ANNALES DE L'I. H. P., SECTION B

JOSEPH GLOVER

Markov functions

Annales de l'I. H. P., section B, tome 27, nº 2 (1991), p. 221-238

<http://www.numdam.org/item?id=AIHPB_1991__27_2_221_0>

Ann. Inst. Henri Poincaré,

Vol. 27, n° 2, 1991, p. 221-238.

Probabilités et Statistiques

Markov functions (*)

by

Joseph GLOVER

University of Florida, Department of Mathematics,
Gainesville, FL 32611, U.S.A.

ABSTRACT. — Let (X_t, P^x) be a Markov process on E with semigroup P_t. Let K be a positive kernel from $(E, \mathscr{E})$ to $(F, \mathscr{F})$, and let (Q_t) be a family of positive kernels from $(F, \mathscr{F})$ to $(F, \mathscr{F})$. Assume: (i) $P_t K = KQ_t$; (ii) $p\sigma\{Kf : f \in p\mathscr{F}\} = \{Kf : f \in p\mathscr{F}\}$; and (iii) there is a function $q>0$ such that $Kq \leqq 1$. Then $\pi(x) = K(x, .)q(.)$ is a Markov function mapping E into the subprobability measures on $F : \pi(X_t)$ is a time homogeneous strong Markov process. Symmetry groups are used to construct such kernels K.

Key words : Markov process, Markov function, symmetry, intertwining.

RÉSUMÉ. — Soit (X_t, P^x) un processus de Markov sur E avec semigroupe P_t. Soit K un noyau positive de $(E, \mathscr{E})$ dans $(F, \mathscr{F})$, et soit (Q_t) une famille de noyaux positives sur $(F, \mathscr{F})$. Supposons que : (i) $P_t K = KQ_t$; (ii) $p\sigma\{Kf : f \in p\mathscr{F}\} = \{Kf : f \in p\mathscr{F}\}$, et (iii) il existe une fonction $q>0$ telle que $Kq \leqq 1$. Alors, $\pi(x) = K(x, .)q(.)$ est une fonction de Markov de E dans les sous-probabilités sur $F : \pi(X_t)$ est un processus de Markov forte homogène. De tels noyaux sont construits avec les groupes de symétries.

Classification A.M.S. : 60 J 25.

(*) Research supported in part by N.S.A. and N.S.F. by grant MDA904-89-H-2037.

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques - 0246-0203
Vol. 27/91/02/221/18/$ 3,80/© Gauthier-Villars

1. INTRODUCTION

Let (X_t, P^x) be a right continuous strong Markov process on a measurable state space $(E, \mathscr{E})$. Let φ be a function mapping E into another state space F: when is $\varphi(X_t)$ a Markov process? Such a φ will be called a Markov function. The roots of this article lie in three different articles, each of which is related to this question: Pitman and Rogers [10], Yor [13], and Glover and Mitro [8].

Let λ be an initial distribution for X. Pitman and Rogers formulated conditions which imply that $\varphi(X_t)$ is a Markov process under P^λ. Previously, Dynkin [3], Kemeny and Snell [9] and Rosenblatt [11] had given conditions implying that $\varphi(X_t)$ is Markov either: (i) under P^λ with λ an invariant measure; or (ii) under P^x for every $x \in E$. The Pitman-Rogers conditions involve not only φ, but also a Markov kernel Λ from F to E. Their conditions were motivated by the following observation. If for each $t>0$ and $A \in \mathscr{E}$,

$$(1.1) \qquad P^\lambda[X_t \in A \mid \varphi(X_s);\ 0 \leqq s \leqq t] = \Lambda(\varphi(X_t), A) \quad a.s.$$

then $\varphi \circ X$ is Markov with transition kernels defined by

$$(1.2) \qquad Q_t f = \Lambda P_t (f \circ \varphi)$$

If we let $\Phi f = f \circ \varphi$, then we can rewrite (1.2) as $Q_t f = \Lambda P_t \Phi f$. Their Theorem (2) follows.

(1.3) THEOREM. – *Suppose there is a Markov kernel* Λ *from* F *to* E *such that:*

(*a*) $\Lambda\Phi = I$, *the identity kernel on* F;

(*b*) *for each* $t \geqq 0$, $Q_t = \Lambda P_t \Phi$ *satisfies* $\Lambda P_t = Q_t \Lambda$.

Let X_t *be Markov with semigroup* P_t *and initial distribution* $\lambda = \Lambda(y, .)$, *where* $y \in F$. *Then* (1.1) *holds and* $\varphi \circ X$ *is Markov with transition semigroup* Q_t.

Our first motivation for this article is the following observation; it may happen that $\varphi \circ X$ is Markov, but there is no kernel satisfying (1.1). Consider the following discrete time example: $E = \{a, b, c, d\}$ and $F = \{a', b', e'\}$; $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(c) = \varphi(d) = e'$. For the semigroup, $P_0 = identity$, $P_1 f(a) = f(c)$, $P_1 f(b) = f(d)$, and $P_1 f(c) = P_1 f(d) = 0$ for every function f. If $\lambda = (\varepsilon_a + \varepsilon_b)/2$, then $\varphi(X_t)$ is Markov under P^λ, and

$$(1.4) \qquad P^\lambda[X_1 = c \mid \varphi(X_0), \varphi(X_1)] = 1_{\{X(0)=a\}} \quad a.s.$$

In particular, the conditional expectation in (1.4) is not a function of $\varphi(X_1) = e'$ *a.s.* (P^λ), so there can be no kernel Λ satisfying the conditions in (1.3). Thus it seems likely that there are more general conditions yielding $\varphi(X_t)$ Markov without (1.1) holding.

If X_t has semigroup P_t and Y_t is another Markov process with semigroup Q_t, then Yor says X_t and Y_t are *intertwined* by a kernel Λ if $\Lambda P_t = Q_t \Lambda$; this is a generalization of conditions (1.3*a*) and (1.3*b*). Using only this analytic relationship, he derives useful probabilistic information about the interwined processes, and it would be helpful to find an explicit probabilistic transformation relating X and Y. In section 2, motivated by this notion of intertwining, we assume: (i) the existence of a positive kernel K from $(E, \mathscr{E})$ to $(F, \mathscr{F})$ and a collection $(Q_t)_{t \geqq 0}$ of positive kernels from $(F, \mathscr{F})$ to $(F, \mathscr{F})$ satisfying $P_t K = K Q_t$ for every $t \geqq 0$. We do not assume that K is a Markov kernel (it may even be infinite), and we do not require (Q_t) to form a semigroup. In addition, we assume that: (ii) if $\mathscr{K} = \sigma\{Kf : f \in \mathscr{F}\}$, then each positive $\mathscr{K}$-measurable function can be written as Kf for some positive function f. This curious condition turns out to be quite natural and verifiable. If there is a function $q > 0$ such that $Kq \leqq 1$, then (i) and (ii) above imply that $t \to K(X_t, dy) q(y)$ is a time homogeneous strong Markov process taking values in the space of subprobability measures on F [*see* (2.2) and the comments following the statement of (2.2) for the precise meaning of the phrase "time homogeneous strong Markov process on $\pi(E)$"]. Thus, if we consider the map $\pi(x) = K(x, .) q(.)$ from E into the subprobabilities on F, then π is a Markov function. This map will not be injective, in general, and one can recover many well known examples of Markov functions; *see* (2.5) for the Brownian motion case.

In section 3, we extend this result to a potential theory setting. There, we assume (ii), but we replace (i) above with: (iii) $UK = KV$, where U is the potential of X and V is another positive kernel. Under some additional technical hypotheses, we obtain the same result: π is a Markov function [*see* (3.8), (3.9), and (3.10)].

