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Fernique type inequalities and moduli of continuity for l^2 -valued Ornstein-Uhlenbeck Processes

by

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ABSTRACT. — We establish Fernique type inequalities and utilize them to study large and small moduli of not necessarily Gaussian processes with values in a separable Banach space. In particular, we prove moduli of continuity results for l^2 -valued Ornstein-Uhlenbeck processes and for their l^2 -norm squared process.

RÉSUMÉ. — On établit des inégalités du type Fernique pour des processus éventuellement non gaussiens à valeurs dans un espace de Banach sépa-

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nable. On utilise ces inégalités pour étudier les lois des familles d'accroissements de ces processus. En particulier, on établit des modules de continuité pour les processus d'Ornstein-Uhlenbeck à valeurs dans l_2 et pour le carré de leur norme.

1. INTRODUCTION

Fernique type inequalities for not necessarily Gaussian processes were established recently by Kalinauskaitė (1986), and Csáki and Csörgő (1990, 1992). The aim and role of these inequalities in studying path properties of processes in general are similar to those for Gaussian and sub-Gaussian processes (cf. Jain and Marcus (1978), Talagrand (1987), Adler (1990)). For example, combining their inequalities for increments of Banach space valued processes with results of Fernique (1989), Csáki and Csörgő (1992) establish moduli of continuity estimates for l^2 -valued Ornstein-Uhlenbeck processes, as well as for the l^2 -norm squared process of these.

The main aim of this exposition is to extend and improve the general results of Csáki and Csörgő (1992) for not necessarily Gaussian processes, as well as their results for the just mentioned l^2 -valued and l^2 -norm squared processes. For related and further results on l^p -valued Ornstein-Uhlenbeck processes we refer to Fernique (1990), Schmuland (1990), and Csáki, Csörgő and Shao (1992).

In Section 2 we state and prove Fernique type inequalities for not necessarily Gaussian processes with values in a separable Banach space. Compared with similar results of Csáki and Csörgő (1992), one of the innovations here is the introduction of the notions of quasi-increasing and quasi-decreasing functions. This leads to improving these earlier results, which are formulated in terms of regularly varying functions.

The inequalities of Section 2 are put to use in Section 3 for proving general theorems on increments, both large and small, of stochastic processes. Most of the existing similar results we are aware of are corollaries of these under weaker growth conditions on the functions figuring in their rates of estimation.

We study path properties of l^2 -valued Ornstein-Uhlenbeck processes in Section 4, and those of their l^2 -norm squared process in Section 5.

The moduli of continuity results in Section 4 are sharp under fairly general growth conditions on the coefficients involved. We establish also

criteria for these coefficients which render the normalizing functions of the moduli of continuity quasi-increasing.

2. INEQUALITIES

The aim of this section is to establish Fernique type inequalities for not necessarily Gaussian processes. The following results are also extensions of inequalities in Csáki and Csörgő (1992).

LEMMA 2.1. — *Let \mathcal{B} be a separable Banach space with norm $\| \cdot \|$ and let $\{ \Gamma(t), -\infty < t < \infty \}$ be a stochastic process with values in \mathcal{B} . Let \mathbf{P} be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is \mathbf{P} -almost surely continuous with respect to $\| \cdot \|$ and that for $|t| \leq t_0$, $0 < x^* \leq x$ and $0 < h \leq h_0$ there exist non-negative monotone non-decreasing functions $\sigma_1(h)$ and $\sigma_2(h)$ such that*

$$(2.1) \quad \mathbf{P} \{ \| \Gamma(t+h) - \Gamma(t) \| \geq x \sigma_1(h) + \sigma_2(h) \} \leq K \exp(-\gamma x^\beta)$$

with some $K, \gamma, \beta > 0$. Then

$$(2.2) \quad \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \| \Gamma(t+s) - \Gamma(t) \| \geq x (\sigma_1(a) + \sigma_1(a, k)) + \sigma_1^*(a, k) + \sigma_2(a) + \sigma_2(a, k) \right\} \leq 4 \left(\frac{T}{a} + 1 \right) K \cdot 2^{2k+1} \exp(-\gamma x^\beta)$$

for any $0 \leq T \leq t_0$, $0 < a \leq h_0$, $x \geq x^*$ and any positive $k \geq 3$, where

$$\begin{aligned} \sigma_1(a, k) &= 2^{3+(1/\beta)} \int_{2^{k-3}}^\infty \frac{\sigma_1(ae^{-z})}{z} dz, \\ \sigma_2(a, k) &= 6 \int_{2^{k-3}}^\infty \frac{\sigma_2(ae^{-z})}{z} dz, \\ \sigma_1^*(a, k) &= 4 \left(\frac{14}{\gamma} \right)^{1/\beta} \beta \int_{2^{(k-2)/\beta}}^\infty \sigma_1(ae^{-z^\beta}) dz. \end{aligned}$$

Proof. — We follow the proof of Lemma 2.1 of Csáki-Csörgő (1992).

For any positive real number t and $k \geq 3$ put $t_j = a \left[t, \frac{2^{2^j}}{a} \right] / 2^{2^j}$, $R = 2^{2^k}$.

We have

$$\begin{aligned} \| \Gamma(t+s) - \Gamma(t) \| &\leq \| \Gamma((t+s)_k) - \Gamma(t_k) \| \\ &\quad + \| \Gamma(t+s) - \Gamma((t+s)_k) \| + \| \Gamma(t) - \Gamma(t_k) \| \end{aligned}$$

$$\begin{aligned} \leq & \|\Gamma((t+s)_k) - \Gamma(t_k)\| + \sum_{j=0}^{\infty} \|\Gamma((t+s)_{k+j+1}) - \Gamma((t+s)_{k+j})\| \\ & + \sum_{j=0}^{\infty} \|\Gamma(t_{k+j+1}) - \Gamma(t_{k+j})\|, \end{aligned}$$

where in the second inequality the a. s. continuity of $\Gamma(\cdot)$ with respect to $\|\cdot\|$ is used. Since

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a(1-(1/R))} |(t+s)_k - t_k| \leq a, \\ & \sup_{0 \leq t \leq T} \sup_{a(1-(1/R)) \leq s \leq a} \left| (t+s)_k - \left(t + a \left(1 - \frac{1}{R} \right) \right)_k \right| \leq 2a \cdot 2^{-2^k}, \\ & \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} |(t+s)_{k+j+1} - (t+s)_{k+j}| \leq a \cdot 2^{-2^{k+j}}, \\ & \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \|\Gamma((t+s)_k) - \Gamma(t_k)\| \\ & \leq \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a(1-(1/R))} \|\Gamma((t+s)_k) - \Gamma(t_k)\| \\ & + \sup_{0 \leq t \leq T} \sup_{a(1-(1/R)) \leq s \leq a} \left\| \Gamma((t+s)_k) - \Gamma\left(\left(t + a \left(1 - \frac{1}{R} \right)\right)_k\right) \right\|, \end{aligned}$$

we get from (2.1) for each $x \geq x^*$ and $x_j \geq x^*$

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a(1-(1/R))} \|\Gamma((t+s)_k) - \Gamma(t_k)\| \geq x \sigma_1(a) + \sigma_2(a) \right\} \\ & \leq 2KR^2 \left(\frac{T}{a} + 1 \right) \exp(-\gamma x^\beta), \\ & \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{a(1-(1/R)) \leq s \leq a} \left\| \Gamma((t+s)_k) - \Gamma\left(\left(t + a \left(1 - \frac{1}{R} \right)\right)_k\right) \right\| \right. \\ & \quad \left. \geq x \sigma_1\left(\frac{2a}{R}\right) + \sigma_2\left(\frac{2a}{R}\right) \right\} \leq 2KR \left(\frac{T}{a} + 1 \right) \exp(-\gamma x^\beta), \\ & \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \|\Gamma((t+s)_{k+j+1}) - \Gamma((t+s)_{k+j})\| \right. \\ & \quad \left. \geq x_j \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) + \sigma_2\left(\frac{a}{2^{2^{k+j}}}\right) \right\} \leq 2K 2^{2^{k+j+1}} \left(\frac{T}{a} + 1 \right) \exp(-\gamma x_j^\beta), \end{aligned}$$

as well as

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \|\Gamma(t_{k+j+1}) - \Gamma(t_{k+j})\| \geq x_j \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) + \sigma_2\left(\frac{a}{2^{2^{k+j}}}\right) \right\} \\ & \leq K 2^{2^{k+j+1}} \left(\frac{T}{a} + 1 \right) \exp(-\gamma x_j^\beta). \end{aligned}$$

Now we put $\gamma x_j^\beta = \gamma x^\beta + 2^{k+j+1}$. Then

$$\sum_{j=0}^{\infty} 2^{2^{k+j+1}} \exp(-\gamma x_j^\beta) = \sum_{j=0}^{\infty} 2^{2^{k+j+1}} e^{-2^{k+j+1}} e^{-\gamma x^\beta} \leq \exp(-\gamma x^\beta).$$

From the definition of x_j , we see that

$$\begin{aligned} x_j &\leq 2^{1/\beta} x + \left(\frac{2}{\gamma}\right)^{1/\beta} 2^{(k+j+1)/\beta}, \\ x \sigma_1\left(\frac{2a}{R}\right) + \sigma_2\left(\frac{2a}{R}\right) + 2 \sum_{j=0}^{\infty} x_j \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) + 2 \sum_{j=0}^{\infty} \sigma_2\left(\frac{a}{2^{2^{k+j}}}\right) \\ &\leq x \left(\sigma_1\left(\frac{2a}{2^{2^k}}\right) + 2^{1+1/\beta} \sum_{j=0}^{\infty} \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) \right) \\ &\quad + 2 \left(\frac{2}{\gamma}\right)^{1/\beta} \sum_{j=0}^{\infty} 2^{(k+j+1)/\beta} \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) + \sigma_2\left(\frac{2a}{R}\right) + 2 \sum_{j=0}^{\infty} \sigma_2\left(\frac{a}{2^{2^{k+j}}}\right), \\ \sigma_1\left(\frac{2a}{2^{2^k}}\right) + 2^{1+(1/\beta)} \sum_{j=0}^{\infty} \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) &\leq (1 + 2^{(1/\beta)+1}) \sum_{j=0}^{\infty} \sigma_1\left(\frac{2a}{2^{2^{k+j}}}\right) \\ &\leq (1 + 2^{(1/\beta)+1}) \sum_{j=0}^{\infty} \int_{2^{2^{k+k-1}}}^{2^{2^{k+j}}} \frac{\sigma_1(2a/2^z)}{z} dz / \ln 2 \\ &\leq 2^{3+(1/\beta)} \int_{2^{k-1}}^{\infty} \frac{\sigma_1(2a/2^z)}{z} dz \\ &\leq 2^{3+(1/\beta)} \int_{2^{k-3}}^{\infty} \frac{\sigma_1(ae^{-z})}{z} dz \\ &= \sigma_1(a, k), \end{aligned}$$

and

$$\begin{aligned} &2 \left(\frac{2}{\gamma}\right)^{1/\beta} \sum_{j=0}^{\infty} 2^{(k+j+1)/\beta} \sigma_1\left(\frac{a}{2^{2^{k+j}}}\right) \\ &\leq \frac{2^{(2/\beta)+1} (2/\gamma)^{1/\beta}}{\beta (2^{1/\beta}-1)} \sum_{j=0}^{\infty} \int_{2^{2^{k+j-1}}}^{2^{2^{k+j}}} z^{(1/\beta)-1} \sigma_1(a2^{-z}) dz \\ &\leq \frac{2^{(1/\beta)+1} (4/\gamma)^{1/\beta}}{2^{1/\beta}-1} \int_{2^{(k-1)/\beta}}^{\infty} \sigma_1(a2^{-z^\beta}) dz \\ &\leq 4 \left(\frac{14}{\gamma}\right)^{1/\beta} \cdot \beta \int_{2^{(k-2)/\beta}}^{\infty} \sigma_1(ae^{-z^\beta}) dz \\ &= \sigma_1^*(a, k), \end{aligned}$$

as well as

$$2 \sum_{j=0}^{\infty} \sigma_2\left(\frac{2}{2^{2^{k+j}}}\right) + \sigma_2\left(\frac{2a}{2^{2^k}}\right) \leq 6 \int_{2^{k-3}}^{\infty} \frac{\sigma_2(ae^{-z})}{z} dz = \sigma_2(a, k).$$

Combining all the above inequalities shows that (2.2) is true. This completes the proof of Lemma 2.1.

Remark 2.1. — Along the lines of the proof of Lemma 2.1, we have that if (2.1) holds for $x \geq \left(c \log \left(\frac{1}{h}\right)\right)^{1/\beta}$, $0 < h \leq h_0 \leq \frac{1}{2}$, $0 < c \leq \frac{1}{\gamma}$, then (2.2) is also true for $x \geq \left(c \log \frac{1}{a}\right)^{1/\beta}$, $0 < a \leq h_0$, where, and in the sequel, $\log x = \log \max(x, e)$ is natural logarithm.

Before stating corollaries and applications of Lemma 2.1, we introduce first the concept of quasi-increasing and quasi-decreasing functions.

