset \mathscr{D}(\mathscr{E}), \mathscr{E}_\alpha(G_\alpha u, v) = (u, v), u \in L^2(D, \mu), v \in \mathscr{D}(\mathscr{E}),$$

where $\mathscr{E}_\alpha(u, v) = \mathscr{E}(u, v) + \alpha(u, v)$, $u, v \in \mathscr{D}(\mathscr{E})$. Moreover,

$$\mathscr{D}(\mathscr{E}) = \{ f \in L^2(D, \mu) : \lim_{\beta \to \infty} \uparrow \mathscr{E}^{(\beta)}(f, f) < \infty \}$$
$$\mathscr{E}(f, g) = \lim_{\beta \to \infty} \mathscr{E}^{(\beta)}(f, g), f, g \in \mathscr{D}(\mathscr{E}),$$

where $\mathscr{E}^{(\beta)}(f, g) = \beta(f - \beta G_\beta f, g)$.

We want now to identify the operator A associated to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ defined by (1), (2). For that, we first introduce the operator L

(3) In fact "subMarkovian" in the usual terminology.

from $C_c^\infty(D)$ into $L^2(D, \mu)$, defined as follows (we use here and henceforth the convention of summation upon repeated indices):

$$Lf = \frac{1}{2} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{1}{2p} \frac{\partial (a_{ij} p)}{\partial x_i} \frac{\partial f}{\partial x_j}.$$

We note that $Lf \in L^2(D, \mu)$ whenever $f \in C_c^\infty(D)$, since the a_{ij} are bounded and $\frac{1}{2p} \frac{\partial (a_{ij} p)}{\partial x_i}$ are bounded on compact subsets of D. We note that

$$Lf = \frac{1}{2p} \frac{\partial}{\partial x_i}\left(a_{ij} p \frac{\partial f}{\partial x_j} \right)$$

and hence for $f, g \in C_c^\infty(D)$, by integration by parts:

$$\begin{aligned}(f, -Lg) &= -\frac{1}{2} \int_D f(x) \frac{\partial}{\partial x_i}\left(a_{ij} p \frac{\partial g}{\partial x_j} \right)(x)\, dx \\ &= \mathscr{E}(f, g).\end{aligned}$$

Thus, $(L, C_c^\infty(D))$ is a negative semi-definite, symmetric operator. We first show:

LEMMA 2.2. – *$(A, \mathscr{D}(A))$ is an extension of $(L, C_c^\infty(D))$.*

Proof. – Let $f \in L^2(D, \mu)$. Then $h = G_1 f \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(h, g) + (h, g) = (f, g)$, $g \in C_c^\infty(D)$, *i.e.*,

$$\int_D \left(\frac{1}{2} a_{ij} \frac{\partial h}{\partial x_i} \frac{\partial g}{\partial x_j} + hg \right)(x)\, p(x)\, dx = \int_D f(x)\, g(x)\, p(x)\, dx.$$

Since $h \in H_{loc}^1(D)$, $a_{ij} p \in W_{loc}^{1,\infty}(D)$, and the support of g is a compact subset of D, we can integrate by parts in the above to obtain:

$$(-Lg + g, h) = (g, f), \qquad g \in C_c^\infty(D).$$

Hence, if L^* denotes the adjoint of L then $h \in \mathscr{D}(L^*)$ and

$$L^* h - h = -f.$$

However, since $h = G_1 f = (I - A)^{-1} f$, we also have:

$$Ah - h = -f.$$

Hence $\mathscr{D}(A) = \text{Range}(G_1) \subset \mathscr{D}(L^*)$ and $A = L^*$ on $\mathscr{D}(A)$, *i.e.* $A \subset L^*$. The result then follows from Lemma 2.3.2 (i) in [9]. □

Let $\mathscr{A}_M(L)$ denote the set of all self-adjoint, negative semi-definite Markovian extensions of L. We have just shown that $A \in \mathscr{A}_M(L)$. We shall next show that A is the maximal element in $\mathscr{A}_M(L)$ in the sense of the next theorem. This theorem is a generalization to our L of Lemma 2.3.4 of [9] which was established for $L = \frac{1}{2}\Delta$. Since the proof for L is similar to that for $\frac{1}{2}\Delta$, we only give details where there is a difference in the argument due to our weak regularity assumptions on the coefficients of L.

For $B \in \mathscr{A}_M(L)$, we shall denote by $(\mathscr{E}_B, \mathscr{D}(\mathscr{E}_B))$ the associated Dirichlet form, and

$$\mathscr{E}_{B,\alpha}(u, v) = \mathscr{E}_B(u, v) + \alpha(u, v).$$

THEOREM 2.3. – *Suppose* $B \in \mathscr{A}_M(L)$, *then* $\mathscr{D}(\mathscr{E}_B) \subset \mathscr{D}(\mathscr{E})$ *and*

$$\mathscr{E}_B(f, f) \geqq \mathscr{E}(f, f), \qquad f \in \mathscr{D}(\mathscr{E}_B).$$

Proof. – By the same reasoning as in the first paragraph of the proof of Lemma 2.3.4 of [9], it suffices to show that for $\alpha > 0$, $\mathscr{N}_\alpha \equiv \{ f \in \mathscr{D}(L^*) : (\alpha - L^*) f = 0 \}$ and $f \in \mathscr{N}_\alpha \cap \mathscr{D}(\mathscr{E}_B)$, we have $f \in H^1_{loc}(D)$ and

$$\mathscr{E}_B(f, f) \geqq \mathscr{E}(f, f).$$

Since $L^* = L$ on $C_c^\infty(D)$, by the definition of $\mathscr{N}_\alpha$, $(L - \alpha) f = 0$ in the sense of distributions. Then by an extension of Weyl's lemma applicable to L (*see* Hörmander [11], Theorem 17.2.7), it follows that $f \in W^{2,2}_{loc}(D)$ and $(L - \alpha) f = 0$ a.e.

Let $\{ G^B_\beta, \beta > 0 \}$ denote the resolvent for B, $G^B_\beta(f^2) = \lim_{n \to \infty} G^B_\beta(f^2 \wedge n)$,

$$\mathscr{E}^{(\beta)}_B(g, g) = \beta(g - \beta G^B_\beta g, g), \qquad \beta > 0, \quad g \in L^2(D, \mu),$$

and extend the definition of $(g, h) = \int_D gh\, d\mu$ to whenever the right member is well defined. Then by the same kind of argument as in the third paragraph of Lemma 2.3.4 of [9], possibly using truncation of f and then passage to the limit, we obtain

$$\mathscr{E}^{(\beta)}_B(f, f) = \frac{\beta}{2}(1, f_\beta),$$

where

$$f_\beta = -(f^2 - \beta G^B_\beta f^2) + 2(f^2 - \beta f G^B_\beta f) + f^2 - f^2 \beta G^B_\beta 1,$$
$$f_\beta(x) = \beta [G^B_\beta (f(x) - f)^2](x) + 2 f^2(x)(1 - \beta G^B_\beta 1)(x) \geqq 0 \quad \text{a.e.}$$

Thus

$$\mathscr{E}^{(\beta)}_B(f, f) \geqq \frac{\beta}{2}(f_\beta, \varphi) \quad \text{for any } 0 \leqq \varphi \leqq 1, \ \varphi \in C_c^\infty(D),$$

and using the first formula above for f_β we have

(9) $$(f_\beta, \varphi) \geqq -(f^2 - \beta G^B_\beta f^2, \varphi) + 2(f^2 - \beta f G^B_\beta f, \varphi)$$

since $(f^2 - f^2 \beta G^B_\beta 1, \varphi) \geqq 0$.

This paragraph takes the place for L of the justification of (2.3.25) for $\frac{1}{2}\Delta$ given in [9]. By Lemma 2.3.2 (i) in [9], $B \subset L^*$, hence

$$L^*(G_\beta^B f) = \beta G_\beta^B f - f$$

and the same holds with $f_n^2 = f^2 \wedge n$ in place of f. Then (9) becomes

$$(f_\beta, \varphi) \geqq \lim_{n \to \infty} (G_\beta^B f_n^2, L\varphi) - 2(G_\beta^B f, L(f\varphi))$$
$$= (f^2, G_\beta^B(L\varphi)) - 2(G_\beta^B f, L(f\varphi)).$$

Here we have used the fact that $f\varphi \in W^{2,2}(D)$ with compact support in D in the first line and the symmetry of G_β^B in the second line. Now,

$$\beta(f_\beta, \varphi) \geqq \beta \int_0^\infty e^{-\beta t}(f^2, P_t^B(L\varphi))\,dt$$
$$-2\beta \int_0^\infty e^{-\beta t}(P_t^B f, L(f\varphi))\,dt,$$

where $\{P_t^B, t \geqq 0\}$ is the semigroup associated with B. Since $L(f\varphi) \in L^2(D, \mu)$ and P_t^B is strongly continuous, $t \to (P_t^B f, L(f\varphi))$ is continuous, and

$$\lim_{\beta \to \infty} \beta \int_0^\infty e^{-\beta t}(P_t^B f, L(f\varphi))\,dt = (f, L(f\varphi)).$$

For the other term, we note that $P_t^B(L\varphi) \to L\varphi$ in μ-measure as $t \downarrow 0$, and is bounded by a constant, hence since $f^2 \in L^1(D, \mu)$,

$$\lim_{\beta \to \infty} \beta \int_0^\infty e^{-\beta t}(f^2, P_t^B(L\varphi))\,dt = (f^2, L\varphi).$$

We have proved that, as $\beta \to \infty$,

$$\liminf_{\beta \uparrow \infty} \beta(f_\beta, \varphi) \geqq (f^2, L\varphi) - 2(f, L(f\varphi)).$$

Now since $f \in \mathcal{N}_\alpha$ and by integration by parts,

$$(f, L(f\varphi)) = \alpha(f, f\varphi)$$
$$(f^2, L\varphi) = 2(fLf, \varphi) + (a\nabla f, \varphi\nabla f)$$
$$= 2\alpha(f, f\varphi) + (a\nabla f, \varphi\nabla f).$$

Thus,

$$\liminf_{\beta \uparrow \infty} \beta(f_\beta, \varphi) \geqq (a\nabla f, \varphi\nabla f).$$

Hence,

$$\begin{aligned}\mathscr{E}_{\mathrm{B}}(f,f) &= \lim_{\beta\to\infty} \mathscr{E}_{\mathrm{B}}^{\beta}(f,f)\\ &= \frac{1}{2}\lim_{\beta\to\infty} \beta(f_\beta, 1)\\ &\geqq \frac{1}{2}\liminf_{\beta\to\infty} \beta(f_\beta, \varphi)\\ &\geqq \frac{1}{2}(a\nabla f, \varphi\nabla f).\end{aligned}$$

Letting $\varphi \uparrow 1$, we deduce finally

$$\mathscr{E}_{\mathrm{B}}(f,f) \geqq \mathscr{E}(f,f). \quad \square$$

3. APPROXIMATION BY PENALIZATION

Let (D, a, p) denote the triplet described in section 1. We shall construct a sequence $\{X^n\}$ of symmetric Markov processes taking values in all of $\mathbb{R}^d$, which under additional assumptions described in Theorem 3.9 converges weakly to the process X associated to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. We shall add one crucial assumption to those made in section 1, which will be supposed to hold only in this section:

(A3) $\partial D = \bar{D} \setminus D$ has zero d-dimensional Lebesgue measure.

We note that when (A3) does not hold, the limit constructed by penalization (which is an "approximation from the exterior") would not need to agree with the limit we shall construct by approximation from the interior in section 4.

We shall use the following lemma, which is proved in Stein [24], p. 171, to define a regularization of the distance to $\bar{D}$.

LEMMA 3.1. – *Let* G *be an open set in* $\mathbb{R}^d$. *For* $x \in \mathbb{R}^d$, *let* $d(x)$ *denote the distance of* x *from* G^c. *There exists a continuous function* $\theta : \mathbb{R}^d \to \mathbb{R}$ *and strictly positive constants* c_1, c_2, c_β, *such that*

(i) $c_1\, d(x) \leqq \theta(x) \leqq c_2\, d(x)$ *for all* $x \in \mathbb{R}^d$,

(ii) $\theta \in C^\infty(G)$ *and for any multi-index* β, *the* β-th *derivative* $\partial^\beta \theta$ *of* θ *satisfies*

$$|\partial^\beta \theta(x)| \leqq c_\beta (d(x))^{1-|\beta|} \textit{for all } x \in G,$$

where $|\beta|$ *denotes the sum of the components of the multi-index* β.

Let us now define the approximation. For this, extend a by I and p by 0 outside of D.

LEMMA 3.2. — *There is a sequence* $\{(D_n, a_n, p_n)\}_{n=1}^{\infty}$ *such that* $\bigcup_n D_n = D$, *and for each* n,

(i) D_n *is open*, $D_n \subset\subset D_{n+1} \subset D$,

(ii) $a_n \in W^{1,\infty}(\mathbb{R}^d)$, a_n *is symmetric and uniformly elliptic on* $\mathbb{R}^d$,

(iii) $p_n \in W^{1,\infty}(\mathbb{R}^d)$, $p_n > 0$ *on* $\mathbb{R}^d$, $\int_{\mathbb{R}^d} p_n\, dx = 1$, *and there is a finite constant* c_n *such that*

$$\frac{|\nabla p_n|}{p_n} \leqq c_n \quad \text{on } \mathbb{R}^d, \tag{10}$$

(iv) $a_n = a$ *and* $\dfrac{\nabla p_n}{p_n} = \dfrac{\nabla p}{p}$ *on* D_n,

(v) $p_n \to p$ *in* $L^1(\mathbb{R}^d)$, *as* $n \to \infty$.