In [8], Glover and Mitro considered the bijections ψ of E such that $\{f \circ \psi : f \in \mathscr{S}\} = \mathscr{S}$, where $\mathscr{S}$ is the collection of excessive functions of X. All of these bijections constitute a group G_A, and they constructed an algorithm which associates to each subgroup H of G_A a function f_H and a time change $\tau(H, t)$ such that $f_H(X_{\tau(H, t)})$ is Markov. That is, f_H is a Markov function for the process $X_{\tau(H, t)}$. This was done under some fairly strong topological and transience hypotheses. In section 4, we define two groups of symmetries, the group G_A defined above and the group G consisting of the bijections ψ of E such that $P_t(f \circ \psi) = (P_t f) \circ \psi$ for every $t \geqq 0$ and for every positive function f. It turns out that we can use these two groups to construct kernels Γ satisfying our assumptions (i), (ii), and (iii) above. Let H be a subgroup of either group, and assume that H is a locally compact topological group with left invariant measure m. (Slightly weaker assumptions are used in section 4.) If we define $K_\varphi(x, .) = \varepsilon_{\varphi(x)}(.)$

for every $\varphi \in \mathrm{H}$ and

$$\Gamma = \int \mathrm{K}_\varphi \, m(d\varphi)$$

then Γ is a positive kernel from E to E. If there is a function $q > 0$ such that $\Gamma q \leqq 1$, then, remarkably enough, Γ satisfies the peculiar assumption (ii) above. If H is a subgroup of G, then Γ also satisfies (i), so $\pi(\mathrm{X}_t)$ is a time homogeneous strong Markov process on $\pi(\mathrm{E})$, where $\pi(x) = \Gamma(x, .)\, q(.)$. If H is a subgroup of G_A, then there is a time change $\tau(t)$ such that $\pi(\mathrm{X}_{\tau(t)})$ is a time homogeneous Markov process provided certain transience assumptions (4.7) and (4.11) hold and provided $\Gamma(fq)$ is finely continuous whenever f is a uniformly continuous function on E. This last condition holds automatically if q is finely continuous and if the left invariant measure m is a probability measure. This result recasts the main result of Glover and Mitro [8]. These assumptions are weaker than the transience and topological conditions they used, although we have assumed the existence of m, which they did not need.

2. MARKOV FUNCTIONS AND KERNELS

Let $(\hat{\mathrm{E}}, \hat{d}_1)$ and $(\hat{\mathrm{F}}, \hat{d}_2)$ be two compact metric spaces with Borel fields $\hat{\mathscr{E}}$ and $\hat{\mathscr{F}}$ and universally measurable sets $\hat{\mathscr{E}}^u$ and $\hat{\mathscr{F}}^u$. We suppose that the state space E of our Markov process is (homeomorphic to) a set in $\hat{\mathscr{E}}$: E is called a Lusin space, and we let $\mathscr{E}$ and $\mathscr{E}^u$ denote the traces of $\hat{\mathscr{E}}$ and $\hat{\mathscr{E}}^u$ on E. Fix a point $\Delta \notin \mathrm{E}$ to serve as cemetery, and let $\mathrm{X} = (\Omega, \mathscr{A}, \mathscr{A}_t, \mathrm{X}_t, \theta_t, \mathrm{P}^x)$ be a right continuous strong Markov process on $(\mathrm{E}, \mathscr{E})$ satisfying the Right Hypotheses [12]. We let P_t and U^α denote the semigroup and resolvent of X, and we assume that they map positive $\mathscr{E}$-measurable functions to positive $\mathscr{E}$-measurable functions. We use a right process since this is a widely accepted framework for studying Markov processes, but many of our results depend only on having a time homogeneous strong Markov process.

Let $\mathrm{F} \in \hat{\mathscr{F}}$ be another Lusin space with trace σ-algebras $\mathscr{F}$ and $\mathscr{F}^u$. Recall that a kernel K from $(\mathrm{E}, \mathscr{E})$ to $(\mathrm{F}, \mathscr{F})$ is a function from $\mathrm{E} \times \mathscr{F}$ into $[0, \infty]$ with the following properties. First, $\mathrm{A} \to \mathrm{K}(x, \mathrm{A})$ is a positive measure for each $x \in \mathrm{E}$, and, second, $x \to \mathrm{K}(x, \mathrm{A})$ is $\mathscr{E}$-measurable for each $\mathrm{A} \in \mathscr{F}$. The kernels encountered in concrete situations are often not subMarkov, so we do *not* require $\mathrm{K}\,1 \leqq 1$. However, we do assume there is an $\mathscr{F}$-measurable function $q > 0$ such that $\mathrm{K} q \leqq 1$ on E. In addition, we suppose the following.

(2.1) HYPOTHESIS. – (i) $\{Kf : f \in p\mathcal{F}\} = p\sigma\{Kf : f \in p\mathcal{F}\}$. (ii) *For each* $t > 0$, *there is a positive kernel* Q_t *from* $(F, \mathcal{F})$ *to* $(F, \mathcal{F})$ *such that* $P_t K = K Q_t$.

The notation in (i) requires some explanation: if $\mathcal{B}$ is a σ-algebra, then $p\mathcal{B}$ customarily denotes the positive $\mathcal{B}$-measurable functions. Similarly, $b\mathcal{B}$ denotes the bounded $\mathcal{B}$-measurable functions. Thus, if $\mathcal{K} = \sigma\{Kf : f \in p\mathcal{F}\}$, the right side of (i) is the collection of positive $\mathcal{K}$-measurable functions, and we are requiring that each $g \in p\mathcal{K}$ can be written as Kf for some $f \in p\mathcal{F}$. Because of (i), $K(x, .)$ cannot be the zero measure. For if this were the case, it would be impossible to find a function f such that $Kf = 1 \in p\mathcal{K}$. The kernels Q_t in (ii) do not form a semigroup necessarily, despite the suggestive notation.

Extend q to all of $\hat{F}$ by defining $q(x) = 1$ for $x \in \hat{F} - F$. Let $\mathcal{M}(\hat{F})$ denote the collection of finite measures on $\hat{F}$ endowed with the Bernoulli topology; so $\mu_n \to \mu$ if and only if $\mu_n(f) \to \mu(f)$ for every $f \in C(\hat{F})$. [Notation: $C(\Xi)$ denotes the collection of real-valued continuous functions defined on a topological space Ξ.] Recall that this topology is the smallest one making the functions $L_f(\mu) = \mu(f)$ continuous functions whenever $f \in C(\hat{F})$. We shall work almost exclusively with $\mathcal{M}^1(\hat{F})$, the collection of subprobability measures on $\hat{F}$. This is a compact set in the Bernoulli topology. If we define

$$\mathcal{C} = \left\{ \prod_{i=1}^{n} L_{f_i} : (f_i) \subset C(\hat{F}),\ n \geqq 0 \right\}$$

then the linear span of $\mathcal{C}$ is an algebra of continuous functions on $\mathcal{M}^1(\hat{F})$ which separates points. [When $n = 0$, the product term is interpreted to be the constant function 1 on $\mathcal{M}^1(\hat{F})$.] By the Stone-Weierstrass theorem, this span is dense in $C(\mathcal{M}^1(\hat{F}))$, and we let $\mathcal{B}(\mathcal{M}^1(\hat{F})) = \sigma(\mathcal{C})$. A monotone class argument can be applied to show that $\mu \to L_f(\mu)$ is $\mathcal{B}(\mathcal{M}^1(\hat{F}))$-measurable whenever $f \in p\hat{\mathcal{F}}$. If we define the map $\pi : E \to \mathcal{M}^1(\hat{F})$ by $\pi(x) = K(x, .)q(.)$, then we can show that π is $\mathcal{E}/\mathcal{B}(\mathcal{M}^1(\hat{F}))$-measurable. Since an elementary set C in $\mathcal{B}(\mathcal{M}^1(\hat{F}))$ is of the form

$$C = \bigcap_{i=1}^{n} \{\mu : L_{f_i}(\mu) \in A_i\}$$

for sequences $(f_i) \subset C(\hat{F})$ and $(A_i) \subset \mathcal{B}(R)$, we need only check that

$$\begin{aligned}\{x : \pi(x) \in C\} &= \bigcap_{i=1}^{n} \{x : L_{f_i}(\pi(x)) \in A_i\} \\ &= \bigcap_{i=1}^{n} \{x : K(f_i q)(x) \in A_i\} \in \mathcal{E}\end{aligned}$$

It follows that $L_f \circ \pi$ is $\mathcal{E}$-measurable whenever $f \in p\hat{\mathcal{F}}$.

