ANNALES DE L'I. H. P., SECTION B

ALEXANDER L. KOLDOBSKY

Generalized Lévy representation of norms and isometric embeddings into L_p-spaces

Annales de l'I. H. P., section B, tome 28, nº 3 (1992), p. 335-353

<http://www.numdam.org/item?id=AIHPB_1992__28_3_335_0>

Ann. Inst. Henri Poincaré,
Vol. 28, n° 3, 1992, p. 335-353. *Probabilités et Statistiques*

Generalized Levy representation of norms and isometric embeddings into L_p-spaces

by

Alexander L. KOLDOBSKY
University of Missouri-Columbia,
Department of Mathematics,
Columbia, MO 65211, U.S.A.

ABSTRACT. – Replacing a measure γ by a distribution in the Levy representation $\|x\|^p = \int_{\mathbb{R}^n} |\langle x, s\rangle|^p \, d\gamma(s)$ of a norm we define the generalized representation which exists and is unique for every finite dimensional Banach space. This representation leads to a criterion of isometric embedability of Banach spaces into L_p-spaces.

Key words : Distribution, Fourier transform, isometric embedding.

RÉSUMÉ. – En remplaçant la représentation Levy la mesure par la distribution nous déterminons la représentation générale Levy qui existe et qui est unique pour n'importe quelle espace de dimension définie. Cette représentation donne les critères de possibilité de placer isométriquement des espaces de Banach dans espaces L_p.

Classification A.M.S. : Primary: 46 E 30. Secondary: 46 F 12, 42 B 10, 60 E 07.

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques - 0246-0203
Vol. 28/92/03/335/19/$ 3,90/© Gauthier-Villars

1. INTRODUCTION

An old question going back to P. Levy is to characterize those Banach spaces which embed linearly and isometrically into some L_p-space. It is well-known that a Banach space is isometric to a subspace of a Hilbert space iff it satisfies the parallelogram law [5], [12]. But as shown by A. Neyman [27], for $p \neq 2$, subspaces of L_p can not be characterized by a finite number of equations of inequalities.

For $p \neq 2$, a popular idea of constructing isometric embeddings into L_p is based on the connection between stable measures and positive definite norm dependent functions which has been discovered by P. Levy [20]. Let $(E, \|\,.\,\|)$ be an n-dimensional Banach space, and assume there exists an even continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(0)=1$ and the function $f(\|x\|)$ is positive definite on $\mathbb{R}^n$. By Bochner's theorem, there exist a probability measure μ on $\mathbb{R}^n$ and a probability measure ν on $\mathbb{R}$ such that $\hat{\mu}=f(\|x\|)$ and $\hat{\nu}=f$ (we denote by $\hat{}$ the Fourier transform). One can easily prove that, for every $x \in \mathbb{R}^n$, $x \neq 0$, the measure ν is the image of the measure μ under the mapping $s \to \langle x, s \rangle / \|x\|$, $s \in \mathbb{R}^n$ (here $\langle x, s \rangle$ is the scalar product). If ν has a finite moment of the p-th order then, for every $x \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |\langle x, s \rangle|^p \, d\mu(s) = \|x\|^p \int_{\mathbb{R}^n} \left| \frac{\langle x, s \rangle}{\|x\|} \right|^p d\mu(s) = \left(\int_{\mathbb{R}} |t|^p \, d\nu(t) \right) . \|x\|^p \quad (1)$$

Now we can embed E isometrically into $L_p([0, 1])$ taking functions from $L_p([0, 1])$ with the joint distribution μ (for details and generalizations of this reasoning *see* [1], [25], [19], [21], [34], [14]).

This method was first used in [1] for constructing isometric embeddings of the spaces l_q^n into L_p where $0<p<q \leqq 2$. The construction was based on the well-known fact that, for $q \in (0, 2]$, the function $\exp(-\|x\|_q^q)$ is positive definite on $\mathbb{R}^n$, where $\|x\|_q = (|x_1|^q + \ldots + |x_n|^q)^{1/q}$.

The authors of [1] have also given a general criterion: For $1 \leqq p \leqq 2$, a Banach space $(E, \|\,.\,\|)$ is isometric to a subspace of L_p iff the function $\exp(-\|x\|^p)$ is positive definite. This result was generalized by J. L. Krivine in [18]: if $p \geqq 1$ and $2r-2<p<2r \leqq 4k$ for some positive integers r and k, then a Banach space E is isometric to a subspace of some L_p-space iff for every choice of $e_1, \ldots, e_n \in E$ and every choice of scalars $\rho_1, \ldots, \rho_n$ with $\sum_{i=1}^{n} \rho_i = 0$, we have

$$(-1)^r \sum \| e_{i_1} + e_{i_2} \pm \ldots \pm e_{i_{2k}} \|^p \rho_{i_1} \ldots \rho_{i_{2k}} \geqq 0$$

where the sum is taken over all choices of integers $1 \leqq i_1, \ldots, i_{2k} \leqq n$ and all choices of signs.

The criterions mentioned above have a disadvantage, namely, it is usually difficult to find a positive definite norm dependent function or to prove the absence of such functions. For instance, the following questions posed by I. J. Schoenberg [30] in 1938 had been open for about fifty years: For which $p>0$ are the functions $\exp(-\|x\|_q^p)$ and $\exp(-\|x\|_\infty^p)$ positive definite, where $q>2$ and $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$? An answer turned out to be the same for both spaces l_q^n and l_∞^n: $p \in (0, 1]$ if $n=2$, and $p \in \varnothing$ if $n \geqq 3$. For the spaces l_q^n, the result was obtained in 1991 [14], and for the spaces l_∞^n the answer was given in 1989 by J. Misiewicz [26]. Moreover, J. Misiewicz proved in [26] that, for $n \geqq 3$, a function $f(\|x\|_\infty)$ is positive definite only if f is a constant function.

One can look at Schoenberg's problems from another point of view. As it was mentioned above, for $0<p\leqq 2$ the function $\exp(-\|x\|^p)$ is positive definite iff the space $(E, \|\,.\,\|)$ is isometric to a subspace of L_p. Obviously for $p>2$ the function $\exp(-\|x\|^p)$ is not positive definite for any norm. L. Dor [2] has proved that for $p \geqq 1$, $q \geqq 1$ the space l_q^n is isometric to a subspace of L_p only in one of the following situations (*a*) $p<q\leqq 2$, (*b*) $q=2$, (*c*) $p=q$, (*d*) $n=2$, $p=1$, q is an arbitrary number. Thus, to give an answer to the first Schoenberg's question it suffices to prove that, for $q>2$, the space l_q^3 is not isometric to a subspace of L_p with $p \in (0, 1)$. Since the space l_∞^2 is not isometric to a subspace of the smooth space L_p with $p>1$, the answer to the second Schoenberg's question will follow from the fact that the space l_∞^3 is not isometric to a subspace of L_p with $p \in (0, 1]$. It is well-known that every two-dimensional Banach space $(E, \|\,.\,\|)$ is isometric to a subspace of L_1 (*see* [4], [10], [23], [33]), and the function $\exp(-\|x\|^p)$ is positive definite for every $p \in (0, 1]$ (note that $\exp(-\|x\|)$ is positive definite and use an easy fact from [31]). For some other results concerned with isometric embeddings into L_p-spaces see [24] and [11].

In this paper we unite the results described above by giving a new general criterion of isometric embeddability into L_p-spaces. We prove that an n-dimensional Banach space $(E, \|\,.\,\|)$ is isometric to a subspace of L_p with $p>0$, $p \neq 2, 4, 6, \ldots$ only if the distribution

$$\gamma(\xi_1, \ldots, \xi_{n-1}) = \frac{1}{(2\pi)^{n-1} \cdot c_p} \cdot (\|x\|^p)^\wedge (\xi_1, \ldots, \xi_{n-1}, 1) \qquad (2)$$

is a finite measure on $\mathbb{R}^{n-1}$ with finite weak moments of the p-th order $\left(i.e. \int_{\mathbb{R}^{n-1}} |\langle x, \xi \rangle|^p d\gamma(\xi) < \infty \text{ for every } x \in \mathbb{R}^{n-1}\right)$. Here $c_p = 2^{p+1} \pi^{1/2} \Gamma((p+1)/2)/\Gamma(-p/2)$ and the Fourier transform is considered in the sense of distributions. In the case where $(\|x\|^p)^\wedge$ is not a regular distribution the expression in the right-hand side of (2) will be clarified (for precise statements see Section 2).

