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Random walks on a tree and capacity in the interval

by

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ABSTRACT. — In this paper we give a geometric criterion for transience of a random walk on a tree. When the tree corresponds to a compact subset Λ of the unit interval in an integer base, transience is independent of the base.

Tight bounds for the logarithmic energy of the resulting harmonic measures are presented, and the points at which their logarithmic potential may become infinite are characterized by a Diophantine approximation condition.

RÉSUMÉ. — On donne un critère géométrique pour savoir si la marche aléatoire sur un arbre est transitoire ou récurrente.

Quand un arbre correspond à un ensemble compact dans l'intervalle [0, 1] en base entière, la récurrence est indépendante de la base.

On obtient des bornes optimales pour l'énergie logarithmique de la mesure harmonique et les points où le potentiel logarithmique est infini sont caractérisés de manière Diophantine.

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1. INTRODUCTION AND MAIN RESULTS

Let T be a tree with finite degree at each vertex, and consider the nearest neighbor, symmetric random walk on T. When is the expected number of visits to a fixed vertex finite, *i.e.*, when is the walk *transient*?

The first condition one thinks of, exponential growth, is neither necessary nor sufficient (*see* examples 0, 1 in section 5). A necessary condition for transience in terms of growth was given by Nash-Williams [N] (*see* also [L1]) but his condition is far from sufficient.

Our main interest in this paper is in trees describing subsets of the interval, but we start with some general transience criteria. Fix an arbitrary vertex v_0 of a tree T, called the root. For two vertices u, w of T define their *meeting height* $(u|w)$ as the distance from the geodesic between u and w to the root v_0.

THEOREM 1. — *For a tree* T *with root* v_0, *the following are equivalent :*

(i) *The symmetric random walk on* T *is transient.*

(ii) *There exists a constant* $C>0$ *such that for any* n, *there are* n *distinct vertices* $u_1, \ldots, u_n$ *with average meeting height less than* C, *i.e.*

$$\binom{n}{2}^{-1} \sum_{i<j} (u_i | u_j) < C. \tag{1.1}$$

(iii) *There exists a constant* $C'>0$, *such that for any finite set of vertices* $v_1, \ldots, v_n$ *there is a vertex* w *which does not lie on the geodesics between the root and* $v_1, \ldots, v_n$, *yet has average meeting height less than* C' *with them:*

$$\frac{1}{n} \sum_{i=1}^{n} (w | v_i) < C'. \tag{1.2}$$

Actually, the theorem is proved for random walks which are not necessarily symmetric, by assigning lengths to the edges of T.

Though theorem 1 is stated solely in terms of the tree T itself, our proof involves the abstract *boundary* ∂T of T equipped with the "Gromov metric" (*see* section 2 or [G], chap. 6] for the definitions).

Now we reverse our viewpoint and *start* from the boundary. Given a compact set $\Lambda \subset [0, 1]$ and an integer $b>1$, we associate to them a tree $T(\Lambda, b)$ as follows. The vertices of this tree are the b-adic intervals $[((j-1)/b^k), j/b^k]$ such that their interiors intersect Λ (with $k \geqq 0$ and $1 \leqq j \leqq b^k$). Two such vertices are connected by an edge if, as intervals, one contains the other and the ratio of their lengths is b. Trees of this type are discussed in [Fu] and [M]. If Λ is the ternary Cantor set C, then $T(C, 3)$ is a binary tree, while $T(C, b)$ looks complicated if b is not a power of 3.

Throughout this paper, Λ always denotes a compact subset of [0, 1], and we start the random walk on $T(\Lambda, b)$ from the vertex v_0 identified with the interval [0, 1]. When this walk is transient, it defines a hitting distribution, called *harmonic measure* on Λ, and denoted $\mu_{\Lambda, b}$.

LEMMA 2. — (I) *Random walk on* $T(\Lambda, b)$ *is transient* iff Λ *has positive logarithmic capacity. In particular, transience of* $T(\Lambda, b)$ *does not depend on the base b.*

(II) *If* $T(\Lambda, b)$ *is transient, the harmonic measure* μ *has finite logarithmic energy, i. e.*

$$I(\mu)=\int_\Lambda \int_\Lambda \log \frac{1}{|x-y|} d\mu(x)\, d\mu(y)<\infty. \tag{1.3}$$

Remarks. — 1. All terms appearing in the theorem are precisely defined in section 2.

2. Theorem 2 implies that if Λ has positive Hausdorff dimension then $T(\Lambda, b)$ is transient. A beautiful refinement of this fact is contained in a recent paper of R. Lyons [L2], theorem 4.3.

3. Part (I) of the theorem is analogous to a classical result of Kakutani concerning plane Brownian motion. This is pursued in section 2.

We turn to study the harmonic measure.

How singular can it be?

PROPOSITION 3. — (i) *If* Λ *has positive Lebesgue measure, then the harmonic measure* $\mu=\mu_{\Lambda, b}$ *for* $T(\Lambda, b)$ *is nonsingular with respect to Lebesgue measure. (In particular,* μ *has Hausdorff dimension* 1.)

(ii) *If* $\Lambda_0 \subset [0, 1]$ *is a compact set of Lebesgue measure zero, then there exists a compact* $\Lambda \subset [0, 1]$ *with* $\Lambda \supset \Lambda_0$, *such that the harmonic measure* $\mu_{\Lambda, b}$ *is supported on a Borel set of Haudsdorff dimension zero. Furthermore,* Λ *can be chosen to have Hausdorff dimension* 1 *in any open interval it intersects.*

In the sense of logarithmic energy, however, μ is spread quite nicely over Λ.

THEOREM 4. — (i) *For any base* $b>1$, *the harmonic measure* $\mu_{\Lambda, b}$ *on* Λ *satisfies*

$$I(\mu_{\Lambda, b}) \leqq 2 (\inf_{\nu_1} I(\nu_1) + \log b) \tag{1.4}$$

where ν_1 *ranges over all probability measures supported on* Λ.

(ii) *Consider dilations* $\alpha\Lambda$ *of* Λ, *where* $0<\alpha<1$. *The map taking* $\alpha \in (0, 1)$ *to the harmonic measure* $\mu_{\alpha\Lambda, b}$ *is continuous in the weak* topology for measures.*

Remarks. — 1. Part (i) extends to the d-dimensional grid (*see* the proof!) with the constant 2 in (1.4) replaced by 2^d. These multiplicative constants are best possible. [The additive constant in (1.4) can be improved, but this will not concern us.] This is of interest primarily when $d=2$, where logarithmic energy is important. By the probabilistic Fatou theorem [KSK], theorem 10-43, the hitting probabilities on the n'th "level" of the tree converge (weakly) to the harmonic measure, and can thus be used to estimate the capacity of a compact plane set, up to a bounded factor.

2. Part (ii) is easy if Λ is a finite union of intervals. For general Λ, the tree $T(\alpha\Lambda, b)$ changes discontinuously with α, and this motivates our interest in (ii). Similar results hold when dilation is replaced by translation.

On the abstract boundary of a tree, one can sharpen the statement that the harmonic measure has finite energy and conclude it has a bounded logarithmic potential (*see* lemma 6 below, or [L2], prop. 4.5). The situation on the interval is different, as the potential may "blow up".

THEOREM 5. – *Let* $\mu=\mu_{\Lambda, b}$ *denote harmonic measure on* Λ. *Define the "singular set" for* Λ, b *by*

$$s(\Lambda, b)=\left\{x\in\Lambda \,\Big|\, \int_{\Lambda} \log\frac{1}{|x-y|}\, d\mu(y)=\infty\right\} \tag{1.5}$$

Then

(i) $s(\Lambda, b)$ *may be uncountable.*

(ii) $s(\Lambda, b)$ *always has zero logarithmic capacity, and consists of transcendental numbers.*

(iii) $\bigcup_{\Lambda} s(\Lambda, b)$ *is a set of Hausdorff dimension zero but positive logarithmic capacity, where the union is over all compact* $\Lambda\subset[0, 1]$.

The rest of the paper is organized as follows.

Section 2 contains background concerning potential theory and random walks. The Dirichlet problem for trees, as well as the connections with Brownian motion are discussed.

In section 3 we prove theorem 1, together with other facts concerning general trees. Theorems 2-5 are proved in section 4.

The flavour of the subject is revealed in the examples, assembled in section 5. For instance, we produce a recurrent tree with exponential growth "above" every vertex (example 2) and show that harmonic measures for different bases on Λ may be mutually singular (example 6).

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2. BACKGROUND, AND THE DIRICHLET PROBLEM ON A TREE

We now extend the setup of section 1 to include non-symmetric random walks.

A *tree* $T=\langle V, E\rangle$ is a connected locally finite, acyclic graph. Suppose that for each edge $(v, w)\in E$ a positive "edge length" (or "resistance") l_{vw} is given with $l_{vw}=l_{wv}$. Use these lengths to define transition probabilities which are inversely proportional:

$$p(v, w)=l_{vw}^{-1}/\sum_{u\in E_v} l_{vu}^{-1} \tag{2.1}$$

where $E_v=\{u\in V\mid (v, u)\in E\}$.

Conversely, one can see that any nearest-neighbor positive transition probabilities $\{p_{vw}\mid (v, w)\in E\}$ may be realized in this manner. [The tree property is crucial here, and allows defining the edge length so as to satisfy formula (2.1), by induction on the distance of an edge from a fixed vertex v_0.]

Définitions: The tree boundary ([G], [Ca])

Let $T=\langle V, E\rangle$ be a tree with given positive edge lengths $\{l_{vw}\mid (v, w)\in E\}$.

1. A *geodesic* in T is a (finite or infinite) nonrepeating sequence of adjacent vertices.

2. Two geodesic rays $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ are called *equivalent* if as sets their symmetric difference is finite. Each equivalence class is called an *end* of T. We refer to the space of ends as the *boundary* ∂T of T.

3. For two vertices v, w their *distance* $d(v, w)$ in T is the sum of the edge lengths along the geodesic connecting them.

4. Any two ends $\xi, \eta\in\partial T$ are "connected" by a unique geodesic $\{u_n\}_{-\infty}^{\infty}$, in the sense that $\{u_{-n}\}_{n=1}^{\infty}\in\xi$ and $\{u_n\}_{n=1}^{\infty}\in\eta$. Similarly an end and a vertex are connected by a unique geodesic ray.

5. Fix a distinguished vertex $v_0\in V$, called the *root* of T. The *meeting height* $(\xi\mid\eta)$, where $\xi, \eta\in V\cup\partial T$, $\xi\neq\eta$ is defined by

$$(\xi\mid\eta)=\min\{d(v_0, w)\mid w \text{ lies on the geodesic connecting } \xi, \eta\}. \tag{2.2}$$

The *Gromov metric* (*see* [G]) on ∂T is defined by

$$\rho(\xi, \eta)=e^{-(\xi\mid\eta)} \quad \text{for } \xi\neq\eta, \qquad \rho(\xi, \xi)=0 \tag{2.3}$$

It will be convenient to use (2.3) also when one of ξ, η is a vertex of T.

6. Assume the random walk $\{Y_n\}_{n\geq 0}$ on T (starting from v_0) is transient. It follows that almost surely, $\{Y_n\}$ converges to a unique (random) end $Y_\infty\in\partial T$. This is easily shown directly, and is also contained in the

result of Cartier [Ca], who shows that the Martin boundary for the random walk may be identified with ∂T. The *harmonic measure* μ (relative to v_0) is defined by

$$\mu(B)=\mathbb{P}[Y_\infty \in B \mid Y_0=v_0]$$

for any Borel set $B \supset \partial T$.