(2.2) THEOREM. – *Assume* (2.1). *Then* $\pi(X_s)$ *is a time homogeneous strong Markov process on* $\pi(E)$ *under* P^x, *for every* $x \in E$.

What is the precise meaning of the assertion in (2.2)? For us, a time homogeneous strong Markov process consists of several ingredients. First, there is a state space [which is $\pi(E)$ in our case] and a measurable structure [which we take to be the trace of $\mathscr{B}(\mathscr{M}^1(\hat{F}))$ on $\pi(E)$]. Second, there is a stochastic process $Y_t = \pi(X_t)$ of random variables : note that each Y_t is $\mathscr{A}_t/\mathscr{B}(\mathscr{M}^1(\hat{F}))$-measurable since π is $\mathscr{E}/\mathscr{B}(\mathscr{M}^1(\hat{F}))$-measurable. The third ingredient is the strong Markov property with respect to a filtration $\mathscr{A}_t$ containing $\sigma\{\pi(X_s) : s \leqq t\}$:

$$P^x[H \circ \pi(X_{t+S}) \mid \mathscr{A}_S] = R_t H(\pi(X_S)) \quad a.s.\ (P^x)$$

for each $(\mathscr{A}_t)$-stopping time S, for each $H \in p\mathscr{B}(\mathscr{M}^1(\hat{F}))$, for each $x \in E$. This is demonstrated in the proof below. The precise formulation of the semigroup R_t requires notation from the proof: it is contained in (2.4).

Some readers may be interested in whether or not $\pi(X_t)$ is a right process. We are not going to discuss this in detail, but we simply point out that some difficulties can crop up in attempting to answer this question, not least of which is understanding the structure of $\pi(E)$ in $\mathscr{M}^1(\hat{F})$. Since π is Borel measurable, $\pi(E)$ is analytic in $\mathscr{M}^1(\hat{F})$ (III-18d, [2]), and hence universally measurable. In general, π is not injective, and therefore $\pi(E)$ may not be Borel. So one nice property of E has already been lost!! If we assume our kernels to be only universally measurable instead of Borel measurable, then we cannot even show $\pi(E)$ is universally measurable in $\mathscr{M}^1(\hat{F})$.

Proof. – Let $H \in p\mathscr{B}(\mathscr{M}^1(\hat{F}))$, and let S be an $(\mathscr{A}_t)$-stopping time. Then

$$(2.3) \quad P^x[H \circ \pi(X_{t+S}) \mid \mathscr{A}_S] = P^{X(S)}[H \circ \pi(X_t)] = P_t[H \circ \pi](X_S) \quad a.s.\ (P^x)$$

Fix a sequence $(f_i) \subset C(\hat{F})$ of positive functions uniformly bounded by 1 and whose linear span is dense in $C(\hat{F})$. Then H can be represented as: $H(\mu) = G(\mu(f_1), \mu(f_2), \ldots)$ for some measurable function $G : [0, 1]^\infty \to R^+$. Since $H \circ \pi = G(K(f_1 q), K(f_2 q), \ldots)$ is $\mathscr{K}$-measurable, there is a function $f \in p\mathscr{F}$ such that $H \circ \pi = Kf$ by (2.1 i). By using (2.1 ii), we may rewrite (2.3) as

$$\begin{aligned} P_t(Kf)(X_S) &= KQ_t f(X_S) = K(qq^{-1} Q_t f)(X_S) \\ &= L_{q^{-1}Q_t f}(\pi(X_S)) \quad a.s.\ (P^x) \end{aligned}$$

That is, $\pi(X_t)$ is a time homogeneous strong Markov process. ■

While we have some latitude in choosing the function f in the proof, $L_{q^{-1}Q_t f} \circ \pi$ is determined uniquely at every point in E. For if h is another function in $p\mathcal{F}$ such that $Kh = H \circ \pi$, we obtain

$$\begin{aligned} L_{q^{-1}Q_t f}(\pi(x)) &= L_{q^{-1}Q_t f}(\pi(X_0)) = P_t K f(X_0) = P_t K h(X_0) \\ &= L_{q^{-1}Q_t h}(\pi(X_0)) = L_{q^{-1}Q_t h}(\pi(x)) \quad a.s.\ (P^x) \end{aligned}$$

for every $x \in E$.

(2.4) COROLLARY. – *For each* $t > 0$ *and* $\pi(x) \in \pi(E)$, *define*

$$R_t H(\pi(x)) = L_{q^{-1}Q_t f}(\pi(x)) = P^x[H \circ \pi(X_t)]$$

Then R_t *is a semigroup mapping* $\mathcal{B}(\mathcal{M}^1(\hat{F}))$*-measurable functions to* $\mathcal{B}(\mathcal{M}^1(\hat{F}))$*-measurable functions.*

Proof. – If H is $\mathcal{B}(\mathcal{M}^1(\hat{F}))$-measurable, then $f \in p\mathcal{F}$, $L_{q^{-1}Q_t f}$ is $\mathcal{B}(\mathcal{M}^1(\hat{F}))$-measurable, and

$$\begin{aligned} R_{t+s} H(\pi(x)) &= L_{q^{-1}Q_{t+s} f}(\pi(x)) = P^x[H \circ \pi(X_{t+s})] \\ &= P^x P^{X(s)}[H \circ \pi(X_t)] = P^x[L_{q^{-1}Q_t f}(\pi(X_s))] \\ &= P^x[R_t H(\pi(X_s))] = R_s R_t H(\pi(x)) \quad ■ \end{aligned}$$

There is a slight additional assumption we can make occasionally about K which enables us to eliminate the arbitrariness in choosing f in the proof of (2.2). The assumption can be verified in some examples. Suppose there is yet another kernel M from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ such that $KMK = K$. Then $Kf = KMKf$, and we obtain $P_t(Kf)(x) = KQ_t(MKf)(x)$. On the other hand, if $Kh = Kf$, then $MKh = MKf$, and this shows there is a unique function of the form MKf such that $K(MKf) = H \circ \pi$.

Two elementary examples may be helpful in absorbing these axiomatics.

(2.5) EXAMPLE. – Let $X_t = (B_t^1, B_t^2, \ldots, B_t^d)$ be Brownian motion on R^d with semigroup $P_t = P_t^1 \ldots P_t^d$. Let GE be the group generated by the rotations, translations and flips about hyperplanes in R^d. If $\varphi \in GE$, then $P_t(f \circ \varphi) = (P_t f) \circ \varphi$ for every $f \in p\mathcal{B}(R^d)$. So if we set $K_1(x, .) = \varepsilon_{\varphi(x)}(.)$, we have $P_t K_1 = K_1 P_t$. It is simple to check that K_1 satisfies (2.1 i), so $\pi(X_t)$ is a Markov process which we can identify with $\varphi(X_t)$ as follows. (Incidentally, there is another kernel M_1 such that $K_1 M_1 = M_1 K_1 =$ *identity* on R^d.) If we define the bijection $\Phi_1 : \pi(R^d) \to R^d$ by setting $\Phi_1(\pi(x)) = \varphi(x)$, then $\Phi_1(\pi(X_t)) = \varphi(X_t)$. A more interesting example may be the one given by

$$K_2(x, dy) = \prod_{i=1}^{d} (\varepsilon_{x_i}(dy_i) + \varepsilon_{-x_i}(dy_i))/2$$