We show in Remark 2 that this criterion does not differ much from the criterion from [1]. Thus, our criterion has a similar disadvantage, namely, it is usually rather difficult to calculate $(\|x\|^p)^\wedge$. Fortunately, formulae for $(\|x\|_q^p)^\wedge$ and $(\|x\|_\infty^p)^\wedge$ have recently been obtained in [14], [15], [16]. In Section 3 we apply these formulae to study isometric embeddings of the spaces l_q^n and l_∞^n into L_p-spaces. In particular, we give concrete expressions for isometric embeddings and prove Schoenberg's conjectures. Besides that, we prove that for every $q > 2$ and $p \in (0, q)$, $p \neq 2k$, $k \in \mathbb{N}$, there exist two operators $T_1 : l_q^n \to L_p([0, 1])$ and $T_2 : l_q^n \to L_p([0, 1])$ such that $\|x\|_q^p = \|T_1 x\|^p - \|T_2 x\|^p$ for every $x \in l_q^n$ (note that the space l_q^n with $q > 2$, $n \geqq 3$ is not isometric to a subspace of L_p). If $q = 2k$, $k \in \mathbb{N}$, this fact is valid for every non-even positive number p.

In the case $n = 2$ the equality (2) can be rewritten in the form $\gamma(\xi) = (1/c_p)(\|e_1 + \xi e_2\|^p)^{(p+1)}$, where in the right-hand side we have the $(p+1)$-th fractional derivative of the function $\|e_1 + \xi e_2\|^p$. This expression is particularly convenient if p is an odd integer. Here one has to calculate ordinary derivatives.

The criterion (2) appears here as a result of generalization of the Levy representation. Let (Ω, σ) be a finite measure space, $p > 0$, E be an n-dimensional subspace of $L_p(\Omega, \sigma)$. Let $f_1, \ldots, f_n$ be a basis in E and μ be the joint distribution of the functions $f_1, \ldots, f_n$ with respect to σ. Then for every $x \in \mathbb{R}^n$ we have

$$\|x\|^p = \left\| \sum_{i=1}^n x_i f_i \right\|^p = \int_\Omega \left| \sum_{i=1}^n x_i f_i(\omega) \right|^p d\sigma(\omega) = \int_{\mathbb{R}^n} |\langle x, s \rangle|^p d\mu(s).$$

These equalities are usually called the Levy representation of the norm in E with the exponent p and the measure μ.

Choosing different bases in E we can get the Levy representation with different measures μ. But if we assume that μ is supported in the unit sphere S^{n-1} in $\mathbb{R}^n$ then such representation is unique (*see* [13], [27], [22], [9] and a more general fact in [17]). It will be more convenient for us to ensure the uniqueness of the Levy representation projecting μ not onto the sphere but onto a hyperplane:

$$\|x\|^p=\int_{\mathbb{R}^n}|\langle x, s\rangle|^p\, d\mu(s)$$

$$=\int_{s_n\neq 0}\left|x_1\frac{s_1}{s_n}+\ldots+x_{n-1}\frac{s_{n-1}}{s_n}+x_n\right|^p|s_n|^p\, d\mu(s)$$

$$+f(x_1, \ldots, x_{n-1})=\int_{\mathbb{R}^{n-1}}|x_1\xi_1-x_2\xi_2-\ldots-x_{n-1}\xi_{n-1}-x_n|^p$$

$$\times d\gamma(\xi_1, \ldots, \xi_{n-1})+(f(x_1, \ldots, x_{n-1}), \quad (3)$$

where γ is the image of the measure $|s_n|^p d\mu(s)$ under the mapping $(s_1, \ldots, s_n)\to(-s_1/s_n, s_2/s_n, \ldots, s_{n-1}/s_n)$ acting from $\mathbb{R}^n\setminus\{s : s_n=0\}$ to $\mathbb{R}^{n-1}$ (thus, γ is a finite measure on $\mathbb{R}^{n-1}$).

The representation (3) was obtained under the assumption that E is a subspace of L_p. But if we allow γ to be a distribution (*see* Definition 1) then such generalized representation exists and is unique for every $p>0$, $p\neq 2, 4, 6, \ldots$ and for every finite dimensional Banach space. Moreover, the distribution γ can be calculated out of the norm. Thus, we get formula (2). Checking if γ is a finite measure on $\mathbb{R}^{n-1}$ with finite weak moments of the p-th order we can verify isometric embeddability of the space into some L_p-space.

2. GENERALIZED LEVY REPRESENTATION

We need some preliminary remarks. As usual, we denote by $S=S(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions, and $S'=S'(\mathbb{R}^n)$ is the space of distributions over S. If Ω is an open subset of $\mathbb{R}^n$ then $\mathscr{D}(\Omega)$ stands for the space of functions from $S(\mathbb{R}^n)$ having compact supports in Ω. We refer the reader to [6] for definitions and facts concerning distributions. Note that the authors of [6] use the multiplier $\exp(itx)$ in the definition of the Fourier transform, and we use $\exp(-itx)$, so the sign in some formulae may differ from that in [6].

We start with a simple fact which could be found in [14] or [17]. We shall, however, give a proof here, because the fact is crucial for further considerations.

LEMMA 1. – *Let* $p>0$, $p\neq 2, 4, 6, \ldots$, $\xi\in\mathbb{R}^n$, $\xi\neq 0$. *Then for every* $\varphi\in S(\mathbb{R}^n)$ *with* $0\notin \operatorname{supp}\varphi$, *we have*

$$\int_{\mathbb{R}^n}|\langle x, \xi\rangle|^p\,\hat{\varphi}(x)\,dx=(2\pi)^{n-1}c_p\int_{\mathbb{R}}|t|^{-1-p}\varphi(t\xi)\,dt \quad (4)$$

Proof. — It is well-known that $(|x|^p)^\wedge(t)=c_p|t|^{-1-p}$, $t\neq 0$, for all $p>0$, $p\neq 2, 4, 6, \ldots$ [6]. By the Fubini theorem

$$\int_{\mathbb{R}^n}|\langle x, \xi\rangle|^p\,\hat{\varphi}(x)\,dx=\int_{\mathbb{R}}|z|^p\left(\int_{\langle x, \xi\rangle=z}\hat{\varphi}(x)\,dx\right)dz$$

$$=\left\langle |z|^p, \int_{\langle x, \xi\rangle=z}\hat{\varphi}(x)\,dx\right\rangle \quad (5)$$

The function $t\to(\hat{\varphi})^\wedge(t\xi)=(2\pi)^n\varphi(-t\xi)$ is the Fourier transform of the function $z\to\int_{\langle x, \xi\rangle=z}\hat{\varphi}(x)\,dx$ (it is a simple property of the Radon transform, *see* [7]). Therefore, we can continue the equality (5):

$$=\frac{1}{2\pi}\langle c_p|t|^{-1-p}, (2\pi)^n\varphi(-t\xi)\rangle=(2\pi)^{n-1}c_p\int_{\mathbb{R}}|t|^{-1-p}\varphi(t\xi)\,dt.$$

Denote by $\mathbb{R}^n_n$ the set $\{x\in\mathbb{R}^n: x_n\neq 0\}$, and, for $p>0$, define a mapping $\tau_p: \mathcal{D}(\mathbb{R}^n_n)\to\mathcal{D}(\mathbb{R}^{n-1})$ by

$$\tau_p(\varphi)(\xi_1, \ldots, \xi_{n-1})=\int_{\mathbb{R}}|t|^{-1-p}\varphi(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)\,dt$$

for every $\varphi\in\mathcal{D}(\mathbb{R}^n_n)$.