Proof. — Let $\bar{\delta}$ denote the function θ from Lemma 3.1 associated with $G = \bar{D}^c$ and

$$q_n(x) = \exp(-n\,\bar{\delta}(x)).$$

Let $\bar{p} \in C_b^{\infty}(\mathbb{R}^d)$ be such that $\bar{p} > 0$ on $\mathbb{R}^d$, and for some $k \in \mathbb{N}$ and $c > 0$,

$$\bar{p}(x) = \begin{cases} 1 & \text{for } x \in B_k, \\ c\exp(-|x|) & \text{for } x \in B_{k+1}^c, \end{cases}$$

where B_l, $l = 1, 2, \ldots$, denotes the ball in $\mathbb{R}^d$ of radius l centered at the origin. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of open subsets of D such that

$$D_n \subset\subset D_{n+1} \quad \text{for all } n, \ \bigcup_n D_n = D.$$

For each n, let $\phi_n \in C_c^{\infty}(\mathbb{R}^d)$ such that $0 \leqq \phi_n \leqq 1$,

$$\phi_n = \begin{cases} 1 & \text{on } \bar{D}_n, \\ 0 & \text{on } D_{n+1}^c, \end{cases}$$

and let $\psi_n = 1 - \phi_n$.

Define

$$a_n = a\,\phi_n + I\,\psi_n \tag{11}$$

$$p_n = k_n(p\,\phi_n + \bar{p} q_n \psi_n) \tag{12}$$

where k_n is a normalization constant:

$$k_n = \left(\int_{\mathbb{R}^d} (p\,\phi_n + \bar{p} q_n \psi_n)\, dx\right)^{-1}.$$

The properties stated in the lemma are easily checked. Note in particular that $k_n \to 1$ as $n \to \infty$ by dominated convergence (we use (A3) here to insure a.e. convergence). □

For each n, by Lemma 5.2.1 and Theorem 5.2.2 of [25], there is a function $\sigma_n : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that $\sigma_n \in W^{1,\infty}(\mathbb{R}^d)$, $\sigma_n(x)$ is symmetric and positive definite for each $x \in \mathbb{R}^d$ and $\sigma_n \sigma_n = a_n$. Let

$$b_n = \frac{1}{2p_n} \nabla . (a_n p_n)$$

and $\mu_n(dx) = p_n(x)\,dx$. We note that $a_n = a$, $b_n = b$ on D_n.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which are defined a d-dimensional Brownian motion $\{W_t, t \geqq 0\}$ and an independent d-dimensional random vector Y^n with law μ_n. Consider the stochastic differential equation

$$\text{(13)} \qquad \begin{cases} dX_t^n = b_n(X_t^n)\,dt + \sigma_n(X_t^n)\,dW_t \\ X_0^n = Y^n. \end{cases}$$

We have the following

THEOREM 3.3. — *Equation* (13) *has a unique strong solution which is a stationary symmetric Markov process with sationary measure* μ_n *and an associated strongly continuous semigroup* $\{P_t^n, t \geqq 0\}$ *on* $L^2(\mathbb{R}^d, \mu_n)$. *The associated Dirichlet form is* $(\mathcal{E}^n, \mathcal{D}(\mathcal{E}^n))$ *where*

$$\text{(14)} \qquad \mathcal{E}^n(g, g) = \int_{\mathbb{R}^d} \nabla g(x) . a_n(x) \nabla g(x) \mu_n(dx),$$

$$\text{(15)} \qquad \mathcal{D}(\mathcal{E}^n) = \{ g \in L^2(\mathbb{R}^d, \mu_n) \cap H^1_{loc}(\mathbb{R}^d) : \mathcal{E}^n(g, g) < \infty \} = H^1(\mathbb{R}^d, \mu_n).$$

Before proving this theorem, we need to establish a technical lemma:

LEMMA 3.4. — *Let* $\alpha_{ij} \in W^{1,\infty}(\mathbb{R}^d)$, $1 \leqq i, j \leqq d$, *and* $\beta_i \in L^\infty(\mathbb{R}^d)$, $1 \leqq i \leqq d$. *Suppose that the matrix* $\alpha(x) = (\alpha_{ij}(x))_{i,j=1}^d$ *is symmetric for each* $x \in \mathbb{R}^d$, *and that there exists* $\bar{\alpha} > 0$ *such that* $\xi' \alpha(x) \xi \geqq \bar{\alpha} |\xi|^2$, *for all* $x, \xi \in \mathbb{R}^d$. *We define*

$$L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(\alpha_{ij} \frac{\partial .}{\partial x_j}\right) + \sum_{i=1}^d \beta_i \frac{\partial .}{\partial x_i}.$$

Let $q : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ *be the unique* ([4]) *measurable function satisfying*

(i) $\int_{\mathbb{R}^d} q(t, y)\,dy = 1$ *for all* $t \geqq 0$,

(ii) $q \in \bigcap_{T>0} L^2((0, T), H^1(\mathbb{R}^d)) \cap C(\mathbb{R}_+, L^2(\mathbb{R}^d))$,

([4]) It is a consequence of our proof that q is unique.

(iii) *for any* $t>0$, $\phi\in C_c^\infty(\mathbb{R}^d)$,

$$(q(t,.),\phi)=(q(0,.),\phi)+\int_0^t (q(s,.),L\phi)\,ds$$

where $(.,.)$ *denotes the usual inner product in* $L^2(\mathbb{R}^d)$.

Moreover, let $\{Y_t, t\geqq 0\}$ *denote a diffusion process in* $\mathbb{R}^d$ *with infinitesimal generator* L *and initial law* $\lambda(dy)=q(0,y)\,dy$. *Then for any* $t>0$, $\lambda_t(dy)\overset{\Delta}{=}q(t,y)\,dy$ *is the law of* Y_t.

Proof. – We note that from the assumptions made on $\{\alpha_{ij},\beta_i\}$, the martingale problem associated to L is well posed. (Indeed, the associated stochastic differential equation even has a unique strong solution (*see* Veretennikov [27]), but we won't use this fact.) Hence the law of the process $\{Y_t, t\geqq 0\}$ is uniquely determined by the conditions given in the statement of the lemma. We shall denote expectation under that law by $\mathbb{E}_\lambda$, and expectation under the law of the same process with the initial condition $Y_0=y$ a.s. will be denoted by $\mathbb{E}_y$, $y\in\mathbb{R}^d$.

The assumption (iii) says that q solves in a weak sense the Fokker-Planck equation:

$$\frac{\partial q}{\partial t}=L^*q,\qquad t>0.$$

For $t>0$ and $\varphi\in C_c^\infty(\mathbb{R}^d)$, let $\{v(s,y):0\leqq s\leqq t, y\in\mathbb{R}^d\}$ denote the unique element of $L^2((0,t),H^1(\mathbb{R}^d))\cap C([0,t],L^2(\mathbb{R}^d))$ which in a similar weak sense solves the backward Kolmogorov equation:

$$\frac{\partial v}{\partial s}+Lv=0,\qquad 0<s<t,$$

$$v(t,y)=\varphi(y),\qquad y\in\mathbb{R}^d,$$

see Dautray-Lions [8], Theorem XVIII.3.1 and 2 for a reference. We first claim that

$$v(0,y)=\mathbb{E}_y[\varphi(Y_t)],\quad \text{a.e. } y.$$

Indeed, this formula is known under additional regularity assumptions on the coefficients α and β, see Bensoussan-Lions [3], Theorem 2.7.4. The claim now follows by taking limits along a regularizing sequence of coefficients, both in the partial differential equation for v (using the methods in [8], Chapter XVIII) and in the martingale problem for Y (*see* Stroock-Varadhan [25], Theorem 11.3.3).

Now, from [8], Theorem XVIII.1.2, $s \to (q(s), v(s))$ is absolutely continuous and

$$\frac{d}{ds}(q(s), v(s)) = \langle v(s), L^* q(s) \rangle - \langle q(s), Lv(s) \rangle$$
$$= 0 \text{ a.e. } s,$$

where $\langle ., . \rangle$ denotes the pairing between $H^1(\mathbb{R}^d)$ and $H^{-1}(\mathbb{R}^d)$. Hence

$$\begin{aligned}\mathbb{E}_\lambda[\varphi(Y_t)] &= \int_{\mathbb{R}^d} \mathbb{E}_y[\varphi(Y_t)]\,\lambda(dy)\\ &= (v(0, .), q(0, .))\\ &= (v(t, .), q(t, .))\\ &= \int_{\mathbb{R}^d} \varphi(y)\,q(t, y)\,dy,\end{aligned}$$

for any $\varphi \in C_c(\mathbb{R}^d)$.

Proof of Theorem 3.3. — Existence and uniqueness of a strong solution follow from the results of Veretennikov [27]. By Lemma 3.4 with $\alpha = \frac{1}{2} a_n$, $\beta = b_n$, $q(t, x) = p_n(x)$, we have that μ_n is an invariant probability measure for X^n.

By identifying the space $L^2(\mathbb{R}^d, \mu_n)$ wih $L^2(\Omega, \sigma(X_0^n), \mathbb{P})$ for each $t \geqq 0$ we can define a bounded linear operator P_t^n on $L^2(\mathbb{R}^d, \mu_n)$ by

$$(P_t^n g)(X_0^n) = \mathbb{E}[g(X_t^n) \mid X_0^n], \qquad \forall g \in L^2(\mathbb{R}^d, \mu_n).$$

The fact that P_t^n is a bounded linear operator with norm one on $L^2(\mathbb{R}^d, \mu_n)$ follows from Jensen's inequality and the conservative property $P_t^n 1 = 1$. The symmetry of $\{X_t^n\}$ follows from the fact that the process obtained by time reversal at a fixed time t is again a solution of a stochastic differential equation with the same coefficients and initial law as in (13), see Pardoux [22]. Note that condition (H2)(ii) there is not needed since a_n is non-degenerate (*see* Remark 2.3 of [22]), and that the Lipschitz continuity of b is assumed there only to ensure existence and uniqueness for the stochastic differential equation.

Since $C_b(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d, \mu_n)$ and $\{P_t^n\}$ is a contraction semigroup, strong continuity on $L^2(\mathbb{R}^d, \mu_n)$ will follow from

$$\| P_t^n g - g \|_{L^2(\mathbb{R}^d, \mu_n)} \to 0 \quad \text{as } t \to 0,$$

for any $g \in C_b(\mathbb{R}^d)$. The latter follows from Jensen's inequality, the continuity of the paths of X^n and bounded convergence.

Let

$$L_n = \frac{1}{2p_n} \nabla . (a_n p_n \nabla .).$$

Since X^n is a solution of (13), by applying Itô's formula we have that for any $g \in C_c^\infty(\mathbb{R}^d)$,

$$g(X_t^n) = g(X_0^n) + \int_0^t (\nabla g . \sigma_n)(X_s^n) . dW_s + \int_0^t (L_n g)(X_s^n) \, ds.$$

Since $\nabla g . \sigma_n$ is bounded, the stochastic integral with respect to W defines a martingale. Hence

$$\frac{1}{t} \mathbb{E}[g(X_t^n) - g(X_0^n) \mid X_0^n] = \frac{1}{t} \mathbb{E}\left[\int_0^t (L_n g)(X_s^n) \, ds \mid X_0^n\right] = \frac{1}{t} \int_0^t P_s^n (L_n g)(X_0^n) \, ds.$$

By the strong continuity of the semigroup $\{P_s^n, s \geqq 0\}$ and boundedness of $P_s^n(L_n g)$, the last entity above converges in $L^2(\Omega, \sigma(X_0^n), \mathbb{P})$ to $(L_n g)(X_0^n)$ as $t \downarrow 0$. It follows that the strong domain of the infinitesimal generator A_n for $\{P_t^n, t \geqq 0\}$ contains $C_c^\infty(\mathbb{R}^d)$ and that

$$A_n g = L_n g \quad \text{for} \quad g \in C_c^\infty(\mathbb{R}^d).$$

Hence, since $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d, \mu_n)$, the Dirichlet form associated to X^n is the unique one extending

$$-\int_{\mathbb{R}^d} (L_n g)(x) g(x) \mu_n(dx), \qquad g \in C_c^\infty(\mathbb{R}^d),$$

which is defined by (14)-(15). □

Since we are only concerned with *stationary* Markov processes, from hereon we restrict attention to processes defined on the time interval [0, 1]. From the definition of X^n we have that

$$(16) \qquad M_t^n \equiv X_t^n - X_0^n - \int_0^t b_n(X_s^n) \, ds = \int_0^t \sigma_n(X_s^n) \, dW_s^n, \ t \in [0, 1],$$

is a continuous martingale with respect to the filtration generated by X^n, which we call the "forward" filtration of X^n, and M^n has mutual variation process

$$(17) \qquad \langle (M_.^n)_i, (M_.^n)_j \rangle_t = \int_0^t (a_n)_{ij}(X_s^n) \, ds \text{ for all } i, j \in \{1, \ldots, d\}.$$

Since X^n is symmetric, it follows that for $\bar{X}_.^n \equiv X_{1-.}^n$,

$$(18) \qquad \bar{M}_t^n \equiv \bar{X}_t^n - \bar{X}_0^n - \int_0^t b_n(\bar{X}_s^n) \, ds, \ t \in [0, 1],$$

is a continuous martingale with respect to the filtration generated by $\bar{X}^n$, called the "blackward" filtration of X^n, and the mutual variation of $\bar{M}^n$ is given by

$$(19) \qquad \langle (\bar{M}^n_\cdot)_i, (\bar{M}^n_\cdot)_j \rangle_t = \int_0^t (a_n)_{ij} (\bar{X}^n_s)\, ds \text{ for all } i, j \in \{1, \ldots, d\}.$$

It can be readily verified by substituting (16) and (18) in the following that we have the decomposition (*cf.* Lyons-Zheng [18], pp. 251-252)

$$(20) \qquad X^n_t = X^n_0 + \frac{1}{2} M^n_t - \frac{1}{2} (\bar{M}^n_1 - \bar{M}^n_{1-t}) \quad \text{for all } t \in [0, 1].$$

Since the mutual variations of M^n and $\bar{M}^n$ given by (17) and (19) are absolutely continuous with respect to dt and have derivatives in t that are uniformly bounded in t and n, it follows (*see* Jacod-Shiryaev [12], Prop. VI.3.26, Thm. VI.3.21) that the laws of

$$\sum_{i=1}^d \langle (M^n_\cdot)_i, (M^n_\cdot)_i \rangle \qquad \text{and} \qquad \sum_{i=1}^d \langle (\bar{M}^n_\cdot)_i, (\bar{M}^n_\cdot)_i \rangle$$

define tight sequences of measures on $C([0, 1], \mathbb{R})$ and consequently by Theorem VI.4.13 of [12] the laws of $\{M^n\}_{n=1}^\infty$ and $\{\bar{M}^n\}_{n=1}^\infty$ are tight.