This kernel also satisfies (2.1 i), but there is no kernel M_2 satisfying $M_2 K_2 =$ *identity*. If we define the bijection $\Phi_2 : \pi(R^d) \to (R^+)^d$ by setting

$\Phi_2(\pi(x)) = (|x_1|, \ldots, |x_d|)$, then $\Phi_2(\pi(X_t)) = (|B_t^1|, \ldots, |B_t^d|)$. An example of an infinite kernel satisfying (2.1 i) and (2.1 ii) is:

$$K_3(x, dy) = \varepsilon_{x_1}(dy_1) \ldots \varepsilon_{x_{d-1}}(dy_{d-1})\, dy_d$$

where dy_d is Lebesgue measure in the dth coordinate. If we set $\Phi_3(\pi(x)) = (x_1, \ldots, x_{d-1})$, then Φ_3 is a bijection from $\pi(R^d)$ to R^{d-1}, and $\Phi_3(\pi(X_t)) = (B_t^1, \ldots, B_t^{d-1})$. If we set $M_3 f(x) = f(x) 1_{[0,1]}(x_d)$, then $K_3 M_3 K_3 = K_3$.

(2.6) Example. – All of the kernels in (2.5) arise from consideration of the geometric symmetries GE of Brownian motion. A general construction procedure involving geometric symmetries will be discussed in section 4. But the temporal symmetries of a process may yield kernels satisfying (2.1) as well. Let X_t be any Markov process on R^d such that $P_t(x, dy) = p_t(x, y)\, dy$ with $P_t 1 = e^{-t}$ and

$$\int p_t(x, y)\, dx = e^{-t}$$

for every $y \in R^d$. Set $K_4(x, dy) = dy$. Then

$$K_4 P_t f = e^{-t} \int f\, dx = P_t K_4 f$$

for every $f \in p\mathscr{B}(R^d)$. The process $\pi(X_t)$ sits at one point for an exponential length of time and then dies.

3. THE POTENTIAL THEORY FORMULATION

We now extend the result in (2.2) to a potential setting; this will be useful in studying symmetries in section 4. Throughout this section, we assume that X is a transient process on E satisfying:

(3.1) Hypothesis. – $\sup\{U 1(x) : x \in E\} = a < 1$.

[Given a transient process $\tilde{X}$, one can reduce to a process X satisfying (3.1) by time change, as we shall do in section 4.] We assume there is a kernel K from $(E, \mathscr{E})$ to $(F, \mathscr{F})$ satisfying:

(3.2) Hypothesis. – (i) $\{Kf : f \in p\mathscr{F}\} = p\sigma\{Kf : f \in p\mathscr{F}\}$.

(ii) *There is a positive kernel* V *from* $(F, \mathscr{F})$ *to* $(F, \mathscr{F})$ *such that* $UK = KV$.

(iii) *There is an* $\mathscr{F}$*-measurable function* $q > 0$ *such that* $Kq \leq 1$ *on* E.

Note that (3.2 i) is the same hypothesis as (2.1 i). Fix an $\mathscr{F}$-measurable function f with $|f| \leq c$. It follows from the resolvent equation (*see*

(V-5.10) in [1]) that

$$\begin{aligned} U^\alpha K(fq) &= \sum_{n=0}^{\infty} (-\alpha)^n (U)^{n+1} K(fq) \\ &= \sum_{n=0}^{\infty} (-\alpha)^n K(V)^{n+1} (fq) \end{aligned}$$

for each $\alpha \leqq 1$. In fact, we need hypothesis (3.1) here to insure that the series converges absolutely. For if $\alpha \leqq 1$,

$$\sum_{n=0}^{\infty} \alpha^n (U)^{n+1} K(|fq|) \leqq ac(1-\alpha a)^{-1}$$

Thus,

$$K\left[\sum_{n=0}^{\infty} \alpha^n (V)^{n+1} (|fq|)\right] = \sum_{n=0}^{\infty} \alpha^n K(V)^{n+1} (|fq|) < \infty$$

so

$$\sum_{n=0}^{\infty} \alpha^n (V)^{n+1} (|fq|) < \infty \quad a.s. \ (K(x, dy))$$

for each $x \in E$. If we define

$$\Lambda_0 = \left\{ z : \sum_{n=0}^{\infty} (V)^{n+1} q(z) < \infty \right\}$$

then $K(x, \Lambda_0^c) = 0$ for each $x \in E$. For each $\alpha \leqq 1$, define a kernel of (possibly signed) measures V^α from $(F, \mathscr{F})$ to $(F, \mathscr{F})$ by setting

$$\begin{aligned} V^\alpha(fq)(x) &= \sum_{n=0}^{\infty} (-\alpha)^n (V)^{n+1} (fq)(x) \qquad \text{if} \quad x \in \Lambda_0 \\ &= 0 \qquad \text{if} \quad x \in \Lambda_0^c \end{aligned}$$

for $f \in b\mathscr{F}$. For each $\alpha \leqq 1$, we have $U^\alpha K = KV^\alpha$. An induction procedure will create a signed kernel V^α from $(F, \mathscr{F})$ to $(F, \mathscr{F})$ for every $\alpha \geqq 0$: suppose we have defined V^α for $\alpha \leqq N$ and that $U^\alpha K = KV^\alpha$ for every $\alpha \leqq N$. A repetition of the argument above yields

$$\sum_{n=0}^{\infty} \alpha^n (U^N)^{n+1} K(|fq|) < \infty$$

for $\alpha \leqq 1$ since $U^N \leqq U^0$. As above, we conclude that

$$\sum_{n=0}^{\infty} \alpha^n (V^N)^{n+1} (|fq|) < \infty \quad a.s. \ (K(x, dy))$$

for each $x \in \mathrm{E}$. Define

$$\Lambda_{\mathrm{N}} = \left\{ z : \sum_{n=0}^{\infty} (\mathrm{V}^{\mathrm{N}})^{n+1} q(z) < \infty \right\}$$

so $\mathrm{K}(x, \Lambda_{\mathrm{N}}^c) = 0$ for each $x \in \mathrm{E}$. For each $\alpha \leqq 1$, set

$$\begin{aligned} \mathrm{V}^{\mathrm{N}+\alpha}(fq)(x) &= \sum_{n=0}^{\infty} (-\alpha)^n (\mathrm{V}^{\mathrm{N}})^{n+1} (fq)(x) \qquad \text{if} \quad x \in \Lambda_{\mathrm{N}} \\ &= 0 \qquad \text{if} \quad x \in \Lambda_{\mathrm{N}}^c \end{aligned}$$

for $f \in b\mathscr{F}$. Since

$$\mathrm{U}^{\mathrm{N}+\alpha} = \sum_{n=0}^{\infty} (-\alpha)^n (\mathrm{U}^{\mathrm{N}})^{n+1}$$

for $\alpha \leqq 1$, we now have $\mathrm{U}^\alpha \mathrm{K} = \mathrm{K}\mathrm{V}^\alpha$ for every $\alpha \leqq \mathrm{N}+1$, and the induction step is complete. Note that V^α may not constitute a positive resolvent, even though the notation is suggestive.

Now let $(\mathrm{H}_i)_{i=1}^n$ be a sequence of positive continuous functions on $\mathscr{M}^1(\hat{\mathrm{F}})$, and let

$$\mathrm{Z} = \prod_{i=1}^{n} \int_0^{\infty} e^{-\alpha_i s} \mathrm{H}_i \circ \pi(\mathrm{X}_s)\, ds \quad . \tag{3.3}$$

where $\alpha_i > 0$ for every $i \leqq n$.

