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# Domains of analytic continuation for the top Lyapunov exponent

by

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ABSTRACT. – For a product of random positive matrices with Markovian dependence, we show the top Lyapunov exponent depends real-analytically on the transition probabilities (under an ergodicity assumption) and determine explicit domains of analytic continuation.

*Key words :* Random matrices, Lyapunov exponent, analytic continuation.

RÉSUMÉ. – On démontre que l'exposant caractéristique maximal d'un produit de matrices positives aléatoires en dépendance Markovienne est une fonction analytique-réelle des probabilités de transition, et on donne explicitement des domaines de prolongement analytique.

## 1. INTRODUCTION

Consider independent identically distributed random variables $X_1$, $X_2, \ldots$ taking finitely many values $\{A_1, \ldots, A_b\}$ in the space of $d \times d$ real matrices. We are concerned with the regularity of the top Lyapunov

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exponent

$$\gamma_1 = \lim_{n \to \infty} \frac{1}{n} \mathrm{E} \log \| X_n X_{n-1} \ldots X_1 \| \tag{1}$$

(where E denotes expectation and $\|.\|$ is a norm on the space of matrices) as a function of the probability parameters

$$p_j = \mathrm{Prob}\,[X_1 = A_j] \qquad (1 \leqq j \leqq b). \tag{2}$$

In [P] it is shown that when $A_1, \ldots, A_b$ are nonsingular matrices and the top Lyapunov exponent is *simple* (*i.e.* $\gamma_1 > \gamma_2$; *see* § 4), then $\gamma_1$ depends real-analytically on $p_1, \ldots, p_b$. Our object here is to obtain explicit domains of analytic continuation for $\gamma_1$.

Such domains provide lower estimates for convergence radii of perturbation series; their usefulness is explained in Kato's book [Ka], section II.3.

This we are able to do only when the matrices $\{A_j\}$ have non-negative entries, using Birkhoff's contraction coefficient $\tau(A)$ which is defined in Section 2.

DEFINITION. — A non-negative matrix is called *row-allowable* if it has no all-zero rows.

Note from Section 2 that $0 \leqq \tau(A) \leqq 1$ for any row-allowable square matrix A, that $\tau(A) < 1$ when A is strictly positive and, most importantly, that $\tau(A)$ may be readily calculated from the entries of A. Our main result is an extension to Markovian random products of the following.

THEOREM 1. — *Let* $F = \{A_1, \ldots, A_b\}$ *be a set of row-allowable non-negative matrices such that at least one* $A_l$ *has strictly positive entries.*

*Let* $\{X_n\}$ *be i.i.d.* F*-valued random variables. Then there is a relatively open domain* $\Omega$ *in the complex hyperplane*

$$H = \left\{ z \in \mathbb{C}^b \;\middle|\; \sum_{j=1}^{b} z_j = 1 \right\}, \tag{3}$$

*defined by*

$$\Omega(F) = \left\{ z \in H \;\middle|\; \sum_{j=1}^{b} \tau(A_j) |z_j| < 1 \right\}, \tag{4}$$

*such that*

(i) $\Omega(F)$ *contains all positive probability vectors* $(p_1, \ldots, p_b)$.

(ii) *The top Lyapunov exponent* $\gamma_1$ *defined in* (1), *as a function of the probability vector* $(p_1, \ldots, p_b)$ *defined in* (2), *may be extended to a complex-analytic function on* $\Omega(F)$.

*Remark.* — Instead of some $A \in F$ being strictly positive, we really need only the sightly weaker condition $\tau(A) < 1$. To obtain *some* domain of analytic continuation, it actually suffices that $\tau(A)<1$ for some matrix A in the semigroup generated by F, as one can replace F by all products of a certain length $r$ thereof, and consider the random product in (1) in blocks of length $r$. The probability parameters for this new random product will be homogenous polynomials of degree $r$ in the original probabilities.

The rest of the paper is organized as follows. Section 2 begins with preliminary material on Hilbert's projective metric. Then we relate Theorem 1 to previous work, most notably that of Ruelle [R], and sketch its proof.

In section 3 random matrix products with Markovian dependence are considered, and an extension of Theorem 1 to that setting is established.

Some methods for estimating Lyapunov exponents are discussed in section 4. Birkhoff's contraction coefficient was already used for this purpose by Wojtkowski [Wojt]. The paper ends with some unsolved problems.

This study was motivated by the application of random matrix products to computing dimensions of intersections of Cantor sets, presented in [KP].

*Remark.* – After the results of the present paper were obtained, I received from P. Bougerol a copy of H. Hennion's preprint [Hen], in which similar methods are used to establish differentiability properties of the top Lyapunov exponent for a random product of i. i. d. non-negative matrices, as a function of the matrix entries. In the same paper, Hennion proves the most general stability result for the top Lyapunov exponent obtained to date, in the context of i. i. d. non-negative matrices. A special case of his result is the fact, that with the assumptions and notation of Theorem 1, $\gamma_1$ depends *continuously* on the (positive) probabilities $p_1, \ldots, p_b$.

## 2. BACKGROUND AND A PROOF OF THEOREM 1

For any vector $x \in \mathbb{R}^d$, denote by $\bar{x}$ the direction it determines in the projective space $\mathbb{P}(\mathbb{R}^d)$. Let

$$\mathbb{P}_+(\mathbb{R}^d)=\{\bar{x}\,|\,x\in\mathbb{R}^d,\ x_j>0 \text{ for } 1\leqq j\leqq b\}.$$

*Hilbert's projective metric* is the metric $h(.,.)$ on $\mathbb{P}_+(\mathbb{R}^d)$ defined by

$$h(\bar{x},\bar{y})=\log \max_{1\leqq i,\,j\leqq d} \frac{x_i y_j}{x_j y_i}. \tag{5}$$