DEFINITION 1. — *A function $f(x)$ on (a, b) (resp. on $[a, b]$) will be called quasi-increasing on (a, b) (resp. on $[a, b]$) if there exists a positive c such that*

$$f(x) \leq cf(y) \quad \text{for all } a < x < y < b \\ \text{(resp. for all } a \leq x < y \leq b).$$

DEFINITION 2. — *A function $f(x)$ on (a, b) will be called quasi-increasing at a (resp. at b) if there exist positive δ and c such that*

$$f(x) \leq cf(y) \quad \text{for all } a < x < y < a + \delta \\ \text{(resp. for all } b - \delta < x < y < b).$$

DEFINITION 3. — *A function $f(x)$ on (a, b) will be called quasi-decreasing if there exists positive c such that*

$$f(x) \geq cf(y) \quad \text{for all } a < x < y < b.$$

The definition of quasi-decreasing at a can be stated in the same way as that of quasi-increasing at a . Clearly, the following two statements are equivalent:

- (i) $f(x)$ is quasi-increasing on (a, b) ,
- (ii) there exist positive constants c_1 and c_2 a non-decreasing function $g(x)$ on (a, b) such that

$$c_1 g(x) \leq f(x) \leq c_2 g(x) \quad \text{for all } a < x < b.$$

Remark 2.2. — By the notion of a slowly varying function, one can see that if $f(x)$ is a regularly varying function at zero with a positive exponent, namely

$$f(x) = x^\alpha L(x), \quad \alpha > 0,$$

where $L(\cdot)$ is slowly varying at zero, then $f(x)/x^{\alpha/2}$ is quasi-increasing at zero.

LEMMA 2.2. — *Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(h)$ and $\sigma_2(h)$ be as in Lemma 2.1 and assume that $\sigma_1(x)/x^\alpha$ and $\sigma_2(x)/x^\alpha$ are quasi-increasing on $(0, h_0)$ for some $\alpha > 0$. Then for any $0 < \varepsilon < 1$ there exists $C = C(\varepsilon, \beta, \gamma, \alpha)$*

such that

$$(2.3) \quad \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(h) + (1 + \varepsilon) \sigma_2(h) \right\} \\ \leq CK \left(\frac{T}{h} + 1 \right) \exp \left(- \frac{\gamma x^\beta}{1 + \varepsilon} \right)$$

for every $x \geq \max \left(1, \frac{x^*}{1 - \varepsilon} \right)$, $0 \leq T \leq t_0$ and $0 < h \leq h_0$.

Proof. — Since $\sigma_1(x)/x^\alpha$ and $\sigma_2(x)/x^\alpha$ are quasi-increasing on $(0, h_0)$, there is a positive c_0 such that

$$(2.4) \quad \sigma_i(ht) \leq c_0 t^\alpha \sigma_i(h), \quad i = 1, 2$$

for all $0 < t \leq 1$. From (2.4) it is easy to find that

$$\sigma_i(h, k) \leq 2^{3+(1/\beta)} c_0 e^{-\alpha(k-3)} \alpha^{-1} \sigma_i(h), \quad i = 1, 2, \\ \sigma_1^*(h, k) \leq 4 \left(\frac{14}{r} \right)^{1/\beta} \beta c_0 e^{-\alpha(k-2)} \alpha^{-1} \sigma_1(h).$$

Hence for $\delta = \min \left(\varepsilon, 1 - \left(\frac{1}{1 + \varepsilon} \right)^{1/\beta} \right)$, we can take k such that

$$\sigma_2(h, k) + \sigma_1(h, k) + \sigma_1^*(h, k) \leq \frac{\delta}{2} (\sigma_1(h) + \sigma_2(h)).$$

By Lemma 2.1 we have

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(h) + (1 + \varepsilon) \sigma_2(h) \right\} \\ \leq \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \right. \\ \left. \geq x(1 - \delta) (\sigma_1(h) + \sigma_1(h, k)) + \sigma_1^*(h, k) + \sigma_2(h, k) \right\} \\ \leq 4K \left(\frac{T}{h} + 1 \right) 2^{2k+1} \exp(-\gamma(x(1 - \delta))^\beta) \\ \leq 4K \left(\frac{T}{h} + 1 \right) 2^{2k+1} \exp \left(- \frac{\gamma x^\beta}{1 + \varepsilon} \right).$$

Now put $C = C(\varepsilon, \beta, \gamma, \alpha) = 4 \cdot 2^{2k+1}$, as desired, and the proof is complete.

Remark 2.3. — Strictly speaking, the constant C in (2.3) depends not only on $\varepsilon, \alpha, \gamma, \beta$ but also on c_0 in (2.4). But, for the sake of convenience, we shall continue writing C for the constant $C = C(\varepsilon, \alpha, \gamma, \beta)$ in the sequel.

LEMMA 2.3. — Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(h)$ and $\sigma_2(h)$ be as in Lemma 2.1 and assume that $\sigma_1(x)\left(\log\frac{1}{x}\right)^\alpha$ and $\sigma_2(x)\left(\log\frac{1}{x}\right)^\alpha$ are quasi-increasing on $\left(0, \frac{1}{2}\right)$ for some $\alpha > \frac{1}{\beta}$, namely, there is a $c_0 > 0$ such that

$$(2.5) \quad \sigma_i(ht) \leq c_0 \sigma_i(h) \left(\log\frac{1}{h}\right)^\alpha / \left(\log\frac{1}{h} + \log\frac{1}{t}\right)^\alpha, \quad i = 1, 2$$

for each $0 < t \leq 1, 0 < h \leq \frac{1}{2}$. Then, for any $\varepsilon > 0$ we have

$$(2.6) \quad \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \right. \\ \left. \geq x \sigma_1(h)(1 + c_1 c_0) + \sigma_2(h)(1 + c_1 c_0) + c_2 c_0 \sigma_1(h) \left(\log\frac{1}{h}\right)^{1/\beta} \right\} \\ \leq 8K \left(\frac{T}{h} + 1\right) \frac{1}{h^{2\varepsilon}} \exp(-\gamma x^\beta)$$

for every $x \geq \max(x^*, 1), 0 < T \leq t_0$ and $0 < h \leq \min\left(e^{-8/\varepsilon}, h_0, \frac{1}{2}\right)$, where

$$c_1 = 2^{3+(1/\beta)} \left(1 + \frac{\varepsilon}{8}\right)^{-\alpha} \left(\frac{1}{\alpha} + \log\left(1 + \frac{8}{\varepsilon}\right)\right), \\ c_2 = 4 \left(\frac{14}{\gamma}\right)^{1/\beta} \beta \frac{\beta\alpha}{\beta\alpha - 1} \left(1 + \frac{\varepsilon}{4}\right)^{-(\beta\alpha - 1/\beta)}$$

Proof. — Put $2^k = \varepsilon \log\frac{1}{h}$ in Lemma 2.1. By (2.5) we have for $i = 1, 2$

$$\sigma_i(h, k) \leq c_0 2^{3+(1/\beta)} \int_{2^{k-3}}^\infty \frac{\sigma_i(h) (\log(1/h))^\alpha}{z (\log(1/h) + z)^\alpha} dz \\ = c_0 2^{3+(1/\beta)} \sigma_i(h) \int_{(\varepsilon/8) \log(1/h)}^\infty \frac{(\log(1/h))^\alpha}{z (z + \log(1/h))^\alpha} dz \\ = c_0 2^{3+(1/\beta)} \sigma_i(h) \int_{\varepsilon/8}^\infty \frac{1}{z (z + 1)^\alpha} dz \\ \leq c_0 2^{3+(1/\beta)} \sigma_i(h) \left(\int_{\varepsilon/8}^{1+(\varepsilon/8)} \frac{1}{z (1+(\varepsilon/8))^\alpha} dz + \int_{1+(\varepsilon/8)}^\infty \frac{dz}{z^{1+\alpha}} \right) \\ \leq c_0 2^{3+(1/\beta)} \sigma_i(h) \left(1 + \frac{\varepsilon}{8}\right)^{-\alpha} \left(\frac{1}{\alpha} + \log\left(1 + \frac{8}{\varepsilon}\right)\right) \\ = c_0 c_1 \sigma_i(h)$$

and

$$\begin{aligned} \sigma_1^*(h, k) &\leq 4 \left(\frac{14}{\gamma}\right)^{1/\beta} \beta c_0 \int_{2^{(k-2)/\beta}}^{\infty} \frac{\sigma_1(h) \log^\alpha(1/h)}{(z^\beta + \log(1/h))^\alpha} dz \\ &\leq 4 \left(\frac{14}{\gamma}\right)^{1/\beta} \beta c_0 \sigma_1(h) \left(\log \frac{1}{h}\right)^{1/\beta} \int_{(\varepsilon/4)^{1/\beta}}^{\infty} \frac{1}{(1+z^\beta)^\alpha} dz \\ &\leq 4 \left(\frac{14}{\gamma}\right)^{1/\beta} \beta c_0 \sigma_1(h) \left(\log \frac{1}{h}\right)^{1/\beta} \left(\int_{(\varepsilon/4)^{1/\beta}}^{\infty} \frac{dz}{(1+(\varepsilon/4))^{1/\beta}} + \int_{(1+(\varepsilon/4))^{1/\beta}}^{\infty} \frac{dz}{z^{\alpha\beta}} \right) \\ &\leq 4 \left(\frac{14}{\gamma}\right)^{1/\beta} c_0 \sigma_1(h) \left(\log \frac{1}{h}\right)^{1/\beta} \frac{\beta^2 \alpha}{\beta\alpha - 1} \left(1 + \frac{\varepsilon}{4}\right)^{-\frac{1}{\beta(\alpha-1)}} \\ &= c_0 \sigma_1(h) \left(\log \frac{1}{h}\right)^{1/\beta} c_2. \end{aligned}$$

Now (2.6) follows from Lemma 2.1.

Clearly, the inequalities (2.3) and (2.6) enable one to study the increments of $\Gamma(\cdot)$ for small h over the interval $(0, 1)$ and as we will see, even to establish some surprising results. The next versions of (2.3) and (2.6) are for the sake of studying large increments of $\Gamma(\cdot)$.

LEMMA 2.4. — Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(h)$ and $\sigma_2(h)$ be as in Lemma 2.1 with $t_0 = h_0 = \infty$. Assume that $\sigma_1(x)/x^\alpha$ and $\sigma_2(x)/x^\alpha$ are quasi-increasing on $(0, \infty)$ for some $\alpha > 0$. Then for any $0 < \varepsilon < 1$ there exists $C = C(\varepsilon, \gamma, \alpha, \beta)$ such that

$$(2.7) \quad \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(a) + (1 + \varepsilon) \sigma_2(a) \right\} \leq CK \left(\frac{T}{a} + 1\right) \exp\left(-\frac{\gamma x^\beta}{1 + \varepsilon}\right)$$

for every $x \geq \max\left(1, \frac{x^*}{1 - \varepsilon}\right)$ and $T, a > 0$.

LEMMA 2.5. — Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(x)$ and $\sigma_2(x)$ be as in Lemma 2.1 with $t_0 = h_0 = \infty$. Assume that $\sigma_1(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $\int_1^\infty \sigma_1(e^{-z^\beta}) dz < \infty$, $\int_1^\infty \sigma_2(e^{-z})/z dz < \infty$. Then, there exist positive numbers c_1 and a_0 such that

$$(2.8) \quad \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \|\Gamma(t+s) - \Gamma(t)\| \geq x c_1 \sigma_1(a) + c_1 \sigma_2(a) + c_1 (\log a)^{1/\beta} \sigma_1(a) \right\} \leq 8KT \exp(-\gamma x^\beta)$$

for every $x \geq \max(1, x^*)$ and $T \geq a \geq a_0$.

Lemma 2.4 is a direct consequence of Lemma 2.2. The proof of (2.8) is similar to that of Lemma 2.3.

3. GENERAL THEOREMS FOR LARGE AND SMALL INCREMENTS

In this section we shall establish general theorems on increments, both large and small, for stochastic processes. Most of the existing similar results we are aware of, follow as corollaries.

THEOREM 3.1. — *Let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in Banach space \mathcal{B} . Let \mathbf{P} be the probability measure generated by $\Gamma(\cdot)$. Let $a_T, b_T, C_T, \sigma_1(T), \sigma_2(T)$ be non-negative continuous functions. Assume that both a_T and b_T are either non-decreasing or non-increasing functions of T and that*

$$(3.1) \quad C_T + \sigma_1(T) + \sigma_1^{-1}(T) \rightarrow \infty \quad \text{as } T \rightarrow \infty,$$

$$(3.2) \quad \mathbf{P} \left\{ \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(T) + \sigma_2(T) \right\} \leq C_T \exp(-\gamma x^\beta)$$

for each

$$(3.3) \quad \left(\frac{1}{\gamma} \left(\log C_T + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} \leq x \\ \leq (1 + \delta) \left(\frac{1}{\gamma} \left(\log C_T + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} + \delta \frac{\sigma_2^2(T)}{\sigma_1(T)}$$

with some $\gamma, \beta, \delta > 0$. Then, we have

$$(3.4) \quad \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad a. s.$$

where $\alpha_T^{-1} = \sigma_1(T) \left(\frac{1}{\gamma} \left(\log C_T + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} + \sigma_2(T)$.