LEMMA 2. — *τ_p maps $\mathcal{D}(\mathbb{R}^n_n)$ onto $\mathcal{D}(\mathbb{R}^{n-1})$. In particular, if $\gamma_1, \gamma_2\in S'(\mathbb{R}^{n-1})$ and $\langle\gamma_1, \tau_p(\varphi)\rangle=\langle\gamma_2, \tau_p(\varphi)\rangle$ for all $\varphi\in\mathcal{D}(\mathbb{R}^n_n)$ then $\gamma_1=\gamma_2$.*

Proof. — Let u be an infinitely differentiable function on $\mathbb{R}$ supported in the segment [1/2, 1] and such that $\int_{\mathbb{R}}|t|^{-1-p}u(t)\,dt=1$. For a given function $\psi\in\mathcal{D}(\mathbb{R}^{n-1})$ define a function φ on $\mathbb{R}^n$ by

$$\varphi(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)=u(t)\,\psi(\xi_1, \ldots, \xi_{n-1})$$

for every $\xi_1, \xi_2, \ldots, \xi_{n-1}, t\in\mathbb{R}$. Then $\varphi\in\mathcal{D}(\mathbb{R}^n_n)$ and $\tau_p(\varphi)=\psi$.

Note that similarly we can get an arbitrary function $\psi\in S(\mathbb{R}^{n-1})$ using a function $\varphi\in S(\mathbb{R}^n)$ supported in $\mathbb{R}^n_n$. Besides that, if Ω is an open subset of $\mathbb{R}^n_n$ and $\Omega_1=\{(x_1/x_n, -x_2/x_n, \ldots, -x_{n-1}/x_n): x\in\Omega\}$ then $\tau_p(\mathcal{D}(\Omega))=\mathcal{D}(\Omega_1)$.

We shall generalize the Levy representation replacing the measure γ in (3) by a distribution.

DEFINITION 1. — We say that an n-dimensional Banach space $E=\mathrm{span}(e_1, \ldots, e_n)$ admits the Levy representation with an exponent $p>0$, $p\neq 2, 4, 6, \ldots$ and a distribution $\gamma\in S'(\mathbb{R}^{n-1})$ if for every function

$\varphi \in S(\mathbb{R}^n)$ with $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n_n)$ we have

$$\int_{\mathbb{R}^n} \|x_1 e_1 + \ldots + x_n e_n\|^p \varphi(x)\,dx$$
$$= \left\langle \gamma(\xi_1, \ldots, \xi_{n-1}), \int_{\mathbb{R}^n} |x_1 \xi_1 - x_2 \xi_2 - \ldots - x_{n-1} \xi_{n-1} - x_n|^p \varphi(x)\,dx \right\rangle \quad (6)$$

The correctness of this definition is ensured by the condition $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n_n)$. Indeed, it follows from (4) that

$$\psi(\xi) = \int_{\mathbb{R}^n} |x_1 \xi_1 - x_2 \xi_2 - \ldots - x_{n-1} \xi_{n-1} - x_n|^p \varphi(x)\,dx$$
$$= \frac{c_p}{2\pi} \int_{\mathbb{R}} |t|^{-1-p} \hat{\varphi}(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)\,dt$$

By Lemma 2 the function ψ belongs to $\mathcal{D}(\mathbb{R}^{n-1})$ and the distribution γ in (6) is unique (if it exists) under a fixed basis in E.

Definition 1 becomes senseless if $p = 2k$, $k \in \mathbb{N}$. In fact, one can open brackets in the right-hand side of (6) and use the connection between differentiation and the Fourier transform to verify that all the moments of the function φ with $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n_n)$ are equal to zero:

$$\int_{\mathbb{R}^n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \varphi(x)\,dx = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} \hat{\varphi}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(0) = 0$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$.

DEFINITION 2. – Let $\gamma \in S'(\mathbb{R}^{n-1})$. Then $|t|^{-1-p}\,d\gamma(\xi)$ denotes the distribution on $\mathcal{D}(\mathbb{R}^n_n)$ satisfying

$$\langle |t|^{-1-p}\,d\gamma(\xi), \varphi \rangle$$
$$= \left\langle \gamma(\xi), \int_{\mathbb{R}} |t|^{-1-p} \varphi(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)\,dt \right\rangle = \langle \gamma, \tau_p(\varphi) \rangle$$

for every function $\varphi \in \mathcal{D}(\mathbb{R}^n_n)$.

It is easy to see that this distribution is even and homogeneous of the order $-n-p$.

Let us point out the connection between the defined Levy representation and the Fourier transform.

THEOREM 1. – *If $p > 0$, $p \neq 2, 4, 6, \ldots$ and an n-dimensional Banach space* E *admits the Levy representation with the exponent p and a distribution*

$\gamma \in S'(\mathbb{R}^{n-1})$ *then:*

(*a*) $(\|x\|^p)^\wedge = (2\pi)^{n-1} c_p |t|^{-1-p} d\gamma(\xi)$ *on* $\mathbb{R}^n_n$;

(*b*) *if, additionaly,* $(\|x\|^p)^\wedge$ *is a continuous function on an open set* $\Omega \subset \mathbb{R}^n_n$ *then the distribution* γ *is a continuous function on the set* $\Omega_1 = \{(x_1/x_n, -x_2/x_n, \ldots, -x_{n-1}/x_n) : x \in \Omega\}$ *and, for every* $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \Omega_1$

$$\gamma(\xi_1, \ldots, \xi_{n-1}) = \frac{1}{(2\pi)^{n-1} c_p} (\|x\|^p)^\wedge (\xi_1, \ldots, \xi_{n-1}, 1).$$

Proof. — Let φ be an arbitrary function from $\mathscr{D}(\mathbb{R}^n_n)$. It follows from (4) that

$$\langle (\|x\|^p)^\wedge, \varphi \rangle = \langle \|x\|^p, \hat{\varphi} \rangle$$

$$= \left\langle \gamma(\xi), \int_{\mathbb{R}^n} |x_1 \xi_1 - x_2 \xi_2 - \ldots - x_{n-1} \xi_{n-1} - x_n|^p \hat{\varphi}(x)\, dx \right\rangle$$

$$= (2\pi)^{n-1} c_p \left\langle \gamma(\xi), \int_{\mathbb{R}} |t|^{-1-p} \varphi(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)\, dt \right\rangle.$$

and the statement (*a*) is proved. Since $(\|x\|^p)^\wedge$ is a homogeneous distribution, (*b*) is an easy consequence of (*a*).

Theorem 1 describes the distribution γ completely in the most important case where $(\|x\|^p)^\wedge$ is a regular distribution. Now we are going to treat the general case where $(\|x\|^p)^\wedge$ may be non-regular. We shall prove that the generalized Levy representation exists for every finite dimensional Banach space and every $p > 0$, $p \neq 2, 4, 6, \ldots$.

For an arbitrary function $\varphi \in S(\mathbb{R}^n)$ and $p > 0$ define the function φ_p on $\mathbb{R}^{n-1}$ by

$$\varphi_p(y_2, \ldots, y_n) = \int_{\mathbb{R}} |x|^{n+p-1} \varphi(x, xy_2, \ldots, xy_n)\, dx. \tag{7}$$

LEMMA 3. — *There exists a number* $k > 0$ *such that*

$$|\varphi_p(y_2, \ldots, y_n)| \leqq k(1 + y_2^2 + \ldots + y_n^2)^{-(n+p)/2}$$

for every $(y_2, \ldots, y_n) \in \mathbb{R}^{n-1}$.