Since $p_n \to p$ in $L^1(\mathbb{R}^d)$, X^n_0 converges weakly to X_0 whose law is μ, where $\mu(dx) = p\,dx$. Thus, individually the laws of $\{M^n\}_{n=1}^\infty$, $\{\bar{M}^n\}_{n=1}^\infty$, and $\{X^n_0\}_{n=1}^\infty$ are tight and consequently the laws of $\{(M^n, \bar{M}^n, X^n_0)\}_{n=1}^\infty$ define a tight sequence of measures on $C([0, 1], \mathbb{R}^{2d}) \times \mathbb{R}^d$. Suppose that $(M, \bar{M}, X_0)$ is a weak limit point of this sequence and define

$$(21) \qquad X_t = X_0 + \frac{1}{2} M_t - \frac{1}{2} (\bar{M}_1 - \bar{M}_{1-t}) \quad \text{for } t \in [0, 1].$$

Then $(M^n\ \bar{M}^n, X^n)$ converges weakly along a subsequence to $(M, \bar{M}, X)$. For notational convenience, without loss of generality, we may suppose that the convergence is along the sequence (not a subsequence). It can be readily verified that X inherits stationarity and symmetry from the X^n. Moreover, since the law of X^n_t is μ_n for each $t \in [0, 1]$ and $\mu_n \Rightarrow \mu$, we deduce that μ is the law of X_t, for each $t \in [0, 1]$. Since $\bar{D}$ is the support of μ, and X almost surely has continuous paths, we deduce that

$$\mathbb{P}(X_t \in \bar{D} \quad \text{for all } t \in [0, 1]) = 1,$$

and moreover, for each $t \in [0, 1]$, $X_t \in D$ a.s., since $\mu(D) = 1$.

In a similar manner to that in the proof of Theorem 3.3, we define for each $t \in [0, 1]$ the bounded linear operator P_t on $L^2(D, \mu)$ by:

$$(P_t g)(X_0) = \mathbb{E}[g(X_t) \mid X_0], \qquad g \in L^2(D, \mu),$$

and remark that P_t is symmetric since $\{X_t\}$ is.

LEMMA 3.5. – *For any* $f, g \in L^\infty(\mathbb{R}^d)$ *and* $t \in [0, 1]$, we have

(i) $\lim_{n \to \infty} \int_D |P_t^n g - P_t g|^2 \, d\mu = 0,$

(ii) $\int_{\mathbb{R}^d} f P_t^n g \, d\mu_n \to \int_D f P_t g \, d\mu$ *as* $n \to \infty$.

Proof. – We first claim that it suffices to prove the lemma for $f, g \in C_b(\mathbb{R}^d)$. To see this, observe that for $h \in L^\infty(\mathbb{R}^d)$ and $g \in C_b(\mathbb{R}^d)$,

$$(22) \quad \int_D |P_t^n h - P_t h|^2 \, d\mu \leqq 4 \left\{ \int_D |P_t(h-g)|^2 \, d\mu + \int_D |P_t^n(h-g)|^2 \, d\mu + \int_D |P_t^n g - P_t g|^2 \, d\mu \right\}.$$

It suffices to consider the first two terms on the right side. Now,

$$(23) \quad \int_D |P_t(h-g)|^2 \, d\mu \leqq \int_D |h-g|^2 \, d\mu,$$

since P_t has norm one on $L^2(D, \mu)$, and

$$\begin{aligned} \int_D |P_t^n(h-g)|^2 \, d\mu &\leqq \int_{\mathbb{R}^d} |P_t^n(h-g)|^2 \, d\mu_n + \int_{\mathbb{R}^d} |P_t^n(h-g)|^2 \, d(\mu - \mu_n) \\ &\leqq \int_{\mathbb{R}^d} |h-g|^2 \, d\mu_n + c \, \|p - p_n\|_{L^1(\mathbb{R}^d)} \\ &\leqq \int_D |h-g|^2 \, d\mu + 2c \, \|p - p_n\|_{L^1(\mathbb{R}^d)} \end{aligned}$$

where $c = (\|h\|_\infty + \|g\|_\infty)^2$. Thus, since $C_b(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d, \mu)$ it suffices to prove (i), and similarly (ii) for $f, g \in C_b(\mathbb{R}^d)$.

Assuming $g \in C_b(\mathbb{R}^d)$, we have for any $h \in C_b(\mathbb{R}^d)$,

$$\begin{aligned} (24) \quad & \left| \int_D h P_t^n g \, d\mu - \int_D h P_t g \, d\mu \right| \\ & \leqq \left| \int_D h P_t^n g \, d(\mu - \mu_n) \right| + \left| \int_D h P_t^n g \, d\mu_n - \int_D h P_t g \, d\mu \right| \\ & \leqq \|h\|_\infty \|g\|_\infty \|p - p_n\|_{L^1(\mathbb{R}^d)} \\ & \quad + |\mathbb{E}[h(X_0^n) g(X_t^n)] - \mathbb{E}[h(X_0) g(X_t)]| \\ & \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

by the convergence of p_n and the weak convergence of X^n to X. It is easily deduced from this that $P_t^n g \to P_t g$ in $L^2(D, \mu)$ weakly, as $n \to \infty$, for each $g \in C_b(\mathbb{R}^d)$. Now (ii) follows from the $L^1(\mathbb{R}^d)$ convergence of p_n to p.

We now show that $P_t^n g$ converges strongly to $P_t g$ in $L^2(D, \mu)$, again for $g \in C_b(\mathbb{R}^d)$. For each $x \in \mathbb{R}^d$, let $\{X_t^{n, x}, t \geqq 0\}$ be the unique strong solution of (13) with initial condition $X^{n, x}(0) = x$. By well posedness of the associated martingale problem (*see* [25], Exercise 6.7.4), the law of $X^{n, x}$ is unique and $x \to \mathbb{E}[g(X_t^{n, x})]$ is measurable. Hence,

$$(P_t^n g)(x) = \mathbb{E}[g(X_t^{n, x})], \quad \text{for } a.e.\ x.$$

Fix $x \in D$ such that the last identity holds and $m \in \mathbb{N}$ such that $x \in D_m$. Let

$$\tau_n = \inf\{s \geqq 0 : X_s^{n, x} \in D_m^c\}.$$

We observe that for $n \geqq m$, the coefficients of equation (13) restricted to D_m do not depend on n. Let $(\tilde{a}, \tilde{b})$ be an extension of (a, b) outside D_m, such that $\tilde{a} \in W^{1, \infty}(\mathbb{R}^d)$ is symmetric and uniformly elliptic, and $\tilde{b}$ is bounded and measurable. Then the associated stochastic differential equation has a unique strong solution $\tilde{X}^x$ starting from x. For $0 < s < t$, let $\tilde{p}(s, x, y)$ denote the associated transition probability density (*see* Theorem 9.2.6 in Stroock-Varadhan [25]), which as a function of y is in $L^2(\mathbb{R}^d)$. Let $\tilde{\tau} = \inf\{u \geqq 0 : \tilde{X}_u^x \in D_m^c\}$. Then,

$$\begin{aligned} P_t^n g(x) &= \mathbb{E}[P_{t-s}^n g(X_s^{n, x})] \\ &= \mathbb{E}[P_{t-s}^n g(X_s^{n, x});\ \tau_n \geqq s] + \xi_n \\ &= \mathbb{E}[P_{t-s}^n g(\tilde{X}_s^x);\ \tilde{\tau} \geqq s] + \xi_n \\ &= \int_{D_m} P_{t-s}^n g(y) \frac{\tilde{p}(s, x, y)}{p(y)} \mu(dy) + \xi_n + \eta_n \end{aligned}$$

where

$$|\xi_n| \leqq \|g\|_\infty \mathbb{P}(\tau_n < s) = \|g\|_\infty \mathbb{P}(\tilde{\tau} < s),$$

and the same bound holds for η_n. Given $\varepsilon > 0$, we can choose $s > 0$ small enough such that $2\|g\|_\infty \mathbb{P}(\tilde{\tau} < s) \leqq \varepsilon$. Then

$$\left| P_t^n g(x) - \int_{D_m} P_{t-s}^n g(y) \frac{\tilde{p}(s, x, y)}{p(y)} \mu(dy) \right| \leqq \varepsilon \tag{25}$$

for any $n \geqq m$. Moreover, from the weak convergence in $L^2(D, \mu)$ of $\{P_{t-s}^n g\}$,

$$\int_D P_{t-s}^n g(y) \frac{\tilde{p}(s, x, y)}{p(y)} 1_{D_m}(y) \mu(dy)$$

converges as $n \to \infty$, since $\dfrac{\tilde{p}(s, x, .)}{p(.)}$ is in $L^2(D_m)$ and hence $\dfrac{\tilde{p}(s, x, .)}{p(.)} 1_{D_m}(.) \in L^2(D, \mu)$. We conclude from this convergence and (25) that $\{P_t^n g(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, for μ-a.e. x in D. Strong convergence in $L^2(D, \mu)$ follows from the bounded convergence theorem, and the strong limit equals the weak limit. □

LEMMA 3.6. — *$\{P_t, 0 \leqq t \leqq 1\}$ is a strongly continuous symmetric Markovian semigroup on $L^2(D, \mu)$.*

Proof. — The symmetry has already been noted and the Markovian property is evident from the definition of P_t. For the semigroup property, since $L^\infty(D, \mu)$ is dense in $L^2(D, \mu)$ and P_t has norm one on $L^2(D, \mu)$, it suffices to show that for each $g, h \in L^\infty(D, \mu)$, and $0 \leqq s, t \leqq 1$ such that $t+s \leqq 1$,

$$\int_D h(P_t P_s g - P_{t+s} g)\, d\mu = 0.$$

But by the symmetry of P_t this is equivalent to

$$\int_D (P_t h P_s g - h P_{t+s} g)\, d\mu = 0. \tag{26}$$

Observe that with P_t^n, P_s^n, P_{t+s}^n, μ_n, $\mathbb{R}^d$ in place of P_t, P_s, P_{t+s}, μ, D, respectively, and the functions h, g extended by 0 outside D, the above holds by the symmetry and semigroup properties of $\{P_u^n, 0 \leqq u \leqq 1\}$. For all n sufficiently large, the left member of (26) can be shown to be within ε of that with the "n-replacement" described above, by invoking Lemma 3.5 and the convergence in $L^1(\mathbb{R}^d)$ of p_n to p, together with the uniform bound $2\|h\|_\infty \|g\|_\infty$ for $P_t^n h P_s^n g - h P_{t+s}^n g$. The strong continuity of $\{P_t, 0 \leqq t \leqq 1\}$ now follows precisely as in the proof of Theorem 3.3. □

THEOREM 3.7. — *The weak limit process X is a stationary symmetric continuous Markov process with stationary measure μ on D and associated strongly continuous semigroup $\{P_t, 0 \leqq t \leqq 1\}$ on $L^2(D, \mu)$.*

Proof. — To prove X is a Markov process with semigroup $\{P_t, 0 \leqq t \leqq 1\}$ on $L^2(D, \mu)$, it suffices to show that

$$\int_D h_0 P_{s_1}(h_1 P_{s_2}(\ldots P_{s_m} h_m)\, d\mu = \mathbb{E}[h_0(X_0) h_1(X_{s_1}) \ldots h_m(X_{s_1+\ldots+s_m})] \tag{27}$$

for all $h_0, h_1, \ldots, h_m \in C_c(D)$ and $s_1, \ldots, s_m > 0$ such that $s_1 + \ldots + s_m \leqq 1$. We extend each of the h_i, $i = 0, 1, \ldots$, to be zero on D^c, so they are in $C_c(\mathbb{R}^d)$. By the analysis of Lemma 3.5, one can show that the left member of (27) equals

$$\lim_n \int_{\mathbb{R}^d} h_0 P_{s_1}^n (h_1 P_{s_2}^n (\ldots P_{s_m}^n h_m)\, d\mu_n.$$

But by the Markov property of X^n, this equals

$$\lim_n \mathbb{E}[h_0(X_0^n)\,h_1(X_{s_1}^n)\ldots h_m(X_{s_1+\ldots+s_m}^n)],$$

which by the weak convergence of X^n equals the right member of (27). □

Under suitable conditions, we now identify the symmetric Markov process X, with strongly continuous Markovian semigroup $\{P_t,\ 0 \leqq t \leqq 1\}$ on $L^2(D, \mu)$, as the one having Dirichlet form

$$\mathscr{E}(g, g) = \frac{1}{2}\int_D \nabla g . a \nabla g \, d\mu \quad \text{for all } g \in \mathscr{D}(\mathscr{E}) \tag{28}$$

where $\mathscr{D}(\mathscr{E}) = \left\{ g \in L^2(D, \mu) \cap H^1_{loc}(D) : \int_D \nabla g . a \nabla g \, d\mu < \infty \right\}$. For this, let $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ denote the Dirichlet form associated with $\{P_t,\ 0 \leqq t \leqq 1\}$. Let $\tilde{A}$ denote the infinitesimal generator of $\{P_t,\ 0 \leqq t \leqq 1\}$ and let $\mathscr{D}(\tilde{A})$ denote its strong domain.

