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Indefinite quadratic functionals of Gaussian processes and least-action paths

by

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ABSTRACT. — A method for finding the law of certain indefinite quadratic functionals of Gaussian processes is presented. This involves establishing a Spectral Theorem for compact operators which are "selfadjoint" with respect to a certain class of indefinite scalar products and proving a version of Mercer's Theorem for their integral kernels. Some applications are given, in particular an application to changing the law of a random harmonic oscillator according to the principle of least action. Some relations between the formulae established here and certain formulae for Feynman path integrals are also discussed.

Key words : Gaussian process, quadratic functional, random harmonic oscillator.

RÉSUMÉ. — On présente une méthode pour calculer la loi de certaines fonctionnelles quadratiques indéfinies de processus gaussiens. Dans ce but, on établit un théorème spectral pour des opérateurs compacts qui sont « auto-adjoints » par rapport avec une certaine classe de produits scalaires indéfinis, d'où on déduit une version du théorème de Mercer pour leurs noyaux. On donne quelques applications, en particulier une application au changement de loi d'un oscillateur harmonique aléatoire, selon le principe de moindre action. On traite de plus les relations entre des formules établies ici et certaines formules pour les intégrales de chemins de Feynman.

1. INTRODUCTION AND PRELIMINARIES

Let Z_t be a continuous Gaussian process (in $\mathbb{R}^d$). We think of Z_t as a function in a suitable Hilbert space with inner product $\langle \,.\,,\,.\, \rangle$ and we are interested in the problem of calculating the law of quadratic functionals of Z of the form

$$\langle Z, SZ \rangle \qquad (1.1)$$

where S is a selfadjoint bounded operator. In particular, we present a method for calculating the Laplace transform

$$\alpha \mapsto \mathbb{E}^z [\exp\{ -\alpha \langle Z, SZ \rangle \}] \qquad (1.2)$$

(here, $\mathbb{E}^z$ denotes expected value for the process Z with $Z_0 = z$).

Initially, we consider only centred Gaussian processes (so $Z_0 = 0$). Later we will use the results for centred Gaussian processes to calculate the Laplace transform for general starting point z.

Our motivation for considering this problem is the following. Let (X_t, V_t) be an Ornstein-Uhlenbeck type model of a simple harmonic oscillator:

$$\left. \begin{array}{l} dX_t = V_t\, dt \\ dV_t = -X_t\, dt + dB_t \end{array} \right\} \qquad (1.3)$$

where B_t is a Brownian motion. We are interested in least-action paths of such a process. Since the Lagrangian of such a harmonic oscillator is $(V_t^2 - X_t^2)/2$, the action of a simple harmonic oscillator is precisely of the form (1.1) with $\langle f, g, \rangle = \displaystyle\int_0^\tau f(s)^{\mathrm{T}} \overline{g(s)}\, ds$ for f, g in the Hilbert space $L^2([0, \tau], \mathbb{C}^2)$ and the operator S is defined by

$$Sf(s) = S\begin{pmatrix} u \\ v \end{pmatrix}(s) = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}.$$

There are many other situations in which one is interested in the law of a quadratic functional of a Gaussian process and we shall consider a few of these as well.

In the case where S is a positive-definite operator, there is already a method for evaluating the Laplace transform (1.2). We begin by briefly reminding the reader how this works. For simplicity we take $S = I$.

Let $c(s, t)$ denote the covariance kernel for the process Z with $Z_0 = 0$:

$$c(s, t) = \mathbb{E}(Z_s Z_t^{\mathrm{T}}).$$

For a continuous process the kernel $c(.,.)$ is jointly continuous. Define the covariance operator $\mathbf{C}$ by

$$\mathbf{C}f(s):=\int_0^\tau c(s,t)f(t)\,dt$$

acting on functions $f\in L^2([0,\tau],\mathbb{C}^d)$ ($\mathbb{C}^d$-valued L^2 functions). We denote by $\langle .,.\rangle$ the usual inner product on L^2:

$$\langle f,g\rangle=\int_0^\tau f(s)^\mathrm{T}\overline{g(s)}\,ds.$$

Strictly speaking, we ought to consider complex-valued functions. However since $\mathbf{C}$ is an integral operator with real kernel, it acts separately on the real and imaginary parts of f, so for practical calculations we need only consider real-valued functions. As is well-known, the covariance operator $\mathbf{C}$ with jointly continuous integral kernel is a positive, selfadjoint, trace-class operator on the Hilbert space $\mathscr{H}=L^2([0,\tau],\mathbb{R}^d)$ (we do not assume positive-definiteness for $\mathbf{C}$ since it may in general have non-trivial null space). Therefore, it has a countable set of real non-negative eigenvalues μ_n with corresponding orthonormal eigenfunctions φ_n. Mercer's theorem states that the continuous integral kernel of a positive selfadjoint compact operator admits an eigenfunction expansion of the form

$$c(s,t)=\sum_n\mu_n\varphi_n(s)\varphi_n(t)^\mathrm{T}. \qquad (1.4)$$

From this we see that the process Z admits a series expansion of the form

$$Z_t=\sum_n\sqrt{\mu_n}\,\xi_n\varphi_n(t) \qquad (1.5)$$

where $\xi_1,\xi_2\ldots$ are i.i.d. $N(0,1)$ random variables. Substituting the expansion (1.5) for Z into (1.2) we see that the Laplace transform may be calculated as

$$\mathbb{E}[e^{-\alpha\langle Z,\,SZ\rangle}]=\mathbb{E}[\exp\{-\alpha\sum_n\mu_n\xi_n^2\}]=\prod_n\sqrt{\frac{1}{1+2\alpha\mu_n}}. \qquad (1.6)$$

Furthermore, we know that the product at (1.6) converges since $\mathbf{C}$ is trace-class (which implies that $\sum\mu_n<\infty$). We can then obtain an explicit formula in terms of α for the product at (1.6) by means of the Hadamard factorisation theorem. (But more on this later.)

The aim of the present work is to extend this method to operators S which are indefinite and obtain eigenfunction expansions which are analogous to (1.4) and (1.5).

Suppose that S has a bounded inverse. We can define an indefinite scalar product $(.\,,.)$ on $\mathcal{H}$ using the selfadjoint operator $J = S^{-1}$:

$$(f, g) = \langle Jf, g \rangle = \langle f, Jg \rangle. \tag{1.7}$$

[Throughout this paper we use $\langle .\,,. \rangle$ to denote the usual positive definite inner product and $(.\,,.)$ to denote the indefinite scalar product.] Note that $(.\,,.)$ is non-degenerate on $\mathcal{H}$: i.e. $(x, y) = 0$ for all $y \in \mathcal{H}$ implies $x = 0$.

Let **K** be the operator on $\mathcal{H}$ defined by

$$\mathbf{K} := J^{-1} \mathbf{C} = S\,\mathbf{C}.$$

This is then an integral operator with integral kernel $k(s, t) = J^{-1} c(s, t)$. The operator **K** is J-selfadjoint, that is, selfadjoint with respect to the scalar product $(.\,,.)$; indeed

$$(\mathbf{K}f, g) = \langle \mathbf{C}f, g \rangle = \langle f, \mathbf{C}g \rangle = (f, \mathbf{K}g)$$

and the same calcultation shows that it is also J-positive: $(\mathbf{K}f, f) \geqq 0$. Because **K** is compact, its spectrum consists of a countable set of eigenvalues of finite multiplicity. Let λ_n be the eigenvalues of **K** with corresponding eigenfunctions e_n. Because **K** is J-selfadjoint, the eigenfunctions e_n are J-orthogonal $[(e_i, e_j) = 0$ if $i \neq j]$. The proof of this is identical to the usual proof from Hilbert space theory. Moreover, $(\mathbf{K}f, f) = \langle \mathbf{C}f, f \rangle = 0$ only if $f \in \ker \mathbf{C}$, since from standard Hilbert space theory we have

$$\mathcal{H} = \ker \mathbf{C} \oplus \overline{\operatorname{Im} \mathbf{C}}$$

and span $\{\varphi_n\} = \operatorname{Im} \mathbf{C}$. Therefore $(\mathbf{K} e_n, e_n) = \lambda_n (e_n, e_n) = \langle \mathbf{C} e_n, e_n \rangle > 0$, so $(e_n, e_n) \neq 0$ and λ_n is real with $\operatorname{sgn}(\lambda_n) = \operatorname{sgn}(e_n, e_n)$. In other words,

$$\left.\begin{array}{c} \text{eigenspaces corresponding to positive}\,(resp.\ negative) \\ \text{eigenvalues are positive}\,(resp.\ negative)\ definite. \end{array}\right\} \tag{1.8}$$

This is a particular case of a general result which is true for a much bigger class of positive operators (*see* Chapter VII of Bognar [1]).

For the moment, let us ignore mathematical rigour and briefly describe how we would like to proceed. By analogy with (1.4), we would like to express the kernel $k(s, t)$ as

$$k(s, t) = \sum_n \frac{\lambda_n}{(e_n, e_n)} e_n(s) \, (J e_n(t))^{\mathrm{T}} \tag{1.9}$$

whence we would obtain a series expansion analogous to (1.5) of the form

$$Z_t = \sum_n \sqrt{\frac{\lambda_n}{(e_n, e_n)}} \xi_n J e_n(t). \tag{1.10}$$

The Laplace transform (1.2) may then be calculated by using

$$\langle Z, SZ \rangle = \sum_n \lambda_n \xi_n^2. \qquad (1.11)$$

Notice that $J e_n$ are precisely the eigenfunctions of the usual adjoint $\mathbf{K}^* = \mathbf{C} \mathbf{J}^{-1}$ of $\mathbf{K}$, so the eigenfunction expansion (1.9) should come as no surprise to anyone familiar with the so-called spectral representation for finite-dimensional matrices which are diagonalisable but not necessarily symmetric.

2. RIGOROUS TREATMENT OF HEURISTIC CALCULATIONS IN SECTION 1

It turns out that by far the biggest hurdle on the way to an analogue of Mercer's Theorem which would give the eigenfunction expansion (1.9) is in proving an analogous Spectral Theorem for compact J-selfadjoint operators. Much of the present theory of indefinite scalar product spaces and spectral theory for selfadjoint operators is due to Krein. While Krein and others have proved the existence of spectral functions for various operators (e. g. see the references given in Bognár [1]), it is incredible that the simplest case — that of a compact selfadjoint operator with countable spectrum — does not seem to have been considered and as yet there is no analogous result to the simplest of all the spectral theorems. However, *assuming* that such a spectral theorem holds, that is, assuming the eigenfunctions e_n of $\mathbf{K}$ span the image of $\mathbf{K}$, it is quite easy to generalise Mercer's Theorem to the case of J-selfadjoint operators. Let us begin by doing so.

