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Limit theorems and variation properties for fractional derivatives of the local time of a stable process

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ABSTRACT. — We obtain limits theorems for the occupation times of 1-dimensional stable Markov processes. These results are refinements of the classical limit theorems of Darling and Kac, and they generalize results obtained by Yamada for Brownian motion. The resulting limit processes are fractional derivatives and Hilbert transforms of the stable local time. We also study the p-variation properties of these limit processes.

Key words : Stable process, local time, fractional derivative, Hilbert transform, limit theorem, occupation time, p-variation.

RÉSUMÉ. — Nous démontrons des théorèmes limites pour les temps d'occupation des processus stables en dimension un. Ces résultats précisent les théorèmes limites classiques de Darling et Kac, et généralisent des résultats dus à Yamada dans le cas du mouvement brownien. Les processus obtenus à la limite sont les dérivées fractionnaires et les transformées de Hilbert des temps locaux. Nous étudions aussi la variation d'ordre p de ces processus limites.

Classification A.M.S. : Primary 60 J 55; secondary 60 J 30, 60 F 05.
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1. INTRODUCTION

We are concerned in this paper with limit theorems for the occupation times of 1-dimensional stable processes, and with certain properties of the limit processes.

To describe our results briefly let $X=(X_t)_{t\geqq 0}$ be a real-valued (strictly) stable process of index $\alpha\in]1,2]$ with $X_0=0$. By an extension ([Bi71], [K81]) of a famous theorem of Darling and Kac [DK57], if $f\in L^1(\mathbb{R})$ then the process

$$(1.1)\qquad \lambda^{-(1-1/\alpha)}\int_0^{\lambda t} f(X_s)\,ds,\qquad t\geqq 0,$$

converges in law as $\lambda\to +\infty$ to the process

$$(1.2)\qquad \int f(x)\,dx\cdot L_t^0,\qquad t\geqq 0,$$

where $(L_t^0)_{t\geqq 0}$ is local time at 0 for X. Now the integral in (1.1) makes sense even if f is only locally in $L^1(\mathbb{R})$, and in this case it is natural to ask if a limit theorem obtains, perhaps after a change in the exponent $(1-1/\alpha)$ and in the limit process (L_t^0). Such limit theorems have been found by Yamada [Y85], [Y86] when X is Brownian motion. (*See* also Kasahara [K77], [K81] and Pitman and Yor [PY86] for related results.) One of our goals is to extend Yamada's results to general stable processes. Typical of the limit theorems we obtain is the following. Consider $f\in L^1_{\text{loc}}(\mathbb{R})$ of the form

$$(1.3)\qquad f(x)=\frac{1}{\Gamma(-\gamma)}\int_0^\infty y^{-1-\gamma}[g(x+y)-g(x)]\,dy$$

where $0<\gamma<(\alpha-1)/2$ and g is a smooth function of compact support. (Thus f is the one-sided fractional derivative of g, of order γ.) Then as $\lambda\to +\infty$,

$$(1.4)\qquad \lambda^{-(1-(1+\gamma)/\alpha)}\int_0^{\lambda t} f(X_s)\,ds \overset{d}{\to}\int g(x)\,dx\cdot H_t^0,$$

where $(H_t^0)_{t\geqq 0}$ is the *fluctuating* continuous additive functional (CAF) of X defined by

$$(1.5)\qquad H_t^0=\frac{1}{\Gamma(-\gamma)}\int_0^\infty y^{-1-\gamma}[L_t^{-y}-L_t^0]\,dy,$$

and where $(L_t^x)_{t\geqq 0}$ is local time at x for X. Concerning the convergence of this integral, see the discussion following (2.20).

The process defined in (1.5) is exemplary of a class of CAF's which is a second focus of our study. Roughly speaking $(H_t^0)_{t\geqq 0}$ is not of finite variation, but it does have zero energy in the sense of Fukushima [F80]; *see* the remark following (4.9). More precisely consider the dyadic p-variation of $(H_t^0)_{0 \leqq t \leqq 1}$

$$V_n^p := \sum_{j=0}^{2^n - 1} |H^0_{(j+1)2^{-n}} - H^0_{j2^{-n}}|^p$$

and define $p_0 = (\alpha - 1)/(\alpha - 1 - \gamma)$. Note that $1 < p_0 < 2$ since $0 < \gamma < (\alpha - 1)/2$. We prove the following

$$(1.6) \qquad V_n^p \to 0 \quad \text{a.s. as } n \to +\infty, \qquad \text{if } p > p_0;$$

$$(1.7) \qquad V_n^{p_0} \to b L_1^0 \quad \text{in probability as } n \to +\infty,$$

where $0 < b < \infty$ is a certain constant. It follows easily from (1.7) that

$$(1.8) \qquad V_n^p \to +\infty \quad \text{in probability as } n \to +\infty, \qquad \text{if } 0 < p < p_0.$$

Actually, (1.6) is a consequence of a recent result of Bertoin [Be90], which implies that the full p-variation of $t \mapsto H_t^0$ is finite on compacts almost surely if and only if $p > p_0$. However, as a by-product of our proof we also obtain convergence in r-th mean ($1 \leqq r < \infty$) in both (1.6) and (1.7). The proof of (1.6) relies on moment estimates, while (1.7) is a consequence of the ergodic theorem coupled with the Darling-Kac theorem.

Very little seems to be known about the distribution of the process (H_t^0). A notable exception is the computation by Biane and Yor [BY87] of the joint Fourier-Laplace transform of (a symmetrized version of) $(H_t^0)_{t \geqq 0}$ in case $\gamma = 0$ [*see* (2.22)] when X is Brownian motion. We have recently extended this result to cover a wide class of symmetric Lévy processes; *see* [FG91]. The general case remains a challenging open problem.

The rest of the paper is organized as follows. Section 2 contains precise definitions and basic facts. In section 3 we prove various limit theorems for functionals of the type (1.1). The variation properties of H_t^0 are studied in section 4.

Throughout we shall use the standard notation for Markov processes (*cf.* [BG68]).

2. LOCAL TIMES AND FRACTIONAL DERIVATIVES

Throughout this paper $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ will denote the canonical realization of a real-valued strictly stable Lévy process, of index $\alpha \in]1, 2]$. Thus $(X_t)_{t \geqq 0}$ is a cadlag process with stationary independent increments, and the Lévy exponent ψ of X, defined by

$$(2.1) \qquad e^{-t\psi(\lambda)} = P^0(e^{i\lambda X_t}), \qquad t \geqq 0, \quad \lambda \in \mathbb{R},$$

takes the special form

$$\psi(\lambda)=a_1|\lambda|^\alpha\left\{1+i\,\mathrm{sgn}(\lambda)\,a_2\tan\left(\frac{\pi\alpha}{2}\right)\right\},\qquad \lambda\in\mathbb{R}. \tag{2.2}$$

Here $a_1>0$ and $a_2\in[-1,1]$ are constants, and $a_2=0$ if $\alpha=2$.

The transition probabilities of X have continuous densities relative to Lebesgue measure:

$$P_t(x,dy)=P^x(X_t\in dy)=p(t,y-x)\,dy,$$

where p is computed from (2.1) and (2.2) by Fourier inversion:

$$p(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}e^{-t\psi(\lambda)}\,d\lambda. \tag{2.3}$$

Clearly (2.2) and (2.3) yield

$$p(t,x)=t^{-1/\alpha}p(1,t^{-1/\alpha}x)\leqq B\,t^{-1/\alpha},\qquad t>0,\quad x\in\mathbb{R}, \tag{2.4}$$

where $0<B<\infty$ is a constant depending only on α and a_1.