Proof. — Since $\varphi \in S(\mathbb{R}^n)$, for every $m \geqq n+p+1$ there exists a number $k_m > 0$ such that $|\varphi(y)| \leqq k_m(1 + |y|)^{-m}$ for all $y \in \mathbb{R}^n$. Then we have

$$|\varphi(y_2, \ldots, y_n)| \leqq k_m \int_{\mathbb{R}} \frac{|x|^{n+p-1}\, dx}{(1 + |x|(1 + y_2^2 + \ldots + y_n^2)^{1/2})^m}$$

$$= k_m \left(\int_{\mathbb{R}} \frac{|z|^{n+p-1}\, dz}{(1 + |z|)^m} \right) (1 + y_2^2 + \ldots + y_n^2)^{-(n+p)/2}.$$

In order to make the general case clear we start with the two-dimensional case where the reasoning and the final result are rather simple.

Let $E = \operatorname{span}(e_1, e_2)$ be a two-dimensional Banach space, $p > 0$, $p \neq 2, 4, 6, \ldots$ As it was mentioned after Lemma 2, for every function $\psi \in S(\mathbb{R})$ there exists a function $\varphi \in S(\mathbb{R}^2)$ with $\operatorname{supp} \hat{\varphi} \subset \mathbb{R}_2^2$ such that

$$\psi(\xi) = \frac{c_p}{2\pi} \int_{\mathbb{R}} |t|^{-1-p} \hat{\varphi}(t\xi, -t)\, dt$$
$$= \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy$$
$$= \int_{\mathbb{R}} |\xi - z|^p \varphi_p(z)\, dz = (|z|^p * \varphi_p)(\xi) \quad (8)$$

for every $\xi \in \mathbb{R}$ [here we changed variables $z = y/x$ and used (4)].

Since $(|z|^p)^\wedge(t) = c_p |t|^{-1-p}$, we can use the connection between the Fourier transform and convolution [6] to verify that $\hat{\psi}(t) = c_p |t|^{-1-p} \hat{\varphi}_p(t)$ for every $t \neq 0$. The functions $\hat{\psi}$ and $\hat{\varphi}_p$ are continuous, so we have $c_p \hat{\varphi}_p(t) = |t|^{p+1} \hat{\psi}(t)$ for all $t \in \mathbb{R}$. That is why $c_p \varphi_p = (|t|^{p+1} \hat{\psi}(t))^\vee = \psi^{(p+1)}$ is the $(p+1)$-th fractional derivative of the function ψ ($\vee$ stands for the inverse Fourier transform).

Let us define the $(p+1)$-th fractional derivative of the function $\|e_1 + te_2\|^p$ as a distribution over $S(\mathbb{R})$ such that for every $\psi \in S(\mathbb{R})$

$$\langle (\|e_1 + te_2\|^p)^{(p+1)}, \psi \rangle = \int_{\mathbb{R}} \|e_1 + te_2\|^p \psi^{(p+1)}(t)\, dt \quad (9)$$

(the convergence of the integral follows from the equality $\psi^{(p+1)} = c_p \varphi_p$ and Lemma 3).

THEOREM 2. – *If* $p > 0$, $p \neq 2, 4, 6, \ldots$ *then every two-dimensional Banach space admits the Levy representation with the exponent* p *and the distribution* $\gamma = (1/c_p)(\|e_1 + te_2\|^p)^{(p+1)}$.

Proof. – For an arbitrary function $\varphi \in S(\mathbb{R}^2)$ with $\hat{\varphi} \in \mathcal{D}(\mathbb{R}_2^2)$ we define a function ψ by (8) and use (9) and the equality $\psi^{(p+1)} = c_p \varphi_p$:

$$\left\langle (\|e_1 + te_2\|^p)^{(p+1)}(\xi), \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy \right\rangle$$
$$= \langle (\|e_1 + te_2\|^p)^{(p+1)}, \psi \rangle = \int_{\mathbb{R}} \|e_1 + te_2\|^p \psi^{(p+1)}(t)\, dt$$
$$= c_p \int_{\mathbb{R}} \|e_1 + te_2\|^p \varphi_p(t)\, dt$$
$$= c_p \int_{\mathbb{R}^2} \|xe_1 + ye_2\|^p \varphi(x, y)\, dx\, dy.$$

We are done.

Let us consider now an arbitrary n-dimensional Banach space $E=\operatorname{span}(e_1, \ldots, e_n)$. Let $p>0$, $p\neq 2, 4, 6, \ldots$ For every function $\psi\in S(\mathbb{R}^{n-1})$ there exists a function $\varphi\in S(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\varphi}\subset\mathbb{R}^n_n$ such that an equality similar to (8) holds:

$$\psi(\xi_1, \ldots, \xi_{n-1})=\frac{c_p}{2\pi}\int_{\mathbb{R}}|t|^{-1-p}\hat{\varphi}(t\xi_1, -t\xi_2, \ldots, -t\xi_{n-1}, -t)\,dt$$
$$=\int_{\mathbb{R}^n}|x_1\xi_1-x_2\xi_2-\ldots-x_{n-1}\xi_{n-1}-x_n|^p\varphi(x)\,dx$$
$$=\int_{\mathbb{R}^n}|\xi_1-y_2\xi_2-\ldots-y_{n-1}\xi_{n-1}-y_n|^p \times|x_1|^{n+p-1}\varphi(x_1, x_1y_2, \ldots, x_1y_n)\,dx_1\,dy_2\ldots dy_n$$
$$=\int_{\mathbb{R}^{n-1}}|\xi_1-y_2\xi_2-\ldots-y_{n-1}\xi_{n-1}-y_n|^p\varphi_p(y_2, \ldots, y_n)\,dy_2\ldots dy_n$$
$$=\int_{\mathbb{R}}|\xi_1-z|^p\,d\sigma(z)=(|z|^p*\sigma)(\xi_1) \quad (10)$$

for all $(\xi_1, \ldots, \xi_{n-1})\in\mathbb{R}^{n-1}$, where σ is the charge of bounded variation which is the image of the charge $\varphi_p(y_2, \ldots, y_n)\,dy_2\ldots dy_n$ under the mapping $(y_2, \ldots, y_n)\to y_2\xi_2+\ldots+y_{n-1}\xi_{n-1}+y_n$ acting from $\mathbb{R}^{n-1}$ to $\mathbb{R}$.

Thus, for all $t\in\mathbb{R}$, we have

$$c_p\hat{\sigma}(t)=|t|^{p+1}(\psi(\xi_1, \ldots, \xi_{n-1}))^\wedge_{\xi_1}(t),$$

where the Fourier transform is computed over the variable ξ_1. Further, $\hat{\sigma}(t)=\hat{\varphi}_p(t\xi_2, \ldots, t\xi_{n-1}, t)$. Therefore, for every $(y_2, \ldots, y_n)\in\mathbb{R}^{n-1}$ we have

$$c_p\varphi_p(y_2, \ldots, y_n)=(|t|^{p+1}(\psi(\xi_1, \xi_2/t, \ldots, \xi_{n-1}/t))^\wedge_{\xi_1}(t))^\vee_{\xi_2, \ldots, \xi_{n-1}, t}(y_2, \ldots, y_n) \quad (11)$$

where the inverse Fourier transform is computed over the variables $\xi_2, \ldots, \xi_{n-1}, t$.

Define a distribution $F\in S'(\mathbb{R}^{n-1})$ similar to (9). For every function $\psi\in S(\mathbb{R}^{n-1})$ put

$$\langle F, \psi\rangle=\int_{\mathbb{R}^{n-1}}\|e_1+y_2e_2+\ldots+y_ne_n\|^p \times(|t|^{p+1}(\psi(\xi_1, \xi_2/t, \ldots, \xi_{n-1}/t))^\wedge_{\xi_1}(t))^\vee{}_{\xi_2, \ldots, \xi_{n-1}, t}(y_2, \ldots, y_n)\,dy_2\ldots dy_n$$

(the convergence of the integral follows from (11) and Lemma 3).

THEOREM 3. — *If $p>0$, $p\neq 2, 4, 6, \ldots$, then every n-dimensional Banach space admits the Levy representation with the exponent p and the distribution* $\gamma=(1/c_p)F$.