Definitions: Potential theory ([C], [W])

Let ν be a finite Borel measure on the compact metric space $\langle X, \rho \rangle$.

1. The *logarithmic potential* of ν is the function

$$\phi_\nu(x)=\int_X \log \frac{1}{\rho(x, y)} d\nu(y)$$

2. The *logarithmic energy* of ν is given by

$$I(\nu)=\int_X \phi_\nu(x)\, d\nu(x)=\int_X \int_X \log \frac{1}{\rho(x, y)} d\nu(x)\, d\nu(y).$$

3. The *logarithmic capacity* of a Borel set $B \subset X$ is defined

$$\operatorname{cap}(B)=\sup\{\nu(B) \mid \nu \text{ is a positive measure with } I(\nu) \leqq 1\}.$$

Note that $\operatorname{cap}(B)>0$ iff B carries a probability measure of finite logarithmic energy.

4. Similarly, B has *positive* α-*capacity* iff some probability measure ν on B satisfies

$$\int_B \int_B \frac{d\nu(x)\, d\nu(y)}{\rho(x, y)^\alpha}<\infty$$

Clearly if B has positive α-capacity for some $\alpha>0$, then $\operatorname{cap}(B)>0$.

Frostman showed in his thesis [FR] that if K is a compact set in $\mathbb{R}^n$ with Hausdorff dimension $>\alpha$, then the α-capacity of K is positive. In particular

$$\text{H-dim}(K)>0 \quad \Rightarrow \quad \operatorname{cap}(K)>0.$$

For a modern exposition, *see* [C] or [K].

The following proposition is crucial in the proof of theorem 1.

Let T be a tree with given positive edge lengths $\{l_{vw}\}$.

Assume

$$\text{For any geodesic ray } \{v_n\}_{n=1}^{\infty}, \qquad \sum_{i=1}^{\infty} l_{v_i v_{i+1}}=\infty \tag{2.4}$$

(Otherwise, the random walk can easily be proven transient, using, for instance, the results of [L1] or [DS].)

PROPOSITION 6. — (i) *The random walk on* T *with transition probabilities defined by* (2.1) *is transient iff* $\text{cap}(\partial T) > 0$ *with respect to the Gromov metric* ρ.

(ii) *If the walk (started at* v_0, *say) is transient, then the harmonic measure* μ *satisfies*

$$\forall \xi \in \partial T, \quad \phi_\mu(\xi) \leqq I(\mu) = \text{cap}(\partial T)^{-1} \tag{2.5}$$

with equality except, possibly, at a set of ξ*'s of capacity zero.*

Remarks. — 1. After the results of this paper were obtained, we were informed that proposition 6 was proved earlier by Russell Lyons [L2].

2. Our proof of proposition 6, given in section 3, is motivated by the relationship between classical potential theory (as in Tsuji [Ts]) and plane Brownian motion.

In 1944 Kakutani [Kak] characterized compact sets of positive logarithmic capacity in the plane as those compact $\Lambda \subset \mathbb{C}$ for which Brownian motion started outside Λ, hits Λ with probability one. *See* [K], 16.5, for proof.

To interpret Kakutani's theorem in terms of the transience-recurrence dichotomy, conformally equip $\mathbb{C}^* \setminus \Lambda$ with the Poincare metric (*i.e.* complete with constant negative curvature). The Brownian paths in $\mathbb{C} \setminus \Lambda$ are the same as for the Euclidean metric; the Brownian particle, however, moves sluggishly when near Λ. It follows that $\text{cap}(\Lambda) > 0$ iff Brownian motion on $\mathbb{C} \setminus \Lambda$ with the Poincare metric is transient.

3. Y. Kifer and F. Ledrappier proved recently [KL] that if the curvature of a simply connected manifold is "sandwiched" between two negative constants, harmonic measure on the sphere at infinity has positive Hausdorff dimension (*i.e.* any Borel set of full measure has positive dimension). The analogous statement for trees, which will be proved in section 3, is

PROPOSITION 7. — *If* T *is a tree with* all degrees greater than 2 *but bounded, and edge lengths "sandwiched" between two positive constants, then harmonic measure on* ∂T *has positive Hausdorff dimension;* (compare with theorem 3).

Actually, we derive an explicit lower bound for the dimension, analogous to [Ki].

We end this section with a discussion of the Dirichlet problem on a tree.

Definition (compare [C], [Ts])

Let $T = \langle V, E \rangle$ be a tree with edge lengths $\{ l_{vw} \}$, and transition probabilities given by (2.1).

(i) A function $h: \mathrm{V} \to \mathbb{R}$ is *harmonic* at $v \in \mathrm{V}$ (*see* [KSK]) if

$$h(v) = \frac{1}{|\mathrm{E}_v|} \sum_{w \in \mathrm{E}_v} p(v, w) h(w).$$

(ii) A boundary point $\xi \in \partial \mathrm{T}$ is called a *regular point for the Dirichlet problem* (in short, a *regular point*) if for any continuous function $\varphi : \partial \mathrm{T} \to \mathbb{R}$ the relation

$$\lim_{v \to \xi} h_\varphi (v) = \varphi (\xi)$$

holds, where h_φ is the unique harmonic function on T for which

$$\operatorname{cap} \{ \eta \in \partial \mathrm{T} \mid \lim_{u \to \eta} h_\varphi (u) \neq \varphi (\eta) \} = 0 \tag{2.6}$$

(For existence and uniqueness of h_ρ, *see* the proof of proposition 8.)

Here $v \to \xi$ means that $\rho (v, \xi) \to 0$.

PROPOSITION 8. — (i) *In the notation above, a point* $\xi \in \partial \mathrm{T}$ *is regular iff* $\lim_{v \to \xi} \mathbb{P} [\exists\, n, \mathrm{Y}_n = v_0 \mid \mathrm{Y}_0 = v] = 0$ *where* $\{ \mathrm{Y}_n \}_{n=0}^{\infty}$ *is the random walk on* T.

(ii) *"Kellogg's theorem"* (see [Ts], *theorem* III.33)

The set of non-regular points in $\partial \mathrm{T}$ *is an* F_σ *set of zero capacity.*

Remarks. — 1. The proposition is proved in section 3. Part (ii) is contained in [L2], proposition 4.5.

2. Example 3 in section 5 shows the non regular points *in the closed support* of the harmonic measure may be uncountable and dense there.

3. GENERAL TREES: PROOFS

Proof of proposition 6

(i) This proof is modeled after [Ts], theorem III.35.

Assume first that cap $(\partial \mathrm{T}) > 0$.

Then some probability measure ν on $\partial \mathrm{T}$ satisfies $\mathrm{I}(\nu) < \infty$. By truncating, we may assume the potential is bounded: $\phi_\nu \leqq \mathrm{M}$ on $\partial \mathrm{T}$. Recall v_0 denotes the "root" of T. For any vertex u of T, define the *"cone" above u*:

$$\mathrm{C}(u) = \{ \xi \in \partial \mathrm{T} \mid \text{The geodesic from } v_0 \text{ to } \xi \text{ passes through } u \}. \tag{3.1}$$

If $v_0, v_1, \ldots, v_n = u$ is the geodesic from the root to u, define:

$$\psi (u) = \sum_{j=1}^{n} \nu [\mathrm{C}(v_j)]\, l_{v_j v_{j-1}}. \tag{3.2}$$

Direct inspection shows, using the additivity of ν, that ψ is harmonic at every vertex *except* v_0. Compute.

$$\phi_\nu(\xi)=\int_{\partial T}\log\frac{1}{\rho(\xi,\eta)}\,d\nu(\eta)=\int_{\partial T}(\xi\mid\eta)\,d\nu(\eta)$$
$$=\int_{\partial T}\left(\sum_{n=1}^{\infty}1_{C(v_n)}(\eta)\,l_{v_n v_{n-1}}\right)d\nu(\eta)$$

where $\{v_n\}_{n=0}^{\infty}$ is the geodesic from v_0 to ξ, and $1_{C(v_n)}$ is the indicator of the cone above v_n. Thus

$$\forall\xi\in\partial T,\quad \phi_\nu(\xi)=\sum_{n=1}^{\infty}\nu[C(v_n)]\,l_{v_n v_{n-1}}. \tag{3.3}$$

Therefore by (3.2), (3.3) ψ is a bounded function, harmonic except at one point, and this is known to imply transience.

Conversely, assume the walk $\{Y_n\}$ on T is *transient*. Define, after T. Lyons [L1], a measure ν on ∂T as follows. For every vertex $w\neq v_0$, denote by $\tilde{w}$ the unique neighbor of w which is closer to v_0. Let

$$\nu[C(w)]=\frac{f(\tilde{w},v_0)-f(w,v_0)}{l_{\tilde{w}w}} \tag{3.4}$$

where $C(w)$ is defined in (3.1) and for any two vertices w, v

$$f(w,v)=\mathbb{P}[\exists n\geq 0,\ Y_n=v\mid Y_0=w].$$

As the cones $\{C(w)\}$ form a basis of closed and open sets for the topology of ∂T, (3.4) defines ν uniquely. Additivity of ν follows from harmonicity of f in its first coordinate w (when $w\neq v_0$). The transience assumption guarantees that f is non-constant and ν is nontrivial.

Using (3.3) and (3.4) we find that

$$\forall\xi\in\partial T,\quad \phi_\nu(\xi)=1-\lim_{w\to\xi}f(w,v_0). \tag{3.5}$$

In particular, ϕ_ν is bounded, so cap$(\partial T)>0$.

(ii) We start by relating the harmonic measure μ to the measure ν defined in (3.4) above. Observe that for any vertex $w\neq v_0$

$$\mathbb{P}[\lim_n Y_n\in C(w)]=\sum_{k=1}^{\infty}P[Y_k=\tilde{w},\ Y_{k+1}=w,\ \forall j>k,\ Y_j\neq\tilde{w}\mid Y_0=v_0]$$

so

$$\mu[C(w)]=g(v_0,\tilde{w})\,p(\tilde{w},w)\,(1-f(w,\tilde{w})). \tag{3.6}$$

Let

$$\pi_w=\sum_{u\in E_w}l_{uw}^{-1}. \tag{3.7}$$

Then

$$\pi_v p(v, w) = \pi_w p(w, v) = l_{vw}^{-1} \tag{3.8}$$

and the expression for g as a sum over individual paths gives

$$\pi_v g(v, w) = \pi_w g(w, v).$$

Therefore

$$g(v_0, \tilde{w}) = \frac{\pi_{\tilde{w}}}{\pi_{v_0}} g(\tilde{w}, v_0) = \pi_{\tilde{w}} f(\tilde{w}, v_0) \frac{g(v_0, v_0)}{\pi_{v_0}}. \tag{3.9}$$

Now (3.4) can be rewritten as

$$\nu[C(w)] = f(\tilde{w}, v_0)[1 - f(w, \tilde{w})] l_{\tilde{w}w}^{-1}.$$

This together with (3.6), (3.8), (3.9) gives

$$\mu[C(w)] = \frac{g(v_0, v_0)}{\pi_{v_0}} \nu[C(w)]. \tag{3.10}$$

Claim:

$$cap\{\xi \in \partial T \mid \phi_\nu(\xi) < 1\} = 0. \tag{3.11}$$

Otherwise, using (3.5) we could find a compact $K \subset \partial T$ with $\operatorname{cap}(K) > 0$ such that

$$\inf_w f(w, v_0) > 0 \tag{3.12}$$

where the infimum is over the subtree T_K of T consisting of the geodesics from v_0 to K. The inequality (3.12) implies that when considering random walk on T_K, the probabilities of reaching v_0 from any vertex are bounded away from zero. Hence T_K is recurrent, contradicting (i). From (3.5), (3.10) and (3.11) we find that

$$\phi_\mu(\xi) < \frac{g(v_0, v_0)}{\pi_{v_0}} \text{ with equality except on a set of capacity zero.} \tag{3.13}$$