First, some simple preliminary results. Throughout this section we use the symbols $\perp$ and $*$ to denote the usual orthogonal complement and adjoint respectively, while the symbols $(\perp)$ and $(*)$ denote respectively orthogonal complement and adjoint with respect to $(.,.)$. It is easy to see that the following relationship holds between the two different kinds of adjoint:

$$A^* = J A^{(*)} J^{-1}.$$

The proof of the following very simple result is left to the reader.

LEMMA 2.1. — *Let* $\mathbf{M}$ *be a subspace of* $\mathcal{H}$. *Then*

$$(J\mathbf{M})^\perp = J^{-1} \mathbf{M}^\perp.$$

COROLLARY 2.1. — *If* $\mathbf{M}$ *is a closed subspace of* $\mathcal{H}$, *then* $\mathbf{M}^{(\perp)(\perp)} = \mathbf{M}$.

Proof. $-$ $M^{(\perp)} = (JM)^{\perp}$, so $M^{(\perp)\,(\perp)} = (J\,(JM)^{\perp})^{\perp} = M^{\perp\perp} = M$. (Here we have used the analogous classical result for $M^{\perp\perp}$ from Hilbert space theory.) $\square$

The result in Corollary 2.1 is proved in much greater generality in Bognár [1].

LEMMA 2.2. $-$ *Let* $A: \mathscr{H} \to \mathscr{H}$ *be a linear operator. Then the following properties hold:*

   (i) $\ker A = (\operatorname{Im} A^{(*)})^{(\perp)}$;
   (ii) $\ker A^{(*)} = (\operatorname{Im} A)^{(\perp)}$;
   (iii) $\overline{\operatorname{Im} A} = (\ker A^{(*)})^{(\perp)}$;
   (iv) $\overline{\operatorname{Im} A^{(*)}} = (\ker A)^{(\perp)}$.

Proof. $-$ (i) Given $x \in \ker A$, $y \in \mathscr{H}$, we have

$$0 = (A\,x,\ y) = (x,\ A^{(*)}\,y)$$

so $\ker A \subset \operatorname{Im} A^{(*)\,(\perp)}$. On the other hand if $v \in \operatorname{Im} A^{(*)\,(\perp)}$, then for all $y \in \mathscr{H}$,

$$0 = (v,\ A^{(*)}\,y) = (A\,v,\ y)$$

and since $(\,.\,,\ .\,)$ is non-degenerate, this implies $A\,v = 0$.

The proofs of (ii)-(iv) are identical to the proofs from the usual Hilbert space theory, provided we have the result of Corollary 2.1. $\square$

In particular, if A is J-selfadjoint then $\ker A^{(\perp)} = \operatorname{Im} A$. However, unless $\ker A$ is non-degenerate [*i.e.* $x \in \ker A$, $(x,\ y) = 0$ for all $y \in \ker A$ imply $x = 0$], it is not true that

$$\mathscr{H} = \ker A \oplus \overline{\operatorname{Im} A}. \qquad (2.1)$$

Now obviously, unless **K** has a non-generate null space [so that the direct sum decomposition (2.1) holds for **K**], its eigenfunctions cannot possibly span its range. We shall return to this point later. For now, let us prove the required analogue of Mercer's Theorem for a J-selfadjoint operator satisfying the necessary conditions.

THEOREM 2.3. $-$ *Let* J *be a bounded selfadjoint operator on the Hilbert space* $\mathscr{H} = L^2\,([0,\ \tau],\ \mathbb{R}^d)$ *and let* $(\,.\,,\ .\,)$ *be an indefinite scalar product defined via* J *as in* (1.7) *and let* K *be J-positive J-selfadjoint integral operator with integral kernel* $k\,(s,\ t)$ *which is continuous on* $[0,\ \tau] \times [0,\ \tau]$. *Let* λ_n, $n = 0,\ 1,\ \ldots$ *denote the (real) eigenvalues of* K *with associated J-orthogonal eigenfunctions* e_n. *Suppose that* $\operatorname{span}\{e_n\} = \operatorname{Im} K$. *Then the kernel* $k\,(s,\ t)$ *admits the eigenfunction expansion*

$$k\,(s,\ t) = \sum_n \frac{\lambda_n}{(e_n,\ e_n)}\,e_n\,(s)\,(J\,e_n\,(t))^{\mathrm{T}}.$$

The series converges uniformly in norm (i.e. the natural norm for square matrices) on $[0,\ \tau] \times [0,\ \tau]$.

Proof. – Since the kernel $k(s, t)$ is real and continuous, the eigenfunctions are real and continuous. Let

$$k_n(s, t) = k(s, t) - \sum_{j=0}^{n} \frac{\lambda_j}{(e_j, e_j)} e_j(s) (J e_j(t))^T$$

and let K_n be the integral operator with integral kernel k_n. By calculating $(K_n f, f)$ for $f \in \ker K$ and $f = e_j$, $j = 0, 1, \ldots$ it is easy to see that $(K_n f, f) \geq 0$ for all $f \in \mathcal{H}$. Hence $k_n(t, t)$ is a J-positive matrix for each t:

$$v^T J k_n(t, t) v \geq 0, \qquad \forall v \in \mathbb{R}^d.$$

[Here, $k_n(t, t) v$ is a $\mathbb{R}^d$ valued-function in $\mathcal{H}$ so $J k_n(t, t)$ makes sense.] Hence

$$0 \leq v^T J k(t, t) v - \sum_{j=0}^{n} \frac{\lambda_j}{(e_j, e_j)} v^T J e_j(t) (J e_j(t))^T v.$$

Since $J e_j(t) (J e_j(t))^T$ is positive-definite in the usual sense, we have, for all $v \in \mathbb{R}^d$

$$0 \leq \sum_{j=0}^{n} \frac{\lambda_j}{(e_j, e_j)} v^T J e_j(t) (J e_j(t))^T v \leq v^T J k(t, t) v.$$

Recall that $\lambda_j (e_j, e_j) > 0$, so each term in the above sum is positive. Hence the above inequality implies

$$\sum_{j=0}^{\infty} \frac{\lambda_j}{(e_j, e_j)} |v^T J e_j(t)|^2 \leq \sup_{t'} v^T J k(t', t') v.$$

Schwartz's inequality implies, for each $v, w \in \mathbb{R}^d$

$$\sum_{j=m}^{n} \left| \frac{\lambda_j}{(e_j, e_j)} v^T J e_j(s) (J e_j(t))^T w \right|$$

$$\leq \left(\sum_{j=m}^{n} \frac{\lambda_j}{(e_j, e_j)} |v^T J e_j(s)|^2 \right)^{1/2} \left(\sum_{j=m}^{n} \frac{\lambda_j}{(e_j, e_j)} |w^T J e_j(t)|^2 \right)^{1/2}.$$

Hence for fixed s and $\varepsilon > 0$, there exists N such that for $n > m \geq N$ and each $v, w \in \mathbb{R}^d$

$$\sum_{j=m}^{n} \left| \frac{\lambda_j}{(e_j, e_j)} v^T J e_j(s) (J e_j(t))^T w \right| \leq \varepsilon A, \qquad \forall t \in [0, \tau]$$

for some constant A. Therefore for each fixed s and for each $v, w \in \mathbb{R}^d$, we have that

$$\sum_{j=0}^{\infty} \frac{\lambda_j}{(e_j, e_j)} v^T J e_j(s) (J e_j(t))^T w \qquad (2.2)$$

converges absolutely and uniformly in t. Since the sum (2.2) is bilinear in v, w, there exists a matrix $\tilde{k}$ such that

$$\tilde{k}(s, t) = \sum_{j=0}^{\infty} \frac{\lambda_j}{(e_j, e_j)} e_j(s) \, (\mathrm{J} \, e_j(t))^{\mathrm{T}}$$

which by virtue of the uniform convergence and the continuity of $e_j(.)$, is continuous in t for fixed s.

Consider now

$$\int_0^{\tau} [k(s, t) - \tilde{k}(s, t)] f(t) \, dt = \mathrm{K} f(s) - \sum_j \frac{\lambda_j}{(e_j, e_j)} (e_j, f) e_j(s).$$

If $f \in \ker \mathrm{K} = \overline{\mathrm{Im} \, \mathrm{K}}^{(\perp)}$, the right-hand side in the above expression is zero since $(e_j, f) = 0$. If $f = e_n$ for some n, the right-hand side is $\lambda_n e_n - \lambda_n e_n = 0$. Hence by virtue of the continuity of $k(s, .) - \tilde{k}(s, .)$, we have $k(s, t) = \tilde{k}(s, t)$ for all s, t.

We now prove the uniform convergence in s and t of the eigenfunction expansion for $k(s, t)$. For all $v \in \mathbb{R}^d$ we have

$$v^{\mathrm{T}} \mathrm{J} \, k(t, t) v = \sum_{j=0}^{\infty} \frac{\lambda_j}{(e_j, e_j)} (v^{\mathrm{T}} \mathrm{J} \, e_j(t))^2.$$

Since the terms of this series are all non-negative, the partial sums form a non-decreasing sequence of continuous functions converging pointwise to the continuous function $v^{\mathrm{T}} \mathrm{J} \, k(t, t) v$. Hence by Dini's Theorem the series converges uniformly in t. For $\varepsilon > 0$, there exists N such that for $n > m \geqq \mathrm{N}$

$$\sum_{j=m}^{n} \frac{\lambda_j}{(e_j, e_j)} |v^{\mathrm{T}} \mathrm{J} \, e_j(t)|^2 < \varepsilon, \qquad \forall \, t \in [0, \tau].$$

An application of Schwartz's inequality as before then gives that there exists N such that for $n > m \geqq \mathrm{N}$

$$\sum_{j=m}^{n} \left| \frac{\lambda_j}{(e_j, e_j)} v^{\mathrm{T}} \mathrm{J} \, e_j(s) \, (\mathrm{J} \, e_j(t))^{\mathrm{T}} w \right| \leqq \varepsilon, \qquad \forall \, s, \, t \in [0, \tau]$$

whence it follows that

$$\left\| k(s, t) - \sum_{j=0}^{n} \frac{\lambda_j}{(e_j, e_j)} e_j(s) \, (\mathrm{J} \, e_j(t))^{\mathrm{T}} \right\| \to 0$$

uniformly on $[0, \tau] \times [0, \tau]$. $\quad\square$

Next we prove that the eigenfunctions of the operator $\mathbf{K}$ from the previous section span the image of $\mathbf{K}$, so that our formal calculations at (1.9)-(1.11) are indeed valid. In view of the motivating example (1.3) and in view of the fact that operators defined via symmetric matrices as in that example are a particularly simple class of selfadjoint operators,

one would ideally like to prove a general analogue of the Spectral Theorem for J-selfadjoint operators, where J is defined via a symmetric matrix, which would allow us to do the calculations at (1.9)-(1.11) for any n-dimensional Gaussian process. It is therefore a great disappointment and a source of immense frustration doing this has proved for the moment to be intractably difficult. Here we prove the existence of a spectral expansion for scalar products of the form

$$(f, g) = \langle f, (I+S)^{-1} g \rangle$$

where S is a *compact* selfadjoint operator. For a Gaussian process Z_t, we shall then be able to find the law of any quadratic functional of the form $\langle Z, (I+S) Z \rangle$.