Proof. — For an arbitrary function $\varphi \in S(\mathbb{R}^n)$ with $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n_n)$ we define a function ψ by (10). It follows from (10) and (11) that

$$\left\langle F(\xi), \int_{\mathbb{R}^n} |x_1 \xi_1 - x_2 \xi_2 - \ldots - x_{n-1} \xi_{n-1} - x_n|^p \varphi(x)\, dx \right\rangle$$
$$= \langle F, \psi \rangle = c_p \int_{\mathbb{R}^{n-1}} \| e_1 + y_2 e_2 + \ldots + y_n e_n \|^p \varphi_p(y)\, dy$$
$$= c_p \int_{\mathbb{R}^n} \| x_1 e_1 + \ldots + x_n e_n \|^p \varphi(x)\, dx,$$

and we are done.

3. ISOMETRIC EMBEDDINGS INTO L_p-SPACES

The Levy representation can be used as a criterion of isometric embeddability of Banach spaces into L_p-spaces. The following theorem is a consequence of (3) and the uniqueness (under a fixed basis) of the distribution γ providing the Levy representation.

THEOREM 4. — *Let $p>0$, $p \neq 2, 4, 6, \ldots$, E be an n-dimensional Banach space which is isometric to a subspace of $L_p(\Omega, \sigma)$ with (Ω, σ) being a finite measure space. Then the distribution γ providing the Levy representation of the norm in E is a finite Borel measure on $\mathbb{R}^{n-1}$ with $\int_{\mathbb{R}^{n-1}} |\langle x, \xi \rangle|^p d\gamma(\xi) < \infty$ for every $x \in \mathbb{R}^{n-1}$.*

Generally speaking the inverse statement is not true. The reason is that the equality (6) is valid only for functions φ with $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n_n)$. However, we prove a fact which can be effectively used as a sufficient condition of isometric embeddability.

THEOREM 5. — *Let $p>0$, $p \neq 2, 4, 6, \ldots$ Let $E = \operatorname{span}(e_1, \ldots, e_n)$ be an n-dimensional Banach space such that the distribution γ providing the Levy representation of the norm in E is a finite Borel measure with finite weak moments of the p-th order. Then there exist a finite measure space (Ω, σ) and a linear operator $T: E \to L_p(\Omega, \sigma)$ such that $\partial^m u / \partial x_n^m \equiv 0$ on $\mathbb{R}^n$, where $m \in \mathbb{N}$ and*

$$u(x_1, \ldots, x_n) = \| x_1 e_1 + \ldots + x_n e_n \|_E^p - \| x_1 T e_1 + \ldots + x_n T e_n \|_{L_p}^p.$$

Proof. — Consider a finite measure space (Ω, σ) and measurable functions $f_1, \ldots, f_{n-1}$ on Ω such that the measure γ is the joint distribution of the functions $-f_1, f_2, \ldots, f_{n-1}$ with respect to σ. Define a linear operator $T: E \to L_p(\Omega, \sigma)$ by $T e_i = f_i$, $1 \leq i \leq n-1$, $T e_n(\omega) = 1$ for all $\omega \in \Omega$.

By the definition of the Levy representation, for each function $\varphi \in S(\mathbb{R}^n)$ with $\hat{\varphi} \in \mathscr{D}(\mathbb{R}^n_n)$ we have

$$\langle \|x_1 e_1 + \ldots + x_n e_n\|^p, \varphi \rangle$$

$$= \left\langle \int_{\mathbb{R}^n} |x_1 \xi_1 - x_2 \xi_2 - \ldots - x_{n-1} \xi_{n-1} - x_n|^p d\gamma(\xi), \varphi(x) \right\rangle$$

$$= \left\langle \int_{\Omega} \left| \sum_{i=1}^{n} x_i \mathrm{T} e_i(\omega) \right|^p d\sigma(\omega), \varphi \right\rangle$$

$$= \langle \|x_1 \mathrm{T} e_1 + \ldots + x_n \mathrm{T} e_n\|^p, \varphi \rangle.$$

Thus, $\langle u, \varphi \rangle = \langle \hat{u}, \hat{\varphi} \rangle = 0$ for every φ with $\hat{\varphi} \in \mathscr{D}(\mathbb{R}^n_n)$, i.e. $\hat{u} = 0$ on $\mathbb{R}^n_n$. Consider the multiplier $g(x_1, \ldots, x_n) = x_n$ on $S(\mathbb{R}^n)$. By Lemma 4 from [8], $g^m \hat{u} = 0$ on $\mathbb{R}^n$ for some $m \in \mathbb{N}$, therefore $\partial^m_u / \partial x^m_n \equiv 0$ on $\mathbb{R}^n$.

Remark 1. — Assume the conditions of Theorem 5 are fulfilled. It is clear that E is isometric to a subspace of L_p if the expression for $\|x\|^p$ does not contain a separate summand which either does not depend on the variable x_n or is a polynomial with respect to x_n. If $\|x\|^p$ contains such a summand then the distribution $(\|x\|^p)^\wedge$ contains a summand which is a linear combination of the δ-function and its derivatives with respect to ξ_n. In this case one has to study the behaviour of the norm on the hyperplane $x_n = 0$.

For example, consider the space with the norm

$$\|(x_1, \ldots, x_n)\| = (\|(x_1, \ldots, x_n)\|^p_q + \|(x_1, \ldots, x_{n-1})\|^p_\infty)^{1/p},$$

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $n \geq 3$ and $1 < p \leq q \leq 2$. This space satisfies the conditions of Theorem 5, because the space l^n_q is isometric to a subspace of L_p and the $\mathbb{R}^n$-Fourier transform of the p-th power of the norm in l^{n-1}_∞ is supported in the hyperplane $\xi_n = 0$. On the other hand, E is not isometric to a subspace of L_p, because the space l^{n-1}_∞ is not smooth. If we take the space l^{n-1}_r, $p \leq r \leq 2$, instead of l^{n-1}_∞ then E is isometric to a subspace of L_p, although $\|x\|^p$ contains a separate summand.

Remark 2. — J. Bretagnolle *et al.* [1] have given a criterion for $1 \leq p \leq 2$: A Banach space is isometric to a subspace of L_p if and only if the function $\|x\|^p$ is negative definite (or, the same, $\exp(-\|x\|^p)$ is a positive definite function). The "if" part was discussed at the beginning of the paper. On the other hand, for an arbitrary finite dimensional subspace of $L_p([0, 1])$, the norm admits the Levy representation with some probability measure μ on $\mathbb{R}^n$. For every $\xi \in \mathbb{R}^n$, $\xi \neq 0$, the function $\exp(-|\langle x, \xi \rangle|^p)$ is positive definite on $\mathbb{R}^n$, since it is the characteristic function of the one-dimensional standard p-stable measure placed on the line $\{t\xi, t \in \mathbb{R}\}$ in $\mathbb{R}^n$. Now it suffices to note that positive definiteness is preserved under multiplication and pointwise limit procedure to verify that $\exp(-\int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p d\mu(\xi))$ is a positive definite function. For $p>2$, this reasoning is not valid because the function $\exp(-|t|^p)$ is not positive definite on $\mathbb{R}$. Thus, for $0<p<2$ the Levy representation criterion and the criterion from [1] are equivalent for trivial reasons. Theorems 4 and 5 extend the criterion from [1] to all positive p's, $p\neq 2k$, $k\in\mathbb{N}$.

Let us pass to examples and applications. First, we shall discuss the well-known fact that every two-dimensional Banach space is isometric to a subspace of L_1.

Example 1. – Let $p=1$ and $E=\operatorname{span}(e_1, e_2)$ be a two-dimensional Banach space, $\|e_1\|=\|e_2\|=1$.

Since $(t^2\hat{\psi})^\vee=-\psi''$ for every $\psi\in S(\mathbb{R})$, the second fractional derivative of the distribution $\|e_1+\xi e_2\|$ is equal to the ordinary derivative $-\|e_1+\xi e_2\|''$. Besides $c_1=-2$, so by Theorem 2 we have $\gamma(\xi)=(1/2)\|e_1+\xi e_2\|''$. It is easy to see that the distribution $(1/2)\|e_1+\xi e_2\|''$ is a probability measure on $\mathbb{R}$ having finite first moment. By Definition 1 and Remark 1,

$$\|xe_1+ye_2\|=\int_{\mathbb{R}}|x\xi-y|\,d\gamma(\xi)+|x|\left(1-\int_{\mathbb{R}}|\xi|\,d\gamma(\xi)\right)$$

for every $x, y\in\mathbb{R}$.