Proof. — Without loss of generality, assume $0 < \delta < \frac{1}{2}$ and both a_T and b_T are non-decreasing. Let $1 < \theta < 1 + \frac{\delta}{2}$. Define

$$\begin{aligned} A_k &= \{ T : \theta^k \leq \sigma_1(T) \leq \theta^{k+1} \}, \quad -\infty < k < \infty \\ A_{k,j} &= \{ T : 2^j \leq C_T \leq 2^{j+1}, T \in A_k \}, \quad j \geq 0 \\ A_{k,j,i} &= \{ T : \theta^i \leq \sigma_2(T) \leq \theta^{i+1}, T \in A_{k,j} \}, \quad -\infty < i < \infty \\ T_{k,j,i} &= \sup \{ T : T \in A_{k,j,i} \}. \end{aligned}$$

Write $a(T) = a_T$ and $b(T) = b_T$. Noting that (3.1) is satisfied and using the continuity of $a_T, b_T, C_T, \sigma_1(T)$ and $\sigma_2(T)$, we have

$$\begin{aligned}
 (3.5) \quad & \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T \|\Gamma(t+s) - \Gamma(t)\| \\
 & \leq \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{T \in A_{k,j}} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T \|\Gamma(t+s) - \Gamma(t)\| \\
 & \leq \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{T \in A_{k,j}} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma)(\log 2^j + \log \log \theta^{|k|}))^{1/\beta} + \sigma_2(T)} \\
 & \leq \max \left\{ \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{i \leq k} \sup_{T \in A_{k,j,i}} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma)(\log(2^j \log \theta^{|k|}))^{1/\beta})}, \right. \\
 & \left. \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{i > k} \sup_{0 \leq t \leq b(T_{k,j,i})} \sup_{0 \leq s \leq a(T_{k,j,i})} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma)(\log 2^j + \log \log \theta^{|k|}))^{1/\beta} + \theta^i} \right\}.
 \end{aligned}$$

We show first that

$$\begin{aligned}
 (3.6) \quad & \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{i > k} \sup_{0 \leq t \leq b(T_{k,j,i})} \sup_{0 \leq s \leq a(T_{k,j,i})} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma)(\log 2^j + \log \log \theta^{|k|}))^{1/\beta} + \theta^i} \\
 & \leq \theta \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{i > k} \sup_{0 \leq t \leq b(T_{k,j,i})} \sup_{0 \leq s \leq a(T_{k,j,i})} \alpha_{T_{k,j,i}} \|\Gamma(t+s) - \Gamma(t)\| \\
 & \leq \theta^2 \quad \text{a.s.}
 \end{aligned}$$

By (3.2), we find

$$\begin{aligned}
 (3.7) \quad & P \left\{ \sup_{j \geq l} \sup_{i > k} \sup_{0 \leq t \leq b(T_{k,j,i})} \sup_{0 \leq s \leq a(T_{k,j,i})} \alpha_{T_{k,j,i}} \|\Gamma(t+s) - \Gamma(t)\| \geq \theta \right\} \\
 & \leq \sum_{j \geq l} \sum_{i > k} C_{T_{k,j,i}} \exp \left(-\gamma \left((\theta - 1) \frac{\sigma_2(T_{k,j,i})}{\sigma_1(T_{k,j,i})} \right. \right. \\
 & \quad \left. \left. + \theta \left(\frac{1}{\gamma} (\log C_{T_{k,j,i}} + \log \log \theta^{|k|}) \right)^{1/\beta} \right)^\beta \right) \\
 & \leq \sum_{j \geq l} \sum_{i > k} C_{T_{k,j,i}} \exp \left(-\gamma \left((\theta - 1) \theta^{i-k-1} \right. \right. \\
 & \quad \left. \left. + \theta \left(\frac{1}{\gamma} (\log C_{T_{k,j,i}} + \log \log \theta^{|k|}) \right)^{1/\beta} \right)^\beta \right) \\
 & \leq \sum_{j \geq l} \sum_{\substack{i > k \\ \theta^{i-k} \leq j^{2/\beta} + k^2}} C_{T_{k,j,i}} \exp(-\theta^\beta (\log C_{T_{k,j,i}} + \log \log \theta^{|k|})) \\
 & \quad + \sum_{j \geq l} \sum_{\theta^{i-k} > j^{2/\beta} + k^2} C_{T_{k,j,i}} \exp(-\gamma (\theta - 1)^\beta \theta^{(i-k-1)\beta})
 \end{aligned}$$

$$\begin{aligned} &\leq K \left(\sum_{j \geq l} (j^{2/\beta} + \log |k|) 2^{-j(\theta^\beta - 1)} (|k| + 1)^{-\theta^\beta} \right. \\ &\quad \left. + \sum_{j \geq l} 2^j \exp \left(-\gamma \left(1 - \frac{1}{\theta} \right)^\beta (j^{2/\beta} + k^2)^\beta \right) \right) \\ &\leq K \left(2^{-l(\theta^\beta - 1)/2} (1 + |k|)^{-(\theta^\beta + 1)/2} + 2^l \exp \left(-\frac{\gamma}{2} (1 - 1/\theta)^\beta (k^2 + l^2) \right) \right), \end{aligned}$$

where K is a constant, depending only on θ , β and γ , and in the last inequality we used the fact that $(j^{2/\beta} + k^2)^\beta \geq \frac{1}{2}(j^2 + k^2)^\beta$. Therefore

$$(3.8) \quad \sum_{l=0}^{\infty} \sum_{|k|=0}^{\infty} P \left\{ \sup_{j \geq l} \sup_{i > k} \sup_{0 \leq t \leq b(\mathbb{T}_{k,j,i})} \sup_{0 \leq s \leq a(\mathbb{T}_{k,j,i})} \alpha_{\mathbb{T}_{k,j,i}} \|\Gamma(t+s) - \Gamma(t)\| > \theta \right\} < \infty.$$

Now (3.6) follows from (3.8) and the Borel-Cantelli lemma.

Next we prove that

$$(3.9) \quad \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{\mathbb{T} \in \mathbb{A}_{k,j}, \sigma_2(\mathbb{T}) \leq \theta^{k+1}} \sup_{0 \leq t \leq b_{\mathbb{T}}} \sup_{0 \leq s \leq a_{\mathbb{T}}} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma) (\log(2^j \log \theta^{k^l})))^{1/\beta}} \leq \theta^2 \quad \text{a.s.}$$

Put

$$\mathbb{T}_{k,j} = \sup \{ \mathbb{T} : \mathbb{T} \in \mathbb{A}_{k,j}, \sigma_2(\mathbb{T}) \leq \theta^{k+1} \}.$$

Then

$$\sigma_2(\mathbb{T}_{k,j}) \leq \theta^{k+1},$$

and

$$\begin{aligned} &\limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{\mathbb{T} \in \mathbb{A}_{k,j}, \sigma_2(\mathbb{T}) \leq \theta^{k+1}} \sup_{0 \leq t \leq b_{\mathbb{T}}} \sup_{0 \leq s \leq a_{\mathbb{T}}} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma) (\log(2^j \log \theta^{k^l})))^{1/\beta}} \\ &\leq \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{0 \leq t \leq b(\mathbb{T}_{k,j})} \sup_{0 \leq s \leq a(\mathbb{T}_{k,j})} \frac{\|\Gamma(t+s) - \Gamma(t)\|}{\theta^k ((1/\gamma) (\log(2^j \log \theta^{k^l})))^{1/\beta} + \sigma_2(\mathbb{T}_{k,j})} \\ &\leq \theta \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{0 \leq t \leq b(\mathbb{T}_{k,j})} \sup_{0 \leq s \leq a(\mathbb{T}_{k,j})} \alpha_{\mathbb{T}_{k,j}} \|\Gamma(t+s) - \Gamma(t)\|. \end{aligned}$$

The rest of the proof of (3.9) follows along the lines of the proof of (3.6), and (3.4) now follows from (3.5), (3.6), (3.9) and the arbitrariness of $\theta > 1$. This completes the proof of Theorem 3.1.

The next theorem indicates that the assumption that both $a_{\mathbb{T}}$ and $b_{\mathbb{T}}$ are either non-decreasing or non-increasing functions of \mathbb{T} can be removed under some additional conditions.

THEOREM 3.2. — *Let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in Banach space \mathcal{B} . Let $a_{\mathbb{T}}, b_{\mathbb{T}}, \sigma_1(\mathbb{T})$ and $\sigma_2(\mathbb{T})$ be non-negative*

continuous functions. Assume that

$$(3.10) \quad \frac{b_T}{a_T} + \sigma_1(T) + \frac{1}{\sigma_1(T)} \rightarrow \infty \quad \text{as } T \rightarrow \infty,$$

$$(3.11) \quad P \left\{ \sup_{0 \leq t \leq b} \sup_{0 \leq s \leq a_T} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(T) + \sigma_2(T) \right\} \leq A \left(1 + \frac{b}{a_T} \right) \exp(-\gamma x^\beta)$$

for each $b \geq b_T$, and

$$\left(\frac{1}{\gamma} \left(\log \left(\frac{b}{a_T} + 1 \right) + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} \leq x \leq (1 + \delta) \left(\frac{1}{\gamma} \left(\log \left(\frac{b}{a_T} + 1 \right) + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} + \delta \frac{\sigma_2(T)}{\sigma_1(T)}$$

with some γ, β, δ and $A > 0$. Then we have

$$(3.12) \quad \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T^* \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad \text{a.s.}$$

where

$$\frac{1}{\alpha_T^*} = \sigma_1(T) \left(\frac{1}{\gamma} \left(\log \left(\frac{b_T}{a_T} + 1 \right) + \log \log \left(\sigma_1(T) + \frac{1}{\sigma_1(T)} \right) \right) \right)^{1/\beta} + \sigma_2(T).$$

Proof. — Let $C_T = 1 + \frac{b_T}{a_T}$. Assume, without loss of generality,

$0 < \delta < \frac{1}{2}$. Let $\theta, A_k, A_{k,j}, A_{k,j,i}$ be as in the proof of Theorem 3.1. Put

$$b(T_{k,j,i}) = \sup \{ b_T : T \in A_{k,j,i} \}$$

and

$$a(T_{k,j,i}^*) = \sup \{ a_T : T \in A_{k,j,i} \}.$$

It is easy to see that $b(T_{k,j,i}) \geq b(T_{k,j,i}^*)$ and

$$2^j \leq \frac{b(T_{k,j,i}^*)}{a(T_{k,j,i}^*)} + 1 \leq \frac{b(T_{k,j,i})}{a(T_{k,j,i}^*)} + 1 \leq \frac{b(T_{k,j,i})}{a(T_{k,j,i})} + 1 \leq 2^{j+1}.$$

Hence, along the way of the proof of Theorem 3.1, we find

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T^* \|\Gamma(t+s) - \Gamma(t)\| \\ & \leq \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_{T \in A_{k,j}} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \alpha_T^* \|\Gamma(t+s) - \Gamma(t)\| \\ & \leq \limsup_{|k|+l \rightarrow \infty} \sup_{j \geq l} \sup_i \sup_{T \in A_{k,j,i}} \sup_{0 \leq t \leq b(T_{k,j,i})} \sup_{0 \leq s \leq a(T_{k,j,i}^*)} \alpha_T^* \|\Gamma(t+s) - \Gamma(t)\| \\ & \leq \theta^2 \quad \text{a.s.} \end{aligned}$$

This, by the arbitrariness of $\theta > 1$, proves (3.12).

Clearly, Theorems 3.1 and 3.2 not only include most existing similar results on increments of stochastic processes as particular cases but also weaken the usual restrictions on a_T and b_T .

Correspondingly, for sequences of discrete parameters we have similar results, as stated in our next two theorems.

THEOREM 3.3. — *Let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in a Banach space \mathcal{B} . Let $a_N, b_N, C_N, \sigma_1(N)$ and $\sigma_2(N)$ be non-negative numbers. Assume that both a_N and b_N are either non-decreasing or non-increasing in N , and that*

$$C_N + \sigma_1(N) + \frac{1}{\sigma_1(N)} \rightarrow \infty, \quad \text{as } N \rightarrow \infty,$$

$$P \left\{ \sup_{0 \leq t \leq b_N} \sup_{0 \leq s \leq a_N} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(N) + \sigma_2(N) \right\} \leq C_N \exp(-\gamma x^\beta)$$

for each

$$\begin{aligned} & \left(\frac{1}{\gamma} \left(\log C_N + \log \log \left(\sigma_1(N) + \frac{1}{\sigma_1(N)} \right) \right) \right)^{1/\beta} \leq x \\ & \leq (1 + \delta) \left(\frac{1}{\gamma} \left(\log C_N + \log \log \left(\sigma_1(N) + \frac{1}{\sigma_1(N)} \right) \right) \right)^{1/\beta} + \delta \frac{\sigma_2(N)}{\sigma_1(N)} \end{aligned}$$

with some $\gamma, \beta, \delta > 0$. Then, we have

$$(3.13) \quad \limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq b_N} \sup_{0 \leq s \leq a_N} \alpha_N \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad \text{a.s.}$$

THEOREM 3.4. — *Let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in a Banach space \mathcal{B} . Let $a_N, b_N, \sigma_1(N)$ and $\sigma_2(N)$ be non-negative numbers. Assume that*

$$\frac{b_N}{a_N} + \sigma_1(N) + \frac{1}{\sigma_1(N)} \rightarrow \infty, \quad \text{as } N \rightarrow \infty,$$

$$\begin{aligned} P \left\{ \sup_{0 \leq t \leq b} \sup_{0 \leq s \leq a_N} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(N) + \sigma_2(N) \right\} \\ \leq A \left(1 + \frac{b}{a_N} \right) \exp(-\gamma x^\beta) \end{aligned}$$

for each $b \geq b_N$ and

$$\begin{aligned} & \left(\frac{1}{\gamma} \left(\log \left(1 + \frac{b}{a_N} \right) + \log \log \left(\sigma_1(N) + \frac{1}{\sigma_1(N)} \right) \right) \right)^{1/\beta} \leq x \\ & \leq (1 + \delta) \left(\frac{1}{\gamma} \left(\log \left(1 + \frac{b}{a_N} \right) + \log \log \left(\sigma_1(N) + \frac{1}{\sigma_1(N)} \right) \right) \right)^{1/\beta} + \delta \frac{\sigma_2(N)}{\sigma_1(N)} \end{aligned}$$

with some $\gamma, \beta, \delta, A > 0$. Then, we have

$$(3.14) \quad \limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq b_N} \sup_{0 \leq s \leq a_N} \alpha_N^* \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad a.s.$$