LEMMA 3.8. – *For any* $g \in C_c^\infty(D)$, *we have* $g \in \mathscr{D}(\tilde{A})$ *and*

$$\tilde{A} g = L g \quad \textit{on } D.$$

Proof. – Let $g \in C_c^\infty(D)$ and $h \in C_b(D)$. Then, after extending g and h by zero outside D, we have

$$\begin{aligned} \mathbb{E}[(g(X_t) - g(X_0))\,h(X_0)] &= \lim_n \mathbb{E}[(g(X_t^n) - g(X_0^n))\,h(X_0^n)] \\ &= \lim_n \mathbb{E}\left[\left(\int_0^t (L_n g)(X_s^n)\,ds\right) h(X_0^n)\right] \\ &= \lim_n \int_0^t \mathbb{E}[(L_n g)(X_s^n)\,h(X_0^n)]\,ds. \end{aligned} \tag{29}$$

Now for n sufficiently large, the support of g will be contained in D_n and then

$$L_n g = L g.$$

Thus the last line of (29) equals

$$\begin{aligned} \lim_n \int_0^t \mathbb{E}[(L g)(X_s^n)\,h(X_0^n)]\,ds &= \lim_n \int_0^t \left(\int_{\mathbb{R}^d} h\,P_s^n(Lg)\,d\mu_n\right) ds \\ &= \int_0^t \left(\int_D h\,P_s(Lg)\,d\mu\right) ds \end{aligned}$$

by Lemma 3.5 (ii). Since h was arbitrary we have

$$P_t g - g = \int_0^t P_s(Lg)\,ds \quad \mu\text{-a.e.}$$

By the strong continuity of P_s, $s \to P_s(Lg)$ is a continuous function from $[0, t]$ into $L^2(D, \mu)$ and so it follows that

$$\lim_{t \downarrow 0} \frac{1}{t}(P_t g - g) = Lg$$

strongly in $L^2(D, \mu)$. □

It now follows from Theorem 2.3 that $\mathscr{D}(\tilde{\mathscr{E}}) \subset \mathscr{D}(\mathscr{E})$ and

$$\tilde{\mathscr{E}}(g, g) \geqq \mathscr{E}(g, g) \quad \text{for all } g \in \mathscr{D}(\tilde{\mathscr{E}}). \tag{30}$$

Before we can state our final result, we need to define:

$$\Lambda = \{ f \in W^{1,\infty}(\mathbb{R}^d) : f \text{ has compact support} \}$$

and Λ_D, the set of restrictions to D of the elements of Λ.

THEOREM 3.9. – *Suppose that*
(A4) Λ_D *is dense in* $\mathscr{D}(\mathscr{E})$ *with the norm* $\|.\|_{\mathscr{D}(\mathscr{E})}$ *defined by*

$$\|f\|^2_{\mathscr{D}(\mathscr{E})} = \mathscr{E}_1(f, f) \equiv (f, f) + \mathscr{E}(f, f).$$

Then $(\mathscr{E}, \mathscr{D}(\mathscr{E})) = (\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$.

Proof. – It suffices to prove that $\mathscr{D}(\mathscr{E}) \subset \mathscr{D}(\tilde{\mathscr{E}})$ and for all $g \in \mathscr{D}(\mathscr{E})$,

$$\mathscr{E}(g, g) \geqq \tilde{\mathscr{E}}(g, g). \tag{31}$$

Since $\mathscr{D}(\mathscr{E})$ equipped with the norm $\|.\|_{\mathscr{D}(\mathscr{E})}$ is complete, it suffices from (A4) to show that $\Lambda_D \subset \mathscr{D}(\tilde{\mathscr{E}})$ and that (31) holds for $g \in \Lambda_D$.

Let $g \in \Lambda_D$. Then g has an extension to $\mathbb{R}^d$ – that we still denote by g – which belongs to Λ. In that sense, $g \in \mathscr{D}(\mathscr{E}^n)$, and from Lemma 1.3.4 of [9], for each $t > 0$,

$$t^{-1} \int_{\mathbb{R}^d} g(g - P_t^n g)\, d\mu_n \leqq \frac{1}{2} \int_{\mathbb{R}^d} \nabla g . a_n \nabla g\, d\mu_n. \tag{32}$$

Taking the limit as $n \to \infty$ in (32), we deduce that

$$t^{-1} \int_D g(g - P_t g)\, d\mu \leqq \frac{1}{2} \int_D \nabla g . a \nabla g\, d\mu < \infty.$$

The result follows by applying Lemma 1.3.4 of [9] again. □

Remark 3.10. – Suppose a is uniformly elliptic, D is bounded and p is bounded above and below by positive constants. Then $\mathscr{D}(\mathscr{E}) = W^{1,2}(D)$ and $\|.\|_{\mathscr{D}(\mathscr{E})}$ is equivalent to $\|.\|_{W^{1,2}(D)}$. In this case, a sufficient condition for (A4) to hold is that D is an $\varepsilon - \delta$ domain, *i.e.*, there is $\varepsilon > 0$ and $\delta > 0$ such that whenever $x, y \in D$ and $|x - y| < \delta$, there is a rectifiable arc $\Gamma \subset D$ joining x to y such that

$$|\Gamma| \leqq \frac{1}{\varepsilon} |x - y|$$

and

$$d(z, D^c) \geqq \frac{\varepsilon |x-z| \, |y-z|}{|x-y|} \quad \text{for all } z \in \Gamma$$

where $|\Gamma|$ denotes the length of Γ. If D is an $\varepsilon-\delta$ domain, then by a theorem of Jones ([13], Theorem 1) any function in $W^{1,2}(D)$ is the restriction to D of some function in $W^{1,2}(\mathbb{R}^d)$, and since Λ is dense in $W^{1,2}(\mathbb{R}^d)$, it follows that Λ_D is dense in $W^{1,2}(D)$. We note from the paper of Jones [13] that an $\varepsilon-\delta$ domain can have a highly non-rectifiable boundary and for any $\alpha \in [d-1, d)$, there is an $\varepsilon-\delta$ domain whose boundary has positive α-dimensional Hausdorff measure.

Remark 3.11. — The results of this section remain true if we replace the sequence $\{(a_n, p_n)\}$ by any other sequence satisfying the properties stated in Lemma 3.2. In particular, we could have made the following choice for p_n:

$$p_n = p\,\phi_n + k_n \bar{p}\,\psi_n$$

where k_n is chosen in such a way that $\int_{\mathbb{R}^d} p_n \, dx = 1$. Accordingly, $k_n \to 0$ as $n \to \infty$ and the term involving ϕ_n creates a large drift towards the interior of D from points x in $D_{n+1} \setminus \bar{D}_n$. In this case, the "strong push" is only exerted inside D, as in the next section, but it is not strong enough so as to keep X^n inside D. When X^n exits D, we count on recurrence to bring it back inside. Note that in the construction used in this section, there is also a strong push on $D_{n+1} \setminus D_n$ due to $p \not\equiv \bar{p}$. However, this push inside can be avoided in the case where D is bounded and p can be extended to an element of $W^{1,\infty}(\mathbb{R}^d)$.

4. APPROXIMATION FROM THE INTERIOR

Let (D, a, p) be the given data as described in section 1. We first approximate a and p by data for a sequence of diffusions defined on all of $\mathbb{R}^d$, by defining (D_n, a_n, p_n) almost as in Lemma 3.2, with the only difference being that q_n and k_n are replaced by 1. It can be verified that all of the properties stated in Lemma 3.2 still hold except possibly for $\int_{\mathbb{R}^d} p_n \, dx = 1$ and for $p_n \to p$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$.

Let δ denote the function θ from Lemma 3.1 associated with G=D. For each $x \in D$, define

$$f_n(x) = \exp\left(\frac{1}{n\,\delta(x)}\right). \tag{33}$$

Note that for each fixed n, $f_n(x) \to +\infty$ as $x \to \partial D$ and for each fixed $x \in D$, $f_n(x) \downarrow 1$ as $n \to \infty$. We extend f_n to D^c by defining $f_n(x) = +\infty$ if $x \in D^c$. Now define

$$\rho_n = \exp(-f_n), \tag{34}$$

where $\exp(-\infty) \equiv 0$. It can be readily verified using Lemma 3.1 that $\rho_n \in C^\infty(D)$ and for each multi-index β, the β-th derivative of ρ_n at $x \in D$ tends to zero as $x \to \partial D$. Combining this with the fact that ρ_n is identically zero outside of D, it follows that ρ_n is a C^∞ function on $\mathbb{R}^d$. Note also that $\rho_n \uparrow e^{-1}$ on D as $n \to \infty$. Let

$$g_n = -\frac{1}{2} a_n \nabla f_n \quad \text{on } D. \tag{35}$$

As in section 3, for each n we define a function $\sigma_n : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that $\sigma_n \in W^{1,\infty}(\mathbb{R}^d)$, $\sigma_n(x)$ is symmetric and positive definite for each $x \in \mathbb{R}^d$, and σ_n satisfies $\sigma_n \sigma_n = a_n$. Let

$$b_n = \frac{1}{2 p_n} \nabla . (a_n p_n),$$

and let μ_n be the probability measure on D that has density $\gamma_n p_n \rho_n$ relative to d-dimensional Lebesgue measure, where

$$\gamma_n = \left(\int_D p_n \rho_n \, dx \right)^{-1}.$$

We note from the definition of p_n that $\gamma_n \to e$ as $n \to \infty$. One can check by dominated convergence that $\gamma_n \rho_n p_n \to p$ in $L^1(D)$ as $n \to \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which are defined a d-dimensional Brownian motion $\{W_t, t \geqq 0\}$ and an independent random vector Y^n taking values in D and having law μ_n. Consider the following stochastic differential equation in D:

$$\begin{cases} dX_t^n = (b_n(X_t^n) + g_n(X_t^n))\, dt + \sigma_n(X_t^n)\, dW_t, \\ X_0^n = Y^n. \end{cases} \tag{36}$$

Note that $|g_n(x)|$ may tend to $+\infty$ as $x \to \partial D$, and so in general (36) does not fall within the realm of the general existence and uniqueness theory for solutions of stochastic differential equations. Nevertheless, the following theorem which guarantees existence of a unique strong solution of (36) that lives forever in D is a consequence of Theorem 5.1 below.

THEOREM 4.1. – *The stochastic differential equation* (36) *has a unique strong solution* $\{X_t^n, t \geqq 0\}$. *Moreover,* X^n *does not exit* D *almost surely, and* X^n *is a stationary, symmetric Markov process with stationary measure* μ_n *and associated strongly continuous semigroup* $\{P_t^n, t \geqq 0\}$ *on* $L^2(D, \mu_n)$.

For each n, let $(\mathcal{E}^n, D(\mathcal{E}^n))$ be the Dirichlet form associated with the strongly continuous symmetric Markovian semigroup $\{P_t^n, t \geqq 0\}$ on $L^2(D, \mu_n)$.

LEMMA 4.2. – *We have* $\mathcal{D}(\mathcal{E}^n) = H^1(D, \mu_n)$ *and*

$$\mathcal{E}^n(g, g) = \frac{1}{2}\int_D \nabla g . a_n \nabla g \, d\mu_n \quad \textit{for all } g \in H^1(D, \mu_n). \tag{37}$$

Proof. – One can show as in section 3 that the strong domain of the infinitesimal generator A_n for $\{P_t^n, t \geqq 0\}$ contains $C_c^\infty(D)$ and that

$$A_n g = L_n g \qquad \text{for} \quad g \in C_c^\infty(D),$$

where $L_n g = \frac{1}{2 p_n \rho_n} \nabla . (a_n p_n \rho_n \nabla g)$. It then follows from Theorem 2.3 that $\mathcal{D}(\mathcal{E}^n) \subset H^1(D, \mu_n)$ and

$$\mathcal{E}^n(g, g) \geqq \frac{1}{2}\int_D \nabla g . a_n \nabla g \, d\mu_n \quad \text{for all } g \in \mathcal{D}(\mathcal{E}^n).$$

Thus it remains to show that $H^1(D, \mu_n) \subset \mathcal{D}(\mathcal{E}^n)$ and that the reverse of the above inequality holds for g in $H^1(D, \mu_n)$. This can be proved in the same manner as Lemma 3.5 of [29] with the observation that since a_n is bounded and uniformly elliptic, whenever (37) holds, the norm $\| . \|_{\mathcal{D}(\mathcal{E}^n)}$ associated with $\mathcal{E}^n$, defined by

$$\| g \|^2_{\mathcal{D}(\mathcal{E}^n)} = \int_D g^2 \, d\mu_n + \mathcal{E}^n(g, g)$$

is equivalent to the norm on $H^1(D, \mu_n)$ given by

$$\| g \|^2_{H^1(D, \mu_n)} = \int_D g^2 \, d\mu_n + \frac{1}{2}\int_D |\nabla g|^2 \, d\mu_n. \quad \square$$

Since we are only concerned with *stationary* Markov processes, from hereon we restrict attention to processes defined on the time interval [0, 1].

In a similar manner to that in section 3, we can show that any subsequence of the sequence $\{X^n, n \in \mathbb{N}\}$ has a weakly convergent subsequence, whose limit $\{X_t, t \in [0, 1]\}$, is a stationary symmetric process with initial law μ and continuous paths in $\bar{D}$. Define a family $\{P_t, 0 \leqq t \leqq 1\}$ of bounded linear operators on $L^2(D, \mu)$ by:

$$(P_t g)(X_0) = \mathbb{E}[g(X_t) \mid X_0], \qquad g \in L^2(D, \mu), \quad t \in [0, 1].$$

LEMMA 4.3. – *For any f, $g \in L^\infty(D)$ and $t \in [0, 1]$, we have*

(i) $\lim_{n \to \infty} \int_D |P_t^n g - P_t g|^2 d\mu = 0,$

(ii) $\int_D f P_t^n g \, d\mu_n \to \int_D f P_t g \, d\mu$ *as* $n \to \infty$.

Proof. – As in Lemma 3.5, it suffices to prove the result for f, $g \in C_b(D)$, and the weak convergence of $P_t^n g$ in $L^2(D, \mu)$ as well as (ii) is proved as there.

For the proof of strong convergence, the only difference from the situation in Lemma 3.5 is that $\tilde{b}$ is replaced by a sequence $\{\tilde{b}_n\}_{n=m}^\infty$ which agrees with $\{b_n + g_n\}_{n=m}^\infty$ on D_m, and which we choose to converge uniformly to $\tilde{b}$, where $\tilde{b} = \frac{1}{2p} \nabla . (ap)$ on D_m.

For each $\varepsilon > 0$, we need to be able to choose s such that

$$\sup_{n \geqq m} \mathbb{P}(\tau_n < s) \leqq \varepsilon.$$

However this follows from Chebyshev's inequality, since from elementary estimates for stochastic differential equations,

$$\sup_{n \geqq m} \mathbb{E}\left(\sup_{0 \leqq u \leqq s} |\tilde{X}_u^{n, x} - x|^2\right)$$

tends to zero as $s \to 0$.

Moreover the transition density $\tilde{p}_n(s, x, y)$ of $\tilde{X}^n$ depends on n, and in order to conclude we only need that

$$\tilde{p}_n(s, x, .) \to \tilde{p}(s, x, .)$$

in $L^2(D)$, which follows from Theorem 11.4.2 in Stroock-Varadhan [25]. □

We can now prove the following in a similar manner to the proof of Theorem 3.7.