Any row-allowable non-negative $d\times d$ matrix A acts naturally on $\mathbb{P}_+(\mathbb{R}^d)$ by

$$A.\bar{x}=\overline{Ax}. \tag{6}$$

The main result of [B] is that when A is strictly positive, it acts as a strict contradiction on $\mathbb{P}_+(\mathbb{R}^d)$. More precisely, define for a row-allowable $d\times d$ matrix $A=(a_{ij})$ the *Birkhoff contraction coefficient*

$$\tau(A)=\sup\left\{\frac{h(A.\bar{x},A.\bar{y})}{h(\bar{x},\bar{y})}\,\middle|\,\bar{x},\bar{y}\in\mathbb{P}_+(\mathbb{R}^d),\bar{x}\neq\bar{y}\right\}. \tag{7}$$

Birkhoff [B] obtained an explicit formula for $\tau$:

$$\tau(A)=[1-\psi(A)^{1/2}]/[1+\psi(A)^{1/2}], \tag{8}$$

where if each column of A is either zero-free or all-zero, then

$$\psi(A)=\min_{i,j,k,l}\frac{a_{ik}\,a_{jl}}{a_{il}\,a_{jk}}$$

(with $k$, $l$ restricted to the zero-free columns, and $1\leq i,j\leq d$) and otherwise $\psi(A)=0$. Recall that we assume A is row-allowable throughout. Another proof of (8) if given in [S], section 3.4.

We now wish to recall the main results of Ruelle [R] and relate them to the present paper. The top Lyapunov exponent $\gamma_1$ is defined in general as follows. Let $\langle Y,\beta,\mu\rangle$ be a probability space, $\sigma:Y\to Y$ a measure preserving map, and M a measurable function from Y to the space of $d\times d$ matrices, with $\int\log^+\|M(y)\|\,d\mu(y)<\infty$. Let

$$\gamma_1(M,\mu)=\lim_{n\to\infty}\frac{1}{n}\int_Y\log\|M(\sigma^{n-1}y)\dots M(\sigma y).M(y)\|\,d\mu(y). \tag{9}$$

Ruelle studies the dependence of $\gamma_1(M,\mu)$ on M and shows, in particular, that if Y is a compact metric space, $\sigma$ and $M_0$ are continuous, and $M_0$ maps Y to the set of *strictly* positive matrices, then $\gamma_1(.,\mu)$ is real analytic in a neighborhood of $M_0$. Actually, the positivity assumption is expressed in [R] in a more invariant form, by requiring that all matrices in the image of $M_0$, map some closed cone in $\mathbb{R}^d$ to its interior.

Without the positivity condition, Le-Page [LP2] has established Hölder continuity for $\gamma_1$ as a function of the *matrix entries* in the i.i.d. case, under rather general assumptions. We are interested in the dependence of $\gamma_1(M,\mu)$ on the *measure* $\mu$, when it is restricted to be a product measure, or more generally, a Markov measure. Let us note that for general matrices, even continuity results are difficult to establish (*see* the paper by Slud in [CKN1] and the references therein). One should always bear in mind the example in [Kif]: if the two matrices $A_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $A_2=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$, taken with probabilities $p_1$ and $p_2=1-p_1$ respectively, are used to define an i.i.d. random product, then $\gamma_1 = 0$ if $0 < p_1 \leqq 1$ but $\gamma_1 = \log 2$ for $p_1 = 0$. Thus the restriction in Theorem 1 to positive probability vectors is necessary. However, if *all* matrices $A_1, \ldots, A_b$ are assumed to be strictly positive, the domain of analytic continuation $\Omega(F)$ in Theorem 1 contains the closed simplex of *all* probability vectors $(p_1, \ldots, p_b)$.

As noted in the remark after Theorem 1, the assumption made there that some $A_i$ is strictly positive may be weakened. However, it cannot be discarded completely. Indeed consider, as in [P], $F = \{A_1, A_2\}$ where $A_j = \begin{pmatrix} a_j & 0 \\ c_j & 1 \end{pmatrix}$ is assigned probability $p_j$, and $a_j$, $c_j > 0$. One finds that

$$\gamma_1 = \max\{p_1 \log a_1 + p_2 \log a_2, 0\}$$

which has a point of non-differentiability as a function of $0 < p_1 < 0$ if

$$a_1 > 1 > a_2.$$

Theorem 1 is considerably easier to prove than its extension discussed in the next section. Though not logically necessary, sketching the proof at this junction might be helpful.

*Proof of theorem* 1 (*sketch*). – Defining $\Omega(F)$ by (4), we only need to verify assertion (ii) of the theorem.

*Step* 1. – For $z = (z_1, \ldots, z_b) \in \Omega(F)$ and any Lipshitz function

$$f: \ \mathbb{P}_+(\mathbb{R}^d) \to \mathbb{C}$$

define

$$(T_z f)(\bar{x}) = \sum_{j=1}^{b} z_j f(A_j . \bar{x}) \qquad \text{for} \quad \bar{x} \in \mathbb{P}_+(\mathbb{R}^d),$$

and check the effect of $T_z$ on the modulus of continuity. If for all $\bar{x}$, $\bar{y} \in \mathbb{P}_+(\mathbb{R}^d)$ we have $|f(\bar{x}) - f(\bar{y})| \leqq m h(\bar{x}, \bar{y})$, then also

$$|(T_z f)(\bar{x}) - (T_z f)(\bar{y})| \leqq \sum_{j=1}^{b} |z_j| . |f(A_j . \bar{x}) - f(A_j . \bar{y})| \leqq m \sum_{j=1}^{b} |z_j| \tau(A_j) h(\bar{x}, \bar{y}).$$

*Step* 2. – For $K \subset \Omega(F)$ compact, let

$$\theta = \max_{z \in K} \sum_{j=1}^{b} |z_j| \tau(A_j) < 1.$$

If $f$ is a Lipshitz function as above, then

$$z \in K \quad \Rightarrow \quad |(T_z^n f)(\bar{x}) - (T_z^n f)(\bar{y})| \leqq \theta^n m h(\bar{x}, \bar{y}).$$

Infer from this and the identity

$$(T_z^{n+1} f)(\bar{x}) - (T_z^n f)\bar{x} = \sum_{j=1}^{b} z_j [(T_z^n f)(A_j . \bar{x}) - (T_z^n f)(\bar{x})]$$

that for each $\bar{x} \in \mathbb{P}_+(\mathbb{R}^d)$, the sequence $(T_z^n f)(\bar{x})$ converges uniformly on compact subsets of $\Omega(F)$. The limit $T_z^\infty f$ does not depend on $\bar{x}$ and is an analytic function of $z \in \Omega(F)$ by Lemma 11 in page 11 of [GRo].