The key result is the following theorem due to Krein.

THEOREM 2.4. − *Let* H *be a compact selfadjoint operator on a Hilbert space. Suppose* H *is either a trace-class or a Hilbert-Schmidt operator. Let* S *be a compact operator and suppose* B *is a compact perturbation of* H *in the sense that* B = (I+S) H. *Suppose further that* B *does not vanish on any non-zero element of the closure of its image. Then the root vectors of* B *span the image of* B.

This theorem is a special case of a more general one (Theorem V.8.1) in Gohberg and Krein [5].

In general, we cannot hope that the eigenvectors of a non-selfadjoint ("selfadjoint" is taken to have the usual Hilbert space meaning here) operator will span the image of that operator; we need to consider other root vectors as well, as in Theorem 2.4 above. However, for J-selfadjoint operators (J here is any bounded operator via which an indefinite scalar product is defined) we have the following result.

LEMMA 2.5. − *Let* λ *be a non-zero eigenvalue of a* J-*positive and* J-*selfadjoint operator* B. *Then the associated root space coincides with the eigenspace:*

$$\ker (B - \lambda I) = \bigcup_{n=0}^{\infty} \ker (B - \lambda I)^n.$$

Proof. − Since $\lambda \neq 0$ the associated eigenspace is either positive-definite or negative-definite by the result (1.8). Now suppose that for some integer $n \geq 2$ and some function f, we have $(B - \lambda I)^n f = 0$ but $(B - \lambda I)^{n-1} f \neq 0$. Then $f_1 := (B - \lambda I)^{n-1} f$ is a non-zero element of the eigenspace associated with λ and we have

$$(f_1, f_1) = ((B - \lambda I)^{n-1} f, (B - \lambda I)^{n-1} f) = ((B - \lambda I)^n f, (B - \lambda I)^{n-2} f) = 0,$$

which contradicts the definiteness of the eigenspace associated with λ.   □

We are now in a position to prove our main result in this section, which is that the calculations (1.9)-(1.11) are valid if the quadratic functional (1.1) arises as a compact perturbation of the identity.

THEOREM 2.6. — *Let Z_t be a continuous $\mathbb{R}^d$-valued centred Gaussian process with co-variance kernel $c(s, t) = \mathbb{E}(Z_s Z_t^T)$. Denote by $\mathbf{C}$ the integral operator with kernel $c(s, t)$ and suppose that $\mathbf{C}$ is a positive-definite. Let S be a compact selfadjoint operator. Then*

$$\mathbb{E}^z[\exp\{-\alpha\langle Z, (I+S)Z\rangle\}] = \det(I + 2\alpha(I+S)\mathbf{C})^{-1/2}, \qquad (2.3)$$

for $1/\lambda_{\max} < \alpha < 1/\lambda_{\min}$, where $\lambda_{\max}$ is the largest (positive) eigenvalue and $\lambda_{\min}$ is the smallest (negative) eigenvalue of $(I+S)\mathbf{C}$ and the determinant is defined as the product of eigenvalues.

Proof. — Since S is compact, the non-zero eigenvalues of $(I+S)$ are bounded away from zero, so if 0 is in the spectrum of $I+S$ it is in the point spectrum. Moreover, the null-space of $I+S$ is at most finite-dimensional. So by discarding a finite-dimensional subspace of $\mathscr{H}$, we may assume we are working on a Hilbert space $\mathscr{H}_1$ on which $(I+S)^{-1}$ exists and is bounded. Let $\mathbf{K} = (I+S)\mathbf{C}$ and let $J = (I+S)^{-1}$. Define a scalar product $(., .)$ via J in the usual way as in (1.7). The operator $\mathbf{K}$ has integral kernel $k(s, t) = J^{-1}c(., t)(s)$, that is the operator J^{-1} acting on the function $c(., t)$ for fixed t, with the result evaluated at s. Moreover, we know from Section 1 that $\mathbf{K}$ is J-selfadjoint and J-positive, and that if λ_n and e_n are respectively the eigenvalues and eigenfunctions of $\mathbf{K}$, then the λ_n are real and the e_n are J-orthogonal.

The condition that $\mathbf{C}$ is positive-definite implies that it has trivial null-space. It is clear that, since $\mathbf{C}$ does not vanish on any non-zero function and since we are insisting that $(I+S)$ also has only trivial null space (by discarding a one-dimensional subspace if necessary), the decomposition (2.1) holds trivially. Therefore $\mathbf{K} = (I+S)\mathbf{C}$ is a compact perturbation of a selfadjoint operator which vanishes only at 0 on $\overline{\operatorname{Im}\mathbf{K}}$, and hence the eigenfunctions e_n span $\operatorname{Im}\mathbf{K}$ by Lemma 2.5 and Theorem 2.4. By Theorem 2.3, the integral kernel $k(s, t)$ has an expansion of the form

$$k(s, t) = \sum_n \frac{\lambda_n}{(e_n, e_n)} e_n(s) J e_n(t).$$

Hence Z_t admits an expansion

$$Z_t = \sum_n \sqrt{\frac{\lambda_n}{(e_n, e_n)}} \xi_n J e_n(t) \qquad (2.4)$$

and

$$\langle Z, J^{-1} Z\rangle = \langle Z, (I+S)Z\rangle = \sum_n \lambda_n \xi_n^2, \qquad (2.5)$$

where ξ_n are i.i.d. $N(0, 1)$ random variables. The result (2.3) now follows. $\square$

The assumption that C is positive-definite is used in the proof of Theorem 2.6 only to show that the direct sum decomposition (2.1) holds for K: in this case (2.1) holds trivially. In general, if C has non-trivial null-space, Theorem 2.6 still holds if we can prove (2.1) directly.

We conclude this section by briefly discussing how one might calculate (1.2) for a non-zero starting point z. So suppose now that $Z_0 = z$ and let $m(t) = \mathbb{E}^z[Z_t]$. Then $\tilde{Z}_t := Z_t - m(t)$ is a centred Gaussian process. We write

$$\langle Z, (I+S)Z \rangle = \langle \tilde{Z}, (I+S)\tilde{Z} \rangle$$
$$+ 2\langle m(.), (I+S)\tilde{Z} \rangle + \langle m(.), (I+S)m(.) \rangle. \quad (2.6)$$

We know how to handle the first and last terms in (2.6) (the last term is just a deterministic function). By means of the eigenfunction expansion (2.4) for the centred Gaussian process $\tilde{Z}$, we may write the second term in (2.6) as

$$\langle m(.), (I+S)\tilde{Z} \rangle = \sum_n \sqrt{\frac{\lambda_n}{(e_n, e_n)}} \xi_n \langle m, e_n \rangle. \quad (2.7)$$

[Recall that $J = (I+S)^{-1}$.] Putting this together with the contributions of ξ_n^2 from the first term in (2.6) and then completing the square, we see that the Laplace transform of (2.6) may be written as the infinite product of Laplace transforms of non-central χ-squared distributions, the non-central χ-squared replacing the central χ-squared distribution in (2.5).

Perhaps a better idea of how such a calculation works in practice can be gained from the detailed example in the next section.

3. DETAILED CALCULATIONS FOR A SPECIFIC EXAMPLE

We return to our motivating example mentioned in Section 1. Let $Z = (X, V)$ satisfy the SDE

$$\left. \begin{array}{l} dX_t = V_t \, dt \\ dV_t = -X_t \, dt + dB_t \end{array} \right\} \quad (3.1)$$

with initial condition $Z_0 = z = (x, v)$. Consider first the case $Z_0 = 0$. For fixed τ, we wish to find the law of

$$\int_0^\tau Z_t^T J Z_t \, dt \quad (3.2)$$

where J is the matrix

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.3)$$

Notice that J is both symmetric and orthogonal. We cannot apply Theorem 2.3 directly since $J = I$ on an infinite-dimensional invariant subspace and so is not a compact perturbation of the identity. However, the process Z does have the following crucial property:

$$X_t = \int_0^t V_s \, ds. \qquad (3.4)$$

Bearing in mind the property (3.4), it is easy to show using integration by parts and Fubini's Theorem that

$$\int_0^\tau Z_s^T J Z_s \, ds = \langle V, (I - A) V \rangle. \qquad (3.5)$$

where A is the compact positive-definite selfadjoint operator defined by

$$A f(s) = \int_0^\tau [\tau - \max(s, t)] f(t) \, dt.$$

The operator $(I - A)$ has eigenvalues

$$1 - \frac{\tau^2}{(n + 1/2)^2 \pi^2}, \qquad n = 0, 1, 2, \ldots$$

Let $c_1(s, t) = \mathbb{E}(V_s V_t)$ be the covariance kernel of V, which defines an integral operator C_1 in the usual way. Set $K_1 := (I - A) C_1$. Now let $J_1 = (I - A)^{-1}$ and define an indefinite scalar product $(., .)_1$ on $\mathscr{H}_1$ via J_1 as before. We can therefore apply Theorem 2.3 to the quadratic functional (3.5). In particular, we have

$$k_1(s, t) = \sum_n \frac{v_n}{(f_n, f_n)_1} f_n(s) J_1 f_n(t)$$

and

$$V_t = \sum_n \sqrt{\frac{v_n}{(f_n, f_n)_1}} \xi_n J_1 f_n(t) \qquad (3.6)$$

where v_n and f_n are the eigenvalues and eigenfunctions of K_1. However, from the point of view of practical calculations, this would mean doing calculations which involve the composition of two integral operators as well as the inverse of an integral operator, and in general one would expect this to be considerably harder to handle than working with the operator $K = J C$ where J is the matrix at (3.3). In any case the calculations (1.9)-(1.11) for a matrix J are more natural, since as we have already said at the beginning of this section, one would like to be able to do calculations along the lines of (1.9)-(1.11) for a general Gaussian process not necessarily satisfying (3.4), and our approach here is in many ways somewhat unsatisfying.

Because of (3.4), it turns out that we *can* in fact apply the results (2.3)-(2.5) to the case where J is the matrix at (3.3)!