(3.4) PROPOSITION. – *Assume* (3.1) *and* (3.2), *and let* T *be an* $(\mathscr{A}_t)$-*stopping time. Then there is a function* $f \in p\mathscr{F}$ *such that*

$$\mathrm{P}^x[\mathrm{Z} \circ \theta_{\mathrm{T}} \mid \mathscr{A}_{\mathrm{T}}] = \mathrm{L}_{q^{-1}\mathrm{V}^\beta f}(\pi(\mathrm{X}_{\mathrm{T}})) \quad a.s.\ (\mathrm{P}^x) \tag{3.5}$$

for every $x \in \mathrm{E}$, *where* $\beta = \alpha_1 + \ldots + \alpha_n$.

Proof. – The left side of (3.5) is $\mathrm{P}^{\mathrm{X}(\mathrm{T})}[\mathrm{Z}]$. It is a standard energy computation to show that

$$\mathrm{P}^x[\mathrm{Z}] = \sum_{\sigma \in \mathrm{S}(n)} g_\sigma \tag{3.6}$$

where $\mathrm{S}(n)$ is the permutation group of $\{1, 2, \ldots, n\}$, and where g_σ is:

$$\begin{aligned} &\mathrm{U}^\beta \mathrm{H}_{\sigma(n)} \circ \pi\, \mathrm{U}^{\alpha_{\sigma(1)} + \cdots + \alpha_{\sigma(n-1)}} \mathrm{H}_{\sigma(n-1)} \circ \pi \times \ldots \\ &\qquad \times \mathrm{U}^{\alpha_{\sigma(1)} + \alpha_{\sigma(2)}} \mathrm{H}_{\sigma(2)} \circ \pi\, \mathrm{U}^{\alpha_{\sigma(1)}} \mathrm{H}_{\sigma(1)} \circ \pi \end{aligned}$$

The rightmost term $\mathrm{H}_{\sigma(1)} \circ \pi$ in the line above can be rewritten as $\mathrm{K} f_{\sigma(1)}$ for some function $f_{\sigma(1)} \in p\mathscr{F}$ by (3.2). Thus

$$\mathrm{U}^{\alpha_{\sigma(1)}} \mathrm{H}_{\sigma(1)} \circ \pi = \mathrm{K}\mathrm{V}^{\alpha_{\sigma(1)}} f_{\sigma(1)} \tag{3.7}$$

Applying (3.2) again, we have

$$\mathrm{H}_{\sigma(2)} \circ \pi\, \mathrm{K}\mathrm{V}^{\alpha_{\sigma(1)}} f_{\sigma(2)\,\sigma(1)}$$

for some function $f_{\sigma(2)\sigma(1)} \in p\mathscr{F}$; so

$$U^{\alpha_{\sigma(1)}+\alpha_{\sigma(2)}} H_{\sigma(2)} \circ \pi\, U^{\alpha_{\sigma(1)}} H_{\sigma(1)} \circ \pi = KV^{\alpha_{\sigma(1)}+\alpha_{\sigma(2)}} f_{\sigma(2)\sigma(1)}$$

Repeating this procedure, we see there are positive functions $f_\sigma = f_{\sigma(n)\sigma(n-1)\,\ldots\,\sigma(1)}$ such that

$$P^x[Z] = \sum_{\sigma \in S(n)} KV^{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(n)}} f_\sigma = KV^\beta f$$

where

$$f = \sum_{\sigma \in S(n)} f_\sigma$$

Thus

$$P^{X(T)}[Z] = KV^\beta f(X_T) = L_{q^{-1}V^\beta f}(\pi(X_T)) \quad a.s.\ (P^x)$$

for each $x \in E$. ■

We now give a general condition which implies that $\pi(X_t)$ is Markov. Let $\mathscr{Z} = \sigma\{Z : Z$ *is of the form* (3.3)$\}$, and let

$$\Pi = \sigma\{H \circ \pi(X_s) : H \in b\mathscr{B}(\mathscr{M}^1(\hat{F})),\ s \geqq 0\}.$$

(3.8) THEOREM. – *Assume* (3.1) *and* (3.2). *If* $\mathscr{Z} = \Pi$, *then* $\pi(X_t)$ *is a time homogeneous strong Markov process on* E.

[Recall the comments following the statement of (2.2).]

Proof. – Let $\mathscr{V}$ be the collection of random variables $Y \in b\Pi$ such that there is a function $H_Y \in b\mathscr{B}(\mathscr{M}^1(\hat{F}))$ with

$$P^x[Y \circ \theta_T \mid \mathscr{A}_T] = H_Y(\pi(X_T)) \quad a.s.\ (P^x)$$

for every $x \in E$, for every $(\mathscr{A}_t)$-stopping time T. Then $\mathscr{V}$ is a vector space. To check that $\mathscr{V}$ is monotone, let Y_n increase to $Y \in b\Pi$, set $H_n = H_{Y(n)}$, and observe that

$$\begin{aligned} 0 \leqq P^x[Y_{n+1} - Y_n &= P^x[H_{n+1} \circ \pi(X_0) - H_n \circ \pi(X_0)] \\ &= H_{n+1} \circ \pi(x) - H_n \circ \pi(x) \quad a.s.\ (P^x) \end{aligned}$$

for every $x \in E$. Therefore, $H_n 1_{\pi(E)}$ increases to a function

$$H\, 1_{\pi(E)} \in b\mathscr{B}(\mathscr{M}^1(\hat{F})),$$

and

$$P^x[Y \circ \theta_T \mid \mathscr{A}_T] = H(\pi(X_T)) \quad a.s.\ (P^x)$$

for every $x \in E$, for every $(\mathscr{A}_t)$-stopping time T. Since $\mathscr{V}$ contains the multiplicative class $\{Z : Z$ *is of the form* (3.3)$\}$, $b\mathscr{Z} \subset \mathscr{V}$. Since $b\mathscr{Z} = b\Pi$, $H \circ \pi(X_t) \in \mathscr{V}$ for each $H \in b\mathscr{B}(\mathscr{M}^1(\hat{F}))$, and we obtain

$$P^x[H \circ \pi(X_{t+T}) \mid \mathscr{A}_T] = G \circ \pi(X_T) \quad a.s.\ (P^x)$$

for some function $G \in b\mathscr{B}(\mathscr{M}^1(\hat{F}))$. ■

(3.9) Corollary. — *Assume* (3.1) *and* (3.2). *If* $\pi(X_t)$ *is a.s. right continuous in* $\mathcal{M}^1(\hat{F})$, *then* $\pi(X_t)$ *is a time homogeneous Markov process on* $\pi(E)$.

Proof. — Let f_n be the continuous fonction on R^+ which is zero on $[0, t] \cup [t+2/n, \infty)$ and has equation

$$\begin{aligned} f_n(s) &= n^2(s-t) \qquad \textit{if} \quad t \leqq s \leqq t+1/n \\ &= -n^2(s-t-2/n) \qquad \textit{if} \quad t+1/n \leqq s \leqq t+2/n \end{aligned}$$

We may approximate f_n uniformly on R^+ by exponentials: choose $p(n) \geqq 0$, $\alpha(n, k) \geqq 0$, and $c(n, k) \geqq 0$ such that $\|f_n - g_n\|_\infty < 1/n$, where

$$g_n(t) = \sum_{k=1}^{p(n)} c(n, j) \exp[-\alpha(n, k)t]$$

If $H \in C(\mathcal{M}^1(\hat{F}))$, then by a.s. right continuity of $\pi(X_t)$,

$$\liminf_{n \to \infty} \int_0^\infty e^{-s} g_n(s) H \circ \pi(X_s)\, ds = e^{-t} G_1$$

where $G_1 = H \circ \pi(X_t)$ *a.s.* (P^x) for each $x \in E$. Thus, if T is an $(\mathcal{A}_t)$-stopping time,

$$P^x[H \circ \pi(X_{t+T}) \mid \mathcal{A}_T] = P^x[G_1 \circ \theta_T \mid \mathcal{A}_T] = H_1 \circ \pi(X_T) \quad a.s.,$$

where $H_1 \in b\mathcal{B}(\mathcal{M}^1(\hat{F}))$ by the preceding proof. ■

Actually, the proof uses only the fact that $P^x[\lim_{s \downarrow t} \pi(X_s) = \pi(X_t)] = 1$ for each fixed t: this is a bit weaker than assuming that $t \to \pi(X_t)$ is a.s. right continuous.