This forces

$$I(\mu) = \frac{g(v_0, v_0)}{\pi_{v_0}}. \tag{3.14}$$

Using [W], theorem 9.1, which translates without difficulty to the tree situation, (3.13) implies that μ is the unique probability measure of minimal energy carried by ∂T, proving (ii). □

Proof of theorem 1

Though the theorem is stated for symmetric random walk, it applies to a tree with arbitrary positive edge lengths satisfying (2.4), and transition probabilities determined accordingly. For any compact metric space

$\langle X, \rho \rangle$ define the *generalized diameters*

$$D_n(X) = \inf_{\{x_1, \ldots, x_n\} \subset X} \binom{n}{2}^{-1} \sum_{i<j} \log \frac{1}{\rho(x_i, x_j)}$$

and the *Tschebyscheff constants*:

$$M_n(X) = \sup_{\{x_1, \ldots, x_n\} \subset X} \inf_{y \in X} \frac{1}{n} \sum_{j=1}^{n} \log \frac{1}{\rho(y, x_j)}.$$

A classical result of Fekete and Szegö asserts that $\{D_n(X)\}_{n=1}^{\infty}$ is an increasing sequence, and

$$\lim_{n \to \infty} D_n(X) = \lim_{n \to \infty} M_n(X) = \operatorname{cap}(X)^{-1} \tag{3.15}$$

(with the convention $0^{-1} = \infty$).

For a proof valid in this generality, *see* Carleson's book [C], theorem 6, p. 37.

Taking $X = \partial T$, and ρ the Gromov metric, we find

$$D_n(\partial T) = \inf_{\{\xi_1, \ldots, \xi_n\} \subset \partial T} \binom{n}{2}^{-1} \sum_{i<j} (\xi_i \mid \xi_j)$$

and

$$M_n(\partial T) = \sup_{\{\xi_1, \ldots, \xi_n\} \subset \partial T} \inf_{\eta \in \partial T} \frac{1}{n} \sum_{j=1}^{n} (\eta \mid \xi_j).$$

Assume that T is transient. By proposition 6, $\operatorname{cap}(\partial T) > 0$ so (3.15) implies the sequences $\{D_n(\partial T)\}_{n=1}^{\infty}$, $\{M_n(\partial T)\}_{n=1}^{\infty}$ are bounded. For each n we can select $\xi_1, \ldots, \xi_n$ in ∂T such that

$$\binom{n}{2}^{-1} \sum_{i<j} (\xi_i \mid \xi_j) < C \quad \text{(for some C).}$$

Choose u_i on the geodesic from v_0 to ξ_i, so that $u_i \neq u_j$ for $i \neq j$. This proves (i) $\Rightarrow$ (ii), and similarly (i) $\Rightarrow$ (iii). The converse requires an additional argument. Assume (ii) of the theorem. We want to replace the vertices $u_1, \ldots, u_n$ given in (1) by boundary points $\xi_1, \ldots, \xi_n$ "above" them so that

$$(\xi_i \mid \xi_j) = (u_i \mid u_j) \qquad \text{for} \quad i \neq j. \tag{3.16}$$

For the original tree T, this is in general impossible. We enlarge T by adding a new geodesic ray (a copy of the positive integers) above every vertex of T. This creates a new tree $\hat{T}$, on the boundary of which (3.16) may be achieved (when u_i are vertices of T). Thus $\{D_n(\partial \hat{T})\}_{n=1}^{\infty}$ is a bounded sequence, so $\operatorname{cap}(\partial \hat{T}) > 0$. As $\partial \hat{T} \setminus \partial T$ is countable, it follows that $\operatorname{cap}(\partial T) > 0$, proving (i). Similarly (iii) $\Rightarrow$ (i). □

Remark. – The vertices $\{u_j\}$ in condition (ii) of theorem 1 cannot, in general be taken to be whole "levels" of the tree. This can be seen by combining examples 0,1 from section 5 (identifying their roots).

We pass to the analogue of the Kifer-Ledrappier theorem. The lower bound obtained is motivated by the recent paper [Ki].

Sharpening of proposition 7

We write explicitly the boundedness assumptions on the degrees and edge lengths:

$$\forall v, w \quad d \leqq d_v - 1 \leqq \mathrm{D}, \qquad l \leqq l_{v,w} \leqq \mathrm{L} \tag{3.17}$$

where d, D are integers, $d \geqq 2$, and l, $\mathrm{L} > 0$.

Under these assumptions we get an explicit bound for the Hausdorff dimension of harmonic measure with respect to the Gromov metric on $\partial \mathrm{T}$:

$$\text{H-dim}(\mu) \geqq \alpha = \frac{1}{\mathrm{L}} \log\left(1 + \frac{l}{\mathrm{L}} \frac{(d-1)^2 \mathrm{D}}{(\mathrm{D}-1) d}\right) > 0. \tag{3.18}$$

Proof. – Denote by v_0 the "root" of T, which is the starting vertex for the random walk. For any vertex $w \neq v_0$ recall $\tilde{w}$ denotes the unique neighbor of w which is closer to v_0. The main burden of the proof is carried by the following

Claim. – Let w_1, w_2 be two vertices with the same "father": $\tilde{w}_1 = \tilde{w}_2 = u$. Then

$$\frac{\mu[\mathrm{C}(w_1)]}{\mu[\mathrm{C}(w_2)]} \geqq \frac{l}{\mathrm{L}} \frac{(d-1)\mathrm{D}}{(\mathrm{D}-1) d} = \beta > 0. \tag{3.19}$$

Proof of claim. – Consider the tree T_1 which consists of the union of the geodesic rays from u to $\mathrm{C}_{(w_1)}$. Similarly define T_2. Using formulas (2.5), (3.7) and (3.14) we have

$$\operatorname{cap}(\partial \mathrm{T}_i) = (l_{u\,w_i} g_i(u, u))^{-1} \quad (i = 1, 2) \tag{3.20}$$

where g_i is Greens function for the tree T_i. This may be used to show

$$\frac{\mu[\mathrm{C}(w_1)]}{\operatorname{cap}(\partial \mathrm{T}_1)} = \frac{\mu[\mathrm{C}(w_2)]}{\operatorname{cap}(\partial \mathrm{T}_2)}. \tag{3.21}$$

Actually (3.21) is a direct consequence of the fact that the potential ϕ_μ is constant a.e. [μ] on $\partial \mathrm{T}$, and formula (3.3) for the potential. Denote by $\mathrm{T}_{d,\mathrm{L}}$ the tree which has degree $d+1$ at all vertices except the root, which has degree 1, with all edges of length L. Similarly, define $\mathrm{T}_{\mathrm{D},l}$. Using the interpretation of capacity in terms of transfinite diameter as in the proof of theorem 1, it is clear that

$$\operatorname{cap}(\mathrm{T}_{d,\mathrm{L}}) \leqq \operatorname{cap}(\mathrm{T}_i) \leqq \operatorname{cap}(\mathrm{T}_{\mathrm{D},l}) \quad (i = 1, 2).$$

Since the harmonic measure on $\partial T_{d,L}$ is determined by symmetry, (3.3) gives the easiest access to the capacity of $T_{d,L}$:

$$\operatorname{cap}(T_{d,L}) = L\frac{d}{d-1}. \tag{3.22}$$

Alternatively (3.22) follows from (3.20) and the well known value, $\frac{d}{d-1}$, of Green's function for a $(d+1)$-regular tree.

Combining (3.21) and (3.22) proves the claim (3.19).

If $w \neq v_0$ is a vertex of T, (3.19) together with (3.17) imply

$$\mu[C(w)] \leqq \frac{1}{1+\beta(d-1)}\mu[C(\tilde{w})]. \tag{3.23}$$

If $v_0, v_1, \ldots, v_n = w$ is the geodesic connecting the root to w, iterating (3.23) gives

$$\mu[C(v_n)] \leqq (1+\beta(d-1))^{-n}. \tag{3.24}$$

Note that

$$\operatorname{diam}[C(v_n)] = \exp\left(-\sum_{j=1}^{n} l_{v_{j-1}v_j}\right) \geqq e^{-nL}. \tag{3.25}$$

From the Hölder condition (3.24), estimating H-dim(μ) is completely standard (*see*, for instance [K], chap. 10). In the definition of Hausdorff measure, we can restrict ourselves to covering with cones. If $\mu(\bigcup_j C_j) > 0$ where C_j are cones, then (3.24) and (3.25) combined give for $\alpha = \frac{1}{L}\log[1+\beta(d-1)]$

$$\sum_j \operatorname{diam}(C_j)^\alpha \geqq \sum_j \mu(C_j) > 0$$

and H-dim$(\mu) \geqq \alpha$ is proved. □

Remark. – If T is a $(d+1)$-regular tree with edge lengths sandwiched between l and L, (3.18) takes the form

$$\text{H-dim}\,\mu \geqq \frac{1}{L}\log\left[1+\frac{l}{L}(d-1)\right]. \tag{3.26}$$

In particular if $l_{vw} \to L$ as $d(v, v_0) \to \infty$ then

$$\text{H-dim}\,\mu \geqq \frac{1}{L}\log d.$$

As in this case

$$\text{H-dim}(\partial T) = \text{H-dim}(\partial T_{d,L}) = \frac{1}{L} \log d$$

inequality (3.26) is actually an equality.

We turn to the Dirichlet problem.

Proof of proposition 8

(i) If T is recurrent there is nothing to prove; assume transcience, with root v_0. Recall the function $f(w, v) = \mathbb{P}[\exists n \geqq 0, Y_n = v \mid Y_0 = w]$.

Using f, define

$$\Omega = \{\xi \in \partial T \mid \lim_{w \to \xi} f(w, v_0) > 0\}.$$

From (3.5) and (3.11) infer that

$$\text{cap}(\Omega) = 0. \tag{3.27}$$

For any vertex v, let μ_v denote the harmonic measure for the random walk started at v. Given a continuous function $\varphi: \partial T \to \mathbb{R}$, define

$$h_\varphi(v) = \int_{\partial T} \varphi(\eta)\, d\mu_v(\eta).$$

Clearly h_φ is harmonic.

Claim:

$$\forall \xi \in \partial T \setminus \Omega, \quad \lim_{w \to \xi} h_\varphi(w) = \varphi(\xi). \tag{3.28}$$

Proof of claim. — Let $\varepsilon > 0$.

By continuity there is a cone $C(w_1)$ containing ξ for which

$$\forall \eta \in C(w_1), \quad |\varphi(\xi) - \varphi(\eta)| < \varepsilon.$$

For any vertex w between w_1 and $C(w_1)$,

$$|h_\varphi(w) - \varphi(\xi)| \leqq \int_{C(w_1)} |\varphi(\eta) - \varphi(\xi)|\, d\mu_w(\eta) + \int_{\partial T \quad C(w_1)} |\varphi(\eta) - \varphi(\xi)|\, d\mu_w(\eta) \leqq \varepsilon + 2\|\varphi\|_\infty \mu_w(\partial T \setminus C(w_1)) \leqq \varepsilon + 2\|\varphi\|_\infty f(w, w_1)$$

Since $\xi \in \partial T \setminus \Omega$ and

$$f(w, v_0) = f(w, w_1) \,.\, f(w_1, v_0),$$

it follows that

$$\limsup_{w \to \xi} |h_\varphi(w) - \varphi(\xi)| \leqq \varepsilon$$

proving the claim.