LEMMA 3.1. − *Let* $Z=(X, V)$ *be the Gaussian process satisfying* (3.1) *and let* J *be as at* (3.3). *Denote by* **C** *the covariance operator of* Z *and let* $\mathbf{K}=\mathbf{J}^{-1}\mathbf{C}=\mathbf{J}\,\mathbf{C}$. *Denote by* $e_n=(e_n^X, e_n^V)$ *the eigenfunction of* **K** *corresponding to eigenvalue* λ_n. *Then*

$$\frac{d}{ds}(\mathbf{J}\,e_n)^X(s)=(\mathbf{J}\,e_n)^V(s).$$

Proof. − We denote by $c(s, t)$ the covariance matrix of Z: $c(s, t)=\mathbb{E}(Z_s Z_t^T)$. Because of the property (3.4), Fubini's Theorem allows us to write $c(s, t)$ in terms of $c_1(s, t)$:

$$c(s, t)=\begin{pmatrix} \int_0^s \int_0^t c_1(u, v)\,du\,dv & \int_0^s c_1(u, t)\,du \\ \int_0^t c_1(s, u)\,du & c_1(s, t) \end{pmatrix}$$

Write $f(t)=(f^X(t), f^V(t))$ and consider functions of the form $g(s)=\mathbf{C}f(s)$. We have

$$g(s)=\int_0^\tau \begin{pmatrix} \int_0^s \int_0^t c_1(u, v)\,du\,dv & \int_0^s c_1(u, t)\,du \\ \int_0^t c_1(s, u)\,du & c_1(s, t) \end{pmatrix}\begin{pmatrix} f^X(t) \\ f^V(t) \end{pmatrix}dt.$$

Differentiating the above expression with respect to s shows that

$$\frac{d}{ds}g^X(s)=g^V(s)$$

and consequently the eigenfunctions e_n exactly reflect the phase-space structure of Z:

$$\frac{d}{ds}(\mathbf{J}\,e_n)^X(s)=(\mathbf{J}\,e_n)^V(s). \quad \square$$

In the present context, Lemma 3.1 implies that

$$e_n^X(t)=-\int_0^t e_n^V(s)\,ds \qquad (3.7)$$

since we have the obvious boundary condition $e_n(0)=0$.

LEMMA 3.2. − *Let* Z, J, K, λ_n *and* e_n *be as is Lemma* 3.1. *The* (2.4) *and* (3.6) *give the same expansion for* V, *and in particular Theorem* 2.3 *holds for the quadratic functional* (3.2).

Proof. − Comparing the two expansions (2.4) and (3.6) for V, we see that we must prove

$$J_1^{-1} e_n^V = f_n, \qquad \lambda_n = \nu_n \tag{3.8}$$

where we have written $e_n = (e_n^X, e_n^V)$. Because of (3.7) the equation

$$K e_n(s) = \lambda_n e_n(s)$$

reads for the e_n^V component

$$\int_0^\tau c_1(s, t) e_n^V(t) dt - \int_0^\tau \int_0^t c_1(s, u) du \int_0^t e_n^V(u) du \, dt = \lambda_n e_n^V(s).$$

An integration by parts shows that

$$\int_0^t c_1(s, u) du \int_0^t e_n^V(u) du$$

$$= \int_0^t c_1(s, u) \int_0^u e_n^V(x) dx \, du + \int_0^t e_n^V(u) \int_0^u c_1(s, x) dx \, du.$$

We calculate

$$\int_0^\tau \int_0^t c_1(s, u) du \int_0^t e_n^V(u) du \, dt$$

$$= \int_{t=0}^\tau \int_{u=0}^t c_1(s, u) \int_{x=0}^u e_n^V(x) dx \, du \, dt$$

$$+ \int_{t=0}^\tau \int_{u=0}^t e_n^V(u) \int_{x=0}^u c_1(s, x) dx \, du \, dt$$

$$= \int_{u=0}^\tau \int_{t=u}^\tau c_1(s, u) \int_{x=0}^u e_n^V(x) dx \, dt \, du$$

$$+ \int_{u=0}^\tau \int_{t=u}^\tau e_n^V(u) \int_{x=0}^u c_1(s, x) dx \, dt \, du$$

$$= \int_{u=0}^\tau (\tau - u) c_1(s, u) \int_{x=0}^u e_n^V(x) dx \, du$$

$$+ \int_{u=0}^\tau (\tau - u) e_n^V(u) \int_{x=0}^u c_1(s, x) dx \, du$$

$$= \int_{x=0}^\tau e_n^V(x) \int_{u=x}^\tau (\tau - u) c_1(s, u) du \, dx$$

$$+ \int_{x=0}^\tau c_1(s, x) \int_{u=x}^\tau (\tau - u) e_n^V(u) du \, dx$$

$$= \int_{x=0}^{\tau} e_n^{\mathrm{V}}(x) \left\{ \int_{u=0}^{\tau} (\tau - \max(x, u)) c_1(s, u) \, du \right.$$

$$\left. - \int_{u=0}^{x} (\tau - x) c_1(s, u) \, du \right\} dx$$

$$+ \int_{x=0}^{\tau} c_1(s, x) \left\{ \int_{u=0}^{\tau} (\tau - \max(x, u)) e_n^{\mathrm{V}}(u) \, du \right.$$

$$\left. - \int_{u=0}^{x} (\tau - x) e_n^{\mathrm{V}}(u) \, du \right\} dx$$

$$= 2 \, \mathbf{C}_1 \, \mathrm{A} \, e_n^{\mathrm{V}}(s) - \int_{u=0}^{\tau} c_1(s, u) \int_{x=u}^{\tau} (\tau - x) e_n^{\mathrm{V}}(x) \, dx \, du$$

$$- \int_{u=0}^{\tau} e_n^{\mathrm{V}}(u) \int_{x=u}^{\tau} (\tau - x) c_1(s, x) \, dx \, du.$$

But the last two terms on the right-hand side are precisely those on the left-hand side, so

$$\int_0^{\tau} \int_0^t c_1(s, u) \, du \int_0^t e_n^{\mathrm{V}}(u) \, du \, dt = \mathbf{C}_1 \, \mathrm{A} \, e_n^{\mathrm{V}}(s).$$

Hence the equation $\mathbf{K} e_n = \lambda_n e_n$ reads for the e_n^{V} component

$$\mathbf{C}_1 \, e_n^{\mathrm{V}}(s) - \mathbf{C}_1 \, \mathrm{A} \, e_n^{\mathrm{V}}(s) = \lambda_n \, e_n^{\mathrm{V}}(s).$$

Therefore

$$\mathrm{J}_1^{-1} \, \mathbf{C}_1 \, \mathrm{J}_1^{-1} \, e_n^{\mathrm{V}} = (\mathrm{I} - \mathrm{A}) \, \mathbf{C}_1 \, (\mathrm{I} - \mathrm{A}) \, e_n^{\mathrm{V}}$$
$$= (\mathbf{C}_1 - \mathrm{A} \, \mathbf{C}_1 - \mathbf{C}_1 \, \mathrm{A} + \mathrm{A} \, \mathbf{C}_1 \, \mathrm{A}) \, e_n^{\mathrm{V}} = (\mathbf{C}_1 - \mathbf{C}_1 \, \mathrm{A}) \, e_n^{\mathrm{V}} - \mathrm{A} \, (\mathbf{C}_1 - \mathbf{C}_1 \, \mathrm{A}) \, e_n^{\mathrm{V}}$$
$$= \lambda_n \, e_n^{\mathrm{V}} - \lambda_n \, \mathrm{A} \, e_n^{\mathrm{V}} = \lambda_n \, \mathrm{J}_1^{-1} \, e_n^{\mathrm{V}},$$

so $f_n = \mathrm{J}_1^{-1} \, e_n^{\mathrm{V}}$ is an eigenfunction of $\mathbf{K}_1 = \mathrm{J}_1^{-1} \, \mathbf{C}_1$ corresponding to eigenvalue λ_n. This proves the result (3.8) and hence the lemma. $\square$

We can now proceed to find the law of (3.2). The SDE (3.1) can be solved explicitly by observing that

$$\exp\left\{ -t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \mathbf{Z}_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mathbf{Z}_t$$

is a local martingale:

$$d\left(\exp\left\{ -t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \mathbf{Z}_t \right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ d\mathbf{B}_t \end{pmatrix}.$$

Hence the covariance matrix $c(s, t)$ may be calculated as

$$
c(s, t) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}
$$

$$
\times \begin{pmatrix} \displaystyle\int_0^{s \wedge t} \sin^2 u \, du & -\displaystyle\int_0^{s \wedge t} \sin u \cos u \, du \\ -\displaystyle\int_0^{s \wedge t} \sin u \cos u \, du & \displaystyle\int_0^{s \wedge t} \cos^2 u \, du \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}
$$

$$
\times \begin{pmatrix} (s \wedge t) - \sin(s \wedge t)\cos(s \wedge t) & -\sin^2(s \wedge t) \\ -\sin^2(s \wedge t) & (s \wedge t) + \sin(s \wedge t)\cos(s \wedge t) \end{pmatrix}
$$

$$
\times \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
$$

We now find the eigenvalues of $\mathbf{K} = \mathbf{J}^{-1} \mathbf{C}$ [recall that we take J to be the matrix at (2.4)]. We have

$$
\lambda_n e_n(s) = \int_0^\tau k(s, t) e_n(t) \, dt
$$

$$
= \int_0^s \frac{1}{2} \begin{pmatrix} -\cos s & -\sin s \\ -\sin s & \cos s \end{pmatrix}
$$

$$
\times \begin{pmatrix} t - \sin t \cos t & -\sin^2 t \\ -\sin^2 t & t + \sin t \cos t \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt
$$

$$
+ \int_s^\tau \frac{1}{2} \begin{pmatrix} -\cos s & -\sin s \\ -\sin s & \cos s \end{pmatrix}
$$

$$
\times \begin{pmatrix} s - \sin s \cos s & -\sin^2 s \\ -\sin^2 s & s + \sin s \cos s \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt. \quad (3.9)
$$

Differentiating the above expression with respect to s gives

$$
\lambda_n e_n'(s) = \frac{1}{2} \begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix}
$$

$$
\times \int_0^s \begin{pmatrix} t - \sin t \cos t & -\sin^2 t \\ -\sin^2 t & t + \sin t \cos t \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt
$$

$$
+ \frac{1}{2} \begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix}
$$

$$
\times \int_s^\tau \begin{pmatrix} s - \sin s \cos s & -\sin^2 s \\ -\sin^2 s & s + \sin s \cos s \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt
$$

$$
+ \begin{pmatrix} 0 & 0 \\ -\sin s & \cos s \end{pmatrix} \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt. \quad (3.10)
$$

Differentiating again gives

$$\lambda_n e_n''(s) = \frac{1}{2} \begin{pmatrix} \cos s & \sin s \\ \sin s & -\cos s \end{pmatrix}$$

$$\times \int_0^s \begin{pmatrix} t - \sin t \cos t & -\sin^2 t \\ -\sin^2 t & t + \sin t \cos t \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt$$

$$+ \frac{1}{2} \begin{pmatrix} \cos s & \sin s \\ \sin s & -\cos s \end{pmatrix}$$

$$\times \int_s^\tau \begin{pmatrix} s - \sin s \cos s & -\sin^2 s \\ -\sin^2 s & s + \sin s \cos s \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt$$

$$+ \begin{pmatrix} \sin s & -\cos s \\ 0 & 0 \end{pmatrix}$$

$$\times \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt + \begin{pmatrix} 0 & 0 \\ -\cos s & -\sin s \end{pmatrix}$$

$$\times \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e_n(s). \quad (3.11)$$