It is well-known ([Bo64], [BG68]) that for each $x\in\mathbb{R}$ there is a local time process $(L_t^x)_{t\geqq 0}$ at x. This is an increasing CAF of X such that the support of the measure $d_tL_t^x$ coincides almost surely with the closure of $\{t: X_t=x\}$. Local time is normalized so that

$$P^x(L_t^y)=\int_0^t p(s,y-x)\,ds.$$

Subject to this normalization there is a version of local time such that $(x,t)\mapsto L_t^x$ is jointly continuous and

$$L_t^x=\frac{d}{dx}\int_0^t 1_{]-\infty,x]}(X_s)\,ds$$

for all $t\geqq 0$, $x\in\mathbb{R}$ almost surely. This fact is due to Trotter [Tr58] when X is Brownian motion ($\alpha=2$), and to Boylan [Bo64] for $1<\alpha<2$. Moreover, a good deal is known about the modulus of continuity of $x\mapsto L_t^x$. Define

$$\left\{\begin{aligned}\delta(u)&=\sup_{|x|\leqq u}\frac{1}{\pi}\int_0^\infty(1-\cos\lambda x)\,\mathrm{Re}\left(\frac{1}{1+\psi(\lambda)}\right)d\lambda,\\ \rho(x)&=\int_0^x[\log(1+u^{-2})]^{1/2}\,d\sqrt{\delta(u)}.\end{aligned}\right. \tag{2.5}$$

Then from the work of Barlow [B85] we know that ρ serves as a modulus of continuity for $x\mapsto L_t^x$ in a sense detailed in Theorem (2.7) below. Note that $\delta(u)\leqq C\,u^{\alpha-1}$, so

$$\rho(u)\leqq C(\beta)\,u^\beta,\qquad \forall\, 0<\beta<(\alpha-1)/2. \tag{2.6}$$

Thus $x \mapsto L_t^x$ is Hölder continuous of any order $\beta < (\alpha - 1)/2$, a fact which follows already from the work of Boylan.

For ease of manipulation in later computations we want to choose as "perfect" a version of local time as possible. The properties of one such version are outlined in the following theorem. The key point (vi) is due to Barlow [B85] as already noted, and the other points follow by well known perfection arguments (e. g. [GK72], [G90]). We omit the proof. As usual $\mathscr{F}^*$ denotes the universal completion of $\mathscr{F}^0 := \sigma\{X_t : t \geqq 0\}$.

(2.7) THEOREM. — *There is a function* $(x, t, \omega) \mapsto L_t^x(\omega)$ *from* $\mathbb{R} \times [0, \infty[\times \Omega$ *to* $[0, \infty[$, *and a set* $\Lambda \in \mathscr{F}^*$ *with* $P^x(\Lambda) = 1$ *for all* $x \in \mathbb{R}$ *and* $\theta_t \Lambda \subset \Lambda$ *for all* $t > 0$ *such that:*

(i) *For each* $T > 0$, $(x, t, \omega) \mapsto L_t^x(\omega)$ *is* $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}([0, T]) \otimes (\mathscr{F}_T \cap \mathscr{F}^*)$-*measurable as a map from* $\mathbb{R} \times [0, T] \times \Omega$ *to* $[0, \infty[$.

(ii) *For each* x *and* ω, $t \mapsto L_t^x(\omega)$ *is continuous and increasing with* $L_0^x(\omega) = 0$, *and the measure* $d_t L_t^x(\omega)$ *is carried by* $\{t : X_t(\omega) = x\}$.

(iii) $L_{t+s}^x(\omega) = L_t^x(\omega) + L_s^x(\theta_t w)$, $\forall s, t \geqq 0$, $\omega \in \Lambda$, $x \in \mathbb{R}$.

(iv) $\displaystyle\int_0^t f(X_s(\omega))\, ds = \int_{\mathbb{R}} f(x) L_t^x(\omega)\, dx$, $\forall t \geqq 0$, $\omega \in \Lambda$ *and all bounded or positive Borel functions* f.

(v) $L_t^x(\omega) = 0$ *whenever* $|x| > \sup\{|X_s(\omega)| : 0 \leqq s \leqq t\}$.

(vi) *For each* t *and* ω *there is a constant* $0 < C(t, \omega) < \infty$ *such that*

$$\sup_{0 \leqq s \leqq t} |L_s^x(\omega) - L_s^y(\omega)| \leqq C(t, \omega)\, \rho(|x - y|), \qquad \forall x, y \in \mathbb{R}.$$

(2.8) *Remarks.* — (*a*) Barlow [B85] has shown that if X is any Lévy process for which 0 is regular for $\{0\}$ and for which

$$P^x(X_t = y \text{ for some } t \geqq 0) > 0 \quad \text{for all } x, y,$$

and if δ is as defined in (2.5), then

$$\sum_n [\delta(2^{-n})/n]^{1/2} < \infty$$

is a sufficient condition for X to have a local time for which ρ defined below (2.5) is a modulus of continuity. Consequently Theorem (2.7) is valid for any such process.

(*b*) It follows immediately from (2.7) (ii) (vi) that $L_t^x(\omega)$ is jointly continuous in (x, t) for each $\omega \in \Omega$.

We now introduce certain "fractional derivative" transforms which play a central role in the sequel. Let $\mathscr{H}(\beta)$ denote the class of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying a global Hölder condition of order β:

$$|f(x) - f(y)| \leqq C(f, \beta)\, |x - y|^\beta, \qquad \forall x, y \in \mathbb{R}.$$

Given $\gamma \in]0, 1[$ we define

(2.9) $$D^\gamma_\pm f(x) = \frac{1}{\Gamma(-\gamma)} \int_0^\infty y^{-1-\gamma} [f(x \pm y) - f(x)]\, dy$$

provided $f \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for some $\beta > \gamma$. The Hölder condition on f ensures that the integral in (2.9) is absolutely convergent. [Note that if $f \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for some $\beta > 0$ then $f(x) \to 0$ as $|x| \to \infty$.] Of course, D^γ_+ and D^γ_- are the familiar one-sided fractional derivatives of order γ. We shall write

(2.10) $$D^\gamma = D^\gamma_+ - D^\gamma_-$$

for the symmetric fractional derivative.

Since y^{-1} is not integrable at $+\infty$, the definition of $D^\gamma_\pm$ must be modified slightly to allow $\gamma = 0$. Accordingly we define

(2.11) $$D^0_\pm f(x) = -\int_0^\infty y^{-1} [f(x \pm y) - 1_{\{0<y<1\}} f(x)]\, dy$$

for $f \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$, $\beta > 0$. [The minus sign in front of the integral is the ghost of $\Gamma(-\gamma)$.] Note that $D^0 = D^0_+ - D^0_-$ is the Hilbert transform (modulo a factor of $1/\pi$). For information on fractional derivatives and the Hilbert transform the reader can consult [HL28], [S70], [T48]. The following two lemmas contain the facts that we will need in the sequel.

(2.12) LEMMA. – *Let* $\{\varphi_t : 0 \leqq t \leqq T\}$ *be a family of functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that for some constants* $\beta > 0$, $0 < C < \infty$,

$$\sup_{0 \leqq t \leqq T} |\varphi_t(x) - \varphi_t(y)| \leqq C |x - y|^\beta, \qquad \forall x, y \in \mathbb{R}$$

and

$$\{x : \varphi_t(x) \neq 0\} \subset [-C, C], \qquad 0 \leqq t \leqq T.$$

Define $f_t = D^\gamma_\pm \varphi_t$, *where* $0 \leqq \gamma < \beta$. *Then there is a constant* $0 < K < \infty$ *such that for all* x, $y \in \mathbb{R}$

(2.13) $$\sup_{0 \leqq t \leqq T} |f_t(x) - f_t(y)| \leqq \begin{cases} K |x-y|^{\beta-\gamma}, & 0 < \gamma < 1, \\ K |x-y|^\beta [1 + |\log|x-y||], & \gamma = 0, \end{cases}$$

and

(2.14) $$\sup_{0 \leqq t \leqq T} |f_t(x)| \leqq K |x|^{-1-\gamma}, \qquad x \neq 0.$$

Proof. – We consider only the case $\gamma = 0$; the argument is easier when $\gamma > 0$ and the reader may consult [HL28]. We shall only prove (2.13); the growth estimate (2.14) is straightforward. Fix $\varphi : \mathbb{R} \to \mathbb{R}$ with $|\varphi(x) - \varphi(y)| \leqq C |x - y|^\beta$ and $\operatorname{supp} \varphi \subset [-C, C]$, and consider

$f=-D^0_+\varphi$. (The argument for D^0_- is the same.) We have

$$f(x)-f(z)=\int_0^1 \frac{1}{y}[\varphi(x+y)-\varphi(x)-\varphi(z+y)+\varphi(z)]\,dy + \int_1^\infty \frac{1}{y}[\varphi(x+y)-\varphi(z+y)]\,dy = I_1+I_2, \quad \text{say.}$$