Alternatively, a harmonic function satisfying (2.6) can be obtained via Perron's method, *i.e.* as the supremum of a suitable class of subharmonic functions (*see* [W], chap. 15). The fact that both procedures generate the same harmonic function h_φ is a consequence of the

Maximum principle [Ts] theorem III.28

If h is a bounded subharmonic function on T, and for all $\xi \in \partial T$ except possibly a set of capacity zero $\limsup_{w \to \xi} h(w) \leqq 0$, *then* for all vertices v, $h(v) \leqq 0$.

Perhaps the simplest proof in our context is to consider the walk $\{Y_n\}_{n \geqq 0}$ on T started at v, $Y_0 \equiv v$, and define the stopping time $\tau_\varepsilon = \inf\{n \mid h(Y_n) \leqq \varepsilon\}$, where $\varepsilon > 0$. As harmonic measure vanishes on sets of capacity zero, τ_ε is finite almost surely. Since $\{h(Y_n)\}_{n \geqq 0}$ is a bounded submartingale,

$$h(v) = h(Y_0) \leqq E[h(Y_{\tau_\varepsilon})] \leqq \varepsilon,$$

completing the proof.

From the discussion above, any boundary point $\xi \in \partial T \setminus \Omega$ is regular for the Dirichlet problem. Conversely, if $\xi \in \Omega$, let $C(w_1)$ be a cone containing ξ with $\mu_{v_0}[C(w_1)] < 1$. The function φ which vanishes in $C(w_1)$ and takes the value 1 in $\partial T \setminus C(w_1)$ is continuous.

$$\liminf_{w \to \xi} h_\varphi(w) \geqq \liminf_{w \to \xi} f(w, v_0)\, \mu_{v_0}[\partial T \setminus C(w_1)] > 0,$$

so ξ is not regular.

(ii) Because of (3.27), (3.28) it remains only to verify Ω is an F_σ-set. Here is a representation as a union of closed sets:

$$\Omega = \bigcup_{k=1}^{\infty} \bigcap_{w} \left\{ C(w) \,|\, f(w, v_0) \geqq \frac{1}{k} \right\}. \quad \square$$

4. TREES FOR SUBSETS OF THE INTERVAL: PROOFS

For a compact $\Lambda \subset [0, 1]$, recall the definition of $T(\Lambda, b)$ given in the introduction. An infinite geodesic ray $\{v_n\}_{n=0}^{\infty}$ from the root $v_0 = [0, 1]$ of $T(\Lambda, b)$ is a nested sequence of closed b-adic intervals. Define a function σ from the abstract boundary $\partial T(\Lambda, b)$ to Λ by

$$\sigma(\xi) = \bigcap_{n=0}^{\infty} v_n \quad \text{if } \{v_n\}_{n=0}^{\infty} \text{ is the geodesic from } v_0 \text{ to } \xi. \quad (4.1)$$

Clearly σ is continuous. It satisfies the Hölder condition

$$|\sigma(\xi)-\sigma(\eta)| \leqq \rho(\xi, \eta)^{\log b} \tag{4.2}$$

where ρ, as usual, denotes the Gromov metric. The map σ is 1-1 except possibly at countably many points which are mapped to b-adic rationals. The harmonic measure μ_{v_0} on $\partial T(\Lambda, b)$ is mapped into $\mu_{v_0}\sigma^{-1}$ which is a measure on Λ (still called harmonic measure). The following lemma is central to the theme of this chapter.

LEMMA 9. — *Let* ν *be a probability measure on* $\partial T(\Lambda, b)$.

(i) *The potentials of* ν *and its image by* σ *are related by*

$$\int_0^1 \log\frac{1}{|x-y|}\, d\nu\sigma^{-1}(y) \geqq \log b \,.\, \int_{\partial T} \log\frac{1}{\rho(\xi, \eta)}\, d\nu(\eta) \tag{4.3}$$

where $x=\sigma(\xi)$.

(ii) *The energies are related by*

$$2\log b\,.\,(I(\nu)+1) \geqq I(\nu\sigma^{-1}) \geqq \log b\,.\,I(\nu). \tag{4.4}$$

(iii) *The multiplicative constants* $2\log b$, $\log b$ *in* (ii) *are best possible.* [Note that in (4.4) the energies are with respect to different metrics.]

Proof. — (i) Formula (4.3), and the right-hand side of (4.4) follow immediately from (4.2).

(ii) It remains to prove the left-hand side of (4.4).

Fix the base $b \geqq 2$, and denote

$$J_k^n = \left[\frac{k-1}{b^n}, \frac{k}{b^n}\right].$$

We need an explicit expression for the energy $I(\nu)$.

$$I(\nu) = \int_{\partial T}\int_{\partial T} \log\frac{1}{\rho(\xi, \eta)}\, d\nu(\xi)\, d\nu(\eta) = \int_{\partial T}\int_{\partial T} (\xi\,|\,\eta)\, d\nu(\xi)\, d\nu(\eta)$$
$$= \int_{\partial T}\int_{\partial T} \Big[\sum_{w\neq v_0} 1_{C(w)}(\xi)\, 1_{C(w)}(\eta)\Big]\, d\nu(\xi)\, d\nu(\eta) = \sum_{w\neq v_0} \nu[C(w)]^2.$$

This can be rewritten

$$I(\nu) = \sum_{n=1}^{\infty}\sum_{k=1}^{b^n} \nu_1(J_k^n)^2 \text{ where } \nu_1 = \nu\sigma^{-1}. \tag{4.5}$$

We turn to compute $I(\nu_1)$ in the interval. Note

$$\log\frac{1}{|x-y|} \leqq \log b\,.\left\lceil \log_b \frac{1}{|x-y|}\right\rceil \leqq \log b\,.\,\sum_{n=0}^{\infty} 1_{\{|x-y| \leqq b^{-n}\}}. \tag{4.6}$$

Now for $n \geqq 2$ we have

$$\{(x, y) \in [0, 1]^2 \,||\, x-y| \leqq b^{-n}\} \subset \bigcup_{r=1}^{b^{n-1}} [(J_r^{n-1} \times J_r^{n-1}) \cup (J_{rb}^n \times J_{rb+1}^n) \cup (J_{rb+1}^n \times J_{rb}^n)\}. \quad (4.7)$$

But product measure $\nu_1 \times \nu_1 = \nu\sigma^{-1} \times \nu\sigma^{-1}$ satisfies

$$(\nu_1 \times \nu_1)(J_{rb}^n \times J_{rb+1}^n) \leqq \frac{1}{2}[\nu_1 (J_{rb}^n)^2 + \nu_1 (J_{rb+1}^n)^2]$$

so for $n \geqq 2$

$$(\nu_1 \times \nu_1)\{(x, y) \,||\, x-y| \leqq b^{-n}\}$$
$$\leqq \sum_{r=1}^{b^{n-1}} (\nu_1 (J_r^{n-1})^2 + \nu_1 (J_{rb}^n)^2 + \nu_1 (J_{rb+1}^n)^2] \leqq 2 \sum_{r=1}^{b^{n-1}} \nu_1 (J_r^{n-1})^2. \quad (4.8)$$

Thus from (4.6) and (4.8) one gets

$$I(\nu_1) = \int_0^1 \int_0^1 \log \frac{1}{|x-y|} d\nu_1(x)\, d\nu_1(y)$$
$$\leqq \log b \,.\left[2 + \int_0^1 \int_0^1 \sum_{n=2}^{\infty} 1_{\{|x-y| \leqq b^{-n}\}}\right]$$
$$\leqq \log b \,.\left[2 + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{b^n} \nu_1 (J_k^n)^2\right].$$

Now use (4.5) to conclude that (4.4) holds.

(iii) Consider the tree $T = T(\Lambda, b)$ where $\Lambda = [b^{-1} - b^{-m}, b^{-1} + b^{-m}]$. The harmonic measure μ on ∂T is mapped by σ to $\mu\sigma^{-1}$, the uniform distribution on Λ. Straightforward computation shows

$$I(\mu\sigma^{-1}) = C_1 + m \log b, \qquad I(\mu) = C_2 + \frac{m}{2}$$

for suitable constants C_1, C_2, independent of m. Thus $2 \log b$ cannot be replaced by a smaller constant in (4.4). Similarly, by considering $\Lambda = [0, b^{-m}]$ it is verified that $\log b$ cannot be replaced by a larger constant in (4.4), □

Proof of theorem 2

If a probability measure ν_1 on Λ satisfies $I(\nu_1) < \infty$, it is neccessarily a continuous measure and there exists a measure ν on $\partial T(\Lambda, b)$ for which $\nu_1 = \nu\sigma^{-1}$. By (4.4), $I(\nu) < \infty$ so proposition 6 shows $T(\Lambda, b)$ is transient. Conversely, if $T(\Lambda, b)$ is transient we know harmonic measure μ on $\partial T(\Lambda, b)$ has finite energy so by (4.4), $I(\mu\sigma^{-1}) < \infty$ as well. □

We pass to the higher dimensional analogue of lemma 9. Start with the unit cube $[0, 1]^d$, partition it into b^d congruent subcubes, and continue partitioning them iteratively. This defines a map σ from the boundary of the b^d-tree (each vertex has b^d "sons") onto $[0, 1]^d$. This σ still satisfies (4.2), with $| . |$ the Euclidean norm.

PROPOSITION 10. — *Let* ν *be a probability measure on the boundary* ∂T *of the* b^d*-tree* T. *The map* σ *sends* ν *to a measure* $\nu\sigma^{-1}$ *on* $[0, 1]^d$ *for which*

$$\log b . (2^d I(\nu)+2) \geqq I(\nu\sigma^{-1}) \geqq \log b . I(\nu) \qquad (4.9)$$

and the multiplicative constants in (4.9) *are best possible.*

Proof. — The key is to generalize relation (4.7) above. Visualize the canonical tiling Γ_n of $[0, 1]^d$ by b^{nd} congruent closed subcubes of edge length b^{-n}. Every d-dimensional cube has $\binom{d}{r} 2^{d-r}$ faces of dimension r for $0 \leqq r \leqq d$, altogether 3^d faces. We shall use the following *observation*.

The 3^d *faces of a d-dimensional cube may be colored in* 2^d *colors so that no two faces of the same color intersect: simply assign each class of parallel faces of the same dimension a unique color.* As every vertex of the cube is contained in 2^d faces, the number of colors cannot be decreased.