*Step* 3. – Apply step 2 to $f_j(\bar{x}) = \log \dfrac{\| A_j x \|}{\| x \|}$. The function

$$\Gamma_\infty(z) = \sum_{j=1}^{b} z_j T_z^\infty f_j$$

is analytic in $\Omega(F)$, and for a positive probability vector $p = (p_1, \ldots, p_b)$, $\Gamma_\infty(p)$ coincides with the top exponent $\gamma_1$ defined by (1) and (2). This is most easily verified by considering Cesáro averages of the sequence

$$\Gamma_n(p) = \sum_{j=1}^{b} p_j (T_p^n f_j)(\bar{x}), \tag{10}$$

for $n \geqq 1$ and fixed $\bar{x} \in \mathbb{P}_+(\mathbb{R}^d)$. ■

## 3. MARKOVIAN RANDOM MATRIX PRODUCTS

In order to extend Theorem 1 to the case in which the factors $X_n$ in the random matrix product (1) are no longer i. i. d., but instead form a matrix-valued, homogenous Markov chain, we need to fix the allowed transitions. Let $U = (u(i,j))_{i,j=1}^b$ be a zero-one matrix which is *irreducible*, *i.e.*, the only nonempty subset $\Lambda$ of $\{1, \ldots, b\}$ satisfying $i \in \Lambda$, $u(i,j) = 1 \Rightarrow j \in \Lambda$, is $\Lambda = \{1, \ldots, b\}$ (*see* [S], section 1.3).

Denote by $S(U)$ the set of $b \times b$ stochastic matrices with the same support as $U$:

$$S(U) = \left\{ P = (p_{ij})_{i,j=1}^b \,\middle|\, p_{ij} \geqq 0, \sum_{j=1}^{b} p_{ij} = 1, p_{ij} > 0 \Leftrightarrow u(i,j) = 1 \right\}. \tag{11}$$

Let $F = \{A_1, \ldots, A_b\}$ be a fixed set of row-allowable non-negative $d \times d$ matrices. For each $P \in S(U)$ denote

$$\gamma_1(P) = \lim_{n \to \infty} \frac{1}{n} \log E \| X_n \ldots X_1 \|, \tag{12}$$

where $\{X_n\}$ is the F-valued stationary Markov process with transition probabilities

$$\operatorname{Prob}[X_{n+1}=A_j \mid X_n=A_i]=p_{ij} \qquad (1 \leqq i, j \leqq b) \tag{13}$$

and initial probabilities

$$\operatorname{Prob}[X_1=A_i]=\pi_P(i) \qquad (1 \leqq i \leqq b). \tag{14}$$

Here $\pi_P$ is the unique probability vector satisfying

$$\pi_P^t P=\pi_P^t. \tag{15}$$

(Throughout the paper a superscript $t$ denotes "transpose".)

THEOREM 2. — *With the notation above, assume that*

(i) U *is irreducible.*

(ii) *For some* $1 \leqq l \leqq b$, *every column of* $A_l$ *is either zero-free or all-zero.*

*Then there is a relatively open subset* $\Omega_1(F, U)$ *of the complex affine space of matrices,*

$$H(U)=\left\{Z=(z_{ij}) \in \mathbb{C}^{b^2} \,\middle|\, \sum_{j=1}^{b} z_{ij}=1 \text{ for } 1 \leqq i \leqq b,\ u(i,j)=0 \Rightarrow z_{ij}=0\right\} \tag{16}$$

*with the properties*

(I) $\Omega_1(F, U)$ *contains* $S(U)$.

(II) *There is a complex-analytic function* $\Gamma_\infty: \Omega_1(F, U) \to \mathbb{C}$, *such that for every* $P \in S(U)$

$$\Gamma_\infty(P)=\gamma_1(P), \tag{17}$$

*where the right hand side was defined in* (12).

(III) $\Omega_1(F, U)$ *is the set of all* $b \times b$ *matrices* $Z=(z_{ij})$ *in* $H(U)$ *such that* 1 *is a simple eigenvalue of* Z *with all other eigenvalues strictly smaller in absolute value, which also satisfy*

$$\rho\left(|z_{ij}|\, \tau(A_j)\right)_{i,j=1}^{b}<1,$$

*where* $\rho(.)$ *denotes spectral radius.*

*Remarks.* — 1. Condition (ii) of the theorem is equivalent to $\tau(A_l)<1$. To obtain real-analyticity of $P \to \gamma_1(P)$ in $S(U)$, it suffices to assume instead that for the set $F_r$ composed of all products of length $r$,

$$A_{l_r} \ldots A_{l_2} A_{l_1} \qquad \text{with} \quad u(l_i, l_{i+1})=1 \text{ for } 1 \leqq i<r,$$

some $A \in F_r$ satisfies $\tau(A)<1$. Indeed, the blocks of length $r$,

$$X_{kr+r} \ldots X_{kr+2} X_{kr+1} \qquad (k \geqq 0)$$

in the product (12) constitute an $F_r$-valued irreducible Markov chain with transition probabilities which are polynomial functions in the $p_{ij}$, so applying the theorem to this new random product gives the desired result.

2. The same device of blocking allows us to pass from any finite-memory homogenous Markov chain to a chain with memory 1.

3. Theorem 1 is the special case of Theorem 2 obtained by taking $u(i,j)=1$ and $p_{ij}=p_j$ for all $i$, $j$. Indeed the only (possibly) nonzero eigenvalue of a matrix with identical rows is the sum of the elements in a row, so taking $z_{ij}=z_j$ for all $1 \leqq i,j \leqq b$ maps the domain $\Omega(\mathrm{F})$ defined in (4) into the domain $\Omega_1(\mathrm{F}, \mathrm{U})$ defined in Theorem 2 (III).