Evidently

$$|I_2|\leqq C|x-z|^\beta \int_1^\infty \frac{1}{y} 1_{A(x,z)}(y)\,dy$$

where $A(x,z)=\{y:|x+y|\leqq C \text{ or } |z+y|\leqq C\}$ has measure $\leqq 4C$. Thus

$$|I_2|\leqq 4C^2|x-z|^\beta, \qquad \forall x,z.$$

Now let $h=|x-z|$ and assume $h\leqq 1/2$. Then

$$\begin{aligned}|I_1|&\leqq \int_0^h \frac{1}{y}[|\varphi(x+y)-\varphi(x)|+|\varphi(z+y)-\varphi(z)|]\,dy\\ &\quad+\int_h^1 \frac{1}{y}[|\varphi(x+y)-\varphi(z+y)|+|\varphi(x)-\varphi(z)|]\,dy\\ &\leqq 2C\int_0^h y^{\beta-1}\,dy+2C|x-z|^\beta\int_h^1 y^{-1}\,dy\\ &=\frac{2C}{\beta}|x-z|^\beta+2C|x-z|^\beta\log\left(\frac{1}{|x-z|}\right).\end{aligned}$$

Combining these estimates with the obvious inequality $|I_1|\leqq 2C/\beta$ we obtain (2.13) (for $\gamma=0$). ■

Remark. – Even if $\varphi\in\mathscr{H}(\beta)$ is merely integrable, it is still true that $D^\gamma_\pm\varphi$ is bounded and continuous provided $0\leqq\gamma<\beta$. In fact when $\gamma>0$ it is easy to see that $D^\gamma_\pm\varphi\in\mathscr{H}(\beta-\gamma)$. When $\gamma=0$ one can argue as in the proof of (2.12): the estimate for I_1 remains valid, while

$$|I_2|\leqq C^{1/2}|x-z|^{\beta/2}\left[2\int_1^\infty y^{-2}\,dy\cdot\|\varphi\|_1\right]^{1/2}$$

by the Cauchy-Schwarz inequality. Thus $D^0_\pm\varphi$ is (Hölder) continuous. A similar application of the Cauchy-Schwarz inequality shows that $D^0_\pm\varphi$ is bounded.

(2.15) Lemma. – *Fix $0\leqq\gamma<1$ and suppose $f, g\in\mathscr{H}(\beta)\cap L^1(\mathbb{R})$ for some $\beta>\gamma$. Then*

$$\int f(x)\,D^\gamma_- g(x)\,dx=\int D^\gamma_+ f(x)\,g(x)\,dx.$$

Proof. — By the preceding remark both $D^\gamma_+ f$ and $D^\gamma_- g$ are bounded and continuous. For $\varepsilon>0$ define $F_\varepsilon(x)$ by the R.H.S. of the + case of (2.9) or (2.11) with the range of integration restricted to $]\varepsilon, \varepsilon^{-1}[$, and let G_ε denote the analogous approximation of $D^\gamma_- g$. It is easily checked that $F_\varepsilon \to D^\gamma_+ f$ (resp. $G_\varepsilon \to D^\gamma_- g$) boundedly and pointwise as $\varepsilon \downarrow 0$, A trivial application of Fubini's theorem reveals that

$$\int f(x)\, G_\varepsilon(x)\, dx = \int F_\varepsilon(x)\, g(x)\, dx.$$

By virtue of the dominated convergence theorem we can now let $\varepsilon \downarrow 0$ to obtain the conclusion of the lemma. ■

The significance of the following consequence of the "switching identity" (2.15) will become apparent in the next section. We should emphasize that $D^0_\pm$ are excluded from consideration here.

(2.16) PROPOSITION. — *Let* D *denote one of the transforms* $D^\gamma_\pm$ $(0<\gamma<1)$, D^γ $(0 \leqq \gamma < 1)$. *Let* $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ *where* $\beta > \gamma$, *and assume that* $f = Dg \in L^1(\mathbb{R})$. *Then* $\int f(x)\, dx = 0$. *When* $\gamma = 0$ *we also have* $\int g(x)\, dx = 0$.

Proof. — If $h: \mathbb{R} \to \mathbb{R}$ and $a>0$ we write h_a for the function $x \mapsto h(ax)$. The crux of the matter is the scaling identity

$$D(h_a) = a^\gamma (Dh)_a. \tag{2.17}$$

For definiteness we assume $D = D^\gamma$; the other cases are entirely similar. By (2.15) and its "dual", if φ is any smooth function of compact support with $\varphi(0) \neq 0$ then

$$\int f(x)\, \varphi_a(x)\, dx = -\int g(x)\, D(\varphi_a)(x)\, dx = -a^\gamma \int g(x)\, (D\varphi)_a(x)\, dx. \tag{2.18}$$

If $0<\gamma<1$ then, since $D\varphi$ is bounded and $f, g \in L^1(\mathbb{R})$, we can let $a \downarrow 0$ in (2.18) to obtain $\int f(x)\, dx = 0$. When $\gamma = 0$, upon letting $a \downarrow 0$ in (2.18) we obtain

$$\varphi(0) \int f(x)\, dx = -D\varphi(0) \int g(x)\, dx.$$

Varying φ we conclude that $\int f(x)\, dx = \int g(x)\, dx = 0$. ■

Remark. — When $D = D^0_\pm$, (2.17) must be replaced by

$$D^0_\pm(h_a) = (D^0_\pm h)_a + \log a \cdot h_a. \tag{2.19}$$

A variant of the above argument now yields the following: if $f = D^{\gamma}_{+} g = D^{\gamma}_{-} h$ for $\gamma \geqq 0$, where $g, h \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for some $\beta > \gamma$, then $\int g(x)\,dx = \int h(x)\,dx$, and both of these integrals vanish if, in addition, $\gamma > 0$. Moreover, if $f = D^0_{\pm} g$ where $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for some $\beta > 0$, then f cannot be integrable.

We close this section by defining the additive functionals that will concern us for the remainder of the paper. By (2.6) and (2.7) (vi) we have

$$\sup_{0 \leqq t \leqq T} |L^x_t(\omega) - L^y_t(\omega)| \leqq C(T, \omega) |x - y|^{\beta} \tag{2.20}$$

for any $\beta \in]0, (\alpha - 1)/2[$. Together with (2.7) (v) this allows us to define for $0 \leqq \gamma < (\alpha - 1)/2$,

$$H^x_t(\gamma \pm) = H^x_t(\gamma \pm; \omega) = D^{\gamma}_{\pm}(L^{\cdot}_t(\omega))(x) \tag{2.21}$$

$$H^x_t(\gamma) = H^x_t(\gamma +) - H^x_t(\gamma -). \tag{2.22}$$

By Lemma (2.12), for each $\omega \in \Omega$, $x \mapsto H^x_t(\gamma \pm; \omega)$ is Hölder continuous of order δ for any $\delta < (\alpha - 1)/2 - \gamma$ and

$$H^x_t(\gamma \pm; \omega) = O(|x|^{-1-\gamma}), \qquad |x| \to \infty, \tag{2.23}$$

both of these statements holding uniformly in t confined to compacts. It follows from (2.7) (iii) that $(H^x_t(\gamma \pm))_{t \geqq 0}$ is an additive functional of X. Moreover, a dominated convergence argument shows that $t \mapsto H^x_t(\gamma \pm; \omega)$ is continuous. Thus $(H^x_t(\gamma \pm))_{t \geqq 0}$ is a CAF of X and $(x, t) \mapsto H^x_t(\gamma \pm)$ is continuous.