Attach to each face F of $Q \in \Gamma_{n-1}$ the slab $S(F, Q)$ consisting of b^r cubes from Γ^n contained in Q, that intersect F, where $r = \dim F$. If $x, y \in [0, 1]^d$ are close: $|x-y| \leqq b^{-n}$, and $x \in Q_x \in \Gamma_{n-1}$, $y \in Q_y \in \Gamma_{n-1}$, then Q_x and Q_y intersect in some common face F, and $x \in S(F, Q_x)$, $y \in S(F, Q_y)$. Therefore, denoting $\nu_1 = \nu\sigma^{-1}$ we have:

$$(\nu_1 \times \nu_1)\{(x, y) \in [0, 1]^d \times [0, 1]^d \,|\, |x-y| \leqq b^{-n}\}$$
$$\leqq \sum_{F} \sum_{Q \cap Q' = F} \nu^1 [S(F, Q)]\, \nu_1 [S(F, Q')]$$

where the outer summation is over all faces F of cubes in Γ_{n-1}. Using

$$\nu_1 (S)\, \nu_1 (S') \leqq \frac{1}{2} [\nu_1 (S)^2 + \nu_1 (S')^2]$$

conclude that

$$(\nu_1 \times \nu_1)\{|x-y| \leqq b^{-n}\}$$
$$\leqq \sum_{Q \in \Gamma_{n-1}} \sum_{F \subset Q} \nu_1 [S(F, Q)]^2 \leqq 2^d \sum_{Q \in \Gamma_{n-1}} \nu_1 (Q)^2 \qquad (4.10)$$

where the right hand inequality follows from the coloring observation above and the fact that faces of Q with the same color are attached to disjoint slabs:

$$F, F_1 \subset Q,\ F \cap F_1 = \phi \ \Rightarrow\ S(F, Q) \cap S(F_1, Q) = \phi.$$

Since (4.5) translates to

$$I(\nu)=\sum_{n=1}^{\infty}\sum_{Q\in\Gamma_n}\nu_1(Q)^2$$

relations (4.6), (4.10) yield

$$I(\nu_1\leqq\log b\,.\left[2+\sum_{n=2}^{\infty}\int_0^1\int_0^1 1_{\{|x-y|\leqq b^{-n}\}}d\nu_1(x)\,d\nu_1(y)\right]$$

$$\leqq\log b\,.\left[2+2^d\sum_{n=2}^{\infty}\sum_{Q\in\Gamma_{n-1}}\nu_1(Q)^2\right]\leqq\log b\,.\,(2+2^d I(\nu)).$$

The right-hand inequality in (4.9) is an immediate consequence of the Hölder condition (4.2). The tightness of the multiplicative constants in (4.9) is verified by considering measures ν_1 uniformly distributed on $[b^{-1}-b^{-m}, b^{-1}+b^{-m}]^d$ (for the left-hand constant $2^d\log b$) and on $[0, b^{-m}]^d$ (for the right-hand constant $\log b$) respectively, where m is a large integer parameter.

We shall apply proposition 10 later; we move on to the harmonic measure.

Proof of proposition 3

(i) We are given a compact $\Lambda\subset[0, 1]$ with $m(\Lambda)>0$, where m denotes Lebesgue measure. Simple probabilistic considerations show harmonic measure $\mu_{\Lambda, b}$ satisfies

$$\mu_{\Lambda, b}(E)\geqq m(E\cap\Lambda)\quad\text{for Borel sets } E\subset[0, 1]. \tag{4.11}$$

Since it suffices to check (4.11) when E is a b-adic interval.

(ii) *See* example 5 in section 5. □

Proof of theorem 4

(i) Is an easy consequence of lemma 9. Let ν_1 be any probability measure of finite logarithmic energy on [0, 1]. In particular ν_1 has no atoms, so there exists a unique probability measure ν on $\partial T(\Lambda, b)$ for which $\nu_1=\nu\sigma^{-1}$ [σ is defined in (4.1)]. Also $\mu_{\Lambda, b}=\mu\sigma^{-1}$ where μ is harmonic measure relative to v_0 on $\partial T(\Lambda, b)$. Recall proposition 6 showed $I(\mu)\leqq I(\nu)$ on the abstract boundary. Now by utilizing both sides of (4.4) we find

$$I(\mu_{\Lambda, b})\leqq 2\log b\,.\,(I(\mu)+1)\leqq 2\log b\,(I(\nu)+1)\leqq 2\log b\left(\frac{I(\nu_1)}{\log b}+1\right)$$

proving (1.4).

Similarly for a compact set $K \subset [0, 1]^d$, the harmonic measure μ_K on K for the random walk on the b^d-tree defined before proposition 10 (starting from the root) satisfies

$$I(\mu_k) \leqq 2^d \inf_{\nu_1} (I(\nu_1) + \log b)$$

where ν_1 runs over probability measures on K. This follows from (4.9) in the same way.

(ii) This proof is divided into three steps, the first of which uses the proof of lemma 9.

For an *atomless* measure ν_1 on [0, 1], and $\alpha > 0$ denote:

$$\nu_\alpha(E) = \nu_1(\alpha^{-1} E), \qquad (\nu_\alpha \sigma)(E') = \nu_\alpha[\sigma(E')]$$

for Borel sets $E \subset [0, 1]$ and $E' \subset \partial T([0, 1], b)$. Thus $\nu_\alpha \sigma$ is a measure on $\partial T([0, 1], b)$.

Step 1. – $I(\nu_\alpha \sigma)$ depends continuously on $\alpha > 0$.

Proof. – From (4.5) we have

$$I(\nu_\alpha \sigma) = \sum_{n=1}^{\infty} \sum_{k=1}^{b^n} \nu_\alpha (J_k^n)^2 \qquad \text{where} \quad J_k^n = \left[\frac{k-1}{b^n}, \frac{k}{b^n}\right], \tag{4.12}$$

Since ν_1 has no atoms, the required continuity will follow once we verify the convergence in (4.12) is uniform in any interval $\alpha \in [b^{-l}, \infty)$. Given $E > 0$ choose n_ε so that

$$\sum_{n=n_\varepsilon}^{\infty} \sum_{k=1}^{b^n} \nu_1(J_k^n)^2 < \varepsilon. \tag{4.13}$$

Since

$$\bigcup_{k=1}^{b^{n+l}} J_k^{n+l} \times J_k^{n+l} \subset \{(x, y) \in [0, 1]^2 \,|\, |x-y| \leqq b^{-n-l}\}$$

we have for $\alpha \geqq b^{-1}$ the inequality

$$\begin{aligned}\sum_{k=1}^{b^{n+l}} \nu_\alpha (J_k^{n+l})^2 &\leqq (\nu_\alpha \times \nu_\alpha) \{(x, y) \,|\, |x-y| \leqq b^{-n-l}\} \\ &= (\nu_1 \times \nu_1) \{(x, y) \,|\, |\alpha x - \alpha y| \leqq b^{-n-l}\} \\ &\leqq (\nu_1 \times \nu_1) \{(x, y) \,|\, |x-y| \leqq b^{-n}\} \leqq 2 \sum_{r=1}^{b^{n-1}} \nu_1 (J_r^{n-1})^2 \end{aligned} \tag{4.14}$$

where the last inequality is precisely (4.8). Combing (4.13) and (4.14) gives

$$\sum_{n>n_\varepsilon+l} \sum_{k=1}^{b^n} \nu_\alpha (J_k^n)^2 < 2\varepsilon \quad (\alpha \geqq b^{-l})$$

which means uniform convergence in (4.12).

Step 2. — For compact $\Lambda \subset [0, 1]$ the capacity on the tree boundary $\text{cap}(\sigma^{-1}(\alpha\Lambda))$ depends continuously on $\alpha \in (0, \infty)$. (Note that for capacity in the interval there is nothing to prove.)

Proof. — It suffices to check

$$\text{If } \alpha_n \to 1 \text{ then } \lim_{n\to\infty} \text{cap}(\sigma^{-1}(\alpha_n \Lambda)) = \text{cap}(\sigma^{-1}\Lambda). \qquad (4.15)$$

Take ν_1 to be harmonic measure $\nu_1 = \mu_{\Lambda, b}$. Using the notation of step 1, ν_α is a probability measure supported on $\alpha\Lambda$. Therefore

$$\liminf_{n\to\infty} \text{cap}(\sigma^{-1}(\alpha_n \Lambda)) \geqq \liminf_{n\to\infty} [I(\nu_{\alpha_n}\sigma)]^{-1}$$
$$= I(\nu_1 \sigma)^{-1} = \text{cap}(\sigma^{-1}\Lambda) \quad (4.16)$$

by step 1.

By the definition of capacity, for each n there exists a positive measure w_n carried by $\alpha_n \Lambda$ for which

$$I(w_n \sigma) \leqq 1, \qquad w_n(\alpha_n \Lambda) > \text{cap}(\sigma^{-1}(\alpha_n \Lambda)) - \frac{1}{n}.$$

It suffices to prove (4.15) under the assumption that the limit there exists (by passing to subsequences). Any weak* limit point w of $\{w_n\}$ is a positive measure on Λ for which

$$I(w\sigma) \leqq 1, \qquad w(\Lambda) \geqq \lim_{n\to\infty} \text{cap}(\sigma^{-1}(\Lambda)) \qquad (4.17)$$

where the left inequality follows from the expression (4.5) for energy. Finally (4.17) and (4.16) combine to complete step 2.

Step 3. — For any interval $J \subset [0, 1]$,

$$\mu_{\alpha\Lambda, b}(J) \text{ depends continuously on } \alpha. \qquad (4.18)$$

(This clearly completes the proof of the theorem.)

Proof. — It suffices to check (4.18) when J is a b-adic interval, $J = J_k^n = \left[\frac{k-1}{b^n}, \frac{k}{b^n}\right]$. This is achieved by induction on n. For $n=0$ there is nothing to show. For $n>0$ formula (3.21), employed in the proof of proposition 7, may be interpreted as meaning that the harmonic measure $\mu_{\alpha\Lambda, b}(J_k^{n-1})$ of a b-adic interval J_k^{n-1} is divided among it's subintervals $\{J_{bk-i}^n \mid 0 \leqq i \leqq b-1\}$ in direct proportion to their (b-adic) capacity when intersected with $\alpha\Lambda$, *i.e.* $0 \leqq i \leqq b-1$, $1 \leqq k \leqq b^{n-1}$ we have

$$\mu_{\alpha\Lambda,b}(J^n_{bk-i}) = Z(\alpha)^{-1}\operatorname{cap}[\sigma^{-1}(\alpha\Lambda \cap J^n_{bk-i})] \cdot \mu_{\alpha\Lambda,b}(J^{n-1}_k)$$

where $Z(\alpha)$ is the "partition function"

$$Z(\alpha) = \sum_{j=0}^{b-1} \operatorname{cap}[\sigma^{-1}(\alpha\Lambda \cap J^n_{bk-j})].$$

Utilizing step 2, this verifies the desired continuity by induction. □

Remarks. — 1. Assume you are viewing a plane fractal $\mathscr{F}$ (with empty interior) on a graphics terminal. The screen is naturally partitioned into four subsquares, and you have the option to enlarge one of them (which intersects $\mathscr{F}$) to fill the whole screen, or to invert such a move and decrease the resolution. At each stage choose randomly among the available options (there are at most five). Theorem 4 (i) may be interpreted to imply that the picture obtained when first "hitting" a preassigned resolution is nicely distributed over $\mathscr{F}$, in the sense that this hitting distribution has minimal energy up to a bounded factor.

2. The limitations of the algorithm suggested by theorem 4 are revealed by comparing the harmonic measure for an interval, $\mu_{[0,1],b}$, which is uniform for any base, with the well known probability measure of minimal energy on [0, 1]. This measure is the asymptotic distribution of the zeros of the Tschebyscheff polynomials for [0, 1], and is given by the density $\dfrac{dx}{\pi\sqrt{x(1-x)}}$.

3. By (3.14), the expected number of visits to the root in $T(\Lambda, b)$ is $g(v_0, v_0) = d_{v_0}/\operatorname{cap}(\sigma^{-1}(\Lambda))$ where d_{v_0} is the degree of the root. Hence the Greens function $g(v_0, v_0)$ for $T(\alpha\Lambda, b)$ has a finite number of discontinuities as a function of α.