Now notice that

$$\begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos s & -\sin s \\ -\sin s & \cos s \end{pmatrix}^2,$$

so therefore, from (3.10) and (3.11) we see that

$$\begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix} \frac{d}{ds} \int_0^\tau k(s,t) e_n(t)\, dt$$

$$+ \begin{pmatrix} -\cos s & -\sin s \\ -\sin s & \cos s \end{pmatrix} \frac{d^2}{ds^2} \int_0^\tau k(s,t) e_n(t)\, dt$$

$$= \begin{pmatrix} \sin s \cos s & \sin^2 s \\ -\cos^2 s & -\sin s \cos s \end{pmatrix}$$

$$\times \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt - \begin{pmatrix} 0 & -\sin s \\ 0 & \cos s \end{pmatrix} e_n(s), \quad (3.12)$$

and also from (3.9) and (3.11)

$$\begin{pmatrix} 0 & \sin s \\ 0 & -\cos s \end{pmatrix} \left[\frac{d^2}{ds^2} \int_0^\tau k(s,t) e_n(t)\, dt + \int_0^\tau k(s,t) e_n(t)\, dt \right]$$

$$= - \begin{pmatrix} \sin s \cos s & \sin^2 s \\ -\cos^2 s & -\sin s \cos s \end{pmatrix}$$

$$\times \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t)\, dt + \begin{pmatrix} 0 & -\sin s \\ 0 & \cos s \end{pmatrix} e_n(s). \quad (3.13)$$

Adding (3.12) and (3.13) yields

$$\begin{pmatrix} -\cos s & 0 \\ -\sin s & 0 \end{pmatrix} e_n''(s) + \begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix} e_n'(s)$$

$$+ \begin{pmatrix} 0 & \sin s \\ 0 & -\cos s \end{pmatrix} e_n(s) = 0. \quad (3.14)$$

If we write

$$e_n(s) = \begin{pmatrix} e_n^X(s) \\ e_n^Y(s) \end{pmatrix}$$

as before, then (3.14) simply says

$$\frac{d}{ds} e_n^X(s) + e_n^Y(s) = 0, \quad (3.15)$$

which we know already from the previous section. Using the equation (3.15) and integration by parts we obtain

$$-\int_s^\tau e_n^Y(t) \sin t \, dt = e_n^X(\tau) \sin \tau - e_n^X(s) \sin s - \int_s^\tau e_n^X(t) \cos t \, dt$$

$$\int_s^\tau e_n^X(t) \sin t \, dt = -e_n^X(\tau) \cos \tau + e_n^X(s) \cos s - \int_s^\tau e_n^Y(t) \cos t \, dt.$$

Hence

$$\begin{pmatrix} \sin s & -\cos s \\ -\cos s & -\sin s \end{pmatrix} \int_s^\tau \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} e_n(t) \, dt$$

$$= \begin{pmatrix} -e_n^X(s) \\ 0 \end{pmatrix} + \begin{pmatrix} e_n^X(\tau) \cos(\tau - s) \\ -e_n^X(\tau) \sin(\tau - s) \end{pmatrix}. \quad (3.16)$$

Adding (3.9) and (3.11) we then obtain the following set of inhomogeneous ODEs for the eigenfunctions:

$$\lambda_n (e_n^{X''}(s) + e_n^X(s)) = -e_n^X(s) + e_n^X(\tau) \cos(\tau - s) \quad (3.17a)$$
$$\lambda_n (e_n^{Y''}(s) + e_n^Y(s)) = -e_n^Y(s) - e_n^X(\tau) \sin(\tau - s) \quad (3.17b)$$

An obvious boundary condition for the ODEs (3.17a)-(3.17b) is [from (3.9)] $e_n(0) = 0$. By taking a suitable linear combination of (3.9) and (3.10), we see that the other boundary condition must be

$$\lambda_n \begin{pmatrix} \cos \tau & \sin \tau \\ \sin \tau & -\cos \tau \end{pmatrix} e_n(\tau) + \lambda_n \begin{pmatrix} \sin \tau & -\cos \tau \\ -\cos \tau & -\sin \tau \end{pmatrix} e_n'(\tau) = 0.$$

Thus the two natural boundary conditions for the ODEs (3.17) are

$$e_n^X(0) = e_n^Y(0) = 0, \quad (3.17c)$$

$$e_n^X(\tau) = \frac{d}{dt} e_n^Y(t) \Big|_{t=\tau}. \quad (3.17d)$$

Equations $(3.17a)$-$(3.17b)$ have the same homogeneous solution and the functions (of s) $e_n^X(\tau)\cos(\tau-s)$ and $-e_n^X(\tau)\sin(\tau-s)$ are particular solutions of $(3.17a)$ and $(3.17b)$ respectively.

We need to consider three cases. Firstly if $\lambda_n>0$ or $\lambda_n<-1$, $(\lambda_n+1)/\lambda_n>0$ so the solution to (3.17) is

$$\left.\begin{aligned} e_n^X(t) &= A\sin x_n t + B\cos x_n t + e_n^X(\tau)\cos(\tau-t) \\ e_n^Y(t) &= A'\sin x_n t + B'\cos x_n t - e_n^X(\tau)\sin(\tau-t) \end{aligned}\right\} \qquad (3.18)$$

where [using $(3.17c)$ and (3.15)]

$$x_n = \sqrt{\frac{\lambda_n+1}{\lambda_n}}$$

$$A = -e_n^X(\tau)\frac{\sin\tau}{x_n}, \qquad B = -e_n^X(\tau)\cos\tau$$

$$A' = -e_n^X(\tau)x_n\cos\tau, \qquad B' = e_n^X(\tau)\sin\tau.$$

Finally, the boundary condition $(3.17d)$ gives the following equation for the eigenvalues λ_n:

$$\frac{\sin\tau}{x_n}\sin(\tau x_n) + \cos\tau\cos(\tau x_n) = 0. \qquad (3.19)$$

[We are of course only interested in the positive solutions of (3.19).]

Next, for $-1<\lambda_n<0$, $(\lambda_n+1)/\lambda_n<0$ so the solution to (3.17) is

$$\left.\begin{aligned} e_n^X(t) &= A\,e^{-y_n t} + B\,e^{y_n t} + e_n^X(\tau)\cos(\tau-t) \\ e_n^Y(t) &= A'\,e^{-y_n t} + B'\,e^{y_n t} - e_n^X(\tau)\sin(\tau-t) \end{aligned}\right\} \qquad (3.20)$$

where [using $(3.17c)$ and (3.15)]

$$y_n = \sqrt{\frac{\lambda_n+1}{-\lambda_n}}$$

$$A = e_n^X(\tau)\left(\frac{\sin\tau}{y_n} - \cos\tau\right)\bigg/2$$

$$B = -e_n^X(\tau)\left(\frac{\sin\tau}{y_n} - \cos\tau\right)\bigg/2$$

$$A' = e_n^X(\tau)(\sin\tau - y_n\cos\tau)/2$$

$$B' = e_n^X(\tau)(\sin\tau + y_n\cos\tau)/2.$$

Again, the boundary condition $(3.17d)$ gives rise to an equation for λ_n as follows:

$$\cos\tau\cosh(\tau y_n) + \frac{\sin\tau}{y_n}\sinh(\tau y_n) = 0. \qquad (3.21)$$

For $\sin\tau\neq0$, (3.21) may be written as

$$y_n\cot\tau = \tanh(-\tau y_n), \qquad (3.22)$$

which has no solution unless

$$\tan \tau < -\frac{1}{\tau}. \tag{3.23}$$

When (3.23) holds, (3.22) has a unique positive solution.

The third case we need to consider is when $\lambda_n = -1$. In this case unless

$$\tau \sin \tau + \cos \tau = 0, \tag{3.24}$$

the equations (3.17) have no solution.

Notice that for $\tau < \pi/2$, we only get positive eigenvalues as we would expect, since we know from the previous section that the operator $\mathbf{K}$ is positive-definite for $\tau < \pi/2$.

Recall from Section 1 that (for $Z_0 = 0$)

$$\mathbb{E}\left[\exp\left\{ -\frac{\alpha}{2} \int_0^\tau Z_t^\mathrm{T} \mathbf{J} Z_t \, dt \right\} \right]$$

$$= \mathbb{E}\left[\exp\left\{ -\frac{\alpha}{2} \sum_n \lambda_n \xi_n^2 \right\} \right] = \prod_n \sqrt{\frac{1}{1 + \alpha \lambda_n}}. \tag{3.25}$$

We are now in a position to calculate the product

$$p(\alpha) := \prod_{n=0}^\infty (1 + \alpha \lambda_n).$$

We know that for $\lambda_n < -1$ or $\lambda_n > 0$, we have $\lambda_n = 1/(x_n^2 - 1)$ where x_n are the positive solutions of (3.19). However even for $-1 < \lambda_n < 0$, we could obtain λ_n by solving an equation of the form (3.19) since (3.21) may be written as (3.19) if we let $x_n = \sqrt{(\lambda_n + 1)/\lambda_n}$ be purely imaginary, in which case we still have $\lambda_n = 1/(x_n^2 - 1)$. Also, since

$$\lim_{x \downarrow 0} \frac{\sin(\tau x)}{x} = \tau$$

we see that (3.19) is consistent with (3.24) even in the case $\lambda_n = -1$. Thus irrespective of the value of λ_n, it is valid to say that λ_n solves (3.19) with $\lambda_n = 1/(x_n^2 - 1)$.

Now the roots of $p(\alpha)$ occur at $\alpha = -1/\lambda_n = 1 - x_n^2$, which because of equation (3.19), coincide with the real roots of

$$f(\alpha) := \sin \tau \frac{\sin(\tau \sqrt{1-\alpha})}{\sqrt{1-\alpha}} + \cos \tau \cos(\tau \sqrt{1-\alpha}).$$

Suppose now that f has other complex roots and let $\zeta = x + iy \in \mathbb{C}$ satisfy

$$\sin \tau \frac{\sin(\tau \zeta)}{\zeta} + \cos \tau \cos(\tau \zeta) = 0.$$

.