Combining Theorem 3.2 with lemmas given in Section 2, we deduce

THEOREM 3.5. — *Let \mathcal{B} be a separable Banach space with norm $\|\cdot\|$ and let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in \mathcal{B} . Let \mathbf{P} be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is \mathbf{P} -almost surely continuous with respect to $\|\cdot\|$ and that there exist non-negative monotone non-decreasing continuous functions $\sigma_1(h)$ and $\sigma_2(h)$ such that*

$$\mathbf{P}\{\|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(h) + \sigma_2(h)\} \leq K \exp(-\gamma x^\beta)$$

for each $t \geq 0, h > 0$ and $x \geq x^* > 0$ with some $K, \gamma, \beta > 0$. Moreover, assume that $\sigma_1(x)/x^\alpha$ and $\sigma_2(x)/x^\alpha$ are quasi-increasing on $(0, \infty)$ for some $\alpha > 0$ and that a_T and b_T are continuous functions with

$$\frac{b_T}{a_T} + \sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \rightarrow \infty \quad \text{as } T \rightarrow \infty.$$

Then we have

$$(3.15) \quad \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \beta_T \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad a.s.$$

where

$$\beta_T^{-1} = \sigma_1(a_T) \left(\frac{1}{\gamma} \left(\log \left(1 + \frac{b_T}{a_T} \right) + \log \log \left(\sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \right) \right) \right)^{1/\beta} + \sigma_2(a_T).$$

Proof. — By Lemma 2.4, for every

$$0 < \varepsilon < 1, \quad b > 0 \quad \text{and} \quad x \geq \max \left(1, \frac{x^*}{1 - \varepsilon} \right)$$

we have

$$\begin{aligned} \mathbf{P} \left\{ \sup_{0 \leq t \leq b} \sup_{0 \leq s \leq a_T} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(a_T) + (1 + \varepsilon) \sigma_2(a_T) \right\} \\ \leq CK \left(1 + \frac{b}{a_T} \right) \exp \left(- \frac{\gamma x^\beta}{1 + \varepsilon} \right). \end{aligned}$$

Consequently, by Theorem 3.2

$$(3.16) \quad \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \sup_{0 \leq s \leq a_T} \beta_T(\varepsilon) \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad a.s.$$

where

$$\beta_T^{-1}(\varepsilon) = \sigma_1(a_T) \left(\frac{1+\varepsilon}{\gamma} \left(\log \left(1 + \frac{b_T}{a_T} \right) + \log \log \left(\sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \right) \right) \right)^{1/\beta} + (1+\varepsilon) \sigma_2(a_T).$$

(3.15) now follows from (3.16) and the arbitrariness of ε .

THEOREM 3.6. — *Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(h)$ and $\sigma_2(h)$ be as in Lemma 2.2 with $t_0 = 1$. Then we have*

$$(3.17) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \theta_h \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad a.s.$$

where $\theta_h^{-1} = \sigma_1(h) \left(\frac{1}{\gamma} \left(\log \frac{1}{h} + \log \log \frac{1}{\sigma_1(h)} \right) \right)^{1/\beta} + \sigma_2(h)$.

THEOREM 3.7. — *Let $\{\Gamma(t), -\infty < t < \infty\}$, $\sigma_1(h)$ and $\sigma_2(h)$ be as in Lemma 2.3 with $t_0 = 1$. Then, there exists a positive C such that*

$$(3.18) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \theta_h \|\Gamma(t+s) - \Gamma(t)\| \leq C \quad a.s.$$

The proofs of Theorems 3.6 and 3.7 are similar to that of Theorem 3.5.

4. l^2 -VALUED ORNSTEIN-UHLENBECK PROCESSES

Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k , i. e., $X_k(\cdot)$ is a stationary mean zero Gaussian process with

$$(4.1) \quad EX_k(s)X_k(t) = \frac{\gamma_k}{\lambda_k} \exp(-\lambda_k |t-s|), \quad k=1, 2, \dots,$$

where $\gamma_k \geq 0, \lambda_k > 0$.

The process $Y(\cdot)$ was introduced by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$(4.2) \quad dX_k(t) = -\lambda_k X_k(t) dt + (2\gamma_k)^{1/2} dW_k(t), \quad k=1, 2, \dots,$$

where $\{W_k(t), -\infty < t < \infty\}_{k=1}^\infty$ are independent Wiener processes. The properties of $Y(\cdot)$ have been extensively studied in the literature. Since $EX_k^2(t) = \gamma_k/\lambda_k$, it is clear that for every fixed t , $Y(t)$ is almost surely in l^2 if and only if

$$E \|Y(t)\|^2 = \sum_{k=1}^\infty \gamma_k/\lambda_k = \Gamma_0 < \infty.$$

In this section we assume throughout that $Y(\cdot) \in l^2$, i. e., that $\Gamma_0 < \infty$, and the Banach space \mathcal{B} is identical with l^2 . Consequently, $\|\cdot\|$ denotes l^2 -norm here.

The continuity properties of $Y(\cdot)$ were investigated by Dawson (1972), Iscoe and McDonald (1986, 1989), Schmuland (1987, 1988 a, 1988 b), Iscoe *et al.* (1990), with a final result due to Fernique (1989), which reads as follows: *for each $x \in \mathbb{R}^+$, let $K(x) = \{k \in \mathbb{N} : \gamma_k > x \lambda_k\}$ and $\lambda(x) = \sup \{\lambda_k : k \in K(x)\}$. Then $Y(\cdot) \in l^2$ is a. s. continuous if and only if*

$\Gamma_0 < \infty$ and $\int ((\log \lambda(x)) \vee 0) dx < \infty$. He showed also that

$$(4.3) \quad \sum_{k=1}^{\infty} (\gamma_k/\lambda_k) (1 + \log(1 + \lambda_k)) < \infty$$

is a sufficient condition for a. s. l^2 continuity of $Y(\cdot)$.

We introduce now the following notations:

$$(4.4) \quad \sigma_k^2(h) = E(X_k(h) - X_k(0))^2 = \frac{2\gamma_k}{\lambda_k} (1 - \exp(-\lambda_k h))$$

$$(4.5) \quad \sigma^2(h) = \sum_{k=1}^{\infty} \sigma_k^2(h) = 2 \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} (1 - \exp(-\lambda_k h))$$

$$(4.6) \quad \sigma^{*2}(h) = \max_{k \geq 1} \sigma_k^2(h)$$

$$(4.7) \quad \Gamma_1 = \sum_{k=1}^{\infty} \gamma_k.$$

Csáki and Csörgő (1992) proved

THEOREM 4. A. — *Assuming that $Y(\cdot)$ is a. s. continuous in l^2 and that $\sigma(h)$ is regularly varying at zero with a positive exponent, we have*

$$(4.8) \quad \limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma(h) (2 \log(1/h))^{1/2}} \leq 1 \quad a.s.$$

If, in particular, we have also $\Gamma_1 < \infty$, then

$$(4.9) \quad \lim_{h \rightarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{(2h\gamma^*)^{1/2} (2 \log(1/h))^{1/2}} = 1 \quad a.s.,$$

where $\gamma^* = \max_k \gamma_k$.

Clearly, it is not known how sharp the upper estimation of (4.8) is in general, since no lower estimation is given when $\Gamma_1 = \infty$. This section is devoted to establish both upper and lower estimations of (4.8). Some surprising results are obtained. In what follows, we always assume that $\Gamma_0 < \infty$, except in Theorem 4.3.

THEOREM 4.1. — *Assuming that $\sigma(h)/h^\alpha$ and $\sigma^*(h)/h^\alpha$ are quasi-increasing on $(0, h_0)$ for some $\alpha > 0$, $h_0 > 0$, we have for each $0 < \varepsilon < 1$*

$$(4.10) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h)(2/(1-\varepsilon^2)\log(1/h))^{1/2} + \sigma(h)/\varepsilon} \leq 1 \quad a.s.$$

THEOREM 4.2. — *Assuming $\sigma^*(h) = o(\sigma(h))$ as $h \rightarrow 0$, we have*

$$(4.11) \quad \limsup_{h \rightarrow 0} \frac{\|Y(t+h) - Y(t)\|}{\sigma(h)} \geq 1 \quad a.s.$$

for every t .

THEOREM 4.3. — *Assuming $\sigma^*(h) < \infty$ for each $h > 0$, we have*

$$(4.12) \quad \liminf_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h} \max_{k \geq 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log(1/h))^{1/2}} \geq 1 \quad a.s.$$

$$(4.13) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \max_{k \geq 1} \frac{|X_k(t+h) - X_k(t)|}{\sigma^*(h)(2\log(1/h))^{1/2}} \geq 1 \quad a.s.$$

Noting that $\|Y(t+s) - Y(t)\| \geq \max_{k \geq 1} |X_k(t+s) - X_k(t)|$, we deduce the following corollaries from the above theorems.

COROLLARY 4.1. — *Assuming that $\sigma(h)/h^\alpha$ is quasi-increasing on $(0, h_0)$ for some $\alpha > 0$, $h_0 > 0$ and that $\sigma^*(h) \left(\log \frac{1}{h}\right)^{1/2} = o(\sigma(h))$ as $h \rightarrow 0$, we have*

$$(4.14) \quad \limsup_{h \rightarrow 0} \frac{\|Y(t+h) - Y(t)\|}{\sigma(h)} = 1 \quad a.s. \text{ for each } t,$$

$$(4.15) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma(h)} = 1 \quad a.s.$$

COROLLARY 4.2. — *Assuming that $\sigma^*(h)/h^\alpha$ is quasi-increasing on $(0, h_0)$ for some $\alpha > 0$, $h_0 > 0$ and that $\sigma(h) = o\left(\sigma^*(h) \left(\log \frac{1}{h}\right)^{1/2}\right)$ as $h \rightarrow 0$, we have*

$$(4.16) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h)(2\log(1/h))^{1/2}} = 1 \quad a.s.$$

$$(4.17) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h)(2\log(1/h))^{1/2}} = 1 \quad a.s.$$

$$(4.18) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \frac{\|Y(t+h) - Y(t)\|}{\sigma^*(h)(2\log(1/h))^{1/2}} = 1 \quad a.s.$$

Results (4.14) and (4.15) are quite contrary to our expectations since the normalizing constant here is just $\sigma(h) = (E \|Y(t+h) - Y(t)\|^2)^{1/2}$.

Before proving the theorems given above, we first state the following proposition, which is a special case of Theorem 2.6 of Csáki, Csörgő and Shao (1991).

PROPOSITION 4.1. - *If $\sigma(h) \leq c \left(\log \frac{1}{h}\right)^{-\theta}$ for some $c, \theta > \frac{1}{2}, 0 < h < \frac{1}{2}$, then $Y(\cdot)$ is a.s. continuous in l^2 .*

Proof of theorem 4.1. - For $0 < \varepsilon < 1$, put $\lambda = \frac{1 - \varepsilon^2}{2(\sigma^*(h))^2}$. Then, for each $x > 0, t > 0$ we have

$$\begin{aligned}
 (4.19) \quad & P \left\{ \|Y(t+h) - Y(t)\| \geq x \sigma^*(h) + \sigma(h)/\varepsilon \right\} \\
 &= P \left\{ \|Y(t+h) - Y(t)\|^2 \geq \left(x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right)^2 \right\} \\
 &\leq \exp \left\{ -\lambda \left(x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right)^2 \right\} \prod_{k=1}^{\infty} \exp(\lambda |X_k(t+h) - X_k(t)|^2) \\
 &= \exp \left\{ -\lambda \left(x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right)^2 \right\} \prod_{k=1}^{\infty} (1 - 2\lambda \sigma_k^2(h))^{-1/2} \\
 &\leq \exp \left\{ -\lambda \left(x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right)^2 \right\} \prod_{k=1}^{\infty} \exp \left(\frac{\lambda \sigma_k^2(h)}{1 - 2\lambda \sigma_k^2(h)} \right) \\
 &\leq \exp \left(-\lambda \left(x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right)^2 + \frac{\lambda \sigma^2(h)}{\varepsilon^2} \right) \\
 &\leq \exp(-\lambda x^2 \sigma^{*2}(h)) \\
 &= \exp \left(-\frac{x^2 (1 - \varepsilon^2)}{2} \right).
 \end{aligned}$$

Put $\beta = 2, \gamma = \frac{1 - \varepsilon^2}{2}, \sigma_1(h) = \sigma^*(h), \sigma_2(h) = \frac{\sigma(h)}{\varepsilon}$ in Theorem 3.6. Applying now Theorem 3.6, we get

$$(4.20) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h) (2/(1 - \varepsilon^2) (\log(1/h) + \log \log(1/\sigma^*(h))))^{1/2} + (\sigma(h)/\varepsilon)} \leq 1, \quad \text{a.s.}$$

On the other hand, it is easy to see that $\sigma_k(2h) \leq 2 \sigma_k(h)$, by Minkowski's inequality. Hence $\sigma_k(h) \geq h \sigma_k\left(\frac{1}{2}\right)$ for each

$$0 < h < \frac{1}{2}, \quad \text{and} \quad \sigma^*(h) \geq h \sigma^*\left(\frac{1}{2}\right)$$

as well. Now (4.10) follows from (4.20). This completes the proof of Theorem 4.1.