It is clear now that E is isometric to a subspace of L_1. In fact, consider a function f_0 on the segment [0, 1] having the distribution γ with respect to Lebesgue measure. Define a function f on [0, 2] as follows: $f=f_0$ on [0, 1] and $f(\omega)=1-\int_{\mathbb{R}}|\xi|\,d\gamma(\xi)$ for every $\omega\in(1, 2]$. Define a linear operator $T:E\to L_1([0, 2])$ by $Te_1=f$ and $Te_2(\omega)=1$ for $\omega\in[0, 1]$, $Te_2(\omega)=0$ for $\omega\in(1, 2]$. Then T is a desired isometry.

Assume $\|e_1+\xi e_2\|''$ is a continuous function on $\mathbb{R}\setminus\{0\}$ without an atom at zero. Then easy calculations show that

$$\|xe_1+ye_2\|=\frac{1}{2}\left(\int_{\mathbb{R}}|x\xi-y|\,\|e_1+\xi e_2\|''\,d\xi\right.$$
$$\left.+|x|\left(\|ze_1+e_2\|'_z(+0)-\|ze_1+e_2\|'_z(-0)\right)\right).$$

If E is a smooth space there is no second summand in the latter formula. For example, for $E=l_q^2$, $q>1$, we have

$$(|x|^q+|y|^q)^{1/q}=\frac{q-1}{2}\int_{\mathbb{R}}|x\xi-y|(1+|\xi|^q)^{1/q-2}|\xi|^{q-2}\,d\xi.$$

If $E=l_1^2$ then γ is the unit mass at zero. If $E=l_\infty^2$ then $\gamma(\xi)=(1/2)(\max(1, |\xi|))''$ is the sum of two 1/2-masses at the points ± 1. We have $\max(|x|, |y|)=(|x+y|+|x-y|)/2$.

Example 2. — Let us consider the spaces l^n_∞, $n \geqq 2$. We shall use the following formula obtained in [15]: Let f be an even continuous functions on $\mathbb{R}$ with power growth at infinity (*i.e.* $\lim_{|t| \to \infty} (f(t)/|t|^\varphi) = 0$ for some $\varphi > 0$). Assume that the distribution $g = (f(t)(\operatorname{sgn} t)^{n-1})^\wedge$ is a continuous function on $\mathbb{R} \setminus \{0\}$. Then

$$(f(\|x\|_\infty))^\wedge(\xi) = \frac{i^{n-1}}{2\xi_1 \ldots \xi_n} \sum_\delta \delta_1 \ldots \delta_n (\delta_1 \xi_1 + \ldots + \delta_n \xi_n) g(\delta_1 \xi_1 + \ldots + \delta_n \xi_n) \quad (12)$$

for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ such that $\xi_k \neq 0$, $1 \leqq k \leqq n$, and $\delta_1 \xi_1 + \ldots + \delta_n \xi_n \neq 0$ for every $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ with $\delta_k = \pm 1$, $1 \leqq k \leqq n$ (Denote by G the set of such vectors ξ). The sum in (12) is taken over all changes of signs.

Let $n = 2$ and $f(t) = |t|^p$, $p > 0$, $p \neq 2k - 1$, $k \in \mathbb{N}$.
Then

$$g(z) = (|t|^p \operatorname{sgn} t)^\wedge(z) = -2i \cos(\pi p/2) \Gamma(p+1) |z|^{-1-p} \operatorname{sgn} z$$

for every $z \neq 0$ (*see* [6]) and we have

$$(\|x\|^p_\infty)^\wedge(\xi_1, \xi_2) = \frac{2}{\xi_1 \xi_2} \cos(\pi p/2) \Gamma(p+1) (|\xi_1 + \xi_2|^{-p} - |\xi_1 - \xi_2|^{-p})$$

for every $(\xi_1, \xi_2) \in G$. By Theorem 1, for every $p \notin \mathbb{N}$, $\xi \in \mathbb{R}$, $\xi \neq 0, -1, 1$, we have

$$\gamma(\xi) = \frac{1}{2\pi c_p} (\|x\|^p)^\wedge(\xi, 1) = -\frac{\Gamma(-p/2)\cos(\pi p/2)\Gamma(p+1)}{2^{p+1}\pi^{1/2}\Gamma((p+1)/2)\pi} \frac{|\xi-1|^{-p} - |\xi+1|^{-p}}{\xi} = \frac{1}{2\pi} \operatorname{cotan}(\pi p/2)(|\xi-1|^{-p} - |\xi+1|^{-p})/\xi.$$

If $p > 1$ then the function γ is not integrable near ± 1, so γ is not a measure. By Theorem 4, the space l^2_∞ is not isometric to a subspace of L_p with $p > 1$.

If $0 < p < 1$ then γ is a positive function bounded near zero and integrable near ± 1. Besides that, γ behaves at infinity like $|\xi|^{-p-2}$. Thus, γ is a measure on $\mathbb{R}$ with finite moment of the p-th order. So for every $p \in (0, 1)$ the space l^2_∞ is isometric to a subspace of L_p, and we have

$$\max^p(|x|, |y|) = \frac{1}{2\pi} \operatorname{cotan}(\pi p/2) \int_{\mathbb{R}} |x\xi - y|^p \frac{|\xi-1|^{-p} - |\xi+1|^{-p}}{\xi} d\xi$$

for every $x, y \in \mathbb{R}$ (it is easy to see that puting $y = 0$ we get an equality, so there are no additional summands in the right-hand side, see Remark 1). Note that the latter formula remains valid for $p \in (-1, 0)$.

For the case $p=1$, see Example 1. If $p=1$ the Fourier transform of the distribution $|x|^p \operatorname{sgn} x$ is supported in zero.

Let us prove now that, for every $p>0$, the space l_∞^n with $n\geqq 3$ is not isometric to a subspace of L_p.

It suffices to treat the case $n=3$. Let $p>0$, $p\neq 2k$, and $f(t)=|t|^p$. Then $g(z)=c_p|z|^{-1-p}$ for every $z\neq 0$.

Hence,

$$(\|x\|_\infty^p)^\wedge(\xi)=\frac{c_p}{2\xi_1\xi_2\xi_3}\sum_\delta \delta_1\delta_2\delta_3|\delta_1\xi_1+\delta_2\xi_2+\delta_3\xi_3|^{-p} \times \operatorname{sgn}(\delta_1\xi_1+\delta_2\xi_2+\delta_3\xi_3)$$

for every $\xi\in G$. By Theorem 1,

$$\gamma(\xi_1,\xi_2)=\frac{-1}{(2\pi)^2\xi_1\xi_2}(|\xi_1+\xi_2+1|^{-p}\operatorname{sgn}(\xi_1+\xi_2+1) - |\xi_1+\xi_2-1|^{-p}\operatorname{sgn}(\xi_1+\xi_2-1)-|\xi_1-\xi_2+1|^{-p}\operatorname{sgn}(\xi_1-\xi_2+1) + |\xi_1-\xi_2-1|^{-p}\operatorname{sgn}(\xi_1-\xi_2-1))$$

and, obviously, γ is not a measure [For instance, $\gamma(3,1)<0$]. By Theorem 4, the space l_∞^3 is not isometric to a subspace of L_p with $p>0$. As it was mentioned in Introduction, this result gives an answer to the second Schoenberg's question.