If either $x=0$ or $y=0$ then this just reduces respectively to the equations (3.19) and (3.21), so we suppose that neither of x and y is zero. For $\sin \tau \neq 0$, the above equation may be written as

$$\zeta \cot \tau + \tan (\tau \zeta) = 0.$$

Writing this in terms of the real and imaginary parts gives

$$x \cot \tau + \frac{\tan (\tau x) \operatorname{sech}^2 (\tau y)}{1 + \tan^2 (\tau x) \tanh^2 (\tau y)} = 0$$

$$y \cot \tau + \frac{\tanh (\tau y) \sec^2 (\tau x)}{1 + \tan^2 (\tau x) \tanh^2 (\tau y)} = 0.$$

Since $x \neq 0$ and $y \neq 0$ we can divide these two equations respectively by x and y to give

$$\frac{\tan (\tau x) \operatorname{sech}^2 (\tau y)}{x} = \frac{\tanh (\tau y) \sec^2 (\tau x)}{y}$$

and, putting $x' = 2 \tau x$, $y' = 2 \tau y$, this implies that

$$\frac{\sin x'}{x'} = \frac{\sinh y'}{y'},$$

which is impossible, since

$$\frac{\sin x'}{x'} \leqq 1 \quad \text{while} \quad \frac{\sinh y'}{y'} \geqq 1$$

and equality holds only when $x' = y' = 0$. This shows that the function $f(\alpha)$ has only those real roots which we have described and no others. Also, since $\sin (\tau x)/x \to \tau$ as $x \downarrow 0$, the function f has a removable singularity at $\alpha = 1$ and so we may regard f as an integral function. Moreover, since the functions $\sin \sqrt{\zeta}$ and $\cos \sqrt{\zeta}$ are both of order $1/2$, $f(\alpha)$ has order $1/2$ and since $f(0) = 1$, the Hadamard factorisation theorem (see Titchmarsh [8]) gives

$$p(\alpha) = f(\alpha)$$

and we have finally arrived at the formula

$$\mathbb{E}\left[\exp \left\{ -\frac{\alpha}{2} \int_0^\tau Z_t^T JZ_t \, dt \right\} \right]$$

$$= \left(\sin \tau \, \frac{\sin (\tau \sqrt{1-\alpha})}{\sqrt{1-\alpha}} + \cos \tau \cos (\tau \sqrt{1-\alpha}) \right)^{-1/2}. \quad (3.26)$$

It is worth emphasising again that (3.26) is only valid for certain values of α [viz. those for which $p(\alpha) \neq 0$] and that, when there are negative as well as positive eigenvalues, the admissible values of α must lie in an

interval around 0, viz.

$$\frac{-1}{\lambda_{max}} < \alpha < \frac{-1}{\lambda_{min}}, \qquad (3.27)$$

where λ_{max} is the largest (positive) eigenvalue and λ_{min} is the smallest (negative) eigenvalue. Moreover, notice from (3.19) that as τ increases, both λ_{max} and λ_{min} increase in modulus, so that (3.27) gives a smaller and smaller admissible range of α.

Let us now see how to calculate the Laplace transform (1.2) for a non-zero starting point $Z_0 = z = (x, v)$. Let $\tilde{Z}$ be defined as in Section 2 with

$$m(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} Z_0.$$

We deal with each of the terms in (2.6) in turn. The first term may be dealt with by means of (3.25). The last term is easily calculated to be

$$\int_0^\tau (x \sin s - v \cos s)^2 - (x \cos s + v \sin s)^2 \, ds$$

$$= (v^2 - x^2) \frac{\sin 2\tau}{2} - 2 xv \sin^2 \tau. \quad (3.28)$$

[Note that (3.28) is precisely the action of the classical least-action path starting at (x, v).] Consider now the second term in (2.6) which, as we have seen, may be written as in (2.7). The integral on the right-hand side of (2.7) may be evaluated by integration by parts:

$$\int_0^\tau m(t)^T e_n(t) \, dt = - \int_0^\tau x (e_n^V(s) \sin s - e_n^X(s) \cos s) \, ds$$

$$+ \int_0^\tau v (e_n^V(s) \cos s + e_n^X(s) \sin s) \, ds$$

$$= x e_n^X(\tau) \sin \tau - v e_n^X(\tau) \cos \tau. \qquad (3.29)$$

If we now normalise the eigenfunctions so that $e_n^X(\tau) = 1$, this then gives

$$\mathbb{E}^z \left[\exp \left\{ -\frac{\alpha}{2} \int_0^\tau (\tilde{Z}_t^T J \tilde{Z}_t + 2 \tilde{Z}_t^T J m(t)) \, dt \right\} \right]$$

$$= \mathbb{E}^z \left[\exp \left\{ -\frac{\alpha}{2} \sum_n \lambda_n \xi_n^2 + \alpha \sum_n \sqrt{\frac{\lambda_n}{(e_n, e_n)}} (v \cos \tau - x \sin \tau) \xi_n \right\} \right]$$

$$= \prod_n \mathbb{E}^z \left[\exp \left\{ -\frac{\alpha \lambda_n}{2} (\xi_n - \gamma_n/\lambda_n)^2 \right\} \right] \exp \left\{ \frac{\alpha \gamma_n^2}{2 \lambda_n} \right\}$$

$$= \prod_n (1 + \alpha \lambda_n)^{-1/2} \exp \left\{ -\frac{\gamma_n^2}{2 \lambda_n^2} [1 - (1 + \alpha \lambda_n)^{-1}] \right\} \exp \left\{ \frac{\alpha \gamma_n^2}{2 \lambda_n} \right\}$$

$$= \prod_n (1 + \alpha \lambda_n)^{-1/2} \exp \left\{ \frac{\alpha^2 \gamma_n^2}{2(1 + \alpha \lambda_n)} \right\}, \qquad (3.30)$$

where we have put

$$\gamma_n = \sqrt{\frac{\lambda_n}{(e_n, e_n)}} \, (v \cos \tau - x \sin \tau).$$

In our calculation (3.30) we have used the formula for the Laplace transform of a non-central χ-squared distribution with 1 degree of freedom and non-centrality parameter γ_n^2 / λ_n^2 (by which we mean the square of the mean of the corresponding Normal distribution), with the Laplace transform being evaluated at the argument $\alpha \lambda_n / 2$. For the moment we do not bother to calculate the scalar product (e_n, e_n) because, except for some special values of τ, the calculation is rather messy.

For α as in (3.27), let $G(\alpha, \tau)$ denote the non-negative function

$$G(\alpha, \tau) = \sum_{n=0}^{\infty} \frac{\lambda_n}{(e_n, e_n)(1 + \alpha \lambda_n)}. \qquad (3.31)$$

(The above sum can easily be shown by direct calculation to converge.) If we put $G(\alpha, 0) = 0$ for all α then $G(., .)$ is actually jointly continuous. Thus, by means of (3.26) and (3.30) we have now arrived at the formula

$$\mathbb{E}^{(x,\,v)} \left[\exp \left\{ - \frac{\alpha}{2} \int_0^\tau Z_t^T J Z_t \, dt \right\} \right]$$

$$= \exp \left\{ - \frac{\alpha}{2} \left[z^T J_z \frac{\sin 2\tau}{2} - 2 xv \sin^2 \tau \right] \right\}$$

$$\times \exp \left\{ \frac{\alpha^2}{2} (v \cos \tau - x \sin \tau)^2 G(\alpha, \tau) \right\}$$

$$\times \left(\sin \tau \frac{\sin(\tau \sqrt{1-\alpha})}{\sqrt{1-\alpha}} + \cos \tau \cos(\tau \sqrt{1-\alpha}) \right)^{-1/2}. \quad (3.32)$$

4. SOME CONNECTIONS WITH FEYNMAN INTEGRALS

In this section we show how the results obtained so far may be interpreted in the context of results on Feynman integrals, in particular the Cameron-Martin formula and the translation formula of Elworthy and Truman [4] for Feynman integrals.

We first recall the essential ingredients for defining the Feynman path integral. Let H be a Hilbert space of paths with inner product $\langle ., . \rangle_H$. The basic class of Feynman-integrable functionals are those functionals

$f: H \to \mathbb{C}$ which are Fourier transforms of complex measures on H with bounded variation. We denote this class of functionals by $\mathscr{F}$ (H); thus $f \in \mathscr{F}$ (H) if and only if

$$f(\gamma) = \int_H \exp\{i \langle \gamma, x \rangle_H\} d\mu_f(x), \qquad (4.1)$$

for some complex measure μ_f. The Feynman integral of $f \in \mathscr{F}$ (H) will be denoted by $\mathbb{F}(f)$. (*See* [4] and the many references given there for how such integrals are constructed.)

In [4], Elworthy and Truman obtained the following Cameron-Martin formula for the Feynman integral:

THEOREM 4.1. − *Let* $L: H \to H$ *be a trace-class and selfadjoint* (with respect to $\langle ., . \rangle_H$!) *linear operator such that* $(I + L)$ *is a bijection. Let* $g: H \to \mathbb{C}$ *be defined by*

$$g(\gamma) = \exp\{i \langle \gamma, L\gamma \rangle_H / 2\} f(\gamma), \qquad \gamma \in H,$$

where $f \in \mathscr{F}$ (H). *Then g is Feynman-integrable and*

$$\mathbb{F}(g) = |\det(I + L)|^{-1/2} \exp\{-i\pi \operatorname{ind}(I + L)/2\}$$

$$\times \int_H \exp\{-i \langle \gamma, (I + L)^{-1} \gamma \rangle_H / 2\} d\mu_f(\gamma). \quad (4.2)$$

Here det is the Fredholm determinant and the index ind (T) of an operator T with countably many eigenvalues of finite multiplicity, is the number of negative eigenvalues of T, counted according to multiplicity.

We also have the following translation formula which is analogous to the familiar Cameron-Martin-Girsanov change of drift formula for Brownian motion. For $a \in H$, let $g_a: H \to \mathbb{C}$ be defined by $g_a(\gamma) = g(\gamma + a)$, $g \in \mathscr{F}$ (H). Then the functional $\exp\{i \langle a, . \rangle_H\} g_a(.)$ is Feynman integrable and

$$\mathbb{F}(\exp\{i \langle a, . \rangle_H\} g_a(.)) = \exp\{-i\|a\|_H^2 / 2\} \mathbb{F}(g). \qquad (4.3)$$

Return now to our original setting and continue with the same notation established in the previous sections. We define a Hilbert space of paths H to be the space of absolutely continuous functions f on $[0, \tau]$ satisfying the boundary conditions

$$f(0) = 0, \qquad (4.4\,a)$$

$$\dot{f}(\tau) = -\int_0^\tau f(u)\, du. \qquad (4.4\,b)$$

[These boundary conditions correspond to $(3.17\,c)$-$(3.17\,d)$]. The inner product on H is given by

$$\langle f, g \rangle_H = \langle f, \mathbf{C}_1^{-1} g \rangle.$$

(Recall that C_1 is the "velocity" covariance operator as defined in Section 2 and $\langle ., . \rangle$ is of course the same as ever.) Let L be the operator $L = C_1(I - A)$. Notice that L is the $\langle \ \rangle$-adjoint of K_1 and is $\langle \ \rangle_H$-selfadjoint. In particular, L has the same eigenvalues as K_1.