Owing to the singularity of $y^{-1-\gamma}$ at 0, each of the CAF's $(H^x_t(\gamma \pm))_{t \geqq 0}$, $(H^x_t(\gamma))_{t \geqq 0}$ is of unbounded variation over any time interval during which X visits x. More precisely, for almost every $\omega \in \Omega$ and all $x \in \mathbb{R}$, $0 \leqq s < t$,

$$L^x_t(\omega) - L^x_s(\omega) > 0 \quad \Rightarrow \quad \mathrm{Var}_{[s,t]}(H^x_{\cdot}(\gamma \pm); \omega) = +\infty, \tag{2.24}$$

and likewise for $H^x_t(\gamma)$. This assertion is a consequence of Theorem (4.3) (*b*) when $0 < \gamma < (\alpha - 1)/2$. Since the case $\gamma = 0$ is not covered by (4.3) (*b*), we shall sketch the proof here, restricting attention to the symmetric case. Thus we shall prove that (2.24) holds [with $H^x_{\cdot}(\gamma \pm; \omega)$ replaced by $H^x_{\cdot}(0; \omega)$] for each $\omega \in \Lambda$ [*see* (2.7)]. There is no loss of generality in assuming $x = s = 0$. Moreover, since $t \mapsto \int_1^{\infty} y^{-1} L^y_t\,dy$ is clearly of bounded variation on finite t-intervals, it suffices to check (2.24) with $H^0_t(0; \omega)$ replaced by the CAF

$$H_t = \int_0^1 y^{-1} [L^y_t - L^{-y}_t]\,dy.$$

It is not hard to check that the variation process $V_t := \mathrm{Var}_{[0,t]}(H_\cdot)$ is increasing, and continuous except perhaps for a single infinite jump which may ocur at $t=0$. The same remarks apply to the positive and negative variation processes (V_t^+), (V_t^-), and of course $V_t = V_t^+ + V_t^-$. Now fix $\omega \in \Lambda$ and $t>0$, and suppose that $V_t(\omega) < \infty$. Then

$$\infty > V_t(\omega) \geqq V_t^+(\omega) \geqq \int_0^t 1_{\{X_s(\omega)>0\}} dV_s^+(\omega). \tag{2.25}$$

But if $[a,b] \subset \{s : s \leqq t;\ X_s(\omega) > 0\}$ then $H_\cdot(\omega)$ is increasing on $[a,b]$ (because $L_\cdot^{-y}(\omega)$ is constant on $[a,b]$ for all $y>0$), so we have

$$V_b^+(\omega) - V_a^+(\omega) = V_b(\omega) - V_a(\omega)$$
$$= H_b(\omega) - H_a(\omega) = \int_0^1 y^{-1} [L_b^y(\omega) - L_a^y(\omega)]\, dy.$$

Consequently

$$1_{\{X_s(\omega)>0\}} dV_s^+(\omega) = \int_0^1 \frac{dy}{y} (dL_s^y(\omega))$$

as measures on $[0,t]$. Hence these measures have the same total mass, so (2.25) implies $\infty > \int_0^1 y^{-1} L_t^y(\omega)\, dy$ which precludes $L_t^0(\omega) > 0$ because $y \mapsto L_t^y(\omega)$ is continuous. This yields (2.24).

3. LIMIT THEOREMS

We present in this section several limit theorems for rescaled additive functionals of the form

$$\lambda^{-p} \int_0^{\lambda t} f(X_s)\, ds, \qquad t \geqq 0, \tag{3.1}$$

for certain $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. (Except for the trivial case $f=0$, all of the f's we consider take both signs.) The method is a simplification of that of Yamada [Y86], but since the proofs are short we give a detailed account.

The key to our limit theorems is the scaling property of stable processes, which is conveniently formulated as follows. For each $c>0$ define a transformation $\Phi_c : \Omega \to \Omega$ by

$$(\Phi_c \omega)(t) = c^{-1/\alpha} \omega(ct), \qquad t \geqq 0. \tag{3.2}$$

Using (2.1) and the fact that $\psi(c^{-1/\alpha}\lambda) = c^{-1}\psi(\lambda)$ [by (2.2)], it is easy to check that

$$\Phi_c(P^x) = P^{x/c^{1/\alpha}}, \qquad \forall x \in \mathbb{R}. \tag{3.3}$$

On the other hand, (2.7) (iv) implies that

$$(3.4) \qquad L_t^x(\Phi_c \omega) = c^{-(1-1/\alpha)} L_{ct}^{xc^{1/\alpha}}(\omega), \qquad \forall t \geqq 0, \quad x \in \mathbb{R},$$

provided $\omega \in \Lambda \cap \Phi_c^{-1}(\Lambda)$. Here Λ is as in Theorem (2.7), hence $P^y(\Lambda) = 1$ for all y and (3.4) holds for almost all $\omega \in \Omega$. When combined, (3.3) and (3.4) yield the following equality in distribution between two-parameter processes:

$$(3.5) \qquad (L_t^x : t \geqq 0, x \in \mathbb{R}; P^y) \stackrel{d}{=} (c^{-(1-1/\alpha)} L_{ct}^{xc^{1/\alpha}} : t \geqq 0, x \in \mathbb{R}; P^{yc^{1/\alpha}}).$$

Here are the main results of this section. We shall denote Lebesgue measure on $\mathbb{R}$ by m, so $m(g) = \int g(x)\,dx$.

(3.6) THEOREM. – *Let* $\gamma \in]0, (\alpha-1)/2[$ *and suppose* $f = D_-^\gamma g$ *where* $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ *for some* $\beta > \gamma$. *Then under the law* P^0,

$$(3.7) \qquad \lambda^{-(1-(1+\gamma)/\alpha)} \int_0^{\lambda t} f(X_s)\,ds \stackrel{d}{\to} m(g)\, H_t^0(\gamma+), \qquad \lambda \to +\infty,$$

where $H_t^0(\gamma+) = D_+^\gamma(L_t^\cdot)(0)$ *is the* CAF *of* X *introduced in section* 2.

Here and elsewhere " $\stackrel{d}{\to}$ " means weak convergence of the laws that the indicated processes induce on the space $C([0, \infty[, \mathbb{R})$, which is equipped with the topology of uniform convergence on compact time sets.

(3.8) THEOREM. – *Suppose* $f = D_-^0 g$ *where* $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ *for some* $\beta > 0$. *Then under* P^0

$$(3.9) \qquad (\lambda^{1-1/\alpha} \log \lambda)^{-1} \int_0^{\lambda t} f(X_s)\,ds \stackrel{d}{\to} -\alpha^{-1} m(g)\, L_t^0$$

and

$$(3.10) \qquad \lambda^{-(1-1/\alpha)} \int_0^{\lambda t} [f(X_s) + \alpha^{-1}(\log \lambda)\, g(X_s)]\,ds \stackrel{d}{\to} m(g)\, H_t^0(0+)$$

as $\lambda \to +\infty$.

(3.11) *Remarks.* – (*a*) As will be evident from the proofs, the limit laws announced in (3.6) and (3.8) are all "linked" in the sense of Pitman and Yor [PY86]. That is, if $f_i = D_-^{\gamma(i)} g_i$, $1 \leqq i \leqq n$, then (writing $p(i) = [1 - (1 + \gamma(i))/\alpha]$) the vector of processes

$$\left(\lambda^{-p(i)} \int_0^{\lambda t} f_i(X_s)\,ds\right)_{1 \leqq i \leqq n}$$

converges in distribution as $\lambda \to +\infty$ to the vector of processes $(m(g_i) H_t^0(\gamma_i+))_{1 \leq i \leq n}$. Similarly, the limit laws of (3.8) are linked with each other and with those of (3.6).

(*b*) Of course the duals to the results of (3.6) and (3.8) (obtained by exchanging +'s and −'s) are equally valid, and they are linked with the announced limit laws. In particular, by subtraction one obtains the appropriate limit theorems when $f = D^\gamma g$, $0 \leq \gamma < (\alpha - 1)/2$.

(*c*) Suppose $f = D_-^\gamma g$ as in (3.6) and assume $f \in L^1(\mathbb{R})$; this happens, for example, if g has compact support. Then by the extended Darling-Kac theorem

$$\text{(3.12)} \qquad \lambda^{-(1-1/\alpha)} \int_0^{\lambda t} f(X_s)\, ds \xrightarrow{d} m(f)\, L_t^0.$$

The apparent conflict between (3.12) and (3.7) is resolved by Proposition (2.16), which tells us that $m(f) = 0$. Thus, in the present case, (3.7) should be viewed as a refinement of the (degenerate) limit law (3.12). Likewise if $f = D_-^\gamma g = D_+^\gamma h$ with $g, h \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for $0 < \gamma < \beta$, then (3.7) and its dual are compatible by virtue of the remark following the proof of (2.16). Similar remarks hold in the context of Theorem (3.8).