4. Observe that unlike the domain $\Omega(\mathrm{F})$ of Theorem 1, $\Omega_1(\mathrm{F}, \mathrm{U})$ is usually *unbounded.*

5. The proof of Theorem 2 will be based on two lemmas, which we state in greater generality than is needed for the present application.

NOTATION AND PRELIMINARIES.

(i) Recall $\mathbb{P}_+(\mathbb{R}^d)$ denotes the positive portion of projective space. Fixing $d$, for $0<\alpha<1$ and integer $b \geqq 1$ we denote by $\mathrm{L}_\alpha^b$ the complex vector space of functions $f:\{1,2,\ldots,b\} \times \mathbb{P}_+(\mathbb{R}^d) \to \mathbb{C}$ which satisfy a Hölder condition with parameter $\alpha$ in the second variable, *i.e.*, all the quantities

$$m_i(f,\alpha)=\sup\left\{\frac{|f(i,\bar{x})-f(i,\bar{y})|}{h(\bar{x},\bar{y})^\alpha} \,\middle|\, \bar{x},\bar{y} \in \mathbb{P}_+(\mathbb{R}^d),\ \bar{x} \neq \bar{y}\right\} \tag{19}$$

for $1 \leqq i \leqq b$, are required to be finite [recall $h(.,.)$ is Hilbert's metric].

(ii) For the remainder of this section, we fix an arbitrary "reference point" $\bar{v} \in \mathbb{P}_+(\mathbb{R}^d)$. For $f \in \mathrm{L}_\alpha^b$ we define

$$\|f\|_\alpha = \max_{1 \leqq i \leqq b} m_i(f,\alpha) + \max_{1 \leqq i \leqq b} |f(i,\bar{v})|. \tag{20}$$

It is readily verified that $\mathrm{L}_\alpha^b$, equipped with the norm $\|.\|_\alpha$, is a Banach space.

(iii) Denote by $\tilde{\mathbb{C}}$ the subspace of $\mathrm{L}_\alpha^b$ consisting of constant functions, and by $\tilde{\mathbb{C}}^b$ the subspace of functions which depend only on the first coordinate:

$$\tilde{\mathbb{C}}^b=\{f \in \mathrm{L}_\alpha^b \mid \forall\, 1 \leqq i \leqq b,\ \forall\, \bar{x},\bar{y} \in \mathbb{P}_+(\mathbb{R}^d),\ f(i,\bar{x})=f(i,\bar{y})\}. \tag{21}$$

(iv) Recall that for any closed subspace W of a Banach space V, the quotient space V/W is also a Banach space with the norm

$$\|v+\mathrm{W}\| = \inf_{w \in \mathrm{W}} \|v+w\|$$

(*see*, for instance, [Ka], section 3.1.8). If T is a bounded linear operator on V for which $\mathrm{T}(\mathrm{W}) \subset \mathrm{W}$, we denote by [T; W] the restriction of T to W and by [T; V/W] the induced linear operator on the quotient space (the same notation is used for a square matrix acting by left multiplication on $\mathbb{C}^b$). The spectrum of T is the union of the spectrum of [T; W] and the spectrum of [T; V/W], when W is finite dimensional. Consequently, under the preceding assumptions,

$$\rho(\mathrm{T})=\max\{\rho([\mathrm{T};\mathrm{W}]),\rho([\mathrm{T};\mathrm{V}/\mathrm{W}])\} \tag{22}$$

where $\rho(.)$ denotes spectral radius.

(v) The quotient norms on $\mathrm{L}_\alpha^b/\tilde{\mathbb{C}}$ and $\mathrm{L}_\alpha^b/\tilde{\mathbb{C}}^b$ may be written more explicitly. Namely, for $f\in\mathrm{L}_\alpha^b$ we have

$$\|f+\tilde{\mathbb{C}}\|=\max_{1\leqq i\leqq b} m_i(f,\alpha)+\frac{1}{2}\max_{1\leqq i,\,j\leqq b}|f(i,\bar{v})-f(j,\bar{v})| \tag{23}$$

and

$$\|f+\tilde{\mathbb{C}}^b\|=\max_{1\leqq i\leqq b} m_i(f,\alpha). \tag{24}$$

LEMMA 3. – *Let* $\mathrm{F}=\{\mathrm{A}_1,\ldots,\mathrm{A}_b\}$ *be a set of row-allowable* $d\times d$ *matrices, with* $\tau(\mathrm{A}_l)<1$ *for some* $1\leqq l\leqq b$, *and let* U *be a* $b\times b$ *irreducible zero-one matrix. For each matrix* $\mathrm{Z}=(z_{ij})\in\mathrm{H}(\mathrm{U})$ [see (16)] *define the operator* $\mathrm{T}_\mathrm{Z}$ *on* $\mathrm{L}_\alpha^b$ (*where* $0<\alpha\leqq 1$) *by*

$$(\mathrm{T}_\mathrm{Z}f)(i,\bar{x})=\sum_{j=1}^{b} z_{ij}f(j,\mathrm{A}_j.\bar{x}). \tag{25}$$

*Then* $\mathrm{T}_\mathrm{Z}$ *preserves* $\tilde{\mathbb{C}}$, *and the spectral radius of the induced action,* $\rho[\mathrm{T}_\mathrm{Z};\mathrm{L}_\alpha^b/\tilde{\mathbb{C}}]$, *is at most*

$$\chi(\mathrm{Z},\alpha)=\max\{\rho(|z_{ij}|\tau(\mathrm{A}_j)^\alpha)_{i,j=1}^b,\rho([\mathrm{Z},\mathbb{C}^b/\Delta_b])\} \tag{26}$$

*where* $\Delta_b$ *is the diagonal* $\{(c,c,\ldots,c)^t\}$ *in* $\mathbb{C}^b$, *which is invariant under left multiplication by* Z.