Proof of theorem 4.2. — It suffices to show that for each $0 < \varepsilon < 1$

$$(4.21) \quad \lim_{h \rightarrow 0} \mathbf{P} \left(\frac{\|Y(t+h) - Y(t)\|}{\sigma(h)} \geq 1 - \varepsilon \right) = 1.$$

Let

$$X = X(h) = \|Y(t+h) - Y(t)\| / \sigma(h).$$

Then $EX^2 = 1$. For each $0 < \delta < 1$, $x \geq (1 - \delta)^{-2}$ we have, by (4.19),

$$(4.22) \quad \begin{aligned} \mathbf{P}(X \geq x) &= \mathbf{P} \left\{ \|Y(t+h) - Y(t)\| \geq \left(x - \frac{1}{1-\delta}\right) \sigma(h) + \frac{\sigma(h)}{1-\delta} \right\} \\ &\leq \mathbf{P} \left\{ \|Y(t+h) - Y(t)\| \geq \delta x \sigma(h) + \frac{\sigma(h)}{1-\delta} \right\} \\ &\leq \exp \left(- \frac{x^2 \delta^2 \sigma^2(h) (1 - (1-\delta)^2)}{2 \sigma^{*2}(h)} \right) \\ &\leq \exp \left(- \frac{x^2 \sigma^2(h) \delta^3}{2 \sigma^{*2}(h)} \right). \end{aligned}$$

Hence, by (4.22)

$$\begin{aligned} 1 &= 2 \int_0^\infty x \mathbf{P}(X \geq x) dx \\ &\leq 2 \int_0^{1-\varepsilon} x dx + 2 \mathbf{P}(X \geq 1 - \varepsilon) \int_{1-\varepsilon}^{(1-\delta)^{-2}} x dx + 2 \int_{(1-\delta)^{-2}}^\infty x \mathbf{P}(X \geq x) dx \\ &\leq (1-\varepsilon)^2 + ((1-\delta)^{-4} - (1-\varepsilon)^2) \mathbf{P}(X \geq 1 - \varepsilon) \\ &\quad + 2 \int_{(1-\delta)^{-2}}^\infty x \exp \left(- \frac{x^2 \delta^3 \sigma^2(h)}{2 \sigma^{*2}(h)} \right) dx \\ &\leq (1-\varepsilon)^2 + ((1-\delta)^{-4} - (1-\varepsilon)^2) \mathbf{P}(X \geq 1 - \varepsilon) + \frac{8 \sigma^{*2}(h)}{\delta^3 \sigma^2(h)}, \end{aligned}$$

and

$$(4.23) \quad 1 \leq (1-\varepsilon)^2 + ((1-\delta)^{-4} - (1-\varepsilon)^2) \liminf_{h \rightarrow 0} \mathbf{P}(X \geq 1 - \varepsilon)$$

by the assumption $\lim_{h \rightarrow 0} \frac{\sigma^*(h)}{\sigma(h)} = 0$. From (4.23) we have

$$\liminf_{h \rightarrow 0} \mathbf{P}(X \geq 1 - \varepsilon) \geq \frac{1 - (1-\varepsilon)^2}{(1-\delta)^{-4} - (1-\varepsilon)^2}$$

from which it follows that

$$\liminf_{h \rightarrow 0} \mathbf{P}(X \geq 1 - \varepsilon) \geq 1$$

by the arbitrariness of δ . This proves (4.21) and (4.11) as well.

Proof of theorem 4.3. – For $1 < \theta < \frac{3}{2}$, let

$$\begin{aligned} A_k &= \{ h : \theta^{-k-1} \leq \sigma^*(h) < \theta^{-k} \}, \\ h_k &= \inf \{ h : h \in A_k \}, \\ \sigma_{j_k}(h_k) &= \sigma^*(h_k), \quad k = 1, 2, \dots \end{aligned}$$

It is easy to see that

$$\begin{aligned} (4.24) \quad & \liminf_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h} \max_{i \geq 1} \frac{|X_i(t+s) - X_i(t)|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \\ & \geq \liminf_{k \rightarrow \infty} \inf_{h \in A_k} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h} \max_{i \geq 1} \frac{|X_i(t+s) - X_i(t)|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \\ & \geq \liminf_{k \rightarrow \infty} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h_k} \max_{i \geq 1} \frac{|X_i(t+s) - X_i(t)|}{\theta^{-k} (2 \log(1/h_k))^{1/2}} \\ & \geq \frac{1}{\theta} \liminf_{k \rightarrow \infty} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h_k} \max_{i \geq 1} \frac{|X_i(t+s) - X_i(t)|}{\sigma^*(h_k) (2 \log(1/h_k))^{1/2}} \\ & \geq \frac{1}{\theta} \liminf_{k \rightarrow \infty} \sup_{0 \leq t \leq 1/2} \sup_{0 \leq s \leq h_k} \frac{|X_{j_k}(t+s) - X_{j_k}(t)|}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} \\ & \geq \frac{1}{\theta} \liminf_{k \rightarrow \infty} \max_{0 \leq t \leq 1/2} \frac{|X_{j_k}(t+h_k) - X_{j_k}(t)|}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} \\ & \geq \frac{1}{\theta} \liminf_{k \rightarrow \infty} \max_{0 \leq l < 1/(2 h_k)} \frac{|X_{j_k}((l+1)h_k) - X_{j_k}(lh_k)|}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}}. \end{aligned}$$

Noting that

$$E(X_{j_k}((l+1)h_k) - X_{j_k}(lh_k))(X_{j_k}((i+1)h_k) - X_{j_k}(ih_k)) \leq 0 \quad \text{for } i \neq l$$

(see (4.2) in Csáki, Csörgő, Lin and Révész (1991)), and using the Slepian lemma, we obtain

$$\begin{aligned} P \left\{ \max_{0 \leq l < 1/(2 h_k)} \frac{X_{j_k}((l+1)h_k) - X_{j_k}(lh_k)}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} \leq \frac{1}{\theta} \right\} \\ \leq \left(\Phi \left(\frac{1}{\theta} \left(2 \log \frac{1}{h_k} \right)^{1/2} \right) \right)^{1/(2 h_k)} \leq (1 - h_k^{1/\theta})^{1/(2 h_k)} \\ \leq \exp \left(- \frac{1}{2} \left(\frac{1}{h_k} \right)^{1-1/\theta} \right) \leq \exp \left(- \frac{1}{2} \theta^{(1/2)k(1-1/\theta)} \right) \end{aligned}$$

for every k sufficiently large. Here, in the last inequality, the following fact is used: since $\sigma^*(h) \geq h \sigma^*\left(\frac{1}{2}\right)$, we have $h_k \leq \theta^{-k} / \sigma^*\left(\frac{1}{2}\right)$.

Consequently,

$$\sum_{k=1}^{\infty} \mathbf{P} \left(\max_{0 \leq l < 1/(2h_k)} \frac{X_{j_k}((l+1)h_k) - X_{j_k}(lh_k)}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} < \frac{1}{\theta} \right) < \infty$$

and

$$(4.25) \quad \liminf_{k \rightarrow \infty} \max_{0 \leq l < 1/(2h_k)} \frac{|X_{j_k}((l+1)h_k) - X_{j_k}(lh_k)|}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} \geq \frac{1}{\theta} \quad \text{a.s.}$$

by the Borel-Cantelli lemma.

Now (4.12) follows from (4.24), (4.25) and the arbitrariness of θ .

The proof of (4.13) is similar to that of (4.12). Noting that

$$\begin{aligned} \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1/2} \max_{i \geq 1} \frac{|X_i(t+h) - X_i(t)|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \\ \geq \liminf_{k \rightarrow \infty} \sup_{0 \leq t \leq 1/2} \frac{|X_{j_k}(t+h_k) - X_{j_k}(t)|}{\sigma_{j_k}(h_k) (2 \log(1/h_k))^{1/2}} \end{aligned}$$

we conclude that (4.13) also holds true by (4.24) and (4.25). This completes the proof of Theorem 4.3.

Proof of corollary 4.1. — For any but fixed $\delta > 0$, put $a(h) = \delta \sup_{0 \leq s \leq h} \frac{\sigma(s)}{(\log(1/s))^{1/2}}$ for $0 < h < \min\left(\frac{1}{2}, h_0\right)$. Noting that $\sigma(s)/s^\alpha$ is quasi-increasing, one can see that there exists a constant c_0 , independent of δ , such that

$$\delta \frac{\sigma(h)}{(\log(1/h))^{1/2}} \leq a(h) = \delta \sup_{0 \leq s \leq h} \frac{s^\alpha}{(\log(1/s))^{1/2}} \frac{\sigma(s)}{s^\alpha} \leq \delta c_0 \frac{\sigma(h)}{(\log(1/h))^{1/2}}$$

for $0 < h < \min\left(\frac{1}{2}, h_0\right)$. Moreover, $a(h)$ is non-decreasing and $a(h)/h^{\alpha/2}$

is also quasi-increasing on $\left(0, \min\left(\frac{1}{2}, h_0\right)\right)$. By the assumption

$\sigma^*(h) = o\left(\frac{\sigma(h)}{(\log(1/h))^{1/2}}\right)$ as $h \rightarrow 0$, we have

$$\sigma^*(h) \leq a(h),$$

provided h is sufficiently small. Now from (4.19), we have

$$\begin{aligned} \mathbf{P} \left\{ \|Y(t+h) - Y(t)\| \geq xa(h) + \frac{\sigma(h)}{\varepsilon} \right\} \\ \leq \mathbf{P} \left\{ \|Y(t+h) - Y(t)\| \geq x\sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right\} \\ \leq \exp\left(-\frac{1-\varepsilon^2}{2}x^2\right). \end{aligned}$$

Whence

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{a(h)((2/(1-\varepsilon^2))\log(1/h))^{1/2} + \sigma(h)/\varepsilon} \leq 1 \quad \text{a.s.}$$

along the lines of the proof of (4.10). This implies

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\delta c_0 \sigma(h)(2/(1-\varepsilon^2))^{1/2} + (\sigma(h)/\varepsilon)} \leq 1 \quad \text{a.s.}$$

Thus we have

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma(h)} \leq 1 \quad \text{a.s.}$$

by the arbitrariness of δ and ε . The last inequality in combination with (4.11) yields (4.14) and (4.15), as desired.

The proof of Corollary 4.2 similar to that of Corollary 4.1.

Remark 4.1. — We note again that in Theorem 4.3 we do not assume the condition $\Gamma_0 < \infty$. When assumed, the latter implies our assumption that $\sigma^*(h) < \infty$.

Remark 4.2. — From the above proofs of our theorems, one can see that if $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ is a sequence of independent stationary Gaussian processes with $EX_k(t) = 0$, $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, where $\sigma_k(h)$ is a non-decreasing continuous function for each $k \geq 1$, then Theorems 4.1 and 4.2 remain true. If, in addition, for any $a < b < c < d$, $k \geq 1$, we have also

$$E(X_k(b) - X_k(a))(X_k(d) - X_k(c)) \leq 0,$$

then, Theorem 4.3 and Corollaries 4.1 and 4.2 also holds true.

In the remaining part of this section we consider under which conditions are $\sigma(h)/h^\alpha$ and $\sigma^*(h)/h^\alpha$ quasi-increasing.

LEMMA 4.1. — Assuming that there exists a constant $C > 0$ such that

$$(4.30) \quad \sum_{k: \lambda_k > 1/h} \frac{\gamma_k}{\lambda_k} \leq Ch \sum_{\lambda_k \leq 1/h} \gamma_k \quad \text{for } 0 < h < \frac{1}{C},$$

we have that $\sigma^2(h)/h^\alpha$ is increasing on $(0, \frac{1}{C})$, where $\alpha = \frac{1}{4(1+C)}$.

Proof. — Let $f(h) = \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} (1 - e^{-\lambda_k h}) / h^\alpha$. Then for $0 < h < \frac{1}{C}$,

$$\begin{aligned} f'(h) &= h^{-1-\alpha} \left(-\alpha \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} (1 - e^{-\lambda_k h}) + \sum_{k=1}^{\infty} \gamma_k h e^{-\lambda_k h} \right) \\ &\geq h^{-1-\alpha} \left(-\alpha \sum_{\lambda_k \leq 1/h} \gamma_k h - \alpha \sum_{\lambda_k > 1/h} \frac{\gamma_k}{\lambda_k} + \sum_{k=1}^{\infty} \gamma_k h e^{-\lambda_k h} \right) \\ &\geq h^{-1-\alpha} \left(-\alpha(1+C) h \sum_{\lambda_k \leq 1/h} \gamma_k + \frac{1}{4} \sum_{\lambda_k \leq 1/h} \gamma_k h \right) \geq 0 \end{aligned}$$

provided $0 < \alpha \leq \frac{1}{4(1+C)}$, as desired.

LEMMA 4.2. — Assuming that $\gamma_i / (1 + \lambda_i)^{1-\alpha}$ is quasi-decreasing for some $0 < \alpha < 1$ and that $1 + \lambda_{i+1} \leq C(1 + \lambda_i)$ for some C and each $i \geq 1$, we have that $\sigma^{*2}(h) / h^\alpha$ is quasi-increasing on $\left(0, \frac{1}{2}\right)$.