4. It is a well known and useful fact, that Hausdorff measure on the interval may be estimated up to a bounded factor, by Hausdorff measure on a tree (*see* [Fu] and [C], chap. 2). By lemma 9 this holds for logarithmic energies as well. We now study this question for potentials. For certain "regular" Cantor sets, It was proved by Ohtsuka (in a different language) that the potential does not blow up when passing from the tree to the interval; *see* [C], section IV, theorem 3. The general case is different.

Preliminaries to the proof of theorem 5

1. $\mathbb{Z}(b) = \bigcup_{n=0}^{\infty} b^{-n}\mathbb{Z}$ denotes the b-adic rationals.

2. For $x \notin \mathbb{Z}(b)$ define the b-adic *approximation exponent:*

$$e_b(x) = \sup\left\{\frac{-\log|x - rb^{-n}|}{n-1} \,\middle|\, n > 1,\ r \in \mathbb{Z}\right\}. \tag{4.19}$$

3. Let

$$S_b = \{x \in [0, 1] \mid x \notin \mathbb{Z}(b),\ e_b(x) = \infty\}.$$

The set S_b consists of numbers which have superb approximations by b-adic rationals, *i.e.* $x \in S_b$ iff there exist sequences $\{n_j\}$, $\{k_j\} \subset \mathbb{N}$, and $\gamma_j \to \infty$ such that

$$0 < |x - k_j b^{-n_j}| < b^{-\gamma_j n_j}. \tag{4.20}$$

For instance, $x = \sum_{j=1}^{\infty} b^{-j!}$ is in S_b. The main part of the proof is establishing

LEMMA 11. – $\bigcup_{\Lambda} s(\Lambda, b) = S_b$, *where the union is over all compact* $\Lambda \subset [0, 1]$. [Recall $s(\Lambda, b)$ is defined in (1.5).]

Proof. – Given $x \in S_b$ there exist sequences $\{k_j\}$, $\{n_j\} \subset \mathbb{N}$, $\gamma_j \to \infty$, satisfying (4.20). Taking integer parts, we may assume

$$\{\gamma_j\} \subset \mathbb{N}.$$

Passing to a subsequence, we way also assume

$$\gamma_1 > 5, \qquad \gamma_{j+1} > 3\gamma_j, \qquad n_{j+1} > n_j \tag{4.21}$$

and that $x - k_j b^{-n_j}$ has constant sign.

Without loss of generality $x > k_j b^{-n_j}$ for all $j \geq 1$.

Denote $\Delta_j = [k_j b^{-n_j} - b^{-\gamma_j n_j}, k_j b^{-n_j}]$ and $\Lambda = \bigcup_{j=1}^{\infty} \Delta_j \cup \{x\}$ (see *Fig.* 1).

Let $\mu = \mu_{\Lambda, b}$ be the harmonic measure for $T(\Lambda, b)$ on Λ. Since harmonic measure for a b-adic interval is uniformly distributed on it (by the symmetry of the corresponding tree) it follows that μ has a representation:

$$d\mu(t) = \sum_{j=1}^{\infty} \frac{m_j}{|\Delta_j|} 1_{\Delta_j}(t)\,dt$$

where $|\Delta_j|$ is the length of Δ_j and $m_j = \mu(\Delta_j)$. Now we estimate the potential of μ at x, using

$$t \in \Delta_j \quad \Rightarrow \quad |x - t| < 2b^{-\gamma_j n_j}.$$

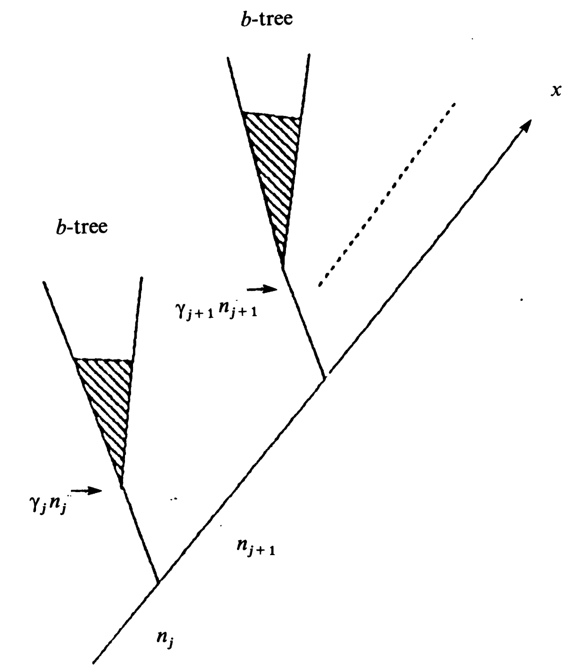

FIG. 1

We get

$$\phi_\mu(x)=\int_0^1 \log \frac{1}{|x-t|}\, d\mu(t)=\sum_{j=1}^{\infty} \frac{m_j}{|\Delta_j|}\int_{\Delta_j} \log \frac{1}{|x-t|}\, dt$$
$$\geqq \sum_{j=1}^{\infty} \frac{m_j}{|\Delta_j|}\int_{\Delta_j} \log \left(\frac{1}{2}\, b^{\gamma_j n_j}\right) dt=\log b.\left(\sum_{j=1}^{\infty} m_j \gamma_j n_j\right)-\log 2$$

which means

$$\phi_\mu(x)=\infty \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} m_j \gamma_i n_j=\infty. \tag{4.22}$$

Our goal now is to verify the right-hand side of (4.22). For this we use the fact that the (b-adic) potential of the harmonic measure $\mu\sigma$ on the tree boundary $\partial T(\Lambda, b)$ is constant a.e. [$\mu\sigma$]. If $\xi \in \sigma^{-1}(\Delta_l)$, $\eta \in \sigma^{-1}(\Delta_j)$ with $j<l$, their meeting height [*see* (2.2)] is $(\xi|\eta)=n_j$ while if $\xi, \eta \in \sigma^{-1}(\Delta_l)$, their meeting height is at least $\gamma_l n_l$. Therefore if $\sigma(\xi) \in \Delta_l$ is not a b-adic rational,

$$\phi_{\mu\sigma}(\xi)=\sum_{j=1}^{l-1}\int_{\Delta_j} n_j\, d\mu+\int_{\Delta_l}\left(\gamma_l n_l+\frac{1}{b}+\frac{1}{b^2}\cdots\right) d\mu+\sum_{r=l+1}^{\infty}\int_{\Delta_r} n_l\, d\mu$$

Since $\phi_{\mu\sigma}(\xi) = \mathrm{I}(\mu\sigma)$ almost everywhere $[\mu\sigma]$, this implies (denoting $\mathrm{M}_l = \sum_{r \geqq l} m_r$) that

$$\forall\, l \geqq 1, \quad \mathrm{I}(\mu\sigma) = \sum_{j=1}^{l-1} n_j m_j + \left(\gamma_l n_l + \frac{1}{b-1}\right) m_l + n_l \mathrm{M}_{l+1}. \tag{4.23}$$

Comparing (4.23) for two consecutive values of l gives:

$$\left(\gamma_l n_l + \frac{1}{b-1}\right) m_l + n_l \mathrm{M}_l = \left(\gamma_{l-1} n_{l-1} + \frac{1}{b-1}\right) m_{l-1} + n_{l-1} \mathrm{M}_l. \tag{4.24}$$

Thus

$$\gamma_l n_l m_l \geqq \gamma_{l-1} n_{l-1} m_{l-1} - n_l \mathrm{M}_l. \tag{4.25}$$

Now (4.24) also implies

$$\gamma_l n_l m_l \leqq \gamma_{l-1} n_{l-1} m_{l-1}$$

and this combined with (4.21) shows

$$m_l < \frac{1}{3} m_{l-1} \quad \text{and consequently} \quad \mathrm{M}_l < 2 m_l. \tag{4.26}$$

Applying (4.23) we know

$$n_l m_l < \frac{\mathrm{I}(\mu\sigma)}{\gamma_l}$$

so that (4.25), (4.26) imply

$$\gamma_l n_l m_l > \gamma_{l-1} n_{l-1} m_{l-1} - 2 \frac{\mathrm{I}(\mu\sigma)}{\gamma_l}. \tag{4.27}$$

Observe that (4.23) for $l=1$ guarantees

$$\gamma_1 n_1 m_1 > \frac{1}{2} \mathrm{I}(\mu\sigma).$$

Since by (4.21), $\sum_{l=2}^{\infty} \frac{1}{\gamma_l} \leqq \frac{1}{10}$, it follows from (4.27) that $\{\gamma_l n_l m_l\}_{l=1}^{\infty}$ is bounded below by a positive constant, and certainly the sum in (4.22) diverges, *i.e.* $x \in s(\Lambda, b)$.

Conversely suppose that $x \notin \mathrm{S}_b$.

We shall verify that for *any* continuous measure μ on $\Lambda \subset [0, 1]$ for which the pullback $\mu\sigma$ has bounded logarithmic potential on $\partial \mathrm{T}(\Lambda, b)$, necessarily

$$\phi_\mu(x) < \infty. \tag{4.28}$$

In particular (4.28) will hold for harmonic measure. The verification is divided into two cases:

Case I: x is not a b-adic rational.

In this case we know that $e_b(x) < \infty$, where $e_b(x)$ is the approximation exponent defined in (4.19). Let $\xi, \eta \in \partial T(\Lambda, b)$ with $\sigma(\xi) = x$, $\sigma(\eta) = y$. Notice that

$$\log \frac{1}{|x-y|} \leqq e_b(x) \log \frac{1}{\rho(\xi, \eta)}. \tag{4.29}$$

Indeed if $(\xi | \eta) = \eta$ (and then $\rho(\xi, \eta) = e^{-n}$), there must be a b-adic rational $\frac{r}{b^{n+1}}$ between x and y. Thus

$$|x-y| \geqq \left| x - \frac{r}{b^{n+1}} \right|.$$

Recalling the definition of $e_b(x)$ in (4.19), this implies (4.29) and consequently (4.28).

Case II: $x = \frac{k}{b^l}$ is a b-adic rational.

In this case there are two points $\xi_1, \xi_2 \in \partial T(\Lambda, b)$ for which $\sigma(\xi_1) = \sigma(\xi_2) = x$. It is easy to see that

$$\forall n > l, \quad \text{if } |y-x| < b^{-n},\ y = \sigma(\eta),$$

then, either $\rho(\xi_1, \eta) < e^{-n}$ or $\rho(\xi_2, \eta) < e^{-n}$ so

$$\log \frac{1}{|x-y|} \leqq \log b . \max \left\{ \log \frac{1}{\rho(\xi_1, \eta)}, \log \frac{1}{\rho(\xi_2, \eta)} \right\}$$

which proves (4.28) in this case as well. This completes the proof of Lemma 11.

Remark. – For $x \in S_b$ the set Λ and the measure $\mu = \mu_{\Lambda, b}$ constructed in the proof above satisfied

$$\phi_{\mu\sigma}(\sigma^{-1})) = \sum_{j=1}^{\infty} n_j m_j = I(\mu\sigma) - \lim_{l \to \infty} \gamma_l n_l m_l < I(\mu\sigma).$$

[*see* (4.23) and the argument leading to it]. As we have already observed in the proof of proposition 8, the strict inequality $\phi_{\mu\sigma}(\xi) < I(\mu\sigma)$ means that ξ is a *non-regular* point for the Dirichlet problem on $\partial T(\Lambda, b)$.