*Proof.* – The $b$-dimensional space $\tilde{\mathbb{C}}^b$ is also $\mathrm{T}_\mathrm{Z}$-invariant. The canonical isomorphism between $\tilde{\mathbb{C}}^b$ and $\mathbb{C}^b$, transports $[\mathrm{T}_\mathrm{Z},\tilde{\mathbb{C}}^b]$ to left multiplication by the matrix Z, and takes $\tilde{\mathbb{C}}$ to the diagonal $\Delta_b$ of $\mathbb{C}^b$. Therefore,

$$\rho([\mathrm{T}_\mathrm{Z};\tilde{\mathbb{C}}^b/\tilde{\mathbb{C}}])=\rho([\mathrm{Z};\mathbb{C}^b/\Delta_b]). \tag{28}$$

It remains to estimate $\rho([\mathrm{T}_\mathrm{Z};\mathrm{L}_\alpha^b/\tilde{\mathbb{C}}^b])$. For $f\in\mathrm{L}_\alpha^b$, $1\leqq i\leqq b$ and $\bar{x},\bar{y}\in\mathbb{P}_+(\mathbb{R}^d)$, we have

$$\begin{aligned}|(\mathrm{T}_\mathrm{Z}f)(i,\bar{x})-(\mathrm{T}_\mathrm{Z}f)(i,\bar{y})|&=\left|\sum_{j=1}^{b} z_{ij}[f(j,\mathrm{A}_j.\bar{x})-f(j,\mathrm{A}_j.\bar{y})]\right|\\&\leqq\sum_{j=1}^{b}|z_{ij}|m_j(f,\alpha)h(\mathrm{A}_j.\bar{x},\mathrm{A}_j.\bar{y})^\alpha\\&\leqq\sum_{j=1}^{b}|z_{ij}|\tau(\mathrm{A}_j)^\alpha m_j(f,\alpha)h(\bar{x},\bar{y})^\alpha,\end{aligned} \tag{29}$$

where $m_j(f,\alpha)$ was defined in (19), $h(.,.)$ is Hilbert's metric, and we have used (7).

From (29) one sees that $T_Z f \in L^b_\alpha$, with

$$m_i(T_Z f, \alpha) \leqq \sum_{j=1}^{b} |z_{ij}| \tau(A_j)^\alpha m_j(f, \alpha). \tag{30}$$

Denote by Q the $b \times b$ non-negative matrix

$$Q = (|z_{ij}| \tau(A_j)^\alpha)^b_{i,j=1}, \tag{31}$$

and by $m(f, \alpha)$ the column vector

$$m(f, \alpha) = (m_1(f, \alpha), \ldots, m_b(f, \alpha))^t. \tag{32}$$

Then (30) may be rewritten as

$$m(T_Z f, \alpha) \leqq Q\, m(f, \alpha), \tag{33}$$

where the inequality should be interpreted coordinatewise. Iterating, we get

$$m(T^n_Z f, \alpha) \leqq Q^n m(f, \alpha). \tag{34}$$

Using the expression (24) for the quotient norm in $L^b_\alpha/\tilde{\mathbb{C}}^b$, (34) implies that

$$\| T^n_Z f + \tilde{\mathbb{C}}^b \| \leqq \| Q^n \|_\infty \| f + \tilde{\mathbb{C}}^b \|, \tag{35}$$

where by $\| Q^n \|_\infty$ we mean the maximal row-sum of $Q^n$ (which is the operator norm of $Q^n$ acting on $\mathbb{R}^d$ with the maximum norm). Therefore,

$$\rho[T_Z;\, L^b_\alpha/\tilde{\mathbb{C}}^b] \leqq \rho(Q). \tag{36}$$

Combining (28) and (36) and recalling (22), the asserted inequality

$$\rho([T_Z;\, L^b_\alpha/\tilde{\mathbb{C}}]) \leqq \chi(Z, \alpha)$$

follows. ■

LEMMA 4. — *Let* F, U *and* $T_Z$ *be as in the preceding lemma, and let* $0 < \alpha \leqq 1$. *Denote by* $\Omega_\alpha(F, U)$ *the set of matrices*

$$\Omega_\alpha(F, U) = \{ Z \in H(U) \mid \chi(Z, \alpha) < 1 \}, \tag{37}$$

*where* H(U) *was defined in* (16) *and* $\chi(Z, \alpha)$ *in* (26). *Then*

(i) *For every* $f \in L^b_\alpha$, $Z \in \Omega_\alpha(F, U)$ *and* $i \in \{1, 2, \ldots, b\}$ *the sequence* $(T^n_Z f)(i, \bar{v})$ *converges. The convergence is uniform on compact subsets of* $\Omega_\alpha(F, U)$ [*and therefore the limit is an analytic function of* $Z \in \Omega_\alpha(F, U)$]. *The limit does not depend on* $i \in \{1, \ldots, b\}$.

(ii) $\Omega_\alpha(F, U)$ *is a relatively open subset of* H(U), *and contains* S(U).

*Proof.* — (i) Fix a compact subset $\mathcal{K}$ of $\Omega_\alpha(F, U)$. Employing continuity of the spectral radius, we may find $\theta$ satisfying

$$\max_{Z \in \mathcal{K}} \chi(Z, \alpha) < \theta < 1. \tag{38}$$

For each $Z \in \mathcal{K}$, there exists $n \geqq 1$ with

$$\| [T^n_Z;\, L^b_\alpha/\tilde{\mathbb{C}}] \| < \theta^n. \tag{39}$$

By compactness, for some $n$ (39) is satisfied for *all* $Z \in \mathscr{K}$, and it follows that

$$\exists c>0, \quad \forall Z \in \mathscr{K}, \quad \forall n \geqq 1, \qquad \| [T_Z^n; L_\alpha^b/\tilde{\mathbb{C}}] \| \leqq c\,\theta^n. \tag{40}$$