THEOREM 4.4. – *The weak limit process* X *is a stationary symmetric continuous Markov process with stationary measure* μ *on* D *and associated strongly continuous semigroup* $\{P_t, 0 \leqq t \leqq 1\}$ *on* $L^2(D, \mu)$.

By a similar proof to that of Lemma 3.8, with the additional observation that $|\nabla f_n|$ goes to zero uniformly on any compact subset of D as $n \to \infty$, it follows that the infinitesimal generator of $\{P_t, 0 \leqq t \leqq 1\}$ is an extension of $(L, C_c^\infty(D))$.

Let $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ denote the Dirichlet form associated to X. As in section 3, $\mathscr{D}(\tilde{\mathscr{E}}) \subset \mathscr{D}(\mathscr{E})$ and

(38) $$\tilde{\mathscr{E}}(g, g) \geqq \mathscr{E}(g, g), \qquad g \in \mathscr{D}(\tilde{\mathscr{E}}).$$

THEOREM 4.5. — *Suppose that for each compact set* $K \subset \mathbb{R}^d$,
(A5) *a is uniformly elliptic on* $K \cap D$,
(A6) $\gamma_n(p_n \rho_n)(x) \leqq c_k p(x)$ *for all* $x \in D \cap K$, $n \in \mathbb{N}$, *for some constant* c_k,
Then, $(\mathscr{E}, \mathscr{D}(\mathscr{E})) = (\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$.

Proof. — By (38), it suffices to show $\mathscr{D}(\mathscr{E}) \subset \mathscr{D}(\tilde{\mathscr{E}})$ and for all $g \in \mathscr{D}(\mathscr{E})$,

$$\mathscr{E}(g, g) \geqq \tilde{\mathscr{E}}(g, g). \tag{39}$$

Since $\mathscr{D}(\tilde{\mathscr{E}})$ is complete with respect to the norm $\|\cdot\|_{\mathscr{D}(\tilde{\mathscr{E}})}$ defined by

$$\|g\|^2_{\mathscr{D}(\tilde{\mathscr{E}})} = (g, g)_{L^2(D, \mu)} + \tilde{\mathscr{E}}(g, g) \quad \text{for all } g \in \mathscr{D}(\tilde{\mathscr{E}}), \tag{40}$$

and (as we will show below).

$$\Lambda \equiv \{ g \in \mathscr{D}(\mathscr{E}) \cap L^\infty(D, \mu) : g \text{ is zero outside } K \cap D \text{ for some compact set } K \text{ (depending on } g\text{) in } \mathbb{R}^d \}$$

is dense in $\mathscr{D}(\mathscr{E})$ with norm $\|\cdot\|_{\mathscr{D}(\mathscr{E})}$, it suffices to show that $\Lambda \subset \mathscr{D}(\tilde{\mathscr{E}})$ and (39) holds there.

To see that Λ is dense in $\mathscr{D}(\mathscr{E})$, let $g \in \mathscr{D}(\mathscr{E})$ and $\Phi_n \in C_b^1(\mathbb{R})$ such that $0 \leqq \Phi_n' \leqq 1$,

$$\Phi_n(y) = \begin{cases} y & \text{for } |y| \leqq n, \\ (n+1)\operatorname{sgn}(y) & \text{for } |y| \geqq n+2. \end{cases}$$

Further, let $\Psi_n \in C_c^\infty(\mathbb{R}^d)$ such that $|\Psi_n(x)| \in [0, 1]$, $|\nabla \Psi_n(x)| \leqq 1$ for all $x \in \mathbb{R}^d$, and

$$\Psi_n(x) = \begin{cases} 1 & \text{for } |x| \leqq n \\ 0 & \text{for } |x| \geqq n+2. \end{cases}$$

Define $h_n = \Psi_n \Phi_n(g)$ and observe that $h_n \in \Lambda$. Moreover, for

$$\Gamma_n \equiv \{ x \in D : |g(x)| \geqq n \} \cup \{ x \in D : |x| \geqq n \},$$

$$\begin{aligned} \|g - h_n\|^2_{\mathscr{D}(\mathscr{E})} &= (g - h_n, g - h_n)_{L^2(D, \mu)} + \mathscr{E}(g - h_n, g - h_n) \\ &\leqq \int_{\Gamma_n} |g - \Psi_n \Phi_n(g)|^2 d\mu \\ &\quad + 2 \int_{\Gamma_n} \{ \nabla g (1 - \Psi_n \Phi_n'(g)) . a \nabla g (1 - \Psi_n \Phi_n'(g)) \\ &\quad + (\nabla \Psi_n \Phi_n(g)) . a (\nabla \Psi_n \Phi_n(g)) \} d\mu \\ &\leqq 4 \int_{\Gamma_n} |g|^2 d\mu + 8 \int_{\Gamma_n} \nabla g . a \nabla g \, d\mu + 2 \int_{\Gamma_n} \|a\|_\infty g^2 d\mu, \end{aligned}$$

where $\|a\|_\infty = \max_{x \in D} \left(\sum_{i, j=1}^d |a_{ij}(x)|^2 \right)^{1/2}$. Since $g \in \mathscr{D}(\mathscr{E})$, the last line above tends to zero as $n \to \infty$, and it follows that Λ is dense in $\mathscr{D}(\mathscr{E})$ with the norm $\|\cdot\|_{\mathscr{D}(\mathscr{E})}$.

We now verify that $\Lambda \subset \mathcal{D}(\tilde{\mathcal{E}})$ and (39) holds for $g \in \Lambda$. For $g \in \Lambda$, since a is uniformly elliptic on the support of g, there is $\lambda_g > 0$ such that

$$\lambda_g \int_D |\nabla g|^2 \, d\mu \leqq \int_D \nabla g \,.\, a \nabla g \, d\mu$$

and so $\nabla g \in L^2(D, \mu)$, and then by assumption (A6), $\nabla g \in L^2(D, \mu_n)$ and also $g \in L^2(D, \mu_n)$, hence $g \in \mathcal{D}(\mathcal{E}^n)$. By Lemma 1.3.4 of [9] and Lemma 4.2, for each $t > 0$,

$$t^{-1} \int_D g(g - P_t^n g) \, d\mu_n \leqq \mathcal{E}^n(g, g) = \frac{1}{2} \int_D \nabla g \,.\, a_n \nabla g \, d\mu_n. \tag{41}$$

It follows form Lemma 4.3 (ii) that the left member of (41) tends as $n \to \infty$ to $t^{-1} \int_D g(g - P_t g) \, d\mu$.

Now a.e. on D, $(\nabla g \,.\, a_n \nabla g) \gamma_n p_n \rho_n \to (\nabla g \,.\, a \nabla g) p$ as $n \to \infty$, where the sequence on the left is dominated by $(\| a \|_\infty + 1) |\nabla g|^2 c_g p \in L^1(D)$ and by (A6), c_g is a constant depending on g. Thus, on letting $n \to \infty$ in (41) we obtain

$$t^{-1} \int_D g(g - P_t g) \, d\mu \leqq \frac{1}{2} \int_D \nabla g \,.\, a \nabla g \, d\mu < \infty. \tag{42}$$

Invoking Lemma 1.3.4 of [9] again, we conclude that $g \in \mathcal{D}(\tilde{\mathcal{E}})$ and $\tilde{\mathcal{E}}(g, g) \leqq \frac{1}{2} \int_D \nabla g \,.\, a \nabla g \, d\mu$, as desired. □

Remark 4.6. – An examination of the above proof shows that all we really need for the result of Theorem 4.5 is that (*a*) there be a common set of functions that is in all of the $\mathcal{D}(\mathcal{E}^n)$ and is dense in $\mathcal{D}(\mathcal{E})$, and (*b*) one can pass to the limit as $n \to \infty$ in (41) to get (42). The reader may verify that another sufficient condition that can be used in place of conditions (A5)-(A6) is that $W^{1,\infty}(D)$ is dense in $\mathcal{D}(\mathcal{E})$ with the norm $\| \cdot \|_{\mathcal{D}(\mathcal{E})}$.

Remark 4.7. – Condition (A6) is really quite weak. It is always satisfied if p is uniformly bounded away from zero on $D \cap K$ for each compact set $K \subset \mathbb{R}^d$. Moreover, since p is bounded away from zero on compact subsets of D, (A6) only puts a lower bound on how rapidly p may go to zero at the boundary of D. In fact, one could even modify our choices of p_n and ρ_n to still keep the same qualitative features but to accommodate more densities p that go to zero at ∂D at a "reasonable rate".

Remark 4.8. – Under the result of Theorem 4.5, any subsequence of $\{X^n\}$ has a further subsequence that is weakly convergent to X, the symmetric Markov process with strongly continuous Markovian semigroup associated with the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, and so it follows that $\{X^n\}$ converges weakly to X. Without the identification provided by Theorem 4.5, we simply know that any subsequence of $\{X^n\}$ has a weakly convergent subsequence whose weak limit is a stationary symmetric Markov process with strongly continuous Markovian semigroup and associated Dirichlet form $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ satisfying

$$C_c^\infty(D) \subset \mathscr{D}(\tilde{\mathscr{E}}) \subset \mathscr{D}(\mathscr{E}), \tilde{\mathscr{E}}(g, g) = (g, -Lg) \text{ for all } g \in C_c^\infty(D),$$

and $\tilde{\mathscr{E}}(g, g) \geqq \mathscr{E}(g, g)$ for all $g \in \mathscr{D}(\tilde{\mathscr{E}})$.

Remark 4.9. — Other approximations from the interior are possible. If we choose the D_n's to have smooth boundaries, we could choose X^n to be a diffusion reflected at the boundary of D_n, as constructed for example in Lions-Sznitman [17]. (Indeed, this is the kind of approximation used by Chen [6] in his contemporaneous, independent work on the existence of Skorokhod-like decompositions for reflecting diffusion processes with non-degenerate a and p.) Alternatively, one could choose the drift of X^n so as to keep this process inside D_n. If $\delta_n(x)$ denotes the distance from $x \in \mathbb{R}^d$ to D_n^c and $f_n = \exp(1/(n\delta_n))$ on D_n, then since ∂D_n is smooth, proving that the additional drift

$$g_n = -\frac{1}{2} a_n \nabla f_n,$$

maintains X^n inside D_n is not too hard to show using f_n as a Lyapounov function, and does not require the involved argument in section 5. However, we feel that our approximation is more natural, since our X^n lives in all of D.

5. CONSTRUCTION OF A DIFFUSION WITH SINGULAR DRIFT

In this section, we carry out the construction of a diffusion with singular drift, which is needed in section 4 for the definition of the approximation from the interior.

Here D will denote the domain in $\mathbb{R}^d$ where the coefficients for our stochastic differential equation are defined. The drift will be only locally bounded in D, and we will need to show that the exit time of the diffusion from D is a.s. infinite. In fact, we shall do that for a stationary and symmetric Markov process, and we shall show that it never hits the "set of nodes", *i.e.*, the set where its stationary density vanishes. We deduce from this the desired result for all starting points x in D of the diffusion. Our result generalizes earlier results of Carlen [4] and Zheng [30] who assume that the diffusion coefficient is the identity matrix. In [10], Fukushima considers a Dirichlet form (1) with $D=\mathbb{R}^d$, uniformly elliptic C^∞ diffusion coefficient a, and density p that may degenerate to zero on a set of points having zero Lebesgue measure. Assuming p is locally bounded on $\mathbb{R}^d$ and satisfies a local finite energy condition, Fukushima uses a potential theoretic method to show that the zero set of p is not reached by the process associated with his Dirichlet form. His proof might be adapted to establish our result in most cases. (We assume less regularity of a, though more of p.) Nevertheless, we believe our method is of independent interest, being more sample path oriented, along the lines of the work of Zheng [30], Meyer-Zheng [20], and Norris [21]. Here we adapt the approach of the latter to our situation.

The following assumptions are adopted throughout this section.

(A1′) $a\equiv(a_{ij})\in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d)$ is symmetric and uniformly elliptic on $\mathbb{R}^d$;

(A2′) $p\in W^{1,\infty}(\mathbb{R}^d)$, $p>0$ on $\mathbb{R}^d$, $p^{-1}|\nabla p|\in L^\infty(\mathbb{R}^d)$, and $\int_{\mathbb{R}^d} p\,dx<\infty$.

We also assume as given a function ρ such that

(A7) (i) $\rho\in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+)$,

(ii) $D\triangleq\{x\in\mathbb{R}^d:\rho(x)>0\}$ is connected,

(iii) $\rho^{-1}|\nabla\rho|^2 p\in L^1(D)$,

(iv) $\rho(x)\leqq c(d(x, D^c))^2$ for all $x\in\mathbb{R}^d$, for some $c>0$,

(v) $\int_D \rho(x)p(x)\,dx=1$

We denote by μ the measure on D defined by $\mu(dx)=\rho(x)p(x)\,dx$. Define

$$b=\frac{1}{2p}\nabla.(ap),\qquad g=\frac{1}{2\rho}a\nabla\rho,$$

so that $b+g=\dfrac{1}{2p\rho}\nabla.(ap\rho)$. Note that (A1′) and (A2′) imply that b is bounded. Let $\sigma\in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d)$ be symmetric, positive definite and such that $\sigma\sigma=a$.

We want to study the equation:

$$dX_t=(b(X_t)+g(X_t))\,dt+\sigma(X_t)\,dW_t, \tag{43}$$

where $\{W_t, t\geqq 0\}$ is a given d-dimensional standard Brownian motion defined on a probablity space $(\Omega, \mathcal{F}, \mathbb{P})$.

The aim of this section is to prove:

THEOREM 5.1. — *For any* $x \in D$, *equation* (43) *has a unique strong solution starting from* x, *which* $\mathbb{P}$-*a.s. does not exit* D *in finite time.*

COROLLARY 5.2. — *If* Y *is a* D-*valued random vector with law* μ *and is independent of* $\{W_t, t \geqq 0\}$, *then the solution of* (43) *with* $X_0 = Y$ *is a stationary symmetric Markov process with an associated strongly continuous semigroup* $\{P_t, t \geqq 0\}$ *on* $L^2(D, \mu)$.