Proof of (3.6). — For this proof only let us set $p = 1 - (1+\gamma)/\alpha$. Then under P^0

$$\text{(3.13)} \quad \lambda^{-p} \int_0^{\lambda t} f(X_s)\, ds = \lambda^{-p} \int f(x)\, L_{\lambda t}^x\, dx, \quad \text{by (2.7)(iv)}$$

$$\stackrel{d}{=} \lambda^{\gamma/\alpha} \int f(x)\, L_t^{x/\lambda^{1/\alpha}}\, dx, \quad \text{by (3.5)}$$

$$= \lambda^{\gamma/\alpha} \int g(x)\, D_+^\gamma (L_t^{\cdot/\lambda^{1/\alpha}})(x)\, dx, \quad \text{by (2.15)}$$

$$= \int g(x)\, H_t^{x/\lambda^{1/\alpha}}(\gamma+)\, dx, \quad \text{by (2.17)}.$$

But for $T > 0$ and $\omega \in \Omega$ we have, by (2.20) and (2.12)

$$\lim_{\lambda \to \infty} \sup_{0 \leq t \leq T} \left| H_t^{x/\lambda^{1/\alpha}}(\gamma+;\omega) - H_t^0(\gamma+;\omega) \right| = 0,$$

and

$$\sup_x \sup_{0 \leq t \leq T} \left| H_t^x(\gamma+;\omega) \right| < \infty.$$

Thus the last integral in (3.13) converges to $\int g(x)\, dx \cdot H_t^0(\gamma+)$ uniformly in $t \in [0, T]$ for each $T > 0$ (and each fixed ω). The theorem is proved. ■

Remark. – The argument used in the above proof gives a very simple proof of the extended Darling-Kac theorem for the stable processes under consideration here.

Proof of (3.8). – In this proof we write $p=1-1/\alpha$. Arguing as in the proof of (3.6) but now using (2.19), we obtain

$$(3.14)\quad \lambda^{-p}\int_0^{\lambda t} f(X_s)\,ds \overset{d}{=} \int g(x)\,H_t^{x/\lambda^{1/\alpha}}(0+)\,dx - \alpha^{-1}\log\lambda \int g(x)\,L_t^{x/\lambda^{1/\alpha}}\,dx$$

and

$$(3.15)\quad \lambda^{-p}\int_0^{\lambda t} [f(X_s)+\alpha^{-1}(\log\lambda)\,g(X_s)]\,ds \overset{d}{=} \int g(x)\,H_t^{x/\lambda^{1/\alpha}}(0+)\,dx.$$

Clearly (3.14) [resp. (3.15)] yields (3.9) [resp. (3.10)]. ■

In the limit theorems (3.6) and (3.8) we required f to be in the range of one of the fractional derivative transforms $D_{\pm}^{\gamma}$. Analogous results can be obtained by replacing the kernel $(y \vee 0)^{-1-\gamma}/\Gamma(-\gamma)$ of $D_{\pm}^{\gamma}$ by a suitable regularly varying function. We will state the resulting limit theorems, leaving the proofs to the interested reader.

For $0 \leqq \gamma < (\alpha-1)/2$ let $k_\gamma : \mathbb{R} \to [0, \infty[$ be a regularly varying function of the form

$$k_\gamma(x) = \begin{cases} l(x)\,x^{-1-\gamma}, & x>0, \\ 0, & x \leqq 0, \end{cases}$$

where l is slowly varying at $+\infty$. Since only the asymptotic behavior of l at $+\infty$ is relevant we may assume with no loss of generality that l is continuously differentiable, $l(x)>0$ for all $x>0$, and $l(0+)=1$; *see* [BGT87], Thm. 1.3.3. Note that $l(x)=o(x^\beta)$ as $x \to +\infty$ for any $\beta>0$ ([BGT 87], Prop. 1.3.6), so when $\gamma>0$, $\int_1^\infty k_\gamma(x)\,dx<\infty$. Consequently, if $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ for some $\beta>\gamma$ and $0<\gamma<(\alpha-1)/2$, then the formulas

$$K_{\pm}^{\gamma}\, g(x) = \frac{1}{\Gamma(-\gamma)} \int_0^\infty k_\gamma(y)\,[g(x \pm y) - g(x)]\,dy$$

define bounded continuous functions.

(3.16) THEOREM. – *Let* $0<\gamma<(\alpha-1)/2$ *and suppose* $f=K_{\pm}^{\gamma}\, g$ *where* $g \in \mathscr{H}(\beta) \cap L^1(\mathbb{R})$ *for some* $\beta>\gamma$. *Then under* P^0 *one has, as* $\lambda \to +\infty$,

$$[\lambda^{1-(1+\gamma)/\alpha}\, l(\lambda^{1/\alpha})]^{-1} \int_0^{\lambda t} f(X_s)\,ds \overset{d}{\to} m(g)\,H_t^0(\gamma \mp)$$

where $H_t^0(\gamma \mp) = D_{\mp}^{\gamma}(L_t^{\cdot})(0)$, *as usual.*

Like $D^0_\pm$, $K^0_\pm$ requires special consideration. We define

$$K^0_\pm g(x) = -\int_0^\infty k_0(y)\,[g(x\pm y) - 1_{\{0<y<1\}}\, g(x)]\,dy,$$

and set

$$q(a) = \int_1^a y^{-1}\, l(y)\,dy, \qquad a>1.$$

Using [BGT87], Thm. 1.3.1, one can check that $l(a)/q(a)\to 0$ as $a\to +\infty$.

(3.17) THEOREM. – *Suppose* $f = K^0_\pm g$ *where* $g\in\mathscr{H}(\beta)\cap L^1(\mathbb{R})$ *for some* $\beta>0$. *Then under* P^0, *as* $\lambda\to+\infty$,

$$[\lambda^{1-1/\alpha} q(\lambda^{1/\alpha})]^{-1}\int_0^{\lambda t} f(X_s)\,ds \xrightarrow{d} -m(g)\,L^0_t$$

and

$$[\lambda^{1-1/\alpha}\, l(\lambda^{1/\alpha})]^{-1}\int_0^{\lambda t} [f(X_s) + q(\lambda^{1/\alpha})\, g(X_s)]\,ds \xrightarrow{d} m(g)\, H^0_t(0\mp).$$

(3.18) *Remark.* – For the reader interested in providing proofs of (3.16) and (3.17) we note that the key extra ingredient is the following observation. Let $\{\varphi_t : 0\leqq t\leqq T\}$ be a family of functions subject to the conditions of Lemma (2.12). If $0\leqq\gamma<\beta$ then as $a\to+\infty$,

$$[l(a)]^{-1}\, a^\gamma K^\gamma_\pm(\varphi_t(\cdot\, a^{-1}))(x) \to D^\gamma_\pm\,\varphi_t(0)$$

uniformly in $(x, t)\in D\times[0, T]$ for each compact $D\subset\mathbb{R}$. This can be proved by a judicious use of Potter's theorem [BGT87], Thm. 1.5.6.

The limit theorems announced in (3.16) and (3.18) are linked with each other and with those announced in (3.6) and (3.8). In particular, one can obtain limit theorems for $f = K^\gamma g := K^\gamma_+ g - K^\gamma_- g$.

The results of this section provide limit theorems for rescaled additive functionals

$$\lambda^{-p}\int_0^{\lambda t} f(X_s)\,ds$$

for p in the range $](1-1/\alpha)/2,\ 1-1/\alpha[$. The Darling-Kac theorem lies at one end of this spectrum ($p = 1-1/\alpha$). At the other end of the spectrum ($p=(1-1/\alpha)/2$) is the "2nd order" limit theorem of Kasahara [K81]. (Actually, Kasahara's theorem is in the spirit of the Darling-Kac theorem and applies in much more generality than the stable context considered here.) The boundary $p=(1-1/\alpha)/2$ seems to be natural – we know of no analogous limit theorems for $p<(1-1/\alpha)/2$. On the other hand, Yamada [Y86] has obtained limit theorems for Brownian motion when $p>1-1/\alpha$

($=1/2$ when $\alpha=2$). Analogous limit theorems obtain in the stable case considered here. These theorems require f to be of the form $I^{\gamma}_{\pm} g$, g continuous and of compact support, where $I^{\gamma}_{\pm}$ are the one-sided fractional *integral* transforms. The reader can consult [Y86] for details.

4. p-VARIATION

We now turn to the variation properties of $t \mapsto H^0_t(\gamma\pm)$. Let (H_t) denote one of the processes $(H^0_t(\gamma\pm))$, $(H^0_t(\gamma))$ and define

(4.1) $$V^p_n(t) = \sum_{j=0}^{t\,2^n-1} |H_{(j+1)\,2^{-n}} - H_{j\,2^{-n}}|^p$$

where $p>0$, $n \in \mathbb{N}$, and t is a dyadic rational of the form $k\,2^{-n}$, $k \in \mathbb{N}$. We complete the definition of $V^p_n(\cdot)$ by linear interpolation on each interval $[(k-1)\,2^{-n}, k\,2^{-n}]$. Given $\gamma \in [0, (\alpha-1)/2[$ set

(4.2) $$p_0 = (\alpha-1)/(\alpha-1-\gamma)$$

so that $1 \leqq p_0 < 2$. The cases $H^0_{\bullet}(0+)$ and $H^0_{\bullet}(0-)$ are excluded from consideration in the following result.