(3.10) Corollary. — *Assume* (3.1) *and* (3.2), *and suppose* $K(fq)$ *is finely continuous on* E *whenever* $f \in C(\hat{F})$. *Then* $\pi(X_t)$ *is a.s. right continuous and a time homogeneous strong Markov process.*

Proof. — $\pi(X_t)$ is right continuous if and only if $L_f(\pi(X_t))$ is right continuous for every $f \in C(\hat{F})$. Since $L_f(\pi(X_t)) = K(fq)(X_t)$, it suffices to have $K(fq)$ finely continuous. ■

[In fact, the proof shows that $\pi(X_t)$ is a.s. right continuous if and only if $K(fq)$ is finely continuous for every $f \in C(\hat{F})$.]

4. SYMMETRIES

Let B denote the collection of bijections $\varphi: E \to E$ such that φ and φ^{-1} are $\mathcal{E}/\mathcal{E}$-measurable, and let G consist of those $\varphi \in B$ such that $P_t(f \circ \varphi) = (P_t f) \circ \varphi$ for every $t > 0$, for every $f \in p\mathcal{E}$. Then G is a group under composition. It might contain only the identity map, or it might be quite rich, as is the case in (2.5).

Let H be a subgroup of G. For each $\varphi \in H$, set $K_\varphi(x, .) = \varepsilon_{\varphi(x)}(.)$, and define $J = \{K_\varphi : \varphi \in H\}$. It is easy to check that: (i) J is a group under composition of kernels which is isomorphic to H; (ii) each $K \in J$ satisfies (2.1); and (iii) $KP_t = P_t K$ for every $K \in J$, for every $t > 0$. We assume the following regularity hypotheses about J.

(4.1) Hypothesis. – *There is a* σ*-algebra* $\mathscr{J}$ *on* J *and a nonzero* σ*-finite measure* $m(dK) = dK$ *on* $(J, \mathscr{J})$ *such that:*

(i) $(K, N) \to KN$ *and* $(K, N) \to K^{-1}N$ *are* $\mathscr{J} \times \mathscr{J} / \mathscr{J}$*-measurable;*

(ii) $(K, x) \to Kf(x)$ *is* $\mathscr{J} \times \mathscr{E} / \mathscr{B}(R)$*-measurable for every* $f \in p\mathscr{E}$*;*

(iii) $\int f(NK)\, dK = \int f(K)\, dK$ *for every* $f \in p\mathscr{J}$.

It is often the case [as in (2.5)] that H and J can be topologized to be locally compact groups. In this case, (4.1 i) and (4.1 iii) are satisfied when $\mathscr{J}$ is taken to be the Borel sets of J; (4.1 ii) is a measure of compatibility between $\mathscr{J}$ and $\mathscr{E}$: *see* (1.3) in [6] for a closely related example. In case J is countable, then (4.1) is always satisfied by taking $\mathscr{J}$ to consist of all subsets of J and m to be counting measure. We remark in passing that if $(J, \mathscr{J})$ is a Lusin measurable space, then (4.1) implies that J can be topologized to be a locally compact topological group by the Mackey-Weil theorem; we do not use this fact in this article: *see* [7].

If $f \in p\mathscr{E}$, then (4.1 iii) allows us to define

$$\Gamma f(x) = \int_J Kf(x)\, dK$$

so Γ is a kernel on $(E, \mathscr{E})$ which enjoys several special properties.

(4.2) Proposition. – $\Gamma(f \Gamma g) = (\Gamma f)(\Gamma g)$ *for every* $f, g \in p\mathscr{E}$.

Proof. – $\Gamma(f \Gamma g) = \int K(f \Gamma g)\, dK = \int (Kf)(K \Gamma g)\, dK$ since each $K = K_\varphi$ for some bijection $\varphi \in H$. But

$$K \Gamma g = \int KNg\, dN = \int Ng\, dN = \Gamma g$$

by (4.1 iii). Thus $\Gamma(f \Gamma g) = (\Gamma f)(\Gamma g)$. ■

(4.3) Proposition. – *If there is a function* $q \in p\mathscr{E}$ *such that* $q > 0$ *and* $\Gamma q \leqq 1$, *then* Γ *satisfies* (2.1).

Proof. – First, let $\mathscr{B}(\Gamma) = \sigma\{\Gamma f : f \in p\mathscr{E}\}$, and let

$$\mathscr{G}_1 = \{g \in b\mathscr{B}(\Gamma) : \Gamma(gfq) = g\Gamma(fq) \text{ for every } f \in b\mathscr{E}\}.$$

Then $\mathcal{Y}_1$ is a monotone vector space containing the constants. If g_1, $g_2 \in \mathcal{Y}_1$, then $\Gamma(g_1 g_2 fq) = g_1 \Gamma(g_2 fq) = g_1 g_2 \Gamma(fq)$, so $\mathcal{Y}_1$ is multiplicative. By (4.2), $\mathcal{Y}_1$ contains all functions of the form $\Gamma(fq)$ for $f \in b\mathcal{E}$. By the monotone class theorem, $\mathcal{Y}_1 = b\mathcal{B}(\Gamma)$.

An application of the monotone convergence theorem shows that $\Gamma(gfq) = g\Gamma(fq)$ for every $g \in p\mathcal{B}(\Gamma)$, for every $f \in p\mathcal{E}$. Thus, if $g \in p\mathcal{B}(\Gamma)$, then $\Gamma(gq/\Gamma(q)) = g\Gamma(q/\Gamma(q)) = g$ since $1/\Gamma(q) \in p\mathcal{B}(\Gamma)$, so (2.1) is satisfied. ■

(4.4) COROLLARY. – *If* $m(\mathrm{J}) = 1$, *then* $\Gamma^2 = \Gamma$.

Proof. – $\Gamma^2 f = \iint \mathrm{KN} f \, d\mathrm{N} \, d\mathrm{K} = \iint \mathrm{N} f \, d\mathrm{N} \, d\mathrm{K}$ *by* (4.1 iii). *This is* $m(1)\Gamma f = \Gamma(f)$. ■

Dynkin [4] developed his theory of sufficient satistics based on Markov kernels Q satisfying $\mathrm{Q}(f\,\mathrm{Q}g) = (\mathrm{Q}f)(\mathrm{Q}g)$, and if Γ is Markov, then his theory is applicable here. However, Γ need not be a Markov kernel, in general.

(4.5) THEOREM. – *If there is a function* $q \in p\mathcal{E}$ *with* $q > 0$ *and* $\Gamma q \leqq 1$, *then* $\pi(\mathrm{X}_t)$ *is a time homogeneous strong Markov process on* $\pi(\mathrm{E})$ *under* P^x, *for every* $x \in \mathrm{E}$, *where* $\pi(x) = \Gamma(x, .)q(.)$.

Proof. – Since $\mathrm{KP}_t = \mathrm{P}_t\mathrm{K}$ for every $\mathrm{K} \in \mathrm{J}$, we obtain $\Gamma\mathrm{P}_t = \mathrm{P}_t\Gamma$. An application of Theorem (2.2) completes the proof. ■

We now identify $\pi(\mathrm{X}_t)$ in another way, as is done in [8]. Define an equivalence relation $\sim$ on E by setting $x \sim y$ if and only if there is a $\varphi \in \mathrm{H}$ such that $\varphi(x) = y$. Each equivalence class $[x]$ is called an H-orbit, and we let $\mathrm{F} = \{[x] : x \in \mathrm{E}\}$ be the collection of all H-orbits. There is a natural surjection $\Phi : \mathrm{E} \to \mathrm{F}$ defined by $\Phi(x) = [x]$, and we would like to compare $\pi(\mathrm{X}_t)$ with $\Phi(\mathrm{X}_t)$.