We begin with the:

Proof of Theorem 5.1. *Step* 1: *Local existence and uniqueness.* — In this step, we prove existence and uniqueness up to the exit time of the solution from D, using rather standard arguments.

Let $\{D_n\}_{n=1}^{\infty}$ denote an increasing sequence of open subsets of D, as defined in Lemma 3.2, and for each n, let g_n be a bounded and measurable mapping from $\mathbb{R}^d$ into itself, which coincides with g on D_n. For each n, we can apply Veretennikov's result [27] to equation (43) with g replaced by g_n and initial condition $X_0 = x$, yielding a unique strong solution $\{X_t^n, t \geqq 0\}$. Let $S_n = \inf\{t \geqq 0 : X_t^n \notin D_n\}$. We deduce from standard arguments the existence of a unique solution $\{X_t, 0 \leqq t < S\}$ of equation (43) starting from x, where $S = \lim_n S_n$, and $X_t = X_t^n$ for $0 \leqq t \leqq S_n$. Since we may consider D as the state space for the process $\{X_t, 0 \leqq t < S\}$, in the following we shall call S the explosion time. (Readers concerned about X being defined on a stochastic interval may define $X_t = \partial$ for $S \leqq t < \infty$, where ∂ is a cemetery point isolated from $\mathbb{R}^d$. This defines a strong Markov process with paths in $D \cup \{\partial\}$.)

Step 2: *A sufficient condition for the explosion time to be a.s. infinite.* — We now show that provided the solution of (43) does not explode (*i.e.* exit D) in finite time a.s. when initialized with an initial law ν on D with respect to which Lebesgue measure on D is absolutely continuous, then for any x in D, the solution of (43) starting from x does not explode a.s. in finite time.

Let $\{X_t, 0 \leqq t < S\}$ be a solution of equation (43) with initial law ν.

We assume that

$$\mathbb{P}(S < \infty) = 0, \tag{44}$$

and we claim that (44) implies that

$$\mathbb{P}(S < \infty \mid X_0 = x) = 0, \qquad \forall x \in D.$$

Indeed, we first note that

$$\mathbb{P}(S < \infty) = \int_D \mathbb{P}(S < \infty \mid X_0 = x)\, \nu(dx),$$

hence

(45) $$\mathbb{P}(S<\infty \mid X_0=x)=0,$$

for ν-a.e. x in D, and therefore for Lebesgue a.e. x in D. Suppose now that there exists $x \in D$ such that $\mathbb{P}(S<\infty \mid X_0=x)>0$. Then there must exist $t>0$ and $n \in \mathbb{N}$ such that

$$\mathbb{P}(t<S_n<S<\infty \mid X_0=x)>0.$$

However

$$\begin{aligned}
&\mathbb{P}(t<S_n<S<\infty \mid X_0=x)\\
&\quad =\mathbb{E}[\mathbb{P}(S<\infty \mid X_t)\, 1_{\{t<S_n\}} \mid X_0=x]\\
&\quad =\mathbb{E}[\mathbb{P}(S<\infty \mid X_t=X_t^n)\, 1_{\{t<S_n\}} \mid X_0^n=x]\\
&\quad \leqq \mathbb{E}[\mathbb{P}(S<\infty \mid X_t=X_t^n)\, 1_{D_n}(X_t^n) \mid X_0^n=x]\\
&\quad =\int_{D_n} \mathbb{P}(S<\infty \mid X_t=y)\, p_t^n(x, y)\, dy,
\end{aligned}$$

where $p_t^n(x, .)$ denotes the transition probability density of the process $\{X_t^n\}$ (starting from x), which exists from Lemma 9.2.2 of Stroock-Varadhan [25]. Now for any y in D_n,

$$\mathbb{P}(S<\infty \mid X_t=y)=\mathbb{P}(S<\infty \mid X_0=y)$$

and the assumption made upon x implies that there is a set of positive Lebesgue measure in D_n such that for all y in that set, $\mathbb{P}(S<\infty \mid X_t=y)>0$, which contradicts the above equality and (45).

Step 3: When initialized with the law μ, *the solution of equation* (43) *does not explode a.s.* – This is the main and last step of the proof. We shall exploit the symmetry of the solution of (43) when initialized with μ (this symmetry is formally recognized in Corollary 5.2), while adapting ideas from Norris [21].

Let us first consider the equation (43) without the unbounded drift g. We shall put it back using a Girsanov transformation a little later.

Given a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{Q})$ on which are defined a d-dimensional Brownian motion $\{V_t, t \geqq 0\}$ (which is also supposed to be an $\{\mathscr{F}_t\}$-martingale) and an $\mathscr{F}_0$-measurable random vector X_0 with law $p(x)\,dx$ on $\mathbb{R}^d$, there exists, again from Veretenikov's result, a unique strong solution $\{X_t, t \geqq 0\}$ of the equation:

(46) $$X_t=X_0+\int_0^t b(X_s)\, ds+\int_0^t \sigma(X_s)\, dV_s, \qquad t \geqq 0.$$

Moreover, by the arguments in the proof of Theorem 3.3, $\{X_t, t \geqq 0\}$ is a stationary symmetric diffusion process.

We let

$$S \overset{\Delta}{=} \inf\{t \geqq 0 : X_t \notin D\}$$

and for $k=1, 2, \ldots$

$$T_k \overset{\Delta}{=} \inf\left\{t \geqq 0 : \int_0^t h(X_s)\,ds \geqq k\right\}$$

where

$$h = \left(\rho^{-2}(a\nabla\rho, \nabla\rho) + \frac{\alpha}{p\rho}\right) 1_D$$

with α some continuous, bounded, and everywhere strictly positive probability density on $\mathbb{R}^d$.

We claim that:

$$\int_0^t h(X_s)\,ds \uparrow +\infty, \text{ as } t \uparrow S\ \mathbb{Q}\text{-a.s. on } \{0<S<\infty\}. \tag{47}$$

For the proof of this, note that $\mathbb{Q}$-a.s. on $\{0<S<\infty\}$, $X_0 \in D$ and $X_S \in D^c$, hence $\rho(X_S)=0$. By a Girsanov transformation and a time change argument using the boundedness and uniform ellipticity of a, one can deduce a Lévy-type bound on the modulus of continuity of X, from which it follows that $\mathbb{Q}$-a.s., for each $t \geqq 0$ there exists c and $\varepsilon > 0$ (ε may depend on t and ω) such that, whenever $|t-s| \leqq \varepsilon$,

$$|X_s - X_t|^2 \leqq c\,|t-s| \log\left(\frac{1}{|t-s|}\right).$$

Hence from (A7) (iv), there exists a random variable ε such that $\varepsilon > 0$ $\mathbb{Q}$-a.s. on $\{0<S<\infty\}$, and a constant $\bar{c}$ such that

$$\rho(X_s) \leqq \bar{c}\,|S-s| \log\left(\frac{1}{|S-s|}\right)$$

on the set $\{0<S<\infty\} \cap \{|S-s| \leqq \varepsilon\}$. Now if $0 \leqq s < S$, $X_s \in D$, hence

$$h(X_s) \geqq \|p\|_\infty^{-1} \frac{\alpha(X_s)}{\rho(X_s)}$$
$$\geqq \|p\|_\infty^{-1} \frac{\alpha(X_s)}{\bar{c}\,|S-s| \log(1/|S-s|)}$$

for $(S-\varepsilon)^+ \leqq s < S < \infty$. Now on this random time interval, X_s lies in a compact set (depending on ω), hence $\alpha(X_s)$ is bounded away from zero on the same random time interval. But

$$\int_{0^+} \left(u \log\left(\frac{1}{u}\right)\right)^{-1} du = +\infty,$$

hence (47) is established. From (47) and the definition of T_k, we have that

$$\int_0^{T_k} h(X_s)\, ds \leqq k,\ T_k \leqq S \qquad \mathbb{Q}\text{-a.s. on } \{0 < S\} \tag{48}$$

and the second inequality is $\mathbb{Q}$-a.s. strict on $\{0 < S < \infty\}$.

We now define

$$Z_t = \exp\left\{\frac{1}{2}\int_0^t (\rho^{-1}\sigma\nabla\rho)(X_s) . dV_s - \frac{1}{8}\int_0^t (\rho^{-2}(a\nabla\rho, \nabla\rho)(X_s))\, ds\right\}, \qquad t \geqq 0,$$

with the convention that $Z_t = 0$ if $X_s \in D^c$ for some $0 \leqq s \leqq t$. From (48) and the fact that $\rho^{-2}(a\nabla\rho, \nabla\rho) \leqq h$ on D, we have

$$\int_0^{T_k} (\rho^{-2}(a\nabla\rho, \nabla\rho))(X_s)\, ds \leqq k, \quad \mathbb{Q}\text{-a.s. on } \{0 < S\}.$$

Hence by standard arguments (*see* e.g. [14], p. 198), $\{Z_{t \wedge T_k}, t \geqq 0\}$ is an L^2-bounded martingale for each $k \in \mathbb{N}$. Let

$$Z_{T_k} = \lim_{t \to \infty} Z_{t \wedge T_k}$$

and define $\mathbb{P}_k$ on $(\Omega, \mathscr{F}_{T_k})$ by

$$\frac{d\mathbb{P}_k}{d\mathbb{Q}} = \rho(X_0) Z_{T_k}.$$

By Girsanov's theorem, under $\mathbb{P}_k$,

$$V_{t \wedge T_k} - \frac{1}{2}\int_0^{t \wedge T_k} (\rho^{-1}\sigma\nabla\rho)(X_s)\, ds, \qquad t \geqq 0, \tag{49}$$

is a Brownian motion stopped at T_k.

Now fix $t > 0$. For $0 \leqq s \leqq t$, define

$$\hat{V}_s = V_t - V_s + \int_s^t (p^{-1}\nabla . (\sigma p))(X_u)\, du,$$
$$\mathscr{G}^s = \sigma\{V_u - V_s;\ s \leqq u \leqq t\}^{\sim}$$

where the last superscript $\sim$ denotes completion with the $\mathbb{Q}$-null sets in $\mathscr{F}$,

$$\mathscr{H}^s = \mathscr{G}^s \vee \sigma(X_t).$$

Then, under $\mathbb{Q}$, $\{\hat{V}_s: 0 \leqq s \leqq t\}$ is a Brownian motion that is a martingale with respect to the backward filtration $\{\mathcal{H}^s: s \in [0, t]\}$ and

$$(50) \qquad X_s - X_t = \int_s^t b(X_u)\,du + \int_s^t \sigma(X_u) * d\hat{V}_u \qquad \text{for} \quad 0 \leqq s \leqq t,$$

where the last integral is a "backward Itô integral" defined as follows:

$$\int_s^t \sigma(X_u) * d\hat{V}_u = \lim_{j \to \infty} \sum_{i=0}^{j-1} \sigma(X_{t^j_{i+1}})(\hat{V}_{t^j_{i+1}} - \hat{V}_{t^j_i})$$

where $s = t^j_0 < t^j_1 < \ldots < t^j_j = t$ and $\sup_{i=0}^{j-1}(t^j_{i+1} - t^j_i) \to 0$ as $j \to \infty$. The claim of the last sentence is made in Theorem 2.2 and Corollary 2.4 of Pardoux [22], with the same minor adaptation as in the proof of our Theorem 3.3.

We now introduce, for each $s \in [0, t]$,

$$\hat{Z}_{st} = \exp\left\{\frac{1}{2}\int_s^t (\rho^{-1}\sigma\nabla\rho)'(X_u) * d\hat{V}_u - \frac{1}{8}\int_s^t (\rho^{-2}(a\nabla\rho, \nabla\rho))(X_u)\,du\right\}$$

with the convention that $\hat{Z}_{st} = 0$ if $X_u \in D^c$ for some $u \in [s, t]$. Let $\hat{Z}_t = \hat{Z}_{0\,t}$. We now claim that

$$(51) \qquad \mathbb{E}^{\mathbb{Q}}[\hat{Z}_t \mid X_t] \leqq 1 \quad \mathbb{Q}\text{-a.s.}$$

To see this, let

$$\hat{S}_k = \sup\left\{s \leqq t : \int_s^t (\rho^{-2}(a\nabla\rho, \nabla\rho))(X_u)\,du \geqq k \ \text{ or } \ X_s \in D^c_k\right\},$$

where the supremum of an empty set is taken to be zero. Then since $\hat{V}$ is a Brownian motion martingale with respect to $\{\mathcal{H}^s: s \in [0, t]\}$, we have

$$(52) \qquad \mathbb{E}^{\mathbb{Q}}[\hat{Z}_{\hat{S}_k \wedge t} \mid X_t] = 1.$$

By considering the cases where $X_s \in D$ for all $s \in [0, t]$, and the alternative, separately, we conclude that

$$\hat{Z}_t \leqq \underline{\lim}_k \hat{Z}_{\hat{S}_k \wedge t}.$$

Then (51) follows from (52) by Fatou's Lemma.