(4.3) THEOREM. – *(a) If* $p>p_0$ *then as* $n \to +\infty$, $V^p_n(t) \to 0$, $\forall\, t \geqq 0$ a.s. P^0 *and in* $L^r(P^0)$, $1 \leqq r < \infty$, *for each fixed* $t \geqq 0$.

(b) If $0<p<p_0$, *then* $V^p_n(t) \overset{P^0}{\to} +\infty$ *as* $n \to +\infty$ *for each* $t>0$. *In particular* $\limsup_{n\to\infty} V^p_n(t) = +\infty$ a.s. P^0 *if* $t>0$. *(Here "*$\overset{P^0}{\to}$*" denotes convergence in* P^0*-probability.)*

(c) If $\gamma>0$ *(so* $p_0>1$*) then for each* $T>0$

$$\sup_{0 \leqq t \leqq T} |V^{p_0}_n(t) - b\,L^0_t| \overset{P^0}{\to} 0 \quad \textit{as } n \to +\infty,$$

where $b = P^m(|H_1|^{p_0}) < \infty$. *Moreover* $V^{p_0}_n(t) \to b\,L^0_t$ *in* $L^r(P^0)$ *for each* $1 \leqq r < \infty$ *and* $t \geqq 0$.

Remark. – As noted in section 1, Bertoin [Be90] has shown that the full p-variation of $H^0_{\bullet}(\gamma\pm)$ is finite on compacts a.s. P^0 if and only if $p>p_0$. (Bertoin is concerned only with Brownian motion, but his argument works just as well in the present context.) In particular, the dyadic p-variation of $H^0_{\bullet}(0\pm)$ is finite (hence zero) for $p>p_0$ ($=1$ when $\gamma=0$), settling a point left untreated by (4.3) (*a*). By the discussion at the end of section 2 the 1-variation of $t \mapsto H^0_t(0\pm)$ is infinite over any interval $[T_1, T_2]$ such that $L^0_{T_2} - L^0_{T_1} > 0$.

For the sake of definiteness we assume in what follows that $H_t = H_t^0(\gamma) = H_t^0(\gamma+) - H_t^0(\gamma-)$, $0 \leqq \gamma < 1$. The proofs for $H_t^0(\gamma\pm)$, $0 < \gamma < 1$, are quite similar. In the sequel we often suppress γ in our notation, writing H_t^x for $H_t^x(\gamma)$.

Everything in this section relies on the joint scaling property of (H_t^x) and (L_t^0). Since we have excluded $H_t^x(0\pm)$ from consideration, the required scaling property results on combining (2.17), (3.3) (3.4) and the analog of (3.4) for (H_t^x). For each $c>0$ we have

(4.4) $$((H_t^x, L_s^0) : t \geqq 0,\ s \geqq 0, x \in \mathbb{R};\ P^y)$$
$$\stackrel{d}{=} ((c^{-(1-(1+\gamma)/\alpha)} H_{ct}^{xc^{1/\alpha}}, c^{-(1-1/\alpha)} L_{cs}^0) : t \geqq 0, s \geqq 0, x \in \mathbb{R};\ P^{yc^{1/\alpha}}).$$

Our first task is to establish the finiteness of certain moments of H_t. This could be accomplished by direct estimation but we prefer to use a variation on Burkholder's inequality due to Bass [Ba87]. The argument given in [Ba87], where X is assumed to be Brownian motion, readily adapts to the case of a general strong Markov process. *See* also Davis [D87] in the context of stable processes.

(4.5) LEMMA. — *Let* $(A_t)_{t \geqq 0}$ *be a continuous increasing* $(\mathcal{F}_t)$*-adapted real-valued process with* $A_0 = 0$. *Assume that:*

(i) $A_{t+s} \leqq A_t + K \cdot A_s \circ \theta_t$, $\forall s, t \geqq 0$, *for some constant* $K > 0$;

(ii) *there is a constant* $q > 0$ *such that* $\lim_{z \to \infty} \sup_{\lambda > 0, x \in \mathbb{R}} P^x(A_\lambda > \lambda^{1/q} z) = 0$.

Then for each $p > 0$ *there is a constant* $0 < C_p < \infty$ *such that*

(4.6) $$P^0(A_t^p) \leqq C_p\, t^{p/q}, \qquad \forall t \geqq 0.$$

Remark. — In fact, (4.6) remains valid (with the same constant C_p) if t is replaced by any stopping time T, $t^{p/q}$ being replaced by $P^0(T^{p/q})$.

(4.7) PROPOSITION. — *Let* $H_t^x = H_t^x(\gamma)$ *where* $0 \leqq \gamma < (\alpha - 1)/2$. *Then*

$$P^0(\sup_{x \in \mathbb{R}} \sup_{0 \leqq s \leqq t} |H_s^x|^p) < \infty, \qquad \forall p > 0.$$

Proof. — We apply (4.5) with $q = \alpha/(\alpha - 1 - \gamma)$ and

$$A_t = \sup_x \sup_{0 \leqq s \leqq t} |H_s^x|, \qquad t \geqq 0.$$

Clearly (A_t) satisfies condition (i) in (4.5), and for any $c>0$

(4.8) $$(A_t : t \geqq 0;\ P^0) \stackrel{d}{=} (c^{-1/q} A_{ct} : t \geqq 0;\ P^0),$$

by (4.4). Also, writing $\tau_y : \Omega \to \Omega$ for the translation $\omega \mapsto \omega(\cdot) + y$, we have $L_t^x(\tau_y \omega) = L_t^{x-y}(\omega)$ by (2.7) (iv) and so

$$H_t^x(\tau_y \omega) = H_t^{x-y}(\omega).$$

Since X is spatially homogeneous [*i.e.*, $\tau_y(P^0)=P^y$], it follows that the P^y-distribution of (A_t) does not depend on y. Thus, by (4.8),

$$\sup_{y,\lambda} P^y(A_\lambda > \lambda^{1/q} z) = \sup_{\lambda} P^0(A_\lambda > \lambda^{1/q} z) = P^0(A_1 > z)$$

so condition (ii) in (4.5) will hold provided $A_1 < \infty$. But

$$A_1 = \sup_x \sup_{0 \leqq t \leqq 1} |D^\gamma(L_t^\cdot)(x)|$$

which is finite by (2.20) and (2.12). Thus (4.5) applies and the proposition follows. ■

(4.9) PROPOSITION. – *$P^m(|H_t|^p) < \infty$ if $p > 1$, and even if $p = 1$ when $\gamma > 0$.*

Proof. – Recall that $H_t = H_t^0$, and note that if $\|.\|_p$ denotes the norm in $L^p(\mathbb{R})$ then

$$P^m(|H_t^0|^p) = \int dx\, P^x(|H_t^0|^p) = \int dx\, P^0(|H_t^x|^p) = P^0(\|H_t^\cdot\|_p^p).$$

Once again we appeal to Lemma (4.5) with $q = \alpha/(\alpha - 1 - \gamma)$ as before, but now

$$A_t = \sup_{0 \leqq s \leqq t} \|H_s^\cdot\|_p.$$

Clearly (A_t) satisfies (i) of (4.5), and owing to the translation invariance of the $L^p(\mathbb{R})$ norm, the P^y-distribution of (A_t) does not depend on y. So condition (ii) of (4.5) will follow by scaling provided $A_1 < \infty$. But this follows since $x \mapsto H_t^x$ is continuous and $O(|x|^{-1-\gamma})$ as $|x| \to +\infty$ uniformly in $t \in [0, 1]$, by (2.20) and (2.12). ■

Remark. – Using (4.4) one easily finds that

$$P^m(H_t^2) = t^{(2\alpha - 1 - 2\gamma)/\alpha} P^m(H_1^2).$$

But $\gamma < (\alpha - 1)/2$ and so $(2\alpha - 1 - 2\gamma)/\alpha > 1$. Therefore (4.9) implies that H_t has zero energy; *i.e.* $t^{-1} P^m(H_t^2) \to 0$. The functionals $H_\cdot^0(0\pm)$ are not covered by (4.4) and (4.9). However, a direct computation shows that $P^m(H_1^0(0\pm)^2) < \infty$, and since

$$P^m(H_t^0(0\pm)^2) = t^{2-1/\alpha} P^m([H_1^0(0\pm) + \alpha^{-1}(\log t) L_1^0]^2),$$

it follows that the functionals $H_\cdot^0(0\pm)$ have zero energy also.