(4.6) PROPOSITION. – *Suppose* $[x] \in \mathcal{E}^u$ *for every* $x \in \mathrm{E}$. *Then* $\pi(x) = \pi(y)$ *if and only if* $[x] = [y]$.

Proof. – If $[x] = [y]$, we may choose $\varphi \in \mathrm{H}$ such that $\varphi(x) = y$. Then

$$\begin{aligned}\Gamma f(x) &= \int \mathrm{K} f(x) \, d\mathrm{K} = \int \mathrm{K}_\varphi(\mathrm{K} f)(x) \, d\mathrm{K} \\ &= \int (\mathrm{K} f)(\varphi(x)) \, d\mathrm{K} = \int \mathrm{K} f(y) \, d\mathrm{K} = \Gamma f(y)\end{aligned}$$

whenever $f \in p\mathcal{E}$. On the other hand, if $[x] \neq [y]$, let $f = 1_{[y]} \in p\mathcal{E}^u$. Then $\int \mathrm{K} f(x) \, d\mathrm{K} = 0$ while $\int \mathrm{K} f(y) \, d\mathrm{K} = m(1)$, so $\Gamma(x, .) \neq \Gamma(y, .)$. ■

By (4.6), we can define a bijection $\rho : \pi(\mathrm{E}) \to \mathrm{F}$ by setting $\rho(\pi(x)) = [x]$. Then $\rho \circ \pi(\mathrm{X}_t) = \Phi(\mathrm{X}_t)$. We note that $[x] \in \mathscr{E}^u$ for every $x \in \mathrm{E}$ if J is a locally compact topological group with Borel field $\mathscr{J}$. For then, define $\xi_x : \mathrm{J} \to \mathrm{E}$ by $\xi_x(\mathrm{K}_\varphi) = \varphi(x)$. This is a Borel map, so $\xi_x(\mathrm{J}) = [x]$ is analytic and hence in $\mathscr{E}^u$ (III-18d, [2]).

Now we turn to a version of the results above which allows time changes to intevene. It is based on the potential formulation in section 3. For the rest of this section, the following transience hypothesis will be in force.

(4.7) Hypothesis. – *There is a function $r \in p\mathscr{E}$ such that $r > 0$ and $\mathrm{U}r \leqq 1$.*

For the technical work ahead, we find it convenient to use a time change of X. Set

$$\mathrm{D}_t = \int_0^t r(\mathrm{X}_s)\,ds$$

$$\sigma_t = \inf\{s : \mathrm{D}_s > t\}$$

If $\mathrm{Y}_t = \mathrm{X}_{\sigma(t)}$, then X and Y have the same collection of excessive functions $\mathscr{S}$. If W^α is the resolvent of Y, then $\mathrm{W}1 = \mathrm{U}r \leqq 1$.

For each $\varphi \in \mathrm{B}$, define $\mathscr{S}_\varphi = \{f \circ \varphi : f \in \mathscr{S}\}$, and let

$$\mathrm{G}_\mathrm{A} = \{\varphi \in \mathrm{B} : \mathscr{S}_\varphi = \mathscr{S}\}.$$

The "A" subscript in G_A represents the word "additive", as in "additive functional". By Hunt's balayage theorem [1] and the Blumenthal-Getoor-McKean theorem ([1], [5]), G_A consists of those $\varphi \in \mathrm{B}$ such that there is a strictly increasing continuous additive functional A_t^φ of Y_t with inverse $\tau(\varphi, t)$ satisfying: $(\varphi(\mathrm{Y}_t), \mathrm{P}^{\varphi^{-1}(x)})$ and $(\mathrm{Y}_{\tau(\varphi, t)}, \mathrm{P}^x)$ are identical in law for every $x \in \mathrm{E}$. It is easy to check that G_A is also a group under composition.

Let H_A be any subgroup of G_A. As we did before, we set $\mathrm{K}_\varphi(x, .) = \varepsilon_{\varphi(x)}(.)$, and we define $\mathrm{J}_\mathrm{A} = \{\mathrm{K}_\varphi : \varphi \in \mathrm{H}_\mathrm{A}\}$.

(4.8) Hypothesis. – *There is a σ-algebra $\mathscr{J}_\mathrm{A}$ on J_A and a nonzero σ-finite measure $m_\mathrm{A}(d\mathrm{K})$ on $(\mathrm{J}_\mathrm{A}, \mathscr{J}_\mathrm{A})$ satisfying the analogues of* (4.1 i), (4.1 ii) *and* (4.1 iii).

If we set

$$\Gamma_\mathrm{A} f = \int_{\mathrm{J}_\mathrm{A}} \mathrm{K} f\, m_\mathrm{A}(d\mathrm{K})$$

then the results in (4.2), (4.3) and (4.4) all hold for Γ_A: (i) $\Gamma_\mathrm{A}(f\,\Gamma_\mathrm{A} g) = (\Gamma_\mathrm{A} f)(\Gamma_\mathrm{A} g)$ for every f, $g \in p\mathscr{E}$; (ii) if there is a function $q \in p\mathscr{E}$ with $q > 0$ and $\Gamma_\mathrm{A} q \leqq 1$, then Γ_A satisfies (2.1); and (iii) $m_\mathrm{A}(\mathrm{J}) = 1$ implies $\Gamma_\mathrm{A}^2 = \Gamma_\mathrm{A}$.

If $\mathrm{K} = \mathrm{K}_\varphi$ with $\varphi \in \mathrm{H}_\mathrm{A}$, we set $\mathrm{A}_t^\mathrm{K} = \mathrm{A}_t^\varphi$ and $\tau(\mathrm{K}, t) = \tau(\varphi, t)$. The next lemma is analogous to (6.2) in [7].

(4.9) LEMMA. — *There is a process* B_t^K *such that:*

(i) *for each* K, B_t^K *and* A_t^K *are indistinguishable; and*

(ii) $(t, K, \omega) \to B_t^K(\omega)$ *is* $\mathscr{B}(R^+) \times \mathscr{J}_A \times \mathscr{A}^0$*-measurable.*

Proof. — For each pair (K, x), define a measure $\rho((K, x), d\omega)$ on $(\Omega, \mathscr{A}^0)$ by setting

$$\rho((K, x), f) = P^x[A_\infty^K f]$$

for every $f \in p\mathscr{A}^0$. Here, $\mathscr{A}^0$ is a countably generated σ-algebra whose canonical Markov completion is $\mathscr{A}$. Suppose for the moment that we can prove that $(K, x) \to \rho((K, x), f)$ is $\mathscr{J}_A \times \mathscr{E}$-measurable. Then Doob's lemma [12] yields a density $C(K, x, \omega) \in \mathscr{J}_A \times \mathscr{E} \times \mathscr{A}^0$ such that $\rho((K, x), f) = P^x[C(K, x, .)f]$ for every $f \in p\mathscr{A}^0$. If we set $C_\infty^K(\omega) = C(K, Y_0(\omega), \omega)$, then C_∞^K is $\mathscr{J}_A \times \mathscr{A}^0$-measurable, and $C_\infty^K = A_\infty^K$ a. s. If we define $C_t^K = C_\infty^K - C_\infty^K \circ \theta_t$, then $C_t^K = A_t^K$ a. s. and $(K, \omega) \to C_t^K(\omega)$ is $\mathscr{J}_A \times \mathscr{A}^0$-measurable for each $t > 0$. Set

$$B_t^K(\omega) = \liminf_{s \downarrow t\ s \in Q} C_s^K(\omega)$$

Then $t \to B_t^K$ is continuous a. s., B_t^K and A_t^K are indistinguishable, and $(t, K, \omega) \to B_t^K(\omega)$ is $\mathscr{B}(R^+) \times \mathscr{J}_A \times \mathscr{A}^0$-measurable.