Proof. — Since $\gamma_i / (1 + \lambda_i)^{1-\alpha}$ is quasi-decreasing, there is a constant C_1 such that for each $i \geq k$

$$(4.31) \quad \frac{\gamma_i}{(1 + \lambda_i)^{1-\alpha}} \leq C_1 \frac{\gamma_k}{(1 + \lambda_k)^{1-\alpha}}.$$

Clearly,

$$(4.32) \quad \frac{1}{3} \max \left\{ \max_{\lambda_i \leq 1/h} \gamma_i h, \max_{\lambda_i > 1/h} \frac{\gamma_i}{\lambda_i} \right\} \leq \sigma^{*2}(h) \leq 2 \max \left\{ \max_{\lambda_i \leq 1/h} \gamma_i h, \max_{\lambda_i > 1/h} \frac{\gamma_i}{\lambda_i} \right\}.$$

Let

$$f(h) = h \max_{\lambda_i \leq 1/h} \gamma_i, \quad i_h = \min \left\{ i : \lambda_i > \frac{1}{h} \right\}.$$

We find

$$\begin{aligned} (4.33) \quad \max_{\lambda_i > 1/h} \frac{\gamma_i}{\lambda_i} &\leq 2 \max_{\lambda_i > 1/h} \frac{\gamma_i}{(1 + \lambda_i)^{1-\alpha}} \cdot \frac{1}{(1 + \lambda_i)^\alpha} \\ &\leq 2 C_1 h^\alpha \frac{\gamma_{i_h-1}}{(1 + \lambda_{i_h-1})^{1-\alpha}} \leq 2 C_1 C h^\alpha \gamma_{i_h-1} / (1 + \lambda_{i_h})^{1-\alpha} \\ &\leq 2 C C_1 h \gamma_{i_h-1} \leq 2 C C_1 h \max_{\lambda_i \leq 1/h} \gamma_i. \end{aligned}$$

Hence, by (4.32), it suffices to show that $f(h)/h^\alpha$ is quasi-increasing. For each $0 < h < s < \frac{1}{2}$ we have

$$\begin{aligned} f(h)/h^\alpha &= h^{1-\alpha} \max_{\lambda_i \leq 1/h} \gamma_i \\ &\leq s^{1-\alpha} \max_{\lambda_i \leq 1/s} \gamma_i + h^{1-\alpha} \max_{1/s < \lambda_i \leq 1/h} \frac{\gamma_i}{(1+\lambda_i)^{1-\alpha}} (1+\lambda_i)^{1-\alpha} \\ &\leq s^{1-\alpha} \max_{\lambda_i \leq 1/s} \gamma_i + h^{1-\alpha} C_1 \frac{\gamma_{i_s-1}}{(1+\lambda_{i_s-1})^{1-\alpha}} \left(1 + \frac{1}{h}\right)^{1-\alpha} \\ &\leq s^{1-\alpha} \max_{\lambda_i \leq 1/s} \gamma_i + 2CC_1 \frac{\gamma_{i_s-1}}{(1+\lambda_{i_s})^{1-\alpha}} \\ &\leq s^{1-\alpha} \max_{\lambda_i \leq 1/s} \gamma_i + 2CC_1 s^{1-\alpha} \gamma_{i_s-1} \\ &\leq (1+2CC_1) s^{1-\alpha} \max_{\lambda_i \leq 1/s} \gamma_i = (1+2CC_1) f(s)/s^\alpha, \end{aligned}$$

as desired.

LEMMA 4.3. — *Assuming*

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{\log(\lambda_i + e)} < \infty,$$

we have $\sigma(h) = o\left(\sigma^*(h) \left(\log \frac{1}{h}\right)^{1/2}\right)$ as $h \rightarrow 0$.

Proof. — Let $\varepsilon = \varepsilon_h = \left(\log \log \frac{1}{h}\right)^{-1}$. Then

$$\begin{aligned} \sigma^2(h) &\leq 2 \sum_{\lambda_k \leq 1/h} \gamma_k h + 2 \sum_{\lambda_k > 1/h} \frac{\gamma_k}{\lambda_k} \\ &\leq 2 \left(\sum_{\lambda_k \leq 1/h} \gamma_k \right) h + 4h \log \frac{1}{h} \sum_{\lambda_k > 1/h} \frac{\gamma_k}{\log(e + \lambda_k)} \\ &\leq 4 \left(\sum_{\lambda_k \geq (1/h)^\varepsilon} \frac{\gamma_k}{\log(e + \lambda_k)} \right) h \log \frac{1}{h} + 4 \left(\sum_{\lambda_k \leq (1/h)^\varepsilon} \frac{\gamma_k}{\log(e + \lambda_k)} \right) 2\varepsilon h \log \frac{1}{h} \\ &\leq 4h \log \frac{1}{h} \sum_{\lambda_k > (1/h)^{\varepsilon_h}} \frac{\gamma_k}{\log(e + \lambda_k)} + 4 \left(\sum_{k=1}^{\infty} \frac{\gamma_k}{\log(e + \lambda_k)} \right) \varepsilon_h h \log \frac{1}{h} \\ &= o\left(h \log \frac{1}{h}\right) \text{ as } h \rightarrow 0, \end{aligned}$$

since $\varepsilon_n \rightarrow 0$, $\left(\frac{1}{h}\right)^{\varepsilon_n} \rightarrow \infty$ and $\sum_{k=1}^{\infty} \frac{\gamma_k}{\log(e + \lambda_k)} < \infty$. On the other hand, it is easy to see that

$$\sigma^{*2}(h) \geq \gamma_1 h$$

as $h \rightarrow 0$. Here, we assume $\gamma_1 > 0$, without loss of generality. Therefore, we have $\sigma(h) = o\left(\sigma^*(h) \left(\log \frac{1}{h}\right)^{1/2}\right)$, as $h \rightarrow 0$.

LEMMA 4.4. — Assume that $1 + \gamma_k$, $1 + \lambda_k$, $k^{1/\alpha}/\gamma_k$ and $\frac{1 + \lambda_k}{\gamma_k}$ are quasi-increasing and that $1 + \lambda_{k+1} \leq \frac{1}{\alpha}(1 + \lambda_k)$, $\lambda_k \leq \frac{1}{\alpha} \exp(k^{1-\alpha})$ for some $0 < \alpha < 1$. Moreover, assume

$$\frac{\sum_{\lambda_k \leq 1/h} \gamma_k}{\log 1/h} \rightarrow \infty \quad \text{as } h \rightarrow 0.$$

Then, we have

$$(4.35) \quad \sigma^*(h) \left(\log \frac{1}{h}\right)^{1/2} = o(\sigma(h)) \quad \text{as } h \rightarrow 0.$$

Proof. — According to the assumption that our functions of interest are quasi-increasing, there exists $C \geq 1$ such that for each $k \leq i$

$$\begin{aligned} 1 + \gamma_k &\leq C(1 + \gamma_i), \\ 1 + \lambda_k &\leq C(1 + \lambda_i), \\ \frac{\gamma_i}{i^{1/\alpha}} &\leq C \frac{\gamma_k}{k^{1/\alpha}}, \\ \frac{\gamma_i}{1 + \lambda_i} &\leq C \frac{\gamma_k}{1 + \lambda_k}. \end{aligned}$$

Along the lines of (4.33), we have

$$(4.36) \quad \begin{aligned} \sigma^{*2}(h) \log \frac{1}{h} &\leq 2 \left(h \max_{\lambda_k \leq 1/h} \gamma_k + \max_{\lambda_k > 1/h} \frac{\gamma_k}{\lambda_k} \right) \log \frac{1}{h} \\ &\leq 4 \left(1 + \frac{C}{\alpha} \right) h \max_{\lambda_k \leq 1/h} \gamma_k \log \frac{1}{h} \\ &\leq 4 \left(1 + \frac{C}{\alpha} \right) C h (1 + \gamma_{k_0}) \log \frac{1}{h}, \end{aligned}$$

where $k_0 = \max \left\{ k : \lambda_k \leq \frac{1}{h} \right\}$. On the other hand, it is clear that

$$\sigma^2(h) \geq \exp(-2C)h \sum_{\lambda_k \leq 2C/h} \gamma_k.$$

So, by (4.36) and the condition $\sum_{\lambda_k \leq 1/h} \gamma_k / \log \frac{1}{h} \rightarrow \infty$, it suffices to show that we have

$$(4.37) \quad \gamma_{k_0} \log \frac{1}{h} = o \left(\sum_{\lambda_k \leq 2C/h} \gamma_k \right).$$

We note that, for each $1 \leq k \leq k_0$, $\lambda_k \leq C(1 + \lambda_{k_0}) \leq C \left(1 + \frac{1}{h} \right) \leq \frac{2C}{h}$. Hence we have

$$\begin{aligned} \sum_{\lambda_k \leq 2C/h} \gamma_k &\geq \sum_{k=1}^{k_0} \gamma_k \geq \frac{\gamma_{k_0}}{C k_0^{1/\alpha}} \sum_{k=1}^{k_0} k^{1/\alpha} \geq \frac{\alpha}{2C} \gamma_{k_0} k_0 \\ &\geq \frac{\alpha \gamma_{k_0}}{2C} \left(\left(\ln \frac{\alpha}{h} \right)^{1/(1-\alpha)} - 1 \right) \geq \frac{\alpha \gamma_{k_0}}{4C} \left(\left(\ln \frac{1}{h} \right)^{1/(1-\alpha)} \right) \end{aligned}$$

as $h \rightarrow 0$. Here, in the last two inequalities, we have used the condition $\lambda_k \leq \frac{1}{\alpha} \exp(k^{1-\alpha})$, which implies $\frac{1}{h} < \lambda_{k_0+1} \leq \frac{1}{\alpha} \exp((k_0+1)^{1-\alpha})$. This proves (4.37) and (4.35) as well.

Remark 4.3. — If $\gamma_k = k^{\theta_1}$, $\lambda_k = k^{\theta_2}$ with $\theta_2 > \theta_1 + 1$, then it is easy to see that the conditions in Lemmas 4.1 and 4.2 are satisfied. That is, we have $\sigma(h)/h^\alpha$ and $\sigma^*(h)/h^\alpha$ are quasi-increasing for some $\alpha > 0$. If, in addition, $\theta_2 > \theta_1 + 1 > 0$, then the conditions in Lemma 4.4 are satisfied. Therefore $\sigma^*(h) \left(\log \frac{1}{h} \right)^{1/2} = o(\sigma(h))$ as $h \rightarrow 0$.

Combining the above lemmas with Corollaries 4.1 and 4.2, we obtain the following results.

COROLLARY 4.3. — Assume that the conditions in Lemmas 4.1 and 4.4 are satisfied. Then (4.14) and (4.15) hold true.

COROLLARY 4.4. — Assume that the conditions in Lemmas 4.2 and 4.3 are satisfied. Then (4.16), (4.17) and (4.18) are true.

5. l^2 -NORM SQUARED PROCESSES

In this section we consider

$$\{ Y(t), -\infty < t < \infty \} = \{ X_k(t), -\infty < t < \infty \}_{k=1}^\infty$$

as in Section 4, *i. e.* $Y(\cdot) \in l^2$, which in turn means that we assume $\Gamma_0 < \infty$ throughout, and study the behaviour of the real valued process

$$(5.1) \quad \chi^2(t) = \| Y(t) \|^2 = \sum_{k=1}^\infty X_k^2(t).$$

Let

$$(5.2) \quad \tilde{\sigma}_k^2(h) = E(X_k^2(t+h) - X_k^2(t))^2 = 4 \left(\frac{\gamma_k}{\lambda_k} \right)^2 (1 - \exp(-2\lambda_k h))$$

$$(5.3) \quad \tilde{\sigma}^2(h) = E(\chi^2(t+h) - \chi^2(t))^2$$

$$= \sum_{k=1}^\infty \tilde{\sigma}_k^2(h) = 4 \sum_{k=1}^\infty \left(\frac{\gamma_k}{\lambda_k} \right)^2 (1 - \exp(-2\lambda_k h))$$

$$(5.4) \quad \tilde{\sigma}^{*2}(h) = \max_{k \geq 1} \tilde{\sigma}_k^2(h).$$

This process was studied by Iscoe and McDonald (1986, 1989), Schmuland (1988 *c*), Csörgő and Lin (1990) and Csáki and Csörgő (1992). Csörgő and Lin (1990) investigated the problem of moduli of continuity for $\chi^2(\cdot)$ under the condition

$$(5.5) \quad \Gamma_2 = \sum_{k=1}^\infty \gamma_k^2 / \lambda_k < \infty$$

and in this case they proved the following results:

- (i) Let $M = \max_{k \geq 1} (\gamma_k^2 / \lambda_k)$, and assume that $T_h \uparrow \infty$ continuously as $h \downarrow 0$.

Then

$$(5.6) \quad \limsup_{h \rightarrow 0} \sup_{|t| \leq T_h} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8 h M)^{1/2} \log(T_h/h)} \leq 1 \quad a.s.$$

- (ii) If, in addition, the continuous non-decreasing function T_h is such that $\log T_h / \log\left(\frac{1}{h}\right) \rightarrow \infty$ as $h \rightarrow 0$, then we have equality to 1 in (5.6) instead of the inequality.

Csáki and Csörgő (1992) obtained a similar result to (5.6):

- (iii) Assume $\tilde{\sigma}(h)$ is regularly varying at zero with a positive exponent.

Then

$$(5.7) \quad \limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}(h) \log 1/h} \leq 1 \quad a.s.$$

(iv) Also, if (5.5) holds, then

$$(5.8) \quad \limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8hM)^{1/2} \log 1/h} \leq 1 \quad a.s.$$

Here, based on our results in Sections 2 and 3, we consider further moduli of continuity for the l^2 -norm squared process $\chi^2(\cdot)$.