Proof of theorem 5

(i) Lemma 11 showed the potential ϕ_μ of harmonic measure could be infinite at a single point. Now we construct Λ for which $s(\Lambda, b)$ is uncountable.

Let $\{l_m\}_{n=1}^{\infty}$ be a rapidly decaying sequence of negative powres of b, say $l_n = b^{(8^{n!})}$. Define sets $\Omega^{(n)}$, $\Delta^{(n)}$ as follows.

$$\left.\begin{aligned} \Omega^{(1)} &= (0, l_1) \cup (1 - l_1, 1), \\ \Delta^{(1)} &= [l_1, l_1 + l_2] \cup [1 - l_1 - l_2, 1 - l_1]. \end{aligned}\right\} \tag{4.30}$$

For $n > 1$ assume $\Omega^{(n-1)}$, $\Delta^{(n-1)}$ have been defined, and $\Omega^{(n-1)}$ open intervals of length l_{n-1}. From each of these intervals, assign the two extreme subintervals (open) of length l_n to $\Omega^{(n)}$, and the neighboring subintervals (closed) of length l_{n+1} to $\Delta^{(n)}$, analogously to (4.30). This defines $\Omega^{(n)}$ and $\Delta^{(n)}$.

Finally, let

$$\Omega = \bigcap_{n=1}^{\infty} \Omega^{(n)}, \quad \Delta = \bigcup_{n=1}^{\infty} \Delta^{(n)}, \qquad \Lambda = \Omega \cup \Delta.$$

See *Fig.* 2

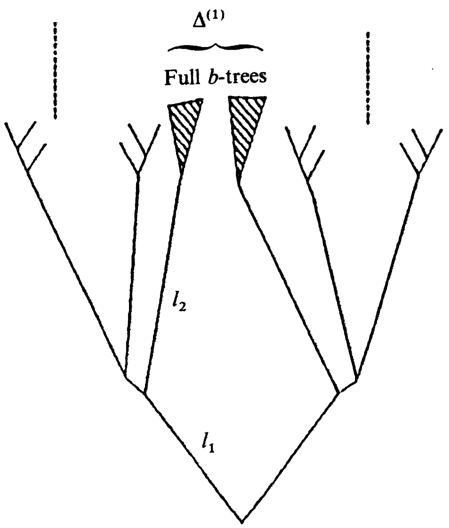

FIG. 2

We claim that

$$s(\Lambda, b) = \Omega. \tag{4.31}$$

Note that Ω is the Cantor set $\bar{\Omega}$ with countably many points removed, and $\operatorname{cap}(\bar{\Omega}) = 0$. As Δ is a countable union of intervals and $s(\Lambda, b)$ cannot contain b-adic rationals by lemma 11, clearly

$$s(\Lambda, b) \subset \Omega$$

and we must show the converse.

Now $\Delta^{(n)}$ is union of intervals of length l_{n+1}:

$$\Delta^{(n)} = \bigcup_{k=1}^{2^n} \Delta_k^{(n)}.$$

By symmetry, they all get the same harmonic measure, $m_n = \mu(\Delta_k^{(n)})$, where $\mu = \mu_{\Lambda, b}$. The measure μ is spread uniformly over each interval $\Delta_k^{(n)}$. As $\operatorname{cap}(\Omega) = 0$, it follows that $\mu(\Omega) = 0$; this is also easy to verify probabilistically. Pick $\xi \in \partial T(\Lambda, b)$ such that $\sigma(\xi) \in \Delta^{(n)}$. Computing potentials as in lemma 11, one finds

$$I(\mu, \sigma) = \phi_{\mu\sigma}(\xi) > 8^{(n+1)!} m_n + \sum_{j=1}^{n-1} 8^{j!} m_j > \frac{1}{3} I(\mu\sigma).$$

The left-hand side implies $m_j < 8^{-(j+1)!} I(\mu\sigma)$ and then the right hand side gives

$$8^{(n+1)!} m_n > \frac{1}{10} I(\mu\sigma). \tag{4.32}$$

Let $x \in \Omega$. Since x is not a b-adic rational, it follows from the construction of Λ that

$$\text{For infinitely many } n, \text{ some } k \text{ satisfies } \Delta_k^{(n)} \subset \{ t \mid |t - x| \leqq 2 l_{n+1} \}. \tag{4.33}$$

Therefore

$$\phi_\mu(x) \geqq \Sigma' m_n \log \frac{1}{2 l_{n+1}} = \Sigma' m_n (8^{(n+1)!} \log b - \log 2)$$

where Σ' indicates the summation is over those n satisfying (4.33). Finally (4.32) shows $\phi_\mu(x) = \infty$, *i.e.* $x \in s(\Lambda, b)$, completing the proof of (4.31).

Remark. – Analogously to the remark before proof of theorem 5 one can verify that if $\sigma(\xi) \in \Omega$ where $\xi \in \partial T(\Lambda, b)$, then

$$\phi_{\mu\sigma}(\xi) = I(\mu) - \lim_{h \to \infty} 8^{(n+1)!} m_n < I(\mu).$$

Consequently, all points of $\sigma^{-1}(\Omega)$ are *non-regular* points for the Dirichlet problem on $\partial T(\Lambda, b)$.

(ii) By Lemma 11, all points of $s(\Lambda, b)$ are Liouville numbers. We verify $\operatorname{cap}(s(\Lambda, b)) = 0$. Recall $\mu = \mu_{\Lambda, b}$ is harmonic measure. Let ν be any measure on [0, 1] with bounded logarithmic potential ϕ_ν. By Fubini,

$$\int_0^1 \phi_\mu \, d\nu = \int_0^1 \int_0^1 \log \frac{1}{|x-y|} \, d\mu(y) \, d\nu(x) = \int_0^1 \phi_\nu \, d\mu < \infty.$$

Therefore $\nu(s(\Lambda, b)) = 0$, as required.

(iii) The fact that $S_b = \bigcup_{\Lambda} s(\Lambda, b)$ (*see* lemma 11) has Hausdorff dimension zero, is immediate from the definition: For any

$$0<\alpha<1, \quad \mathscr{C}_N = \{ (kb^{-n} - b^{-2n/\alpha}, kb^{-n} + b^{-2n/\alpha}) \mid n \geqq N, 0 \leqq k \leqq 2^n \},$$

defines a sequence of covers of S_b which shows S_b has vanishing α-dimensional Hausdorff measure.

Consider the subset

$$A = \left\{ \sum_{n=1}^{\infty} \sum_{k=n!+1}^{2n!} \varepsilon_k b^{-k} \mid \varepsilon_k \in \{0, 1, \ldots, b-1\} \right\} \tag{4.34}$$

of S_b.

The distribution ν supported on A, obtained by taking the $\{\varepsilon_k\}$ in (4.34) as independent symmetrical random variables, has bounded b-adic logarithmic potential when pulled back to the tree boundary $\partial T(A, b)$:

$$\phi_{\nu\sigma}(\xi) \leqq \sum_{n=1}^{\infty} (n+1)!\, 2^{-n!} < \infty \quad \text{for all boundary points } \xi.$$

By lemma 9, $I(\nu) < \infty$ in the interval.

This show $\operatorname{cap}(S_b) > 0$.

5. EXAMPLES

Example 0

A transient tree with polynomial growth.

This example is well known. Fix $1 < \alpha < 2$. Start with a binary tree, and replace each edge between "level" k and "level" $k+1$ by a segment of $[\alpha^k]$

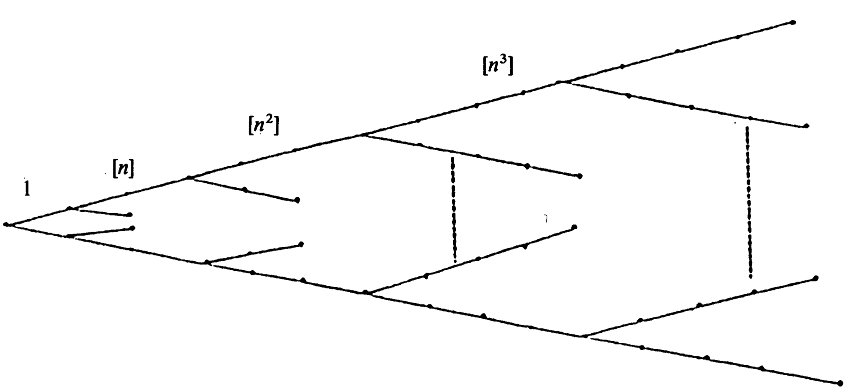

FIG. 3

edges (see *Fig.* 3), thereby constructing a tree T_0. From level k of the original binary tree the probability of hitting level $k+1$ before hitting level $k-1$ tends to $\frac{2}{2+\alpha}$ as $k \to \infty$. Thus the random walk on these levels has a positive "drift", which implies transience. The growth function of T_0 (*i.e.* the number of vertices in a ball of radius n around the root) grows like n^β where $\beta = \alpha \frac{\log 2}{\log \alpha} > 2$.

Example 1

A recurrent tree with exponential growth. (This example appears in [L2].) Let T_1 be the tree, depicted in Figure 4, with 2^n verticles in level n.

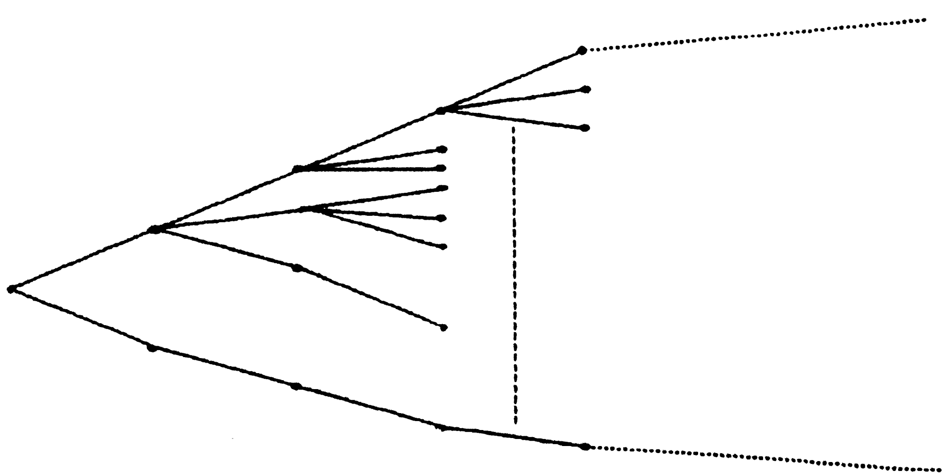

FIG. 4

The root is connected to both vertices in level 1. For $n \geqq 1$, $1 \leqq k \leqq 2^{n-1}$, the k'th vertex in level n has three sons numbered $3k-2$, $3k$ in level $n+1$, and for $2^{n-1}+1 \leqq k \leqq 2^n$, vertex k has a unique son numbered $k+2^n$ in level $n+1$. Observe that for any geodesic ray ξ from the root except the top ray, all but finitely many of the vertices on ξ have just one "son" in the next level. Hence the boundary ∂T_1 is countable (simply map each boundary point to the last vertex on its ray which has a "brother"). By proposition 6, T_1 is recurrent; it is also easy to give a direct probabilistic argument. This example is closely related to the well known fact that many countable sets in [0, 1] have positive box dimension (*cf.* [Ta]).

Example 2

A recurrent tree with exponential growth above any vertex. (This example is due to B. Weiss.)