So we need only check that $(K, x) \to P^x[A_\infty^K f]$ is $\mathscr{J}_A \times \mathscr{E}$-measurable. If Z_t is the $(\mathscr{A}_t^0)$- predictable projection of f ([10], p. 209), then

$$P^x[A_\infty^K f] = P^x \int Z_t \, dA_t^K \tag{4.10}$$

so it will suffice to check that $(K, x) \to P^x \int Z_t \, dA_t^K$ is jointly measurable whenever Z_t is a bounded positive $(\mathscr{A}_t^0)$-predictable process. By the monotone class theorem, it suffices to check this for $Z_t = 1_{(0, T]}$ for T a finite $(\mathscr{A}_t^0)$-optional time since these processes generate all $(\mathscr{A}_t^0)$-predictable processes. In this case, (4.10) reduces to

$$P^x[A_T^K] = P^x[A_\infty^K] - P^x[P^{X(T)}[A_\infty^K]].$$

But if $K = K_\varphi$, then

$$P^x[A_\infty^K] = P^x \int 1_E(Y_{\tau(K, t)}) \, dt = P^{\varphi^{-1}(x)} \int 1_E \circ \varphi(Y_t) \, dt$$
$$= (W\,1)(\varphi^{-1}(x)) = K^{-1} W\,1(x)$$

By (4.8), $(K, x) \to P^x[A_\infty^K]$ is $\mathscr{J}_A \times \mathscr{E}$-measurable, so $P^x[A_T^K]$ is $\mathscr{J}_A \times \mathscr{E}$-measurable. ■

With this measurability result at our command, we can define a diffuse homogeneous random measure $\kappa(dt)$ by:

$$\int_0^\infty Z_t(\omega)\,\kappa(dt)=\int_{J_A}\int_0^\infty Z_t(\omega)\,dB_t^{K^{-1}}\,m_A(dK)$$

and we set

$$W_\kappa f(x)=P^x\int f(Y_s)\,\kappa(ds)$$

Now

$$W_\kappa\Gamma_A=\int W_{B^{K^{-1}}}\Gamma_A\,m_A(dK)=\int W_{B^{K^{-1}}}K\,\Gamma_A\,m_A(dK)$$

since $K\Gamma_A=\Gamma_A$. If $K=K_\varphi$, then

$$W_{B^{K^{-1}}}Kf(x)=P^x\int_0^\infty f\circ\varphi\,(Y_{\tau(\varphi^{-1},\,t)}\,dt$$
$$=P^{\varphi(s)}\int f(Y_t)\,dt=KWf(x)$$

Therefore, $W_\kappa\Gamma_A=\Gamma_A W\Gamma_A$.

(4.11) HYPOTHESIS. – *There is a function* $q\in p\mathscr{E}$ *with* $q>0$ *such that* $\Gamma_A W\Gamma_A q\leqq 1$.

This hypothesis is a transience hypothesis analogous to, but weaker than, (6.1) in [8]. It implies $W_\kappa\Gamma_A q\leqq 1$, also. Set $g=\Gamma_A q$ and

$$B_t=\int_0^t g(Y_s)\,\kappa(ds)$$

Then B_t is a strictly increasing continuous additive functional with finite potential: $W_B 1=W_\kappa g=W_\kappa\Gamma_A q\leqq 1$. Also,

$$W_B\Gamma_A f=W_\kappa g\Gamma_A f=W_\kappa\Gamma_A(gf)$$
$$=\Gamma_A W\Gamma_A gf=\Gamma_A W_g\Gamma_A f$$

by (4.2), where $W_g h=W(gh)$. Thus $W_B\Gamma_A=\Gamma_A V_1$, where $V_1=W_g\Gamma_A$. Let $\gamma_t=\inf\{s:B_s>t\}$, and set $Z_t=Y_{\gamma(t)}$: Z_t has potential $V=W_B$ satisfying $V1\leqq 1$ and $V\Gamma_A=\Gamma_A V_1$. Thus Proposition (3.4) holds for Z_t.

(4.12) THEOREM. – *Assume* (4.7), (4.8), *and* (4.11). *If* $\Gamma_A(fq)$ *is finely continuous whenever* $f\in C(\hat{F})$, *then* $\pi(Z_t)$ *is a.s. right continuous and is a time homogeneous strong Markov process.*

Proof. – Apply (3.10). ■

(4.13) COROLLARY. – *If* (4.7), (4.8) *and* (4.11) *hold, and if* $\Gamma_A(fq)$ *is finely continuous whenever* $f\in C(\hat{F})$, *then there is a strictly increasing continuous additive functional* I_t *of* X_t *with inverse* ν_t *such that* $\pi(X_{\nu(t)})$ *is a right continuous time homogeneous strong Markov process.*

There is at least one situation in which the fine continuity hypothesis of (4.13) can be easily verified.

(4.14) PROPOSITION. — *If q can be chosen to be bounded and finely continuous, and if $m_A(J_A)=1$, then $\Gamma_A(fq)$ is finely continuous whenever $f \in C(\hat{F})$.*

Proof. — Let (T_n) be a sequence of optional times decreasing to T. Then

$$\begin{aligned}\lim_{n\to\infty}(fq)(X_{T(n)}) &= \lim_{n\to\infty}\int K(fq)(X_{T(n)})\,m_A(dK)\\ &= \int \lim_{n\to\infty} K(fq)(X_{T(n)})\,m_A(dK)\end{aligned} \tag{4.15}$$

by the dominated convergence theorem. Recall that $K=K_\varphi$ for some $\varphi \in H_A$ having the property that $\mathcal{S}_\varphi=\mathcal{S}$. It follows that $g\circ\varphi$ is finely continuous whenever g is finely continuous, so (4.15) is

$$\int K(fq)(X_T)\,m_A(dK)=\Gamma_A(fq)(X_T) \quad \blacksquare$$

REFERENCES

[1] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory*, Academic Press, New York, 1968.
[2] C. DELLACHERIE and P. A. MEYER, *Probabilites et Potentiel*, Chapitres I-IV, Hermann, Paris, 1975.
[3] E. B. DYNKIN, *Markov Processes*, Vol. **I**, Springer-Verlag, Berlin, 1965.
[4] E. B. DYNKIN, Sufficient Statistics and Extreme Points, *Ann. Proba.*, Vol. **6**, 1978, pp. 705-730.
[5] P. J. FITZSIMMONS, R. K. GETOOR and M. J. SHARPE, The Blumenthal-Getoor-McKean Theorem Revisited, *Seminar on Stochastic Processes*, 1989, Birkhauser, Boston, 1990.
[6] J. GLOVER, Symmetries of Excessive Measures of Markov Processes, *Math. Z.*, Vol. **204**, 1990, pp. 1-12.
[7] J. GLOVER, Symmetry Groups and Translation Invariant Representations of Markov Processes, *Ann. Proba.*, 1991 (to appear).
[8] J. GLOVER and J. MITRO, Symmetries and Functions of Markov Processes, *Ann. Proba.*, Vol. **18**, 1990, pp. 655-668.
[9] J. G. KEMENY and J. L. SNELL, *Finite Markov Chains*, Van Nostrand, Princeton, 1960.
[10] J. PITMAN and L. C. G. ROGERS, Markov Functions, *Ann. Proba.*, Vol. **9**, 1981, pp. 573-582.
[11] M. ROSENBLATT, *Markov Processes. Structure and Asymptotic Behavior*, Springer-Verlag, Berlin, 1971.
[12] M. J. SHARPE, *General Theory of Markov Processes*, Academic Press, San Diego, 1988.
[13] M. YOR, *Intertwinings of Bessel Processes*, 1990 (unpublished manuscript).

(*Manuscript received June* 20, 1990.)