Recall from (2.4) that $p(t, x) \leqq B t^{-1/\alpha}$ for $t > 0$ and $x \in \mathbb{R}$. The constant B appears in the next result.

(4.10) PROPOSITION. – *Let $F \geqq 0$ be $\mathscr{F}_1$ measurable and such that $P^m(F^k) < \infty$ for each $k \in \mathbb{N}$. Let $\beta = 1 - 1/\alpha$. Define*

$$\Phi_n = n^{-\beta} \sum_{j=1}^{n} F \circ \theta_j; \qquad n \in \mathbb{N}.$$

Then $P^0(\Phi_n^k) < \infty$ *for* $n, k \in \mathbb{N}$ *and*

$$\limsup_{n \to \infty} P^0(\Phi_n^k) \leqq k!\,[BP^m(F)\,\Gamma(\beta)]^k/\Gamma(k\beta+1) \tag{4.11}$$

for $k \in \mathbb{N}$.

Proof. – The proof goes by induction on k. First note that

$$P^x(F \circ \theta_j) = \int p(j, y-x)\,P^y(F)\,dy \leqq B j^{-1/\alpha} P^m(F)$$

for $j \geqq 1$ and $x \in \mathbb{R}$. Suppose $k=1$. Then

$$P^0(\Phi_n) = n^{-\beta} \sum_{j=1}^{n} P^0(F \circ \theta_j) \leqq B n^{-\beta} P^m(F) \sum_{j=1}^{n} j^{-1/\alpha}$$

and since $\sum_{j=1}^{n} j^{-1/\alpha} \sim n^\beta/\beta$ as $n \to +\infty$, this establishes (4.11) when $k=1$.

Suppose (4.11) is valid for $1 \leqq k < K$ and all F satisfying the hypotheses in (4.10). Then

$$\begin{aligned} P^0(\Phi_n^K) &= n^{-K\beta} \sum P^0(F \circ \theta_{j_1} \dots F \circ \theta_{j_K}) \\ &= K!\, n^{-K\beta} \sum_{1 \leqq j_1 < \dots < j_K \leqq n} P^0(F \circ \theta_{j_1} \dots F \circ \theta_{j_K}) + n^{-K\beta} \sum_{l=1}^{K-1} A(l) \end{aligned} \tag{4.12}$$

where the sum after the first equality is taken over all K-tuples $(j_1, \dots, j_K)$ from $\{1, 2, \dots, n\}$ and $A(l)$ is the sum over all such K-tuples with exactly l distinct elements. Let $G = \sup_{1 \leqq k \leqq K} F^k$. Then $P^m(G) < \infty$ and

$$A(l) \leqq C(l, K) \sum_{1 \leqq j_1 < \dots < j_l \leqq n} P^0(G \circ \theta_{j_1} \dots G \circ \theta_{j_l}),$$

where $C(l, K)$ is the number of K-tuples of l distinct elements such that each element appears at least once in each K-tuple. By the induction hypothesis $n^{-\beta l} A(l)$ is bounded and hence $(l < K)$

$$n^{-K\beta} \sum_{l=1}^{K-1} A(l) \to 0 \quad \text{as } n \to +\infty. \tag{4.13}$$

If $j_1 < j_2 < \ldots < j_K$, then because F is $\mathscr{F}_1$ measurable

$$\begin{aligned} P^0(F\circ\theta_{j_1}\ldots F\circ\theta_{j_K}) &= P^0(F\circ\theta_{j_1}\ldots F\circ\theta_{j_{K-1}} P^{X(j_{K-1})}(F\circ\theta_{j_K - j_{K-1}})) \\ &\leqq B(j_K - j_{K-1})^{-1/\alpha} P^m(F)\, P^0(F\circ\theta_{j_1}\ldots F\circ\theta_{j_{K-1}}) \\ &\vdots \\ &\leqq [BP^m(F)]^K [j_1(j_2-j_1)\ldots(j_K-j_{K-1})]^{-1/\alpha}. \end{aligned}$$

But one may readily check that as $n\to\infty$

$$\begin{aligned} &\sum_{1\leqq j_1<\ldots<j_K\leqq n} [j_1(j_2-j_1)\ldots(j_{K-1}-j_K)]^{-1/\alpha} \\ &\sim \int\ldots\int_{0<x_1<\ldots<x_K\leqq n} [x_1(x_2-x_1)\ldots(x_K-x_{K-1})]^{-1/\alpha}\, dx_1\ldots dx_K \\ &= n^{K\beta}\int\ldots\int_{0<t_1<\ldots<t_K\leqq 1} [t_1(t_2-t_1)\ldots(t_K-t_{K-1})]^{-1/\alpha}\, dt_1\ldots dt_K \\ &= n^{K\beta}(\Gamma(\beta))^K/\Gamma(K\beta+1). \end{aligned}$$

Combining this with (4.12) and (4.13) completes the induction, establishing (4.10). ■

Remarks. – A more careful analysis actually shows that

$$\lim_{n\to\infty} P^0(\Phi_n^k) = k!\,[p(1,0)\,P^m(F)\,\Gamma(\beta)]^k/\Gamma(k\beta+1).$$

In fact we may regard $((\theta_j)_{j\geqq 1}, P^0)$ as a Markov chain with state space $(\Omega, \mathscr{F}_1)$, and then the above limit is just the moment calculation necessary to prove the discrete version of the Darling-Kac theorem. However, we do not impose on F the strong restriction (A′) on page 452 of [DK57]. Of course we are dealing with a chain over X, for which a scaling relationship is valid.

Proof of (4.3) (*a*). – Fix $p>p_0$. It suffices to consider one fixed $t>0$, say $t=1$. We shall prove that

$$\sum_{n=1}^{\infty} P^0([V_n^p(1)]^k) < \infty, \qquad \forall k\in\mathbb{N}. \tag{4.14}$$

This is more than enough to yield both the a.s. and the L^r convergence of $V_n^p(1)$ to 0. To see (4.14) note that under P^0 we have, by (4.4) for each $n\in\mathbb{N}$

$$V_n^p(1) \overset{d}{=} (2^{-n})^{p(1-(1+\gamma)/\alpha)} \sum_{j=0}^{2^n-1} |H_{j+1}-H_j|^p = (2^{-n})^{\delta}(2^{-n})^{1-1/\alpha}\left[F + \sum_{j=1}^{2^n-1} F\circ\theta_j\right]$$

where $\delta=(p-p_0)\,(1-(1+\gamma)/\alpha)>0$ and $F=|H_1|^p$. Now $P^0(F^k)<\infty$ and $P^m(F^k)<\infty$ for all $k\geqq 1$ by (4.7) and (4.9) respectively. Because $F\in\mathscr{F}_1$

we can apply (4.10) to obtain

$$\limsup_{n \to \infty} \mathrm{P}^0([\mathrm{V}_n^p(1)]^k)/(2^{-n})^{\delta k} < \infty$$

which easily implies (4.14). ∎

For the proofs of parts (*b*) and (*c*) of Theorem (4.3) we require the Chacon-Ornstein ergodic theorem in the following form. The shift operator θ_1 is a measure preserving transformation of the σ-finite measure space $(\Omega, \mathscr{F}^0, \mathrm{P}^m)$. According to the Chacon-Ornstein theorem if F, $\mathrm{G} \in \mathrm{L}^1(\mathrm{P}^m)$ with $\mathrm{G} > 0$, then, as $n \to +\infty$,

$$(4.15) \qquad \sum_{j=1}^{n} \mathrm{F}(\theta_j) \Big/ \sum_{j=1}^{n} \mathrm{G}(\theta_j) \to \frac{\mathrm{P}^m(\mathrm{F})}{\mathrm{P}^m(\mathrm{G})} \quad \text{a.s. } \mathrm{P}^m.$$

Acutally, (4.15) obtains provided θ_1 and $(\Omega, \mathscr{F}^0, \mathrm{P}^m)$ satisfy the following two conditions (*cf.* [Re75], p. 112, 118):

(4.16) If $\mathrm{Y} \in \mathrm{L}^1(\mathrm{P}^m)$ is positive and $\Gamma := \{\sum_{n \geq 1} \mathrm{Y} \circ \theta_n = \infty\}$, then either

$\mathrm{P}^m(\Gamma) = 0$ or $\mathrm{P}^m(\Omega \setminus \Gamma) = 0$.