It suffices to establish (i) under the assumption

$$\|f\|_\alpha < 1.$$

Then (24) and (40) imply that

$$\forall\, 1 \leqq j \leqq b, \quad m_j(T_Z^n f, \alpha) \leqq c\,\theta^n \tag{41}$$

and

$$\forall\, i, j \in \{1, \ldots, b\}, \quad |(T_Z^n f)(i, \bar{v}) - (T_Z^n f)(j, \bar{v})| \leqq 2\,c\,\theta^n. \tag{42}$$

Now fix $i \in \{1, 2, \ldots, b\}$ and observe that

$$(T_Z^{n+1} f)(i, \bar{v}) - (T_Z^n f)(i, \bar{v}) = \sum_{j=1}^{b} z_{ij} [(T_Z^n f)(j, A_j . \bar{v}) - (T_Z^n f)(i, \bar{v})]. \tag{43}$$

Let

$$c_1 = \max_{1 \leqq j \leqq b} h(A_j . \bar{v}, \bar{v})^\alpha,$$

and infer from (41) that for $1 \leqq j \leqq b$,

$$|(T_Z^n f)(j, \bar{v}) - (T_Z^n f)(j, A_j . \bar{v})| \leqq c_1\, c\,\theta^n.$$

In conjunction with (42) this yields

$$|(T_Z^n f)(j, A_j . \bar{v}) - (T_Z^n f)(i, \bar{v})| \leqq (c_1 + 2)\, c\,\theta^n. \tag{44}$$

Thus, denoting

$$c_3 = \max \left\{ \sum_{j=1}^{b} |z_{ij}| \;\middle|\; Z = (z_{ij}) \in \mathscr{K},\, 1 \leqq i \leqq b \right\}$$

we find from (43), (44) that for all $Z \in \mathscr{K}$,

$$|(T_Z^{n+1} f)(i, \bar{v}) - (T_Z^n f)(i, \bar{v})| \leqq c_3 (c_1 + 2)\, c\,\theta^n,$$

ensuring the required uniform convergence of $(T_Z^n f)(i, \bar{v})$. The inequality (42) implies the limit does not depend on $i$. By [GRo], lemma 11, p. 11, the limit is a holomorphic function of $Z$ in $\Omega_\alpha(F, U)$.

(ii) $\Omega_\alpha(F, U)$ is open, since the spectral radius of a matrix depends continuously on its entries. Let

$$P = (p_{ij})_{i, j=1}^{b} \in S(U).$$

Since $P$ is an irreducible stochastic matrix, the Perron-Frobenius theorem [S], theorem 1.5, asserts that 1 is a simple eigenvalue of $P$, with all other eigenvalues of modulus less than 1. Therefore, since $\Delta_b$ is the right eigenspace of P corresponding to the eigenvalue 1,

$$\rho([\mathrm{P};\ \mathbb{C}^b/\Delta_b]) < 1. \tag{45}$$

Recalling that $\tau(\mathrm{A}_l) < 1$, we observe that the inequality

$$p_{ij}\,\tau(\mathrm{A}_j)^\alpha \leqq p_{ij} \qquad (1 \leqq i, j \leqq b),$$

is strict for $i = l$ and at least one value of $j$. Invoking the Perron-Frobenius theorem again,

$$\rho\,(p_{ij}\,\tau(\mathrm{A}_j)^\alpha)_{i,\,j=1}^b < \rho(\mathrm{P}) = 1.$$

In conjunction with (45), this proves that $\mathrm{P} \in \Omega_\alpha(\mathrm{F}, \mathrm{U})$. ■

*Proof of theorem* 2. – First, note that the definitions of $\Omega_1(\mathrm{F}, \mathrm{U})$ in the statement of the theorem and in (37) agree, so we need only to establish assertion (II) of the theorem.

Recall that for every $f \in \mathrm{L}_1^b$ and $\mathrm{Z} \in \Omega_1(\mathrm{F}, \mathrm{U})$, Lemma 4 guarantees the existence of

$$\mathrm{T}_\mathrm{Z}^\infty f = \lim_{n \to \infty} (\mathrm{T}_\mathrm{Z}^n f)(k, \bar{v}), \tag{46}$$

where the limit does not depend on $k \in \{1, \ldots, b\}$, and is an analytic function of $\mathrm{Z} \in \Omega_1(\mathrm{F}, \mathrm{U})$.

We apply this to the functions $f_{ij} \in \mathrm{L}_1^b$ defined by

$$f_{ij}(k, \bar{x}) = \log \frac{\|\mathrm{A}_j x\|_\infty}{\|x\|_\infty}\,\delta_{ik} \qquad (1 \leqq k, j \leqq b) \tag{47}$$

where we have equipped $\mathbb{R}^b$ with the maximum norm $\|.\|_\infty$, and $\delta_{ik}$ is the Kronecker $\delta$.

The inequality

$$\log \frac{\|\mathrm{A}_j y\|_\infty}{\|\mathrm{A}_j x\|_\infty} \leqq \log \max_{1 \leqq r \leqq b} \frac{x_r}{y_r} \leqq h(\bar{x}, \bar{y}),$$

valid for vectors $x, y \in \mathbb{R}^d$ with positive components and $\|x\|_\infty = \|y\|_\infty = 1$, ensures that $f_{ij} \in \mathrm{L}_1^b$.

DEFINE:

$$\Gamma_\infty(\mathrm{Z}) = \sum_{i=1}^b \sum_{j=1}^b z_{ij}\,\mathrm{T}_\mathrm{Z}^\infty f_{ij}. \tag{48}$$

We know that $\Gamma_\infty$ is analytic in $\Omega_1(\mathrm{F}, \mathrm{U})$; it remains to verify (17).

Let $\mathrm{P} \in \mathrm{S}(\mathrm{U})$ and let $\{\mathrm{K}_n\}_{n \geqq 0}$ denote a Markov chain with state space $\{1, \ldots, b\}$, transition matrix P and initial distribution $\pi_\mathrm{P}$. Then the random matrices $\mathrm{X}_n = \mathrm{A}_{\mathrm{K}_n}$ satisfy (13) and (14).

It is convenient at this junction to take the vector $v \in \mathbb{R}^d$, which was used to define the norm in $\mathrm{L}_\alpha^b$, to be the all-ones vector $v = (1, \ldots, 1)^t$.