Observe now that the action of the harmonic oscillator may be written as

$$\int_0^\tau Z_t^T J Z_t \, dt = \langle V, (I - A) V \rangle = \langle V, LV \rangle_H,$$

so the Laplace transform of the action as at (3.25) is just $\mathbb{E}[\exp\{-1/2 \langle V, \alpha LV \rangle_H\}]$. Moreover the product on the right-hand side of (3.25) $\prod (1 + \alpha \lambda_n)^{-1/2}$ is nothing but $\det(I + \alpha L)^{-1/2}$, since L is has a continuous kernel and so the Fredholm determinant is the same as the characteristic determinant. Notice also that the range of admissible values of α ensures that $(I + \alpha L)$ is positive-definite, so its index is zero. Therefore, the formula (3.26) is the "Cameron-Martin formula" for Wiener integrals which is the exact analogue of the formula (4.2) for Feynamn integrals (with $f \equiv 1$). It is also clear from the way (3.32) was arrived at that it corresponds exactly to the translation formula (4.3) for Feynman integrals. The formula (4.2) is itself analogous to the classical Cameron-Martin formula [2] for the Wiener integral of linear transformations of Brownian motion. In fact Cameron and Martin [2] were able to use their formula to obtain the Laplace transform of $\int_0^1 B_s^2 \, ds$.

Pushing the analogy with Feynman integrals a little further, observe that while we do not have an explicit representation for $\langle ., . \rangle_H$ as we do not know what C_1^{-1} is explicitly, we do have a natural *reproducing kernel* for H: this is simply the covariance kernel $c_1(s, t)$, for by definition, it has the property that

$$\langle c_1(s, .), f(.) \rangle_H = f(s), \qquad \forall f \in H. \qquad (4.5)$$

These connections become even more transparent if we consider the very simple example of Brownian motion started at 0 and run up to time 1.

Let $G(s, t) = s \wedge t$, $s, t \in [0, 1]$, be the covariance kernel. The covariance operator has eigenvalues

$$\frac{1}{(n + 1/2)^2 \pi^2}, \qquad n = 0, 1, 2 \ldots$$

and eigenfunctions φ_n satisfying $\varphi_n(0) = 0$ and $\dot{\varphi}_n(1) = 0$. The natural Hilbert space of paths H is the space of absolutely continuous functions on [0, 1] satisfying these same boundary conditions, with inner product

$$\langle f, g \rangle_H = \langle \dot{f}, \dot{g} \rangle = -\langle f, \ddot{g} \rangle.$$

(The inner product on H is thus the one naturally associated with the Hamiltonian of the free particle. The derivatives are to be interpreted in the weak L^2 sense.) Now $G(s, t)$ is precisely the Green's function of $-d^2/dt^2$ satisfying the boundary conditions $G(s, 0) = 0$ and $\dot{G}(s, 1) = 0$ and these same boundary conditions for $f \in H$ mean that $G(s, t)$ is actually the inverse kernel of $-d^2/dt^2$. Hence we see that the covariance kernel $G(s, t)$ is once again the reproducing kernel on H. (Because our integrals are Wiener integrals rather than Feynman integrals, the boundary conditions are different; in particular the starting point and not the final point is fixed. This explains why the reproducing kernel is not the more usual $1 - s \vee t$ encountered in the Feynman-integral formulation of quantum mechanics.)

Finally, it is worth pointing out that — unlike the case of the free particle — in the case of our phase-space picture of the harmonic oscillator, the natural inner product on H (defined via the inverse of the covariance operator) is not the one which one might at first expect, namely the inner product

$$\langle \dot{f}, \dot{g} \rangle + \langle f, g \rangle \qquad (4.6)$$

associated with the Hamiltonian of the harmonic oscillator. That this is not the inner product defined by C_1^{-1} is implied by (among other things) the rather strange boundary condition at $(4.4b)$. The explanation for this difference lies in the fact that, while (4.6) is the natural inner product to put on paths in *configuration space*, our sample paths are realisations of the *velocity* process (*i.e.* essentially paths in phase-space).

5. AN APPLICATION TO LEAST-ACTION PATHS

In this section we see how one could use a Cameron-Martin-Girsanov type change of measure to construct paths of the process $Z_t = (X_t, V_t)$ satisfying (3.1) which have least action. The deterministic harmonic oscillator corresponding to (3.1) has period 2π, so we are mainly interested in the case $\tau = 2\pi$. Let $\mathbb{P} = \mathbb{P}^z$ be the law of $Z_{\tau \wedge \cdot}$ and $\mathscr{F}_t = \sigma\{Z_s, s \leq (\tau \wedge t)\}$ be its natural filtration. Define a functional $\rho(t) = \rho_\alpha(t)$ by

$$\rho_\alpha(t) = \mathbb{E}^z \left[\exp\left\{ -\frac{\alpha}{2} \int_0^{2\pi} Z_s^T J Z_s \, ds \right\} \bigg| \mathscr{F}_t \right]. \qquad (5.1)$$

(The above expectation is taken under $\mathbb{P}$.) Thus $\rho(t)$ is actually a $\mathbb{P}$-martingale. Let us first try a change of measure of the following form. We define a measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathscr{F}_t} = \frac{\rho_\alpha(t)}{\rho_\alpha(0)}.$$

We leave the reader to prove the following two simple results (or else see Jacod and Shiryaev [6]).

LEMMA 5.1. — Y_t *is a local $\mathbb{Q}$-martingale if and only if $\rho(t) Y_t$ is a local $\mathbb{P}$-martingale.*

LEMMA 5.2. — *Let $\mathscr{G}$ be the generator of Z. Then for $f \in D(\mathscr{G})$ and $t < 2\pi$,*

$$f(Z_t) - f(Z_0) - \int_0^t \mathscr{G} f(Z_s) \, ds - \int_0^t \rho(s)^{-1} \, d[\rho(.), f(Z.)]_s$$

is a local $\mathbb{Q}$-martingale.

The Markov property for Z shows that

$$\rho_\alpha(t) = \exp \left\{ -\frac{\alpha}{2} \int_0^t Z_s^T J Z_s \, ds \right\} \mathbb{E}^z \left[\exp \left\{ -\frac{\alpha}{2} \int_t^{2\pi} Z_s^T J Z_s \, ds \right\} \middle| \mathscr{F}_t \right]$$

$$= \exp \left\{ -\frac{\alpha}{2} \int_0^t Z_s^T J Z_s \, ds \right\} \mathbb{E}^{Z_t} \left[\exp \left\{ -\frac{\alpha}{2} \int_0^{2\pi-t} Z_s^T J Z_s \, ds \right\} \right]$$

$$= \exp \left\{ -\frac{\alpha}{2} \int_0^t Z_s^T J Z_s \, ds \right\} H_\alpha(X_t, V_t, t), \quad \text{(say)}, \qquad (5.2)$$

where $H_\alpha(X_t, V_t, t)$ is given by (3.32) with $\tau = 2\pi - t$. Therefore, for $f = f(Z_t)$ we have

$$d[\rho, f]_t = d\rho \, df = \exp \left\{ -\frac{\alpha}{2} \int_0^t Z_s^T J Z_s \, ds \right\} dH \, df$$

$$= \exp \left\{ -\frac{\alpha}{2} \int_0^t Z_s^T J Z_s \, ds \right\} \frac{\partial H_\alpha}{\partial v} \frac{\partial f}{\partial v} \, dt.$$

Hence, from Lemma 5.2, the change of measure induces an additional time-dependent drift griven by

$$H_\alpha(X_t, V_t, t)^{-1} \frac{\partial H_\alpha}{\partial v} \frac{\partial}{\partial v}. \qquad (5.3)$$

We can calculate the drift (5.3) induced by $\mathbb{Q}$ using the formula (3.32) for the function $H_\alpha(x, v, t)$. Differentiating (3.32) gives

$$H_\alpha(x, v, t)^{-1} \frac{\partial H_\alpha}{\partial v} = \alpha x \sin^2(2\pi - t) - \alpha v \sin(2\pi - t) \cos(2\pi - t)$$

$$+ \alpha^2 v \cos^2(2\pi - t) G(\alpha, 2\pi - t)$$
$$- \alpha^2 x \sin(2\pi - t) \cos(2\pi - t) G(\alpha, 2\pi - t)$$
$$= (x \sin t + v \cos t)(\alpha^2 G(\alpha, 2\pi - t) \cos t + \alpha \sin t), \quad (5.4)$$

where G is the function defined at (3.31).

For $\tau = 2\pi$, it is easily seen from (3.19) that $x_n = n/2 + 1/4$ [and $\lambda_n = 1/(x_n^2 - 1)$ as before]. It is also easy to calculate, assuming the normalisation $e_n^X(2\pi) = 1$,

$$(e_n, e_n) = \frac{\pi}{\lambda_n},$$

$$G(\alpha, 2\pi) = -\frac{\tan(2\pi\sqrt{1-\alpha})}{\alpha\sqrt{1-\alpha}}. \tag{5.5}$$

Moreover the interval (3.27) now becomes $-9/16 < \alpha < 7/16$.

One puzzling aspect of our calculations is that, since α is constrained to lie in a finite interval, there seems to be *two* extremal paths rather than just the one extremal predicted by classical mechanics: as $\alpha \to 7/16$ the action is minimised while as $\alpha \to -9/16$ the action is "maximised"! Also, from (5.5) we see that as α approaches the two critical values, the drift (5.4) induced by $\mathbb{Q}$ is infinite at $t = 0$. The explanation for this unexpected behaviour lies in the observation that, as $\alpha \to 7/16$, the measure $\mathbb{Q}$ "reweights" the sample paths of Z in favour of those paths which achieve a small (*i.e.* very negative) action, while as $\alpha \to -9/16$, $\mathbb{Q}$ favours those paths which achieve a large (*i.e.* very positive) action; in either case, actions which are large in modulus are favoured. But from (3.28), we see that the classical least-action path for simple harmonic motion has *zero* action over one period. Hamilton's principle merely says that the classical path is the one which achieves the extremal of the action, not necessarily the minimum of the action.

In view of this observation, let us try to change the measure via the *modulus* of the action. We define a measure $\widetilde{\mathbb{Q}}$ such that

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{r_\alpha(t)}{r_\alpha(0)},$$

where r_α is the functional

$$r_\alpha(t) = \mathbb{E}^z[\exp\{-\alpha|S(\tau) - S_{cl}(\tau, z)|\} | \mathscr{F}_t], \tag{5.6}$$

where $S(\tau)$ is the action up to time τ and $S_{cl}(\tau, z)$ is the action realised by the classical path started at z up to time τ. For $\tau = 2\pi$, we have $S_{cl}(2\pi, z) = 0$.