Given a rapidly increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, construct a tree T_2 as follows (see *Fig.* 5).

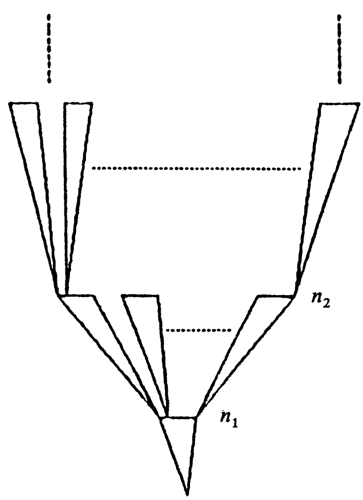

FIG. 5

The first n_1 levels are as in T_1, with root v_0. Above each vertex v in level n_1, add a copy of T_1 with v as it's root, up to level n_2 and continue in the same manner. Since T_1 is recurrent, the n_i may be chosen so that the probability of the random walk returning to it's origin v_0 before hitting level n_j, tends to 1 as $i \to \infty$.

In view of theorem 2, T_2 provides an easily accessible example of a compact set $\Lambda \subset [0, 1]$ with vanishing logarithmic capacity, and positive box dimension in every open interval which intersects Λ.

Example 3

Non-regular points for the Dirichlet problem.

In proposition 8 it is shown that for any tree T, the set of non-regular points has vanishing logarithmic capacity. It is easy to construct nonregular points by adding a recurrent tree to a transient tree. All the ends of the recurrent tree will be nonregular points, *outside* the support of harmonic measure μ on ∂T. To construct a nonregular point inside supp (μ), consider an infinite geodesic ray (a copy of the positive integers), $\{v_n\}_{n=\infty}^{\infty}$. Choose a subsequence $\{n_j\}$ of $\mathbb{N}$; to each v_{n_j} attach an additional long geodesic segment of length N_j, followed by a full 3-tree $\tau_{j,3}$ (each vertex has 3 sons). This defines a tree T_3. If these lengths N_j increase rapidly enough, then the probability of reaching v_o, starting from any v_n is bounded away from zero. The end ξ of $\{v_n\}_{n=0}^{\infty}$ is thus, by proposition 8, a nonregular point. Similarly, one may construct uncountably many nonregular points in the support of harmonic measure. Trees of this type were given explicitly in the proofs of lemma 11 and theorem 5 (see *Fig.* 1, 2).

We now iterate these constructions.

Let $T_3^{(1)}$ denote T_3 with every full 3-tree $\tau_{j,3}$ replaced by a full binary tree $\tau_{j,2}$. To each vertex w of $\tau_{j,2}$ (for every j) add a copy of $T_3^{(1)}$ with w as its root, thus defining $T_3^{(2)}$. Continuing in the same manner, we get a sequence $T_3^{(1)}, T_3^{(2)}, T_3^{(3)}, \ldots$ of trees which converge to a tree $T_3^{(\infty)}$. The harmonic measure μ for this tree has the whole boundary $\partial T_3^{(\infty)}$ for its closed support, and clearly the nonregular points are dense there. By starting from the tree in the theorem 5 instead of $T_3^{(1)}$, this construction yields an uncountable dense set of nonregular points.

Example 4

A "misguided" harmonic measure.

Start with a full binary tree. To every vertex w add a copy $T_{0,w}$ of T_0 with w as its root, where T_0 denotes a fixed transient tree of polynomial growth. This defines T_4. By Borel-Cantelli, with probability one the random walk on T_4 converges to an end of $T_{0,w}$ for some w. Thus, even though ∂T_4 has positive Hausdorff dimension, the harmonic measure there has dimension zero.

T_4 is analogous to a classical example of Kakutani in the plane. We now describe this construction in the interval.

Let $\mathscr{C}$ be the ternary Cantor set in [0, 1]. Then $T(\mathscr{C}, 3)$ is a binary tree. Let $\Gamma \subset [0, 1]$ be such that $T(\Gamma, 3) = T_0$. To every complementary interval $(a, a+c)$ of $\mathscr{C}$, add a dilated copy $a+c\Gamma$ of Γ. This defines a compact set $\Lambda^* \supset \mathscr{C}$ for which $T(\Lambda^*, 3) = T_4$. The motivation for transferring the construction to the interval is the possibility of changing bases.

PROPOSITION 12. – *For any base b, the harmonic measure $\mu_{\Lambda^*, b}$ satisfies $\mu_{\Lambda^*, b}(\mathbf{C}) = 0$, and consequently* H-dim $(\mu_{\Lambda^*, b}) = 0$.

Proof. – For $b=3$, this is contained in the discussion above. To obtain the general case consider any open interval J which intersects $\mathscr{C}$. Recall the standard construction

$C=\bigcap_{n=1}^{\infty} C^{(n)}$, where $C^{(n)}$ is a union of 2^n closed intervals of length 3^{-n} each,

$$C^{(n)}=\bigcup_{k=1}^{2^n} C_k^{(n)}.$$

Let

$$n_0=\min\{n\geqq 1 \mid J\supset C_k^{(n)} \text{ for some } k\}.$$

It is easily verified that J intersects at most four of the intervals $\{C_k^{(n_0)}\}_{k=1}^{2^{n_0}}$, and J contains a dilated copy of Γ

$$J\supset a+3^{-n_0-1}\Gamma \quad \text{for an appropriate } a.$$

From theorem 4 (step 2) and the expression (3.21) relating harmonic measure and capacity, it follows that the ratio

$$\mu_{\Lambda^*, b}(a+3^{-n_0-1}\Gamma)/\mu_{\Lambda^*, b}(J)$$

is bounded away from zero, uniformly in J. Applying Borel-Cantelli concludes the proof. □

Example 5

More on harmonic measure.

We extend the previous example to prove theorem 3 (ii). We are given $\Lambda_0\subset[0, 1]$ with $m(\Lambda_0)=0$, where m denotes Lebesgue measure. Think of $T(\Lambda_0, b)$ as a sub-tree of the full b-tree $T([0, 1], b)$. For every vertex w of $T(\Lambda_0, b)$ with $j<b$ sons, add $b-j$ copies of T_0 with w as their root. The resulting tree is still a subtree of $T([0, 1], b)$ and can be represented as $T(\Lambda, b)$. The set Λ is obtained from Λ_0 by adjoining countably many dilated copies of the set Γ defined in example 4. Since $\mu_{[0, 1], b}=m$, the random walk on $T(\Lambda_0, b)$ hits infinitely many vertices which have less than b sons, with probability 1. Again, Borel-Cantelli shows $\mu_{\Lambda, b}(\Lambda_0)=0$ which implies

$$\text{H-dim}(\mu_{\Lambda, b})=0. \tag{5.1}$$

Now we iterate this construction to obtain a compact set $\Lambda_\infty\supset\Lambda_0$ with local Hausdorff dimension 1, *i.e.*

$$\text{H-dim}(\Lambda_\infty\cap J)=1 \quad \text{for any open interval J which intersects } \Lambda_\infty, \tag{5.2}$$

such that (5.1) still holds for $\mu_{\Lambda_\infty, b}$. Without loss of generality

$$\text{H-dim}(\Lambda_0)=1.$$

Call the set Λ constructed above Λ_1. To every vertex in $T(\Lambda_1, b)\setminus T(\Lambda_0, b)$ adjoin a long geodesic segment followed by a copy of $T(\Lambda_1, b)$, where the lengths of these segments increase very rapidly. This defines a tree $T(\Lambda_2, b)$. To every vertex w in $T(\Lambda_2, b)$ such that the cone above w has dimension zero, adjoin an (enormous) geodesic segment followed by a copy of $T(\Lambda_1, b)$, thus creating $T(\Lambda_3, b)$, etc. Let Λ_∞ denote the closure of $\bigcup_{n=1}^{\infty} \Lambda^n$. If the segments referred to above were chosen long enough, the random walk on $T(\Lambda_\infty, b)$ will visit only finitely many trees $T(\Lambda_n, b)$, with probability one. Therefore the random walk will converge to an end of one of the countably many copies of T_0 adjoined in the construction. The requirement (5.2) is clearly satisfied.

Extending proposition 12, one can produce a set $\Lambda' \supset \Lambda_0$ such that $\text{H-dim}(\mu_{\Lambda', b}) = 0$ for all $b > 1$.

Example 6

Changing bases and singular measures.

Let $\Lambda = \left\{ \sum_{n=1}^{\infty} a_n 10^{-2n} \,\middle|\, \varepsilon_n \in \{0, 1, 2, \ldots, 90\} \right\}$. $T(\Lambda, 10)$ and $T(\Lambda, 100)$ are depicted in Figure 6.

We claim that the harmonic measures $\mu_{10} = \mu_{\Lambda, 10}$ and $\mu_{100} = \mu_{\Lambda, 100}$ are relatively singular. Let $\varepsilon_n(x) = [10^{2n} x]$ denote the n'th digit of x in base 100.

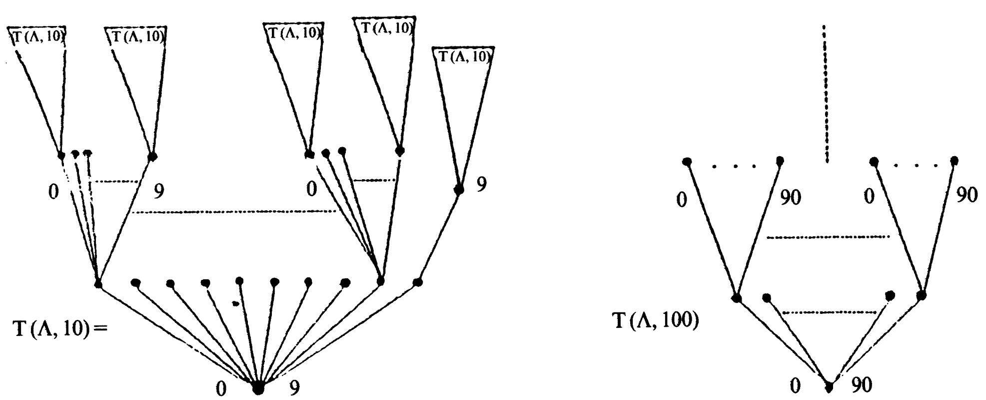

FIG. 6

According to μ_{100}, the digits $\{\varepsilon_n\}_{n=1}^{\infty}$ are i.i.d. symmetric random variables:

$$\mu_{100}[\varepsilon_n = j] = \frac{1}{91} \quad \text{for} \quad 0 \leqq j \leqq 90. \tag{5.3}$$

As every vertex in level 2 of $T(\Lambda, 10)$ has above it at least a full 9-tree, it is easily seen that

$$\mu_{10}[\varepsilon_1 = 90] > \frac{1}{40}. \tag{5.4}$$

The self similarity of Λ implies that $\{\varepsilon_n\}_{n=1}^{\infty}$ are i.i.d. also under μ_{10}, so (5.3), (5.4) show μ_{10} and μ_{100} are singular. Note that μ_{10}, μ_{100} both have positive Hausdorff dimension.

Question 1

Consider the ternary Cantor set $\mathscr{C}$. Is $\mu_{\mathscr{C},2}$ singular relative to Cantor measure $\mu_{\mathscr{C},3}$?

Question 2

If $\Lambda \subset [0, 1]$ is compact, and H-dim$(\mu_{\Lambda, b}) > 0$, does it follow that H-dim$(\mu_{\Lambda, b'}) > 0$ for all $b' > 1$?

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