If we equip $\mathbb{R}^d$ with the maximum norm $\|.\|_\infty$ and the space of $d\times d$ matrices with the corresponding operator norm, then for every non-negative matrix A,

$$\|A\|=\|Av\|_\infty.$$

The definition (12) of the top Lyapunov exponent may be rewritten

$$\gamma_1(P)=\lim_{N\to\infty}\frac{1}{N}E\log\|X_N\ldots X_1 v\|_\infty$$

$$=\lim_{N\to\infty}\frac{1}{N}(\Gamma_0(P)+\Gamma_1(P)+\ldots+\Gamma_{N-1}(P)) \quad (49)$$

where we *define*, for $n\geqq 0$

$$\Gamma_n(P)=E\log\frac{\|X_{n+1}\ldots X_1 v\|_\infty}{\|X_n\ldots X_1 v\|_\infty}. \quad (50)$$

The proof will be complete once we check that

$$\Gamma_n(P)\xrightarrow{n\to\infty}\Gamma_\infty(P). \quad (51)$$

By conditioning on $K_1,\ldots,K_n$ we find that (50) is equivalent to

$$\Gamma_n(P)=E\left[\sum_{i=1}^{b}\sum_{j=1}^{b}p_{ij}f_{ij}(K_n,X_n\ldots X_1.\bar{v})\right]. \quad (52)$$

For any $f\in L_1^b$, one easily verifies by induction on $n\geqq 0$ that

$$E[f(K_n,X_n\ldots X_1.\bar{v})\,|\,K_0=k]=(T_P^n f)(k,\bar{v})$$

for all $k\in\{1,\ldots,b\}$.

Therefore

$$E[f(K_n,X_n\ldots X_1.\bar{v})]=\sum_{k=1}^{b}\pi_P(k)(T_P^n f)(k,\bar{v}).$$

Thus, from (53) we get

$$\Gamma_n(P)=\sum_{i=1}^{b}\sum_{j=1}^{b}p_{ij}\left[\sum_{k=1}^{b}\pi_P(k)\,T_P^n f_{ij}(k,\bar{v})\right],$$

which implies (51) and, in view of (49), also

$$\gamma_1(P)=\Gamma_\infty(P). \quad \blacksquare$$

## 4. ESTIMATES OF LYAPUNOV EXPONENTS

In the context of i.i.d. random products of invertible matrices, Guivarc'h and Raugi [GR] found general criteria for simplicity of the top

Lyapunov exponent, *i.e.*, for the inequality $\gamma_1 > \gamma_2$ to hold (*see*, for instance, [CKN2] for the definition of all Lyapunov exponents $\gamma_1, \gamma_2, \ldots$). A special case of their results is the following assertion:

Assume that the invertible matrices $\{A_1, \ldots, A_b\}$ define a strongly irreducible action on $\mathbb{R}^d$ (*i.e.*, no finite union of proper subspaces is invariant under all $A_i$) and, for some $1 \leqq l \leqq b$, the matrix $A_l$ has a unique simple eigenvalue of magnitude $\rho(A_l)$.

Then the i.i.d. random matrix product, defined by attaching to $A_1, \ldots, A_b$ corresponding positive probabilities $p_1, \ldots, p_b$, satisfies

$$\gamma_1 > \gamma_2.$$

However, their methods do not yield lower bounds for the difference $\gamma_1 - \gamma_2$.

QUESTION 1. — For the i.i.d. random product mentioned in the previous paragraph, can one obtain lower estimates for $\gamma_1 - \gamma_2$ expressed explicitly in terms of the probabilities $p_1, \ldots, p_b$ and the entries and spectra of the matrices $A_1, \ldots, A_b$?

If the matrices $A_1, \ldots, A_b$ are non-negative and *allowable*, *i.e.*, without all-zero rows or columns, and we assume that at least one $A_l$ is strictly positive (the invertibility assumption may be dropped), such estimates may be given; namely, we then have the inequality

$$\gamma_1 - \gamma_2 \geqq \sum_{j=1}^{b} p_j \log \frac{1}{\tau(A_j)}. \tag{53}$$

This assertion is contained in the following general (and easy) estimate, which slightly extends a result of Wojtkowski.

PROPOSITION 5. — *Let* $\{X_n\}_{n \geqq 1}$ *be an ergodic stationary stochastic process, taking values in the space of allowable non-negative* $d \times d$ *matrices, such that* $E[\log^+ \| X_1 \|] < \infty$. *If, with positive probability, all entries of* $X_1$ *are positive, then the two top Lyapunov exponents* $\gamma_1, \gamma_2$ *for the random product of the* $X_i$ *differ and, moreover,*

$$\gamma_1 - \gamma_2 \geqq E\left[\log \frac{1}{\tau(X_1)}\right]. \tag{54}$$

[*If the right hand side is infinite,* (54) *should be interpreted as asserting that* $\gamma_2 = -\infty$.]

*Proof.* — Of the different (equivalent) definitions of the Lyapunov exponents $\gamma_1, \gamma_2, \ldots$ for the process $\{X^n\}$, the most useful in the present context is the following (*see* [CKN2] for the equivalence of the different definitions, and for the proof of the assertion below which is part of Oseledec's theorem). Denote

$$S_n = X_n X_{n-1} \ldots X_1, \tag{55}$$

and let $B_n$ be the unique positive-definite matrix satisfying

$$B_n^{2n} = S_n^t S_n, \tag{56}$$

where $S_n^t$ is the transpose of $S_n$. Then with probability 1, the matrices $B_n$ tend to a (random) limit matrix B, with eigenvalues

$$e^{\gamma_1} \geqq e^{\gamma_2} \geqq \ldots \geqq e^{\gamma_d}. \tag{57}$$

For any symmetric $d \times d$ matrix A, denoted by

$$\lambda_1(A) \geqq \lambda_2(A) \geqq \ldots \geqq \lambda_d(A)$$

the eigenvalues of A.