First, a reminder of the following old trick. Let X be any random variable and for $\alpha > 0$, let $\varphi(\theta\alpha) = \mathbb{E}(e^{-i\theta\alpha X})$ be the characteristic function of X. We can integrate φ against the Cauchy density $[\pi(1 + \theta^2)]^{-1}$ to obtain

$$\int_{-\infty}^{\infty} \frac{\varphi(\theta\alpha)}{\pi(1 + \theta^2)} \, d\theta = \mathbb{E} \int_{-\infty}^{\infty} \frac{e^{-i\theta\alpha X}}{\pi(1 + \theta^2)} \, d\theta = \mathbb{E}[e^{-\alpha|X|}].$$

We can apply this to $X = S(2\pi)$ and $\varphi(\theta\alpha) = \rho_{i\theta\alpha}(t) = \rho(i\theta\alpha, t)$. This gives, using (5.2) and (3.32) and putting $u = 2\pi - t$,

$$
\begin{aligned}
r_\alpha(t) &= \int \frac{\rho(i\theta\alpha, t)}{\pi(1+\theta^2)} d\theta \\
&= \int d\theta \left[\frac{1}{\pi(1+\theta^2)} \right] \exp\{-i\theta\alpha S(t)\} \exp\{-i\theta\alpha S_{cl}(u, Z_t)\} \\
&\quad \times \exp\left\{ \frac{-\theta^2\alpha^2}{2} (V_t \cos u - X_t \sin u)^2 G(i\theta\alpha, u) \right\} \\
&\quad \times \left(\sin u \frac{\sin(u\sqrt{1-i\theta\alpha})}{\sqrt{1-i\theta\alpha}} + \cos u \cos(u\sqrt{1-i\theta\alpha}) \right)^{-1/2}. \quad (5.7)
\end{aligned}
$$

As before, the new measure $\tilde{Q}$ induces a drift given by

$$
r_\alpha(X_t, V_t, t)^{-1} \frac{\partial r_\alpha}{\partial v} \frac{\partial}{\partial v}. \quad (5.8)
$$

It is perhaps asking rather too much to expect to be able to calculate this drift explicitly since it is difficult actually to perform the integration at (5.7). We can however, show that the drift (5.8) has a finite limit as $\alpha \to \infty$.

Differentiating the expression (5.7) with respect to v under the integral sign gives

$$
\frac{\partial r_\alpha(t)}{\partial v} = \int \frac{\rho(i\theta\alpha, t) D(\theta\alpha, X_t, V_t, t)}{\pi(1+\theta^2)} d\theta, \quad (5.9)
$$

where, from (5.4),

$$
D(\theta\alpha, X_t, V_t, t) = (X_t \sin t + V_t \cos t)(-\theta^2\alpha^2 G(i\theta\alpha, 2\pi - t)\cos t + i\theta\alpha \sin t).
$$

The function G may be written (using the same notation as in Section 3)

$$
G(i\theta\alpha) = \sum_n \frac{\lambda_n}{(e_n, e_n)(1+i\theta\alpha\lambda_n)} = \sum_n \frac{\lambda_n(1-i\theta\alpha\lambda_n)}{(e_n, e_n)(1+(\theta\alpha\lambda_n)^2)}.
$$

Considering the real and imaginary parts of D separately, we see therefore that asymptotically (as $\alpha \to \infty$)

$$
D(\theta\alpha) \sim (X_t \sin t + V_t \cos t)(A + i\theta\alpha B)
$$

where A and B depend only on t and not on θ or α; for example,

$$
-A = \lim_{x \to \infty} x^2 \sum_n \frac{\lambda_n}{(e_n, e_n)(1+x^2\lambda_n^2)}.
$$

By the Dominated Convergence Theorem, the integral (5.9) is then asymptotically

$$\frac{\partial r_\alpha(t)}{\partial v} \sim (X_t \sin t + V_t \cos t)$$

$$\times \left\{ A \int \frac{\rho(i\theta\alpha, t)}{\pi(1+\theta^2)} \, d\theta + i B \int \frac{\theta\alpha\rho(i\theta\alpha, t)}{\pi(1+\theta^2)} \, d\theta \right\}. \quad (5.10)$$

Consider now the second term in braces $\{\ \ \}$ in (5.10), which is clearly real (and finite). Observed that

$$\int \frac{i\theta\alpha\rho(i\theta\alpha, t)}{\pi(1+\theta^2)} \, d\theta = \int \frac{i\alpha x \rho(ix, t)}{\pi(\alpha^2 + x^2)} \, dx \quad (5.11)$$

and that also

$$\int \frac{\rho(i\theta\alpha, t)}{\pi(1+\theta^2)} \, d\theta = \int \frac{\alpha\rho(ix, t)}{\pi(\alpha^2 + x^2)} \, dx \quad (5.12)$$

so that (5.11) and (5.12) have identical asymptotic behaviours as $\alpha \to \infty$. This shows that the drift (5.8) has a finite limit of the form $(X_t \sin t + V_t \cos t) F(t)$ as $\alpha \to \infty$.

Although we shall not actually prove this here, it is reasonable to expect that the corresponding processes with the additional drift (5.8) (under the law $\tilde{\mathbb{Q}}$) will converge weakly as $\alpha \to \infty$ to a process whose law induces an additional drift equal to the limit of the drift at (5.8).

This idea of integrating the Fourier transform of the action against the Cauchy density is very similar to a "method of stationary phase". Roughly speaking, if we replace α by $i\theta$ in (5.1) and let $\theta \to \infty$, then the dominant contribution to the expectation will come those paths which achieve extremals of the action. Of course, the main difficulty with this idea is that if we were to change the measure directly via the Fourier transform of the action, the resultant measure is complex-valued. What we have done here is essentially to get around this by taking the Cauchy transform of the complex-valued density to obtain one which is real-valued.

6. ANOTHER APPLICATION

We end with one final application of the techniques presented here for calculating laws of quadratic functionals of Gaussian processes, which, although very simple, is nevertheless interesting and has practical applications to polymer physics. The problem is to find the distribution of

$$\int_0^1 B_s^2 \, ds - \left(\int_0^1 B_s \, ds \right)^2 \quad (6.1)$$

where B is a BM $(\mathbb{R})$. For a detailed discussion of how this problem arises in applications to polymers in elongational flows, we refer the reader to the work of Jansons, Chan and Rogers [7]. It is worth emphasising however that there again, it is the idea of change of measure to favour paths of minimum energy that is the underlying theme of this work.

The functional (6.1) may be written as $\langle B, (I-M) B \rangle$ where M is the operator on $L^2 ([0, 1])$ defined by

$$M f (t) = \text{const.} = \int_0^1 f (s) \, ds$$

The operator M is compact and selfadjoint, so we are in the situation of Section 2. Furthermore, M has eigenvalues 0 and 1, so $(I - M)$ is a positive operator (but not positive-definite). The situation here is thus much easier than that in the preceding sections. No doubt one can apply the techniques from before directly to this problem, but the present situation admits one further simplification: let us consider first the distribution of

$$\langle B, (I - \rho M) B \rangle, \qquad 0 < \rho < 1. \tag{6.2}$$

The functional (6.2) is itself of interest in polymer physics and there is even a physical interpretation to the value of ρ (*see* [7]). The operator $(I - \rho M)$ is then positive-definite with positive-definite inverse and we can calculate the Laplace transform

$$\alpha \mapsto \mathbb{E} \left[\exp \left\{ - \alpha \langle B, (I - \rho M) B \rangle \right\} \right]$$

using only standard Hilbert space theory.

THEOREM 6.1. $-$ *Let* B *be a real-valued Brownian motion. Then*

$$\mathbb{E} \left[\exp \left\{ - \alpha \langle B, (I - \rho M) B \rangle \right\} \right]$$
$$= \left((1 - \rho) \cosh \sqrt{2 \alpha} + \rho \, \frac{\sinh \sqrt{2 \alpha}}{\sqrt{2 \alpha}} \right)^{-1/2}. \tag{6.3}$$

Proof. $-$ Define a new inner product on $L^2 ([0, 1])$ as follows:

$$\langle f, g \rangle_1 : = \langle f, J_1 g \rangle$$

where $J_1 = (I - \rho M)^{-1}$. Note that $\langle ., . \rangle_1$ is a true inner product in the Hilbert space sense. Define the covariance operator C via the covariance kernel $c (s, t) = s \wedge t$ and define the operator $K : = (I - \rho M) C$ which is an integral operator with kernel $k (s, t) = (I - \rho M) c (., t) (s)$. Using the usual standard arguments, we see that K is J_1-positive-definite, J_1-selfadjoint, etc. and therefore it has a countable set of real positive eigenvalues λ_n with corresponding J_1-orthonormal eigenfunctions e_n and the kernel $k (s, t)$ admits the expansion

$$k (s, t) = \sum_n \lambda_n e_n (s) [J_1 e_n] (t).$$

Hence, as before, the distribution of (6.1) may be calculated as

$$\mathbb{E}\left[\exp\left\{-\alpha\left\langle B, (I-\rho M) B\right\rangle\right\}\right]=\prod_n (1+2\alpha\lambda_n)^{-1/2}$$

and all that remains is to find the eigenvalues λ_n.

Differentiating twice as in (3.9)-(3.11) gives the following very simple ODE

$$\lambda_n e_n''(t) = -e_n(t),$$

whence immediately $e_n(t) = A\cos\omega_n t + B\sin\omega_n t$, where $\omega_n = \lambda_n^{-1/2}$. It is easy to deduce that

$$e_n'(1) = 0, \qquad\qquad (6.4a)$$

$$e_n(0) = A = -\frac{\rho}{\lambda_n}\int_0^1 e_n(s)(s-s^2/2)\,ds. \qquad\qquad (6.4b)$$

The boundary conditions (6.4) show (after a few easy calculations) that the eigenvalues are given by $\lambda_n = \omega_n^{-2}$, where ω_n are the positive solutions to

$$(1-\rho)\,\omega\cos\omega + \rho\sin\omega = 0. \qquad\qquad (6.5)$$

An argument identical to that used in Section 3 then gives the required result. $\square$

Letting $\rho\uparrow 1$ in (6.3) gives the desired Laplace transform

$$\mathbb{E}\left[\exp\left\{-\alpha\left\langle B, (I-M) B\right\rangle\right\}\right]=\left(\frac{\sinh\sqrt{2\alpha}}{\sqrt{2\alpha}}\right)^{-1/2}, \qquad (6.6)$$

$$\alpha > -\pi^2/2.$$

Notice also that letting $\rho\downarrow 0$ in (6.3) gives

$$\mathbb{E}\left[\exp\left\{-\alpha\int_0^1 B_s^2\,ds\right\}\right]=\sqrt{\frac{1}{\cosh\sqrt{2\alpha}}},$$

a result which has been known for a long time. (*See*, for example, Cameron and Martin [2].)

Just recently, Donati-Martin and Yor [3] have given a more elegant non-computational approach to the law of (6.1), by exploiting links between this problem and Lévy's famous stochastic area formula.

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