THEOREM 5.1. — Assume that $\tilde{\sigma}(h)/h^\alpha$ is quasi-increasing on $(0, \alpha)$ for some $\alpha > 0$ and that $\tilde{\sigma}^*(h) \left(\log \frac{1}{h}\right)^{1/2} = o(\tilde{\sigma}(h))$ as $h \rightarrow 0$. Then, we have

$$(5.9) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}(h) (2 \log 1/h)^{1/2}} \leq 1 \quad a.s.$$

Proof. — For given $0 < \varepsilon < \frac{1}{4}$, for each $0 < x < \frac{\sqrt{\varepsilon} \tilde{\sigma}(h)}{(1-\varepsilon) \tilde{\sigma}^*(h)}$, let $\lambda = \frac{(1-\varepsilon)x}{\tilde{\sigma}(h)}$. Clearly, we have $0 < \lambda^2 \tilde{\sigma}^{*2}(h) < \varepsilon$ and for each $0 < s \leq h$

$$(5.10) \quad \begin{aligned} &P \{ |\chi^2(t+s) - \chi^2(t)| \geq x \tilde{\sigma}(h) \} \\ &\leq 2 \exp(-\lambda x \tilde{\sigma}(h)) \prod_{k=1}^{\infty} E \exp(\lambda (X_k^2(t+s) - X_k^2(t))) \\ &\leq 2 \exp(-\lambda x \tilde{\sigma}(h)) \prod_{k=1}^{\infty} (1 - \lambda^2 \tilde{\sigma}_k^2(h))^{-1/2} \\ &\leq 2 \exp\left(-\lambda x \tilde{\sigma}(h) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\lambda^2 \tilde{\sigma}_k^2(h)}{1 - \lambda^2 \tilde{\sigma}_k^2(h)}\right) \\ &\leq 2 \exp\left(-\lambda x \tilde{\sigma}(h) + \frac{\lambda^2 \tilde{\sigma}^2(h)}{2(1-\varepsilon)}\right) = 2 \exp\left(-\frac{1-\varepsilon}{2} x^2\right). \end{aligned}$$

We now prove that there is a positive C such that

$$(5.11) \quad P \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} |\chi^2(t+s) - \chi^2(t)| \geq x \tilde{\sigma}(h) \right\} \leq C \frac{1}{h} \exp\left(-\frac{1-2\varepsilon}{2} x^2\right)$$

for each $\left(\log \frac{1}{h}\right)^{1/2} \leq x \leq 4 \left(\log \frac{1}{h}\right)^{1/2}$, provided h is small enough.

Since $\tilde{\sigma}(h)/h^\alpha$ is quasi-increasing on $(0, \alpha)$, there is a constant $c_0 > 0$ such that for each $0 < s < h < \alpha$,

$$(5.12) \quad \tilde{\sigma}(s) \leq c_0 \left(\frac{s}{h}\right)^\alpha \tilde{\sigma}(h).$$

Similarly to the proof of Lemma 2. 1, by (5. 10) we have

$$(5. 13) \quad \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} |\chi^2(t+s) - \chi^2(t)| \geq x(\tilde{\sigma}(h)) \right. \\ \left. + \tilde{\sigma}(h2^{-k}) + 2 \sum_{j=0}^{\infty} x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \\ \leq \frac{8 \cdot 2^{2k}}{h} \exp\left(-\frac{1-\varepsilon}{2}x^2\right) + \frac{4}{h} \sum_{j=0}^{\infty} 2^{k+j+1} \exp\left(-\frac{1-\varepsilon}{2}x_j^2\right)$$

for each $k \geq 1, 0 < x_j \leq \frac{\sqrt{\varepsilon}}{(1-\varepsilon)} \frac{\tilde{\sigma}(h/2^{k+j+1})}{\tilde{\sigma}^*(h/2^{k+j+1})}$. Let $x_j^2 = x^2 + 2(j+1)$. Then

$$x_j \tilde{\sigma}^*\left(\frac{h}{2^{k+j+1}}\right) / \tilde{\sigma}\left(\frac{h}{2^{k+j+1}}\right) = \frac{(x^2 + 2j + 2)^{1/2} \tilde{\sigma}^*(h/2^{k+j+1})}{\tilde{\sigma}(h/2^{k+j+1})} \\ \leq (16 \log \frac{1}{h} + 2j + 2)^{1/2} \tilde{\sigma}^*\left(\frac{h}{2^{k+j+1}}\right) / \tilde{\sigma}\left(\frac{h}{2^{k+j+1}}\right) \\ \leq 16 \left(\log \frac{2^{k+j+1}}{h}\right)^{1/2} \tilde{\sigma}^*\left(\frac{h}{2^{k+j+1}}\right) / \tilde{\sigma}\left(\frac{h}{2^{k+j+1}}\right) \\ \rightarrow 0, \text{ as } h \rightarrow 0, \text{ uniformly in } j \geq 0.$$

That is to say, we have $x_j \leq \frac{\sqrt{\varepsilon} \tilde{\sigma}(h/2^{k+j+1})}{(1-\varepsilon) \tilde{\sigma}^*(h/2^{k+j+1})}$ for each $j \geq 0$, provided h is small enough. So, from (5. 13) and (5. 12) we obtain

$$\sum_{j=0}^{\infty} 2^{k+j+1} \exp\left(-\frac{1-\varepsilon}{2}x_j^2\right) \leq 2^{k+2} \exp\left(-\frac{1-\varepsilon}{2}x^2\right),$$

and

$$2 \sum_{j=0}^{\infty} x_j \tilde{\sigma}(h2^{-k-j}) \leq 2 \sum_{j=0}^{\infty} xc_0 \tilde{\sigma}(h) 2^{-(k+j)\alpha} \\ + \sum_{j=0}^{\infty} c_0(j+1) \tilde{\sigma}(h) 2^{-(k+j)\alpha} \\ \leq \frac{2c_0 2^{-k\alpha}}{2^\alpha - 1} x \tilde{\sigma}(h) + \frac{4c_0 2^{-k\alpha}}{2^\alpha - 1} \tilde{\sigma}(h).$$

Consequently, we have (5.11), provided we let k be sufficiently large. By (5.11) and using Theorem 3.1, we have

$$(5.14) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}(h)((2/(1-2\varepsilon))(\log(1/h) + \log \log(1/\tilde{\sigma}(h))))^{1/2}} \leq 1 \quad \text{a.s.}$$

from which it follows that (5.9) holds true on account of $\liminf_{h \rightarrow 0} \frac{\tilde{\sigma}^2(h)}{h} > 0$ and the arbitrariness of ε . This completes the proof of Theorem 5.1.

THEOREM 5.2. — Assume that $\tilde{\sigma}^*(h)/h^\alpha$ is quasi-increasing on $(0, \alpha)$ for some $\alpha > 0$ and that $\tilde{\sigma}(h) \leq c \tilde{\sigma}^*(h) \left(\log \frac{1}{h}\right)^{1/2}$ as $h \rightarrow 0$. Then, we have

$$(5.15) \quad \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}^*(h) \log(1/h)} \leq \theta \quad \text{a.s.}$$

where $\theta = \left(1 - \frac{3c}{4}\right)^{-3/2}$ if $0 < c \leq 1$, and $\theta = 4(1 + c^4)^{1/2}$ if $c > 1$.

Proof. — Let $\varepsilon = \frac{2}{3}c$ if $0 < c \leq 1$ and $\varepsilon = \frac{c^4}{1 + c^4}$ if $c > 1$, $\lambda^2 = \frac{1 - \varepsilon}{\tilde{\sigma}^{*2}(h)}$.

Then, similarly to (5.10),

$$(5.16) \quad \begin{aligned} &P\{|\chi^2(t+h) - \chi^2(t)| \geq x \tilde{\sigma}^*(h)\} \\ &\leq 2 \exp(-\lambda x \tilde{\sigma}^*(h) + \lambda^2 \tilde{\sigma}^2(h)/2\varepsilon) \\ &\leq 2 \exp\left(-\sqrt{1-\varepsilon}x + \frac{1-\varepsilon}{2\varepsilon}c^2 \log \frac{1}{h}\right) \\ &\leq 2 \exp\left(-\sqrt{1-\varepsilon}\left(x - \frac{\sqrt{1-\varepsilon}}{2\varepsilon}c^2 \log \frac{1}{h}\right)\right) \leq 2 \exp\left(-\frac{x}{\theta}\right) \end{aligned}$$

for each $x \geq \log \frac{1}{h}$. Using Remark 2.1, (5.16) and Theorem 3.1, it follows that (5.15) is true.

According to the relationship that for $|t| \leq T$ we have

$$|\chi^2(t+h) - \chi^2(t)| \leq 2 \|Y(t+h) - Y(t)\| \sup_{|t| \leq T} \|Y(t)\|,$$

one can see that

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\sigma(h)} \leq 2 \sup_{0 \leq t \leq 1} \|Y(t)\| \quad \text{a.s.}$$

under the conditions of Corollary 4.1, and

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \leq 2 \sup_{0 \leq t \leq 1} \|Y(t)\| \quad \text{a.s.}$$

under the conditions of Corollary 4.2. We do not know how sharp are our upper estimations in (5.9) and (5.15).

We conclude with two conjectures.

CONJECTURE 5.1. – *Under the conditions of Theorem 5.1 we have*

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}(h) (2 \log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

CONJECTURE 5.2. – *Assume that $\tilde{\sigma}^*(h)/h^\alpha$ is quasi-increasing on $(0, \alpha)$ for some $\alpha > 0$ and that $\tilde{\sigma}(h) = o\left(\tilde{\sigma}^*(h) \left(\log \frac{1}{h}\right)^{1/2}\right)$ as $h \rightarrow 0$. Then, there is no function $\theta(h)$ such that*

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \theta(h) |\chi^2(t+s) - \chi^2(t)| = 1 \quad \text{a.s.}$$

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6. APPENDIX

Proof of lemma 2.5. – Let $a \geq 8$, put $2^{k+1} = \log_2 a$ in Lemma 2.1, where \log_2 is logarithm to the base 2. Then, by Lemma 2.1, we have

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} \|\Gamma(t+s) - \Gamma(t)\| \geq x(\sigma_1(a) \right. \\ & \qquad \qquad \qquad \left. + \sigma_1(a, k)) + \sigma_1^*(a, k) + \sigma_2(a) + \sigma_2(a, k) \right\} \\ & \leq 4(T/a + 1) K 2^{2^{k+1}} \exp(-\gamma x^\beta) \\ & = 4(T/a + 1) K \exp(-\gamma x^\beta) \leq 8TK \exp(-\gamma x^\beta), \end{aligned}$$

by the fact that $a \leq T$, where

$$\sigma_1(a, k) = 2^{3+(1/\beta)} \int_{2^{k-3}}^\infty \frac{\sigma_1(ae^{-z})}{z} dz,$$

$$\sigma_2(a, k) = 6 \int_{2^{k-3}}^{\infty} \frac{\sigma_2(ae^{-z})}{z} dz,$$

$$\sigma_1^*(a, k) = 4(14/\gamma)^{1/\beta} \beta \int_{2^{(k-2)/\beta}}^{\infty} \sigma_1(ae^{-z^\beta}) dz.$$

Noting that $2^{k-3} = \frac{\log a}{16 \log 2}$, $2^{(k-2)/\beta} = \left(\frac{\log a}{8 \log 2}\right)^{1/\beta}$, we have

$$\begin{aligned} & \int_{2^{k-3}}^{\infty} \frac{\sigma_1(ae^{-z})}{z} dz = \int_{\log a/(16 \log 2)}^{\infty} \frac{\sigma_1(ae^{-z})}{z} dz \\ & \leq \int_{2 \log a}^{\log a/16} \frac{\sigma_1(ae^{-z})}{z} dz + \int_{\log a}^{\infty} \frac{\sigma_1(e^{-(z-\log a)})}{z} dz \\ & \leq \int_{\log a/16}^{\log a} \frac{\sigma_1(a)}{(\log a/16)} dz + \int_{\log a}^{\infty} \frac{\sigma_1(e^{-z})}{z} dz \\ & \leq 32 \sigma_1(a) + \int_{\log a}^{\infty} \frac{\sigma_1(e^{-z})}{z} dz \\ & = 32 \sigma_1(a) + \int_{\log \log a}^{\infty} \sigma_1(e^{-e^z}) dz \\ & \leq 32 \sigma_1(a) + \int_1^{\infty} \sigma_1(e^{-z^\beta}) dz \leq 33 \sigma_1(a), \end{aligned}$$

provided a is sufficiently large for $\sigma_1(a) \rightarrow \infty$ as $a \rightarrow \infty$ and $\int_1^{\infty} \sigma_1(e^{-z^\beta}) dz < \infty$. Similarly, we have

$$\begin{aligned} \int_{2^{k-3}}^{\infty} \frac{\sigma_2(ae^{-z})}{z} dz & \leq 32 \sigma_2(a) + \int_{\log a}^{\infty} \frac{\sigma_2(e^{-z})}{z} dz \\ & \leq 32 \sigma_2(a) + \int_1^{\infty} \frac{\sigma_2(e^{-z})}{z} dz \leq 32 \sigma_2(a) + \sigma_1(a) \end{aligned}$$

for every big enough a .