We will now prove that for each $t \geqq 0$,

$$(53) \qquad \rho(X_0) Z_t = \hat{Z}_t \rho(X_t) \quad \mathbb{Q}\text{-a.s.}$$

Since (53) is trivially satisfied on the set $\{S \leqq t\}$ from the definitions of Z_t and $\hat{Z}_t$, by comparing the exponents in (53), we see that it suffices to show that $\mathbb{Q}$-a.s. on the set $\{S>t\}$,

$$\log\rho(X_t)+\frac{1}{2}\int_0^t(\rho^{-1}\sigma\nabla\rho)'(X_s)*d\hat{V}_s$$
$$=\log\rho(X_0)+\frac{1}{2}\int_0^t(\rho^{-1}\sigma\nabla\rho)(X_s).dV_s.$$

Now we have roughly speaking the following two facts:

(*a*) the "forward" Itô formula applied to X_t satisfying (46) yields that on $\{S>t\}$

$$\begin{aligned}(54)\qquad &\log\rho(X_t)-\log\rho(X_0)\\ &=\int_0^t(\rho^{-1}b.\nabla\rho)(X_s)\,ds+\int_0^t(\rho^{-1}\sigma\nabla\rho)(X_s).dV_s\\ &+\frac{1}{2}\sum_{i,j=1}^d\int_0^t\left(a_{ij}\frac{\partial^2\log\rho}{\partial x_i\partial x_j}\right)(X_s)\,ds,\end{aligned}$$

and

(*b*) the "backward" Itô formula applied to X_t satisfying (50) with $s=0$ yields that on $\{S>t\}$

$$\begin{aligned}(55)\qquad \log\rho(X_0)-\log\rho(X_t)=&\int_0^t(\rho^{-1}b.\nabla\rho)(X_s)\,ds\\ &+\int_0^t(\rho^{-1}\nabla\rho\sigma)(X_s)*d\hat{V}_s\\ &+\frac{1}{2}\sum_{i,j=1}^d\int_0^t\left(a_{ij}\frac{\partial^2\log\rho}{\partial x_i\partial x_j}\right)(X_s)\,ds.\end{aligned}$$

Note that (54) [resp. (55)] is not obtained by a direct application of Itô's formula since $\log\rho\notin C^2(\mathbb{R}^d)$, but rather as follows. For any n, there exists a function $g_n\in C^2(\mathbb{R}^d)$ which coincides with $\log\rho$ on D_n. We apply the forward (resp. backward) Itô formula to g_n. This gives (54) [resp. (55)] on the set $\{S_n>t\}$ for each n, hence on the set $\{S>t\}=\bigcup_n\{S_n>t\}$, where $S_n=\inf\{t\geqq 0:X_t\notin D_n\}$.

Subtracting (55) from (54) yields the desired result.

Letting $\mathbb{E}_k$ denote expectation under $\mathbb{P}_k$, we can now conclude that for $t \geqq 0$ fixed,

$$
\begin{aligned}
k\,\mathbb{P}_k(\mathrm{T}_k \leqq t) &\leqq \mathbb{E}_k\left[\int_0^{\mathrm{T}_k \wedge t} h(\mathrm{X}_s)\,ds\right] \\
&= \int_0^t \mathbb{E}_k[h(\mathrm{X}_s)\,1_{\{s \leqq \mathrm{T}_k\}}]\,ds \\
&= \int_0^t \mathbb{E}^{\mathbb{Q}}[h(\mathrm{X}_s)\,\rho(\mathrm{X}_0)\,\mathrm{Z}_{s \wedge \mathrm{T}_k}\,1_{\{s \leqq \mathrm{T}_k\}}]\,ds \\
&\leqq \int_0^t \mathbb{E}^{\mathbb{Q}}[h(\mathrm{X}_s)\,\rho(\mathrm{X}_0)\,\mathrm{Z}_s]\,ds \\
&= \int_0^t \mathbb{E}^{\mathbb{Q}}[h(\mathrm{X}_s)\,\rho(\mathrm{X}_s)\,\hat{\mathrm{Z}}_s]\,ds, \text{ by (53)} \\
&\leqq \int_0^t \mathbb{E}^{\mathbb{Q}}[h(\mathrm{X}_s)\,\rho(\mathrm{X}_s)]\,ds, \text{ by (51)} \\
&= t\left\{\int_{\mathrm{D}} (\rho^{-1}(a\nabla\rho, \nabla\rho))(x)\,p(x)\,dx + \int_{\mathrm{D}} \alpha(x)\,dx\right\}, \\
&\qquad \text{by the stationarity of X,} \\
&< \infty,
\end{aligned}
$$

from (A1′), (A7) (iii).

This shows that for each $t \geqq 0$,

(56) $$\mathbb{P}_k(\mathrm{T}_k \leqq t) \to 0 \quad \text{as } k \to \infty.$$

For each $t \geqq 0$, $\mathrm{A} \in \mathscr{F}_t$, define

$$\mathbb{P}(\mathrm{A}) = \lim_{k \to \infty} \mathbb{P}_k(\mathrm{A} \cap \{\mathrm{T}_k > t\}).$$

It follows from (56) that this defines a probability measure on $(\Omega, \bigcup_t \mathscr{F}_t)$ (*see* Stroock-Varadhan [25], p. 35). Then $\mathrm{T} \triangleq \lim_k \mathrm{T}_k$ satisfies

(57) $$\mathbb{P}(\mathrm{T} = +\infty) = 1.$$

Now let

$$\mathrm{W}_t = \begin{cases} \mathrm{V}_t - \dfrac{1}{2}\displaystyle\int_0^t (\rho^{-1}\sigma\nabla\rho)(\mathrm{X}_s)\,ds & \text{for all } t \geqq 0, \text{ on } \{\mathrm{T} = \infty\}; \\ 0 \quad \text{for all } t \geqq 0, \text{ on } \{\mathrm{T} < \infty\}. \end{cases}$$

From (49), $\{W_{t \wedge T_k}, t \geqq 0\}$ is a continuous $\mathbb{P}$-martingale with mutual variation

$$\langle W^i_{\cdot \wedge T_k}, W^j_{\cdot \wedge T_k} \rangle_t = \delta_{ij}(t \wedge T_k), \qquad t \geqq 0, \quad i, j = , \ldots, d.$$

Consequently $\{W_t, t \geqq 0\}$ is a Brownian motion under $\mathbb{P}$, and $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t (b+g)(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \qquad t \geqq 0, \tag{58}$$

where X_0 has distribution μ under $\mathbb{P}$. The fact that X does not exit D $\mathbb{P}$-a.s. follows from (48), (57) and the fact $\mathbb{P}(X_0 \in D) = 1$. □

We can now proceed with the:

Proof of Corollary 5.2. – We only need to prove stationarity and symmetry, since strong continuity of the semigroup follows in a similar manner to that in the proof of Theorem 3.3. We shall use the notations from the proof of Theorem 5.1.

We first note that for $A \in \mathscr{F}_t$,

$$\begin{aligned} \mathbb{P}(A) &= \lim_{k \to \infty} \mathbb{P}_k(A \cap \{T_k > t\}) \\ &= \lim_{k \to \infty} \mathbb{E}^{\mathbb{Q}}(\rho(X_0) Z_t;\, A \cap \{T_k > t\}) \\ &= \mathbb{E}^{\mathbb{Q}}(\rho(X_0) Z_t;\, A) \end{aligned}$$

by monotone convergence, since $Z_t = 0$ $\mathbb{Q}$-a.s. on $(\bigcup_k \{T_k > t\})^c$. Hence $\mathbb{P}|_{\mathscr{F}_t} \ll \mathbb{Q}|_{\mathscr{F}_t}$, and

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_t} = \rho(X_0) Z_t. \tag{59}$$

Let now $\phi, \psi \in C_b(D)$. From (59), (53), we deduce

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\phi(X_0)\psi(X_t)] &= \mathbb{E}^{\mathbb{Q}}[\phi(X_0)\psi(X_t)\rho(X_0) Z_t] \\ &= \mathbb{E}^{\mathbb{Q}}[\phi(X_0)\psi(X_t)\rho(X_t)\hat{Z}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[\phi(\bar{X}_t)\psi(\bar{X}_0)\rho(\bar{X}_0)\hat{Z}_t] \end{aligned} \tag{60}$$

where $\bar{X}_s = X_{t-s}$, $0 \leqq s \leqq t$. Now we rewrite (50) as

$$\bar{X}_s = X_t + \int_0^s b(\bar{X}_r)\, dr + \int_0^s \sigma(\bar{X}_r)\, d\bar{V}_r, \qquad 0 \leqq s \leqq t,$$

with $\bar{V}_\cdot = -\hat{V}_{t-\cdot}$, and also

$$\hat{Z}_t = \exp\left\{\frac{1}{2}\int_0^t (\rho^{-1}\sigma\nabla\rho)(\bar{X}_s) \,.\, d\bar{V}_s - \frac{1}{8}\int_0^t (\rho^{-2}(a\nabla\rho, \nabla\rho))(\bar{X}_s)\, ds\right\},$$

i.e., $\hat{Z}_t$ is the same functional of $(\bar{X}, \bar{V})$ as Z_t is of (X, V). Now, under $\mathbb{Q}$, V and $\bar{V}$ are both standard Wiener processes and X (resp. $\bar{X}$) is a strong solution of a stochastic differential equation with coefficients (b, σ) driven by V (resp. by $\bar{V}$) and with initial condition X_0 independent of V (resp. X_t independent of $\bar{V}$), where X_0 and X_t have the same law. Hence, under $\mathbb{Q}$, the law of (X, V) equals the law of $(\bar{X}, \bar{V})$, and the last line of (60) equals

$$\mathbb{E}^{\mathbb{Q}}[\phi(X_t)\psi(X_0)\rho(X_0)Z_t] = \mathbb{E}^{\mathbb{P}}[\phi(X_t)\psi(X_0)].$$

We have proved that

$$\mathbb{E}^{\mathbb{P}}[\phi(X_0)\psi(X_t)] = \mathbb{E}^{\mathbb{P}}[\phi(X_t)\psi(X_0)],$$

which establishes the symmetry property, as well as stationarity (choose $\psi \equiv 1$). □

6. SEMIMARTINGALE REPRESENTATION

In this section we give sufficient conditions for the Markov process X constructed in section 4 to be a semimartingale and we give a form for the decomposition in this case. For this, in addition to conditions on D, we need to impose more stringent assumptions on a and p than previously encountered. Accordingly, we shall assume henceforth that a and p have extensions to all of $\mathbb{R}^d$ (again denoted by a and p) such that

(A1″) $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $a \in C^1(\mathbb{R}^d)$, a is symmetric, bounded and locally elliptic on $\mathbb{R}^d$,

(A2″) $p : \mathbb{R}^d \to \mathbb{R}$, $p \in C^1(\mathbb{R}^d)$, $p > 0$ on D and $\int_D p(x)\,dx = 1$.

In this case, we could have taken simpler modifications a_n, p_n of a, p than those defined at the beginning of section 4. In particular, suppose that for each n, $B_n \equiv \{x \in \mathbb{R}^d : |x| < n\}$, $D_n \equiv D \cap B_n$, $\phi_n \in C^\infty(\mathbb{R}^d)$ such that $0 \leqq \phi_n \leqq 1$ and

$$\phi_n = \begin{cases} 1 & \text{on } \bar{B}_n, \\ 0 & \text{on } B_{n+1}^c. \end{cases}$$

Define $\psi_n = 1 - \phi_n$ and let a_n, p_n be defined by equations (11)-(12) with the ϕ_n, ψ_n of this section and with $k_n = 1$, $q_n = 1$ (as in section 4). Then the proofs of section 4 go through with only minor modification using these (D_n, a_n, p_n). One point to note is that sometimes the D_n are used for two purposes, firstly as the sets where $a_n = a$ and $b_n = b$, and secondly as convenient sets whose closures are compact sets in D which expand to fill all of D. For this second use, one can use other bounded open sets with compact closures in D that cover all of D.

Assumption. — We shall assume in this section that $\{X^n\}$ is the sequence of processes as defined in section 4 but with D_n, a_n, p_n as defined above. Moreover, conditions (A5)-(A6) of Theorem 4.5 will be assumed to hold, so that $\{X^n\}$ converges weakly to X, the process associated with the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Using conditions of Meyer-Zheng [19] for convergence of semi-martingales to semimartingales, we will identify conditions under which X is a semimartingale. In the following, B_m is defined as at the beginning of this section and

$$b \equiv \frac{1}{2p} \nabla . (ap).$$

Observe that by the assumptions (A1″) and (A2″) on a and p, b is continuous.

THEOREM 6.1. — *Suppose that for each* $m \geqq 1$,

$$\liminf_{n \to \infty} \int_{D \cap B_m} |\nabla \exp(-f_n)| \, dx < \infty. \tag{61}$$

Then X *is a continuous semimartingale, with decomposition relative to the filtration* $\{\mathscr{F}_t^X\}$ *generated by* X *of the form*

$$X_t = X_0 + M_t + \int_0^t b(X_s) \, ds + V_t, \; t \in [0, 1], \tag{62}$$

where M *is a martingale relative to* $\{\mathscr{F}_t^X\}$ *with mutual variation process:* $\langle M_i, M_j \rangle_t = \int_0^t a_{ij}(X_s) \, ds$, *and* V *is a continuous* $\{\mathscr{F}_t^X\}$*-adapted process of bounded variation such that for each* $v \in C_c^2(\mathbb{R}^d, \mathbb{R}^d)$,

$$\mathbb{E}\left[\int_0^1 v(X_t) . dV_t\right] = -\frac{1}{2}\int_D \operatorname{div}(avp) \, dx. \tag{63}$$

Remark 6.2. — Condition (61) is slightly weaker than condition (4.1) of Williams-Zheng [29]. However, an examination of the proof of Lemma 4.1 in [29] reveals that one only needs a subsequence of $\{X^n\}$ for which (4.1) in [29] holds, and the existence of such is guaranteed by our condition (61) (5). We also note from Theorem 4.1 of [29] that a sufficient geometric condition for (61) to hold for each m is that the boundary of D be *locally of finite* $(d-1)$*-dimensional upper Minkowski content*, *i.e.*, for each m,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \nu\{x \in \mathbb{R}^d : d(x, \partial D \cap B_m) \leqq \varepsilon\} < \infty, \tag{A8}$$

(5) This was also observed by Z. Q. Chen [6].

where ν denotes Lebesgue measure on $\mathbb{R}^d$. In fact, if one restricts x to be in D in (A8), this condition still implies that (61) holds. (One might paraphrase the condition in this case by saying that the $(d-1)$-dimensional upper Minkowski content measured from *within* D is finite).

The following lemma will play an important role in our proof of Theorem 6.1. Here M^n, $\bar{M}^n$ are defined from X^n, $\bar{X}^n$ by (16) and (18) with $b_n + g_n$ in place of b_n there.

LEMMA 6.3. — *Suppose* $(M^n, \bar{M}^n, X^n)$ *converges weakly to* $(M, \bar{M}, X)$ *along a subsequence. Then* M *is a martingale with respect to the forward filtration generated by* X *and* $\bar{M}$ *is a martingale with respect to the backward filtration generated by* X. *Moreover,*

$$(64) \qquad \langle M_i, M_j \rangle_t = \int_0^t a_{ij}(X_s)\, ds, \qquad t \in [0, 1],$$

$$(65) \qquad \langle \bar{M}_i, \bar{M}_j \rangle_t = \int_0^t a_{ij}(\bar{X}_s)\, ds, \qquad t \in [0, 1].$$

Remark 6.4. — To avoid problems with null sets, we take X to be defined on a *complete* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the forward filtration generated by X is $\{\mathcal{F}_t^X \equiv \sigma\{X_s : 0 \leqq s \leqq t\}^{\sim}, t \in [0, 1]\}$, where $\sim$ denotes augmentation by the $\mathbb{P}$-null sets in $\mathcal{F}$. Similarly, the backward filtration of X is defined with $\bar{X}_{\cdot} = X_{1-\cdot}$ in place of X.