Example 3. – Let us consider the problem of isometric embedding of the spaces l_q^n into L_p-spaces. A formula for the Fourier transform of the function $\|x\|_q^p$ was obtained in [14], [16]: for every $q>0$, $p\in(-n, nq)$ such that $p/q\notin\mathbb{N}\cup\{0\}$ and for every $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$ with non-zero coordinates, we have

$$(\|x\|_q^p)^\wedge(\xi)=\frac{q}{\Gamma(-p/q)}\int_0^\infty t^{n+p-1}\prod_{k=1}^n \gamma_q(t\xi_k)\,dt,$$

where $\gamma_q(t)=(\exp(-|z|^q))^\wedge(t)$, $t\in\mathbb{R}$. By Theorem 2, for every $p\in(0, nq)$, $p\neq 2k$, $k\in\mathbb{N}$, $p/q\notin\mathbb{N}$, the distribution γ providing the Levy representation of the norm in l_q^n with the exponent p has the form

$$\gamma(\xi_1,\ldots,\xi_{n-1})=\frac{q\Gamma(-p/2)}{(2\pi)^{n-1}2^{p+1}\pi^{1/2}\Gamma((p+1)/2)\Gamma(-p/q)} \times\int_0^\infty t^{n+p-1}\gamma_q(t)\prod_{k=1}^{n-1}\gamma_q(t\xi_k)\,dt \quad (13)$$

for every $(\xi_1,\ldots,\xi_{n-1})\in\mathbb{R}^{n-1}$, $\xi_k\neq 0$ (if q is an even integer (13) remains valid for all $p>0$, $p\neq 2k$, $k\in\mathbb{N}$).

The properties of the function γ_q are well-known [28], [29], [32], [3]. If $0<q<2$ then γ_q is an even positive function on $\mathbb{R}$ which is equal (up to a constant) to the density of the standard q-stable measure. For $q>2$, the function γ_q is not positive. The behaviour of the functions γ_q at infinity is as follows: for every $q>0$,

$$\lim_{t\to\infty} t^{1+q}\gamma_q(t)=2\Gamma(q+1)\sin(\pi q/2).$$

If q is an even integer γ_q decreases exponentially at infinity. Thus, for $\alpha\in(-1, q)$, the integral $S_q(\alpha)=\int_{\mathbb{R}}|t|^\alpha\gamma_q(t)\,dt$ converges absolutely. For $\alpha\geqq q$, this integral diverges if $q\neq 2k$, $k\in\mathbb{N}$, and it converges if $q=2k$, $k\in\mathbb{N}$. The numbers $S_q(\alpha)$ can easily be calculated [32], [14]: if $\alpha\in(-1, q)$ $\alpha\neq 0, 2, \ldots, 2[q/2]$, then

$$S_q(\alpha)=\frac{2^{\alpha+2}\pi^{1/2}\Gamma(-\alpha/q)\Gamma((\alpha+1)/2)}{q\Gamma(-\alpha/2)}. \tag{14}$$

Let us go back to the distribution γ providing the Levy representation. Consider the integral

$$\begin{aligned}\mathscr{I}(\alpha_1, \ldots, \alpha_{n-1})&=\int_{\mathbb{R}^{n-1}}|\xi_1|^{\alpha_1}\ldots|\xi_{n-1}|^{\alpha_{n-1}}\\ &\times\left(\int_0^\infty t^{n+p-1}\gamma_q(t)\prod_{k=1}^{n-1}\gamma_q(t\xi_k)\,dt\right)d\xi_1\ldots d\xi_{n-1}\\ &=\int_0^\infty t^{n+p-1}\gamma_q(t)\left(\prod_{k=1}^{n-1}\int_{\mathbb{R}}|\xi_k|^{\alpha_k}\gamma_q(t\xi_k)\,d\xi_k\right)dt\\ &=S_q(\alpha_1)\ldots S_q(\alpha_{n-1}).S_q(-\alpha_1-\ldots-\alpha_{n-1}+p).\end{aligned} \tag{15}$$

If all the numbers $\alpha_1, \ldots, \alpha_{n-1}, -\alpha_1-\ldots-\alpha_{n-1}+p$ belong to the interval $(-1, q)$ then all the integrals in (15) converge absolutely and the Fubini theorem is applicable. If $q=2k$, $k\in\mathbb{N}$, then the interval $(-1, q)$ may be replaced here by the half-line $(-1, \infty)$.

LEMMA 3. — *Let $p>0$, $p\neq 2k$, $k\in\mathbb{N}$ and either $0<p<q$ or q is an even integer. Then the distribution γ is a charge in $\mathbb{R}^{n-1}$ such that its variation is bounded and has finite weak moments of the order p.*

Proof. — Note that all the integrals remain convergent if we replace γ_q by its variation $|\gamma_q|$ everywhere in (15) (We shall use the notation $|\mathscr{I}|$ instead of $\mathscr{I}$). Now the desired result follows from the fact that $|\mathscr{I}|(\alpha_1, \ldots, \alpha_{n-1})<\infty$ for $\alpha_1=\ldots=\alpha_{n-1}=0$ and for $(\alpha_1, \ldots, \alpha_{n-1})$ such that one of the α's is equal to p and others are equal to zero.

Thus, if $p>0$, $p\neq 2k$, $k\in\mathbb{N}$ and either $0<p<q$ or $q=2k$, $k\in\mathbb{N}$, then the norm in the space l_q^n admits the Levy representation with the charge γ:

$$\|x\|_q^p=\frac{q\,\Gamma(-p/2)}{(2\pi)^{n-1}2^{p+1}\pi^{1/2}\Gamma((p+1)/2)\Gamma(-p/q)}\times\int_{\mathbb{R}^{n-1}}|x_1\xi_1-x_2\xi_2-\ldots-x_{n-1}\xi_{n-1}-x_n|^p\times\left(\int_0^\infty t^{n+p-1}\gamma_q(t)\prod_{k=1}^{n-1}\gamma_q(t\xi_k)\,dt\right)d\xi_1\ldots d\xi_{n-1}\quad(16)$$

One can easily verify the absence of additional summands (*see* Remark 1) puting $x_n=0$ and using the reasoning at the beginning of the paper with $f(t)=\exp(-|t|^q)$.

Let $0<p<q\leqq 2$. Then γ_q is a positive function. Since the numbers $\Gamma(-p/2)$ and $\Gamma(-p/q)$ are both negative, γ is a measure on $\mathbb{R}^{n-1}$ and the equality (16) gives an isometric embedding of the space l_q^n into L_p (*see* Theorem 5).

Let $q>2$, $0<p<2$, $n=3$. Put $\alpha_1=\alpha_2=-1+\delta$ in (15), where $\delta\in(0, 1)$ and $(2+p-\min(4, q))/2<\delta<p/2$. Then $\alpha_1, \alpha_2\in(-1, 0)$ and $-\alpha_1-\alpha_2+p\in(2, \min(4, q))$. It follows from (15) and (14) that $\mathcal{I}(\alpha_1, \alpha_2)<0$. Hence, the charge γ given by (13) is not a measure, and, by Theorem 4 the space l_q^3 is not isometric to a subspace of L_p with $0<p<2$. This result gives an answer to the first Schoenberg's question (*cf.* [14]).

Example 4. — Let E be an n-dimensional Banach space and $p>0$, $p\neq 2k$, $k\in\mathbb{N}$. Assume that the corresponding distribution γ is a charge on $\mathbb{R}^{n-1}$ such that its variation $|\gamma|$ is bounded and has finite weak p-th moments. Then $\gamma=\gamma_1-\gamma_2$ is the difference of two measures on $\mathbb{R}^{n-1}$. Let $\gamma_1(\mathbb{R}^{n-1})=a$, $\gamma_2(\mathbb{R}^{n-1})=b$. The spaces E_1 and E_2 with the norms

$$\|x\|_i^p=\int_{\mathbb{R}^{n-1}}|x_1\xi_1-x_2\xi_2-\ldots-x_{n-1}\xi_{n-1}-x_n|^p\,d\gamma_i(\xi),\quad i=1, 2$$

are isometric to subspaces of $L_p([0, a)]$ and $L_p([0, b)]$, respectively. Besides that $\|x\|_E^p=\|x\|_1^p-\|x\|_2^p$ for every $x\in E$. Since E_1 and E_2 are isometric to subspaces of $L_p([0, 1])$, we have proved the following fact.