We estimate $\sigma_1^*(a, k)$ below. We have

$$\begin{aligned} & \int_{2^{(k-2)/\beta}}^{\infty} \sigma_1(ae^{-z^\beta}) dz \\ & = \int_{(\log a/(8 \log 2))^{1/\beta}}^{(2 \log a)^{1/\beta}} \sigma_1(ae^{-z^\beta}) dz + \int_{(2 \log a)^{1/\beta}}^{\infty} \sigma_1(ae^{-z^\beta}) dz \\ & \leq \int_{(\log a/(8 \log 2))^{1/\beta}}^{(2 \log a)^{1/\beta}} \sigma_1(a) dz + \int_{(2 \log a)^{1/\beta}}^{\infty} \sigma_1(e^{-z^\beta - \log a}) dz \\ & \leq (2 \log a)^{1/\beta} \sigma_1(a) + \int_{(\log a)^{1/\beta}}^{\infty} \sigma_1(e^{-z^\beta}) (z^\beta + \log a)^{1/\beta - 1} z^{\beta - 1} dz \end{aligned}$$

$$\begin{aligned}
&\leq (2 \log a)^{1/\beta} \sigma_1(a) + \int_{(\log a)^{1/\beta}}^{\infty} \sigma_1(e^{-z^\beta}) (z^\beta + \log a)^{1/\beta} z^{-1} dz \\
&\leq (2 \log a)^{1/\beta} \sigma_1(a) + 2^{1/\beta} \int_{(\log a)^{1/\beta}}^{\infty} \sigma_1(e^{-z^\beta}) dz \\
&\leq (2 \log a)^{1/\beta} \sigma_1(a) + 2^{1/\beta} \int_1^{\infty} \sigma_1(e^{-z^\beta}) dz \\
&\leq (3 \log a)^{1/\beta} \sigma_1(a)
\end{aligned}$$

for every big enough a , by the assumption $\sigma_1(a) \rightarrow \infty$ and $\int_1^{\infty} \sigma_1(e^{-z^\beta}) dz < \infty$ again. Taking now

$$c_1 = \left(2^{3+1/\beta} + 4 \left(\frac{14}{\gamma} \right)^{1/\beta} \beta \right) (32 + 3^{1/\beta}),$$

we arrive at the desired result.

Proof of theorem 3.6. — By Lemma 2.2, for every $0 < \varepsilon < 1$, $0 \leq h \leq h_0$ and $x \geq \max\left(1, \frac{x^*}{1-\varepsilon}\right)$ we have

$$\begin{aligned}
\mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(h) + (1+\varepsilon) \sigma_2(h) \right\} \\
\leq \text{CK} \frac{1}{h} \exp\left(-\frac{\gamma x^\beta}{1+\varepsilon}\right).
\end{aligned}$$

Consequently, by Theorem 3.1,

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \theta_h(\varepsilon) \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \quad \text{a.s.},$$

where

$$\theta_h^{-1}(\varepsilon) = \sigma_1(h) \left(\frac{1+\varepsilon}{\gamma} \left(\log \frac{1}{h} + \log \log \left(\sigma_1(h) + \frac{1}{\sigma_1(h)} \right) \right) \right)^{1/\beta} + (1+\varepsilon) \sigma_2(h),$$

which yields (3.17) by the fact that $\sigma_1(h) \leq \sigma_1(1) < \infty$ and the arbitrariness of ε .

Proof of theorem 3.7. — Put $\varepsilon = 1$ in Lemma 2.3. By Lemma 2.3, for every $0 < h \leq \min\left(e^{-8}, h_0, \frac{1}{2}\right)$, $x \geq \max(x^*, 1)$, we have

$$\begin{aligned}
\mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \|\Gamma(t+s) - \Gamma(t)\| \geq x \sigma_1(h) (1 + c_1 c_0) \right. \\
\left. + \sigma_2(h) (1 + c_2 c_0) + c_2 c_0 \sigma_1(h) \left(\log \frac{1}{h} \right)^{1/\beta} \right\} \\
\leq 18 K h^{-3} \exp(-\gamma x^\beta).
\end{aligned}$$

Therefore, by Theorem 3. 1,

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \theta_h^* \|\Gamma(t+s) - \Gamma(t)\| \leq 1 \text{ a.s.,}$$

where

$$\theta_h^{*-1} = \sigma_1(h)(1 + c_1 c_0) \left(\frac{1}{\gamma} \left(\log h^{-3} + \log \log \left(\sigma_1(h)(1 + c_1 c_0) + \frac{1}{\sigma_1(h)(1 + c_1 c_0)} \right) \right) \right)^{1/\beta} + \sigma_1(h)(1 + c_2 c_0) + c_2 c_0 \sigma_1(h) \left(\log \frac{1}{h} \right)^{1/\beta}.$$

Take $C = (1 + c_1 c_0 + c_2 c_0) \left(1 + \frac{1}{\gamma} \right)^{1/\beta}$. Clearly, we have

$$\limsup_{h \rightarrow 0} \theta_h^{*-1} / \theta_h^{-1} \leq C.$$

Putting the above inequalities together, we conclude that (3.18) holds true.

Proof of corollary 4.2. – Since

$$\|Y(t+s) - Y(t)\| \geq \max_{k \geq 1} |X_k(t+s) - X_k(t)|,$$

it suffices to show that

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \leq 1 \text{ a.s.}$$

by Theorem 4.3.

For any but fixed $\delta > 0$, put

$$a(h) = \delta \sup_{0 \leq s \leq h} \sigma^*(s) \left(\log \frac{1}{s} \right)^{1/2} \text{ for } 0 < h < \min \left(\frac{1}{2}, h_0 \right).$$

Noting that $\sigma^*(h)/h^\alpha$ is quasi-increasing, one can see that there exists a constant c_0 , independent of δ , such that

$$\delta \sigma^*(h) \left(\log \frac{1}{h} \right)^{1/2} \leq a(h) = \delta \sup_{0 \leq s \leq h} s^\alpha \left(\log \frac{1}{s} \right)^{1/2} \frac{\sigma^*(s)}{s^\alpha} \leq \delta c_0 \sigma^*(h) \left(\log \frac{1}{h} \right)^{1/2}$$

for $0 < h < \min \left(\frac{1}{2}, h_0 \right)$. Moreover, $a(h)$ is non-decreasing and $a(h)/h^{\alpha/2}$ is quasi-increasing on $\left(0, \min \left(\frac{1}{2}, h_0 \right) \right)$. By the assumption

$$\sigma(h) = o \left(\sigma^*(h) \left(\log \frac{1}{h} \right)^{1/2} \right)$$

as $h \rightarrow 0$, we have

$$\sigma(h) \leq a(h),$$

provided that h is sufficiently small. Now, from (4.19), for every $0 < \varepsilon < 1$, we have

$$\begin{aligned} & \mathbf{P} \left\{ \left\| Y(t+h) - Y(t) \right\| \geq x \sigma^*(h) + \frac{a(h)}{\varepsilon} \right\} \\ & \leq \mathbf{P} \left\{ \left\| Y(t+h) - Y(t) \right\| \geq x \sigma^*(h) + \frac{\sigma(h)}{\varepsilon} \right\} \\ & \leq \exp \left(- \frac{x^2 (1 - \varepsilon^2)}{2} \right). \end{aligned}$$

Therefore, we have

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h) ((2/(1-\varepsilon^2)) \log(1/h))^{1/2} + a(h)/\varepsilon} \leq 1 \quad \text{a.s.}$$

along the lines of the proof of (4.10). This implies

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h) (2/(1-\varepsilon^2) \log(1/h))^{1/2} + ((\delta c_0 \sigma^*(h))/\varepsilon) (\log(1/h))^{1/2}} \leq 1 \quad \text{a.s.}$$

which yields

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h) (2 \log(1/h))^{1/2}} \leq 1 \quad \text{a.s.}$$

by the arbitrariness of δ and ε (let $\delta \rightarrow 0$ first and then $\varepsilon \rightarrow 0$), as desired.

Proof of (5.13). – We proceed along the lines of the proof of Lemma

2.1. Given $k \geq 1$, $0 < h \leq \alpha$, let $t_j = \left[\frac{t 2^j}{h} \right] h 2^{-j}$. We have

$$\begin{aligned} & |\chi^2(t+s) - \chi^2(t)| \\ & \leq |\chi^2((t+s)_k) - \chi^2(t_k)| + |\chi^2(t+s) - \chi^2((t+s)_k)| + |\chi^2(t) - \chi^2(t_k)| \\ & \leq |\chi^2((t+s)_k) - \chi^2(t_k)| + \sum_{j=0}^{\infty} |\chi^2((t+s)_{k+j+1}) - \chi^2((t+s)_{k+j})| \\ & \quad + \sum_{j=0}^{\infty} |\chi^2(t_{k+j+1}) - \chi^2(t_{k+j})|. \end{aligned}$$

Noting that for $0 \leq t \leq 1$, $0 \leq s \leq h$,

$$\begin{aligned} & (t+s)_k - t_k \in \{h 2^{-k} i, i=0, 1, \dots, 2^k+1\}, \\ & (t+s)_{k+j+1} - (t+s)_{k+j} = 0 \quad \text{or} \quad h 2^{-(k+j+1)}, \end{aligned}$$

and

$$\tilde{\sigma}(h2^{-k}(2^k+1)) \leq \tilde{\sigma}(h) + \tilde{\sigma}(h2^{-k}),$$

by Minkowski's inequality, we obtain

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} |\chi^2(t+s) - \chi^2(t)| \geq x(\tilde{\sigma}(h) \right. \\ & \qquad \left. + \tilde{\sigma}(h2^{-k})) + 2 \sum_{j=0}^{\infty} x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \\ & \leq \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} |\chi^2((t+s)_k) - \chi^2(t_k)| \geq x(\tilde{\sigma}(h) + \tilde{\sigma}(h2^{-k})) \right\} \\ & \quad + \sum_{j=0}^{\infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} (|\chi^2((t+s)_{k+j+1}) - \chi^2((t+s)_{k+j})| \right. \\ & \qquad \left. + |\chi^2(t_{k+j+1}) - \chi^2(t_{k+j})|) \geq 2x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \\ & \leq \left(\frac{1}{h2^{-k}} + 1 \right) \sum_{i=1}^{2^{k+1}} \mathbb{P} \left\{ |\chi^2(ih2^{-k}) - \chi^2(0)| \geq x \tilde{\sigma}(h2^{-k}(2^k+1)) \right\} \\ & \quad + \sum_{j=0}^{\infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq 1+h} |\chi^2(t_{k+j+1}) - \chi^2(t_k)| \geq x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \\ & \leq 2 \frac{2^k 2^{k+1}}{h} \sum_{i=1}^{2^{k+1}} \mathbb{P} \left\{ |\chi^2(ih2^{-k}) - \chi^2(0)| \geq x \tilde{\sigma}(h2^{-k}i) \right\} \\ & \quad + \sum_{j=0}^{\infty} \left(\frac{1+h}{h2^{-(k+j+1)}} + 1 \right) \mathbb{P} \left\{ |\chi^2(h2^{-(k+j+1)}) \right. \\ & \qquad \left. - \chi^2(0)| \geq x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \\ & \leq 2 \frac{2^k 2^{k+1}}{h} \sum_{i=1}^{2^{k+1}} \mathbb{P} \left\{ |\chi^2(ih2^{-k}) - \chi^2(0)| \geq x \tilde{\sigma}(h2^{-k}i) \right\} \\ & \quad + \frac{2}{h} \sum_{j=0}^{\infty} 2^{k+j+1} \mathbb{P} \left\{ |\chi^2(h2^{-(k+j+1)}) - \chi^2(0)| \geq x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\}. \end{aligned}$$

For each $j \geq 0$, $0 < x_j \leq \frac{\sqrt{\varepsilon}}{1-\varepsilon} \frac{\tilde{\sigma}(h2^{-(k+j+1)})}{\tilde{\sigma}^*(h2^{-(k+j+1)})}$, by (5.10) we have

$$\mathbb{P} \left\{ |\chi^2(h2^{-(k+j+1)}) - \chi^2(0)| \geq x_j \tilde{\sigma}(h2^{-(k+j+1)}) \right\} \leq 2 \exp \left(- \frac{1-\varepsilon}{2} x_j^2 \right).$$

Notice that for $x \leq 4 \left(\log \frac{1}{h} \right)^{1/2}$,

$$\begin{aligned} \frac{x \tilde{\sigma}^*(h 2^{-k} i)}{\tilde{\sigma}(h 2^{-k} i)} &\leq 4 \left(\log \frac{1}{h} \right)^{1/2} \tilde{\sigma}^*(h 2^{-k} i) / \tilde{\sigma}(h 2^{-k} i) \\ &\leq 8 \left(\log \frac{1}{h 2^{-k} i} \right)^{1/2} \tilde{\sigma}^*(h 2^{-k} i) / \tilde{\sigma}(h 2^{-k} i) \\ &\rightarrow 0 \text{ as } h \rightarrow 0, \text{ uniformly in } k \geq 1, 1 \leq i \leq 2^k + 1, \end{aligned}$$

by the assumption that $\tilde{\sigma}^*(h) \left(\log \frac{1}{h} \right)^{1/2} = o(\tilde{\sigma}(h))$ as $h \rightarrow 0$. Therefore, we can use (5.10) and obtain

$$P \left\{ \left| \chi^2(ih 2^{-k}) - \chi^2(0) \right| \geq x \tilde{\sigma}(hi 2^{-k}) \right\} \leq 2 \exp \left(- \frac{1-\varepsilon}{2} x^2 \right).$$

Putting the above inequalities together, we obtain (5.13).

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