Remark 6.5. — We note that when one has (M^n, X^n) converging weakly to (M, X) along a subsequence, and each M^n is a martingale with respect to the forward filtration of X^n, it does not follow in general that M is a martingale with respect to X. For example consider $X^n = \frac{1}{n} B$, $M^n = B$ where B is a Brownian motion.

Proof of Lemma 6.3. — For simplicity we suppose $(M^n, \bar{M}^n, X^n)$ converges weakly to $(M, \bar{M}, X)$, not just along a subsequence. We will prove the results for M, the arguments for $\bar{M}$ being similar. Since M^n is a martingale with respect to the filtration generated by X^n, it follows from the weak convergence that M is a local martingale with respect to the filtration generated by X and M (M is included here to ensure adaptedness).

Since $\tau_r \equiv \inf\{t \geqq 0 : |X_t| \geqq r \text{ or } |M_t| \geqq r\}$ is a non-decreasing function of r, the (random) set of points of discontinuity of $r \to \tau_r$ is at most countable and hence has zero Lebesgue measure. Fubini's theorem then yields that for a.e. $r \in \mathbb{R}_+$, τ_r is $\mathbb{P}$-a.s. a continuous functional of (X, M) (*cf.* Kurtz [15], p. 13-14). Thus we can choose a sequence of reals $r \to \infty$ such that for each r in the sequence and

$$\tau_r^n = \inf\{t \geqq 0 : |X_t^n| \geqq r \quad \text{or} \quad |M_t^n| \geqq r\},$$

$(M^n_{\cdot\wedge\tau^n_r}, X^n_{\cdot\wedge\tau^n_r})$ converges weakly to $(M_{\cdot\wedge\tau_r}, X_{\cdot\wedge\tau_r})$ as $n\to\infty$. Since $a_n=a$ on $\{x\in\mathbb{R}^d: |x|\leqq r\}$ for $n>r$, and a_{ij} is continuous on $\mathbb{R}^d$, we also have

$$\langle M^n_i, M^n_j\rangle_{\cdot\wedge\tau^n_r}=\int_0^{\cdot\wedge\tau^n_r} a_{ij}(X^n_s)\,ds$$

converges weakly to $\int_0^{\cdot\wedge\tau^n_r} a_{ij}(X_s)\,ds$. It follows from [19], Theorem 12, that the mutual variation of $M_{\cdot\wedge\tau_r}$ is given by (64) with $t\wedge\tau_r$ in place of t for all $t\in[0, 1]$. Letting $r\to\infty$ yields the mutual variation of M. Since a is bounded, it follows that M is an L^2-martingale, not just a local martingale. It remains to show that M is adapted to $\{\mathcal{F}^X_t\}$.

Let $\{U_m\}_{m=1}^\infty$ be an increasing sequence of open subsets of D such that for each m, $\bar{U}_m$ is a compact set in D and $\bigcup_m U_m=D$. Observe that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t 1_{\partial D}(X_s)\,dM_s\right|^2\right]&=\mathbb{E}\left[\int_0^t (1_{\partial D}\operatorname{Tr}(a))(X_s)\,ds\right]\\&=\int_0^t \mathbb{E}[(1_{\partial D}\operatorname{Tr}(a))(X_s)\,ds]\\&=0,\end{aligned}$$

where $\operatorname{Tr}(a)$ is the trace of a and the last equality comes from $\mathbb{P}(X_s\in\partial D)=\mathbb{P}(X_0\in\partial D)=0$ for all s. Since the stochastic integral is continuous, $\int_0^{\cdot} 1_{\partial D}(X_s)\,dM_s\equiv 0$ $\mathbb{P}$-a.s.

Thus, $\mathbb{P}$-a.s.,

$$\text{(66)}\qquad M_t=\int_0^t 1_D(X_s)\,dM_s=\lim_m \int_0^t 1_{U_m}(X_s)\,dM_s\quad\text{for all } t\in[0, 1].$$

Fix $i\in\{1, \ldots, d\}$ and $m\in\mathbb{N}$. Let $u\in C^2_c(D)$ such that $u(x)=x_i$ on $\bar{U}_m$. Then by Itô's formula applied to u and X^n, we have

$$\text{(67)}\qquad u(X^n_t)=u(X^n_0)+\int_0^t \nabla u(X^n_s).\,dM^n_s+\int_0^t L_n u(X^n_s)\,ds,$$

where $L_n=\frac{1}{2p_n\rho_n}\nabla.(a_n p_n\rho_n\nabla.)$. Since u has compact support, its support is contained in B_n for all $n\geqq n_0$, some n_0. Since $a_n=a$, $p_n=p$ on B_n, for $n\geqq n_0$,

$$L_n u=\frac{1}{2p}\nabla.(ap\nabla u)-\frac{1}{2}\nabla f_n.a\nabla u$$

where ∇f_n tends to zero uniformly on the support of u as $n \to \infty$. Also, ∇u, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$, a and b are continuous and bounded on the support of u, so it follows that we may take weak limits in (67) (*cf.* Kurtz-Protter [16]) to obtain

$$(68) \qquad u(\mathrm{X}_t) = u(\mathrm{X}_0) + \int_0^t \nabla u(\mathrm{X}_s) \, . \, d\mathrm{M}_s + \int_0^t \mathrm{L} u(\mathrm{X}_s) \, ds.$$

Rearrangement of this equation yields that

$$\left\{ \mathrm{N}_t \equiv \int_0^t \nabla u(\mathrm{X}_s) \, . \, d\mathrm{M}_s, \; t \in [0, 1] \right\}$$

is adapted to $\{ \mathscr{F}_t^{\mathrm{X}}, t \in [0, 1] \}$. But since $u(x) = x_i$ on U_m, we have

$$\int_0^t 1_{\mathrm{U}_m}(\mathrm{X}_s) \, d\mathrm{M}_s^i = \int_0^t (1_{\mathrm{U}_m} \nabla u)(\mathrm{X}_s) \, . \, d\mathrm{M}_s$$
$$= \int_0^t 1_{\mathrm{U}_m}(\mathrm{X}_s) \, d\mathrm{N}_s \in \mathscr{F}_t^{\mathrm{X}}.$$

Letting $m \to \infty$ and i range over $\{1, \ldots, d\}$, from (66) we conclude that M is $\{ \mathscr{F}_t^{\mathrm{X}} \}$-adapted. □

Proof of Theorem 6.1. – Fix $i \in \{1, \ldots, d\}$, $m \in \mathbb{N}$ and let $u \in \mathrm{C}_c^2(\mathbb{R}^d)$ such that $u(x) = x_i$ for all $x \in \mathrm{B}_m$ and $\operatorname{supp}(u) \subset \mathrm{B}_{m+1}$. Consider $n \geqq m+1$ and observe that for such n, $a_n = a$, $b_n = b$ on $\operatorname{supp}(u)$. It follows from Itô's formula that

$$(69) \quad u(\mathrm{X}_t^n) = u(\mathrm{X}_0^n) + \int_0^t \nabla u(\mathrm{X}_s^n) \, . \, d\mathrm{M}_s^n + \int_0^t ((b + g_n) \, . \, \nabla u)(\mathrm{X}_s^n) \, ds$$
$$+ \frac{1}{2} \int_0^t \sum_{i,j=1}^d \left(a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) (\mathrm{X}_s^n) \, ds.$$

We first consider the term $\int_0^t (g_n \, . \, \nabla u)(\mathrm{X}_s^n) \, ds$ in (69). Observe that by the stationarity of X^n,

$$(70) \quad \mathbb{E}\left[\int_0^1 | g_n \, . \, \nabla u | (\mathrm{X}_s^n) \, ds \right] = \frac{\gamma_n}{2} \int_{\mathrm{D}} | \nabla u \, . \, a \nabla f_n | p \, \rho_n \, dx$$
$$\leqq \frac{\gamma_n}{2} \| p a \nabla u \|_{\mathrm{B}_{m+1}, \infty} \int_{\mathrm{D} \cap \mathrm{B}_{m+1}} | \nabla \exp(-f_n) | \, dx,$$

where $\| . \|_{\mathrm{B}_{m+1}, \infty}$ denotes the supremum norm on B_{m+1}. Since $\gamma_n \to e$ as $n \to \infty$, condition (61) implies that the lim $\inf_{n \to \infty}$ of the left member of (70) is finite. Hence, by Meyer-Zheng [19], Corollary 9, there is a subsequence of $\left\{\int_0^{\cdot} (g_n . \nabla u)(X_s^n)\, ds\right\}_{n=1}^{\infty}$ that converges weakly (relative to the pseudo-path topology on the Skorokhod space $D([0, 1], \mathbb{R})$) to a bounded variation process. By section 4 we know that there is a further subsequence of that subsequence along which (M^n, X^n) converges weakly to (M, X) (with the uniform topology on $C([0, 1], \mathbb{R}^{2d})$), where by Lemma 6.3, M is a martingale relative to the forward filtration $\{\mathcal{F}_t^X\}$ of X. It follows from Kurtz-Protter [16], Theorem 2.2, that since $a, b, u, \nabla u, \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ are continuous and bounded on the support of u,

$$(71)\quad \left(u(X_\cdot^n), \int_0^{\cdot} \nabla u(X_s^n) . dM_s^n, \int_0^{\cdot} (b . \nabla u)(X_s^n)\, ds, \int_0^{\cdot} \sum_{i, j=1}^{d} \left(a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}\right)(X_s^n)\, ds\right)$$

converges weakly along the same subsequence as (M^n, X^n) to (71) with the indices n removed. It follows that the weak limit V^u of $\left\{\int_0^{\cdot} (g_n . \nabla u)(X_s^n)\, ds\right\}_{n=1}^{\infty}$ along this subsequence is a continuous bounded variation process adapted to $\{\mathcal{F}_t^X\}$. Combining the above, we conclude that for all $t \in [0, 1]$,

$$(72)\quad u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) . dM_s + \int_0^t (b . \nabla u)(X_s)\, ds + V_t^u + \frac{1}{2} \int_0^t \sum_{i, j=1}^{d} \left(a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}\right)(X_s)\, ds.$$

By stopping X at $\tau_m = \inf\{t \geqq 0 : X_t \notin B_m\}$, from (72) we obtain a decomposition of $X_{t \wedge \tau_m}^i$. Note from this that the last term in (72) is zero for $t \leqq \tau_m$. On letting $m \to \infty$ and i range over $\{1, \ldots, d\}$, we obtain by consistency that

$$(73)\qquad X_t = X_0 + M_t + \int_0^t b(X_s)\, ds + V_t, \quad t \in [0, 1],$$

where V is a continuous, $\{\mathcal{F}_t^X\}$-adapted process with paths of bounded variation. Noting the mutual variation of M from Lemma 6.3, we see that it remains to show (63).

For $v \in C_c^2(\mathbb{R}^d, \mathbb{R}^d)$, by using (73) to reexpress dV_t, we have

$$(74) \qquad \int_0^1 v(X_t) . dV_t = \int_0^1 v(X_t) \circ d(X_t - X_0) - \int_0^1 v(X_t) \circ dM_t - \int_0^1 v(X_t) . b(X_t)\, dt,$$

where $\circ$ denotes the Stratonovitch integral. By Lemma 6.3, $X_. - X_0$ is the difference of a forward and backward martingale with respect $\dot{X}$ and hence by Lyons-Zheng [18], (4.5), so too is $t \to \int_0^t v(X_s) \circ d(X_s - X_0)$ where the martingale initial values are zero. It follows that the expectation of this process vanishes at each t. Hence,

$$(75) \qquad \mathbb{E}\left[\int_0^1 v(X_t) . dV_t\right] = -\mathbb{E}\left[\int_0^1 v(X_t) \circ dM_t\right] - \mathbb{E}\left[\int_0^1 v(X_t) . b(X_t)\, dt\right]$$

where using the Stratonovitch-Itô conversion (e.g., *see* Protter [23], p. 216),

$$(76) \qquad \mathbb{E}\left[\int_0^1 v(X_t) \circ dM_t\right] = \mathbb{E}\left[\int_0^1 v(X_t) . dM_t\right] + \mathbb{E}\left[\frac{1}{2} \sum_{i,j=1}^d \int_0^1 \left(\frac{\partial v_i}{\partial x_j} a_{ij}\right)(X_s)\, ds\right].$$

The first term on the right of (76) is zero. Thus, combining (75)-(76) with the definition of b and stationarity of X, we have

$$\mathbb{E}\left[\int_0^1 v(X_t) . dV_t\right] = -\frac{1}{2} \int_D \sum_{i,j=1}^d \left(\frac{\partial v_i}{\partial x_j} a_{ij} p + v_i \frac{\partial (a_{ij} p)}{\partial x_j}\right) dx$$

which simplifies to (63). □

Remark 6.6. – If a, p, ∂D are smooth, the right member of (63) is equal by the divergence theorem to

$$\frac{1}{2} \int_{\partial D} n . avp\, d\sigma$$

where n is the inward unit normal to ∂D and σ denotes surface measure on ∂D. This is consistent with X having *conormal* reflection (*i.e.*, reflection in the direction an) at the boundary of D. For in this case,

$$V_t = \int_0^t (an)(X_s)\, dL_s,$$

where L is a one-dimensional, continuous, non-decreasing process adapted to X that increases only when X is on ∂D. The process L is called the local time of X on ∂D. Its Revuz measure is concentrated on ∂D and has density $\frac{1}{2}p$ with respect to the surface measure σ. Consequently,

$$\mathbb{E}\left[\int_0^t v(X_t) . dV_t\right] = \mathbb{E}\left[\int_0^1 (v . an)(X_t)\, dL_t\right] = \frac{1}{2}\int_{\partial D} v . anp\, d\sigma.$$

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