LEMMA 4. — *If* E *and* p *are as above there exist two operators* $T_1: E\to L_p([0, 1])$ *and* $T_2: E\to L_p([0, 1])$ *such that*

$$\|x\|_E^p=\|T_1x\|^p-\|T_2x\|^p\quad \textit{for every}\quad x\in E.$$

Lemma 3 and Lemma 4 show that the space l_q^n can be renormed, so that it becomes embeddable into some L_p-space.

THEOREM 6. — *Let* $n \geqq 2$, $p>0$, $p \neq 2k$, $k \in \mathbb{N}$, *and either* $p<q$ *or* $q=2k$, $k \in \mathbb{N}$. *Then there exist two operators* $T_1 : l_q^n \to L_p([0, 1])$ *and* $T_2 : l_q^n \to L_p([0, 1])$ *such that* $\|x\|_q^p = \|T_1 x\|^p - \|T_2 x\|^p$ *for every* $x \in l_q^n$.

Example 5. — Let $q>3$, $p=3$. Consider the space l_q^2. By Theorem 2, the distribution γ providing the Levy representation of the norm with the exponent $p=3$ is equal to $(1/c_3)(\|e_1+\xi e_2\|^3)^{(4)}$. So, we have to compute the forth derivative of the function $(1+|\xi|^q)^{3/q}$. One can easily verify that

$$\gamma(\xi)=64(q-3)(q-1)(1+|\xi|^q)^{3/q-4}|\xi|^{q-4} \times ((q-2)|\xi|^{2q}+(5-4q)|\xi|^q+q-2).$$

The function γ is integrable near zero, since $q>3$, and it decreases at infinity like $|\xi|^{-1-q}$. Thus, γ is a charge with bounded variation having finite weak moments of the third order. On the other hand, it is clear that γ is not a measure. So, the space l_q^2 is not isometric to a subspace of L_3, but it can be renormed in the sense of Theorem 6.

REFERENCES

[1] J. BRETAGNOLLE, D. DACUNHA-CASTELLE and J. L. KRIVINE, Lois stables et spaces L_p, *Ann. Inst. H. Poincaré. Ser. B.*, Vol. **2**, 1966, pp. 231-259.

[2] L. DOR, Potentials and Isometric Embeddings in L_1, *Israel J. Math.*, Vol. **24**, 1976, pp. 260-268.

[3] M. V. FEDORYUCK, *Assymptotics. Integrals, series*, Nauka, Moscow, 1987.

[4] T. S. FERGUSON, A Representation of the Symmetric Bivariate Cauchy Distribution, *Ann. Math. Stat.*, Vol. **33**, 1962, pp. 1256-1266.

[5] M. FRECHET, Sur la définition axiomatique d'une classe d'espaces vectoriel distances applicables vectoriellement sur l'espace de Hilbert, *Ann. Math.*, Vol. **36**, 1935, pp. 705-718.

[6] I. M. GELFAND and G. E. SHILOV, *Generalized functions* 1, Fizmatgiz, Moscow, 1959.

[7] I. M. GELFAND, M. I. GRAEV and N. Ya. VILENKIN, *Generalized Functions* 5, Academic Press, New York, 1966.

[8] E. A. GORIN and A. L. KOLDOBSKY, On Potentials of Measures in Banach Spaces, *Siberian Math. J.*, Vol. **28**, 1987, pp. 46-58.

[9] R. GRZASLEWICZ, Plane Sections of the Unit Ball of L_p, *Acta Math. Hung.*, Vol. **52**, 1988, pp. 219-225.

[10] C. S. HERZ, A Class of Negative Definite Functions, *Proc. Am. Math. Soc.*, Vol. **14**, 1963, pp. 670-676.

[11] W. B. JOHNSON, B. MAUREY, G. SCHECHTMAN and L. TZAFRIRI, Symmetric structures in Banach spaces, *Memoires of the Am. Math. Soc.*, Vol. **217**, 1979, 298 p.

[12] P. JORDAN and J. VON NEUMANN, On Inner Products Linear Metric Spaces, *Ann. Math.*, Vol. **36**, 1935, pp. 719-723.

[13] M. KANTER, The L_p-norm of Sums of Translates of a Function, *Trans. Am. Math. Soc.*, Vol. **179**, 1973, pp. 35-47.

[14] A. L. KOLDOBSKY, Schoenberg's Problem on Positive Definite Functions, Algebra & Analysis, *Leningrad Math. J.*, Vol. **3**, No. 3, 1991, pp. 78-85.

[15] A. L. KOLDOBSKY, *A Formula for the Fourier Transform of Maximum Dependent Functions*, preprint.

[16] A. L. KOLDOBSKY, The Fourier Transform Technique for Convolution Equations in Infinite Dimensional l_q-spaces, *Math. Ann.*, Vol. **291**, 1991, pp. 403-407.
[17] A. L. KOLDOSKY, Convolution Equations in Certain Banach Spaces, *Proc. Am. Math. Soc.*, Vol. **111**, 1991, pp. 755-765.
[18] J. L. KRIVINE, Plongment des espaces normes dans les L_p pour $p>2$, *C. R. Acad. Sci. Paris*, T. **261**, 1965, pp. 4307-4310.
[19] Yu. G. KURITSYN, Multidimensional Versions and Two Schoenberg's Problems, *Stability Problems for Stochastic Models*, Moscow, 1989, pp. 72-79 (Russian).
[20] P. LEVY, *Théorie de l'addition de variable aléatoires*, Gauthier-Villars, Paris, 1937.
[21] W. LINDE, Infinitely Divisible and Stable Measures on Banach Spaces, *Teubner-Texte für Mathematik*, Vol. **58**, Teubner, Leipgiz, 1983.
[22] W. LINDE, Moments of Measures on Banach Spaces, *Math. Ann.*, Vol. **258**, 1982, pp. 277-287.
[23] J. LINDENSTRAUSS, On the Extension of Operators with Finite Dimensional Range,*Illinois J. Math.*, Vol. **8**, 1964, pp. 488-499.
[24] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces, *Lect. Notes Math.*, Vol. **338**, 1973.
[25] J. MISIEWICZ, On Norm Dependent Positive Definite Functions, *Bull. Acad. Sci. Georgian SSR*, Vol. **130**, 1988, pp. 253-256 (Russian).
[26] J. MISIEWICZ, Positive Definite Functions on l_∞, *Statist. and Probab. Lett.*, Vol. **8**, 1989, pp. 255-260.
[27] A. NEYMAN, Representation of L_p-Norms and Isometric Embedding in L_p-Spaces, *Israel J. Math.*, Vol. **48**, 1984, pp. 129-138.
[28] G. POLYA, On the Zeroes of an Integral Function Represented by Fourier's Integral, *Messenger Math.*, Vol. **52**, 1923, pp. 185-188.
[29] G. POLYA and G. SZEGO, *Aufgaben und lehrsatze aus der analysis*, Springer-Verlag, Berlin-New York, 1964.
[30] I. J. SCHOENBERG, Metric Spaces and Positive Definite Functions, *Trans. Am. Math. Soc.*, Vol. **44**, 1938, pp. 522-536.
[31] I. J. SCHOENBERG, Metric Spaces and Completely Monotone Functions, *Ann. Math.*, Vol. **39**, 1938, pp. 811-841.
[32] V. M. ZOLOTAREV, *One-Dimensional Stable Distributions*, Nauka, Moscow, 1983.
[33] D. YOST, L_1 Contains Every Two-Dimensional Normed Space, *Ann. Polonici Math.*, Vol. **49**, 1988, pp. 17-19.
[34] I. AHARONI, B. MAUREY and B. S. MITYAGIN, Uniform Embeddings of Metric Spaces and of Banach Spaces into Hilbert Spaces, *Israel J. Math.*, Vol. **52**, 1985, pp. 251-265.

(*Manuscript received July* 11, 1990;
revised October 30, 1991.)