For allowable matrices A with non-negative entries, Hopf proved in 1963 [Hop] the basic inequality

$$\lambda_1(A) \geqq \lambda_2(A)/\tau(A). \tag{58}$$

(Here the symmetry assumption on A is not required if $\lambda_2$ is interpreted as the second largest absolute value of an eigenvalue. *See* [S], theorems 2.10 and 3.13 for an exposition).

The definition (55) of $S_n$ implies

$$\tau(S_n) \leqq \prod_{j=1}^{n} \tau(X_j)$$

so that applying (58) to $A = S_n^t S_n$ gives

$$\lambda_1(S_n^t S_n) \geqq \lambda_2(S_n^t S_n) \left[\prod_{j=1}^{n} \frac{1}{\tau(X_j)}\right]^2.$$

Since

$$\lambda_i(S_n^t S_n) = \lambda_i(B_n)^{2n},$$

taking logarithms we see that

$$\log[\lambda_1(B_n)] \geqq \log[\lambda_2(B_n)] + \frac{1}{n} \sum_{j=1}^{n} \log \frac{1}{\tau(X_j)}.$$

Passing to the limit, Birkhoff's ergodic theorem implies

$$\log \lambda_1(B) \geqq \log \lambda_2(B) + E\left[\log \frac{1}{\tau(X_1)}\right],$$

which, in view of (57), completes the proof. ■

*Remark.* – For $2 \times 2$ matrices, (54) was already noted by Wojtkowski [Wojt], and used by him to estimate from below the entropy of a billiard system. The main reason for including Proposition 5 here is to spark off

QUESTION 2. — Can the relation between the *maximal* domain of analytic continuation for $\gamma_1$ (in the sense of Theorems 1, 2) and the gap $\gamma_1 - \gamma_2$, be made explicit?

We turn now to the excruciating problem of the subject:

QUESTION 3. — Devise reasonably general and effective algorithms for explicit calculation (or at least approximation) of Lyapunov exponents.

For the remainder of the discussion, we restrict attention to i.i.d. random products, as these seem sufficiently elusive.

In 1963 Furstenberg [F] found a formula for the top Lyapunov exponent, valid under quite general hypotheses:

$$\gamma_1 = \iint \log \frac{\|Mx\|}{\|x\|}\, d\mu_1(M)\, d\nu(\bar{x}), \tag{59}$$

where $\mu_1$ is the distribution of each factor in the i.i.d. product, and $\nu$ is a $\mu_1$-stationary measure on projective space (*see* [F] or [BL] for details). This formula is fundamental for establishing positivity of $\gamma_1$ [for $SL(d, \mathbb{R})$-valued random matrices] but its utility for computation is limited by one's knowledge of the stationary measure $\nu$. Thus far, Lyapunov exponents have been calculated explicitly only under very special circumstances ([CN], [Key1], [KP], [Let]). The central limit theorems in [FK] and [LP1] allow Monte-Carlo estimation of $\gamma_1$, which suffers from the usual drawbacks (slow convergence and unreliability). Since the definition (1) of $\gamma_1$ provides approximations to $\gamma_1$ from above, E. Key has suggested using supermultiplicative functions to obtain lower bounds [Key2]. For instance, if $X_i$ are i.i.d. positive $d \times d$ matrices and $S_n$ is defined by (55), then

$$\frac{1}{n} E[\log(\operatorname{Per}(S_n))^{1/d}], \tag{60}$$

where Per denotes the permanent, are approximations converging to $\gamma_1$ from below.

This provides rigorous bounds, but the convergence is usually exceedingly slow $\left[\text{the error of the } n\text{'th approximation (60) seems to be of order } \Omega\left(\frac{1}{n}\right)\right]$.

In the setting of Theorem 1, *i.e.*, when each of the factors $X_n$ in the random product is chosen from the allowable non-negative matrices $A_1, \ldots, A_b$ with probabilities $p_1, \ldots, p_b$, an approach suggested by (59) is to use the approximations $\Gamma_n(p)$, defined in (10).

These give exponential convergence

$$|\Gamma_n(p) - \gamma_1| \leq \frac{C}{1-\theta}\theta^n \tag{61}$$

where $\theta = \sum_{j=1}^{b} p_j \tau(A_j) < 1$ [assuming $\tau(A_l) < 1$] and

$$C = \sum_{j=1}^{b} p_j h(\bar{x}, A_j . \bar{x})$$

[$\bar{x} \in \mathbb{P}_+(\mathbb{R}^d)$ is arbitrary].

Note, however, that the computational effort required to calculate $\Gamma_n(p)$ is also exponential in $n$ [the same holds for the approximations (60)].

Le-Page [LP1] proved (61), with *some* uncomputable $\theta < 1$, for invertible $A_1, \ldots, A_b$, replacing the assumption that $A_i$ are non-negative by "strong irreducibility" and "the contraction property" (*see* [BL] for definitions and details). In that context, no sharpening of the inequality $\theta < 1$ is known (this is closely related to question 1 above).

We feel the problem of calculating Lyapunov exponents deserves attention comparable to that allotted to its deterministic analogue, the eigenvalue problem (e. g. [Wil]). Also suggested by the discussion above is

QUESTION 4. – Can simplicity, stability and regularity results for Lyapunov exponents be obtained *without* assuming the matrices involved to be non-negative or invertible? For instance, we conjecture that for an i. i. d. random product defined by (2), the condition $\gamma_1 > \gamma_2$ suffices to guarantee (local) real-analytic dependence of $\gamma_1$ on $(p_1, \ldots, p_b)$.

Finally, we mention two intriguing convexity conjectures due to E. Key.

QUESTION 5 ([Key 1], [Key 2]). – Denote by $f(p)$ the top Lyapunov exponent for an i. i. d. random product where each factor equals A with probability $p$ and B with probability $1-p$. Key conjectured that:

(i) If A, B are normal $d \times d$ matrices, then $f$ is a convex function in the interval $(0, 1)$.

(ii) If $A = B^t$, then $f$ is concave in $(0, 1)$.

Because of Theorem 1 and [P], these conjectures can (usually) be studied by examining the derivatives of $f$.

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