(4.17) There is at least one positive $\mathrm{Y} \in \mathrm{L}^1(\mathrm{P}^m)$ such that

$$\mathrm{P}^m\Big(\sum_{n \geq 1} \mathrm{Y} \circ \theta_n < \infty\Big) = 0.$$

Now since X has independent increments, (4.16) is an easy consequence of Kolmogorov's 0-1 law. As for (4.17) let $g \in \mathrm{L}^1(\mathbb{R})$ be a bounded positive function with compact support and $m(g) > 0$. Then by the discrete time form of the Darling-Kac theorem [DK57], for each $x \in \mathbb{R}$ we have

$$(4.18) \qquad \mathrm{P}^x\Big(n^{-(1-1/\alpha)} \sum_{j=1}^{n} g(\mathrm{X}_0) \circ \theta_j \leq z\Big) \to \mathrm{M}_{1-1/\alpha}(az), \qquad n \to +\infty,$$

where $0 < a < \infty$ is a constant and $\mathrm{M}_{1-1/\alpha}$ is the Mittag-Leffler distribution with parameter $1 - 1/\alpha$. But $\mathrm{M}_{1-1/\alpha}$ is concentrated on $]0, \infty[$ and $n^{-(1-1/\alpha)} \to 0$ as $n \to +\infty$, so (4.17) holds for $\mathrm{Y} = g(\mathrm{X}_0)$.

Proof of (4.3) (*c*). — To prove the convergence in probability assertion it suffices to show

$$(4.19) \qquad \mathrm{V}_n^{p_0}(t) \xrightarrow{\mathrm{P}^0} b\,\mathrm{L}_t^0, \qquad n \to +\infty, \quad \forall t \in \mathrm{D}$$

where $\mathrm{D} = \{k 2^{-m} : k, m \in \mathbb{N}\}$ denotes the set of dyadic rationals. In fact, if (4.19) holds and if $\mathrm{N}_1 \subset \mathbb{N}$ is an infinite sequence then by the Cantor diagonal procedure we can find an infinite sequence $\mathrm{N}_2 \subset \mathrm{N}_1$ such that

$$(4.20) \qquad \lim_{n \in \mathrm{N}_2} \mathrm{V}_n^{p_0}(t) = b\,\mathrm{L}_t^0, \qquad \forall t \in \mathrm{D}, \quad \text{a.s. } \mathrm{P}^0.$$

Since D is dense in $[0, \infty[$ and since both $V_n^{p_0}(t)$ and L_t^0 are continuous increasing functions of t, the qualifier "$\forall t \in D$" in (4.20) can be replaced by "$\forall t \geqq 0$". But if a sequence of increasing functions converges pointwise to a continuous (increasing) function then the convergence occurs uniformly on compacts. Thus (4.20) implies

$$\lim_{n \in N_2} \sup_{0 \leqq t \leqq T} |V_n^{p_0}(t) - b L_t^0| = 0, \qquad \forall T > 0, \quad \text{a.s. } P^0,$$

and this yields the first assertion in (4.3) (c) since N_1 was arbitrary.

By scaling it is enough to prove (4.19) for $t=1$. As a special case of (4.4), if $n \in \mathbb{N}$ then

$$(H_{j2^{-n}}, L_{k2^{-n}}^0 : j \geqq 1, k \geqq 1; P^0) \stackrel{d}{=} (c_n H_j, d_n L_k^0 : j \geqq 1, k \geqq 1; P^0),$$

where $c_n = (2^{-n})^{1-(1+\gamma)/\alpha}$, $d_n = (2^{-n})^{1-1/\alpha}$. Consequently

$$(4.21) \quad (V_n^{p_0}(1), L_1^0; P^0) \stackrel{d}{=} \left(c_n^{p_0} \sum_{j=0}^{2^n-1} |H_{j+1} - H_j|^{p_0}, d_n L_{2^n}^0; P^0 \right)$$
$$= \left(d_n \sum_{j=0}^{2^n-1} |H_1|^{p_0} \circ \theta_j, d_n \sum_{j=0}^{2^n-1} L_1^0 \circ \theta_j, n \geqq 1; P^0 \right).$$

Now fix a bounded positive function $g \in L^1(\mathbb{R})$ with $m(g) > 0$, and note that $b = P^m(|H_1|^{p_0}) < \infty$ by (4.9) since $p_0 > 1$. We claim that as $n \to +\infty$,

$$(4.22) \quad \sum_{j=0}^{2^n-1} (|H_1|^{p_0} - b L_1^0) \circ \theta_j \Big/ \sum_{j=0}^{2^n-1} g(X_0) \circ \theta_j \to 0, \quad \text{a.s. } P^0.$$

Indeed $P^m(L_1^0) = 1$, so $P^m(|H_1|^{p_0} - b L_1^0) = 0$ hence the convergence in (4.22) occurs a.s. P^m by the ergodic theorem (4.15). Moreover the proof of (4.17) shows that $\sum_{j=0}^{\infty} g(X_j) = +\infty$ a.s. P^x for all x. Let us write Γ for the ω-set on which either the convergence asserted in (4.22) fails or $\sum_{j=0}^{\infty} g(X_j) < \infty$. Clearly $\theta_1^{-1} \Gamma = \Gamma$ and $P^m(\Gamma) = 0$, so

$$P^0(\Gamma) = P^0(\theta_1^{-1} \Gamma) = \int p(1, y) P^y(\Gamma) \, dy = 0,$$

proving (4.22). On the other hand we know from (4.18) that the P^0-law of $d_n \sum_{j=0}^{2^n-1} g(X_0) \circ \theta_j$ converges weakly to a distribution concentrated on $]0, \infty[$. By Slutsky's theorem [Du91], (4.22) therefore implies that as

$n \to +\infty$,

$$(4.23) \qquad d_n \sum_{j=0}^{2^n-1} (|H_1|^{p_0} - b L_1^0) \circ \theta_j \xrightarrow{d} 0$$

under P^0. In view of (4.21), (4.23) yields (4.19) for $t=1$.

It remains to show that $V_n^{p_0}(t) \to b L_t^0$ in $L^r(P^0)$ as $n \to +\infty$ for each r with $1 \leqq r < \infty$. As before, it suffices to prove this for $t=1$. Since the convergence in P^0-probability of $V_n^{p_0}(1)$ to $b L_1^0$ has already been established, we need only show that for each $r \geqq 1$ the random variables $(V_n^{p_0}(1))^r$, $n \in \mathbb{N}$, are uniformly integrable under P^0. But this follows immediately from (4.10) since by (4.4), under P^0

$$V_n^{p_0}(1) \overset{d}{=} (2^{-n})^{1-1/\alpha} \left[|H_1|^{p_0} + \sum_{j=1}^{2^n-1} |H_1|^{p_0} \circ \theta_j \right],$$

and since $P^0(|H_1|^{kp_0}) + P^m(|H_1|^{kp_0}) < \infty$ for all $k \in \mathbb{N}$ by (4.7) and (4.9). ■

Proof of (4.3) (*b*). – Because of the discussion at the end of section 2, we need only consider the case $\gamma > 0$. Fix $t > 0$. By (4.3) (*c*) if $N_1 \subset \mathbb{N}$ is any subsequence then there is a subsequence $N_2 \subset N_1$ such that

$$\lim_{n \in N_2} V_n^{p_0}(t) = b L_t^0 > 0 \quad \text{a.s. } P^0.$$

But then a simple real variable argument shows that

$$\lim_{n \in N_2} V_n^p(t) = +\infty \quad \text{a.s. } P^0$$

if $0 < p < p_0$. It follows that $V_n^p(t) \to +\infty$ in P^0-probability. ■

Remark. – The proof of (4.3) (*c*) allows us to close a gap left open in the statement of (4.9). Namely, $P^m(|H_1|^{p_0}) = \infty$ if $\gamma = 0$ ($p_0 = 1$ if $\gamma = 0$). For if $P^m(|H_1|^{p_0})$ were finite for $\gamma = 0$, then the argument used to prove (4.3) (*c*) would yield the convergence in P^0-probability of $(V_n^1(1))_{n \geqq 1}$ to a finite limit, and this would violate the fact that $H_\bullet^0(0)$ has infinite 1-variation, as noted at the end of section 2.

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