Annales de l'I. H. P., section B

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Superprocesses and projective limits of branching Markov process

<http://www.numdam.org/item?id=AIHPB_1991__27_1_91_0>

Echanges Annal

Superprocesses and projective limits of branching Markov process

by

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ABSTRACT. — We present an a. s. limit theorem to construct Watanabe-Dawson "superprocesses" starting from a projective system of branching Markov processes.

RÉSUMÉ. — Cet article présente un théorème de limite presque sûre permettant d'obtenir des super-processus de Watanabe-Dawson à partir d'un système projectif de processus de branchement markoviens.

Branching Markov processes are generally considered as measure valued processes. However, as stressed by Neveu [N1], even in the case of the simplest Galton-Watson processes, the genealogy of the points can be also usefully described and a formalization in terms of trees of paths can be given as explained in the following. We will consider a family of such processes indexed by $\tau \in \mathbb{R}^+$. The paths start at a point x with a number of points following a Poisson law of mean $\psi \tau^{-1/\beta}$, and points are moving as independent Markov processes of semigroup P_t and branching at exponential times of expectation τ according to a reproduction law $\{q\}$

defined by the generating function :

$$\hat{q}(u) = \sum q_k u^k = \frac{1}{1+\beta}(1-u)^{1+\beta} + u,$$

β being a parameter in $]0, 1]$.

Denote $\tilde{Z}_t^{(\tau)}$ the induced multiplet-valued process and $\tilde{Z}_{t,s}^{(\tau)}$ the multiplet of the ancestors of $\tilde{Z}_t^{(\tau)}$ at time $s < t$. One can check, as in [N2] that $(\tilde{Z}_{t+h,t}^{(\tau)}, t \in \mathbb{R}^+)$ and $(\tilde{Z}_t^{(\tau+h)}, t \in \mathbb{R}^+)$ have the same distributions.

Taking a projective limit, we can define the processes simultaneously in such a way that the above identities hold almost surely, for all τ and $h > 0$.

By a martingale argument, it appears that the measure $\psi^{-1} \varepsilon^{1/\beta} \tilde{Z}_{t-\varepsilon}^{\varepsilon} P_\varepsilon$ converges a. s. towards a measure μ_t as $\varepsilon \downarrow 0$, and that $(\mu_t, t > 0)$ is a Markov process.

If $U_t f(x) = -\text{Log}(E(e^{-\mu_t(f)}))$, we have

$$U_t f = P_t f - \frac{\psi^{-\beta}}{1+\beta} \int_0^t P_s((U_{t-s}(f))^{1+\beta}) \, ds$$

We recover a class of the measure valued process defined by Watanabe and Dawson, called superprocesses by Dynkin and studied by many authors : *see* [W], [Da], [D], [F1], [I], [P], [LG], [R].

When $\beta = 1$, this construction is closely related to the one given by Le Gall in [LG1], using the Brownian excursion to generate the branching system *cf.* [NP], [LG2].

These results have been announced in [LJ].

Remarks : (*a*) Considering a general semi-group P_t instead of, for example, diffusions, has some drawbacks (measurability problems need more care) but also advantages. It allows to apply the theory to path-valued processes.

(*b*) The three first paragraphs introduce basic notions and technical (mostly measure-theoretical) preliminaries.

(*c*) Proofs of intuitively obvious properties have been only briefly sketched.

1. PATHS WITH EXPONENTIAL LIFETIME

Let E be a Polish space, with $\mathscr{E}$ its Borel σ-field. Let $P_t(x, dy)$ be a semi-group of markovian kernels on E. Let F be the space of mappings ω from a finite interval $]S(\omega), T(\omega)]$ into E. Denote X_t its coordinates $(\omega(t) = X_t(\omega))$. Let $\mathscr{F}^0$ be the σ-field generated by the sets $\{T \geqq t > S\} \cap \{X_t \in A\}$, $(t \in \mathbb{R} \cdot \mathscr{A} \in \mathscr{E})$ and set $\mathscr{F} = \mathscr{F}^0 \vee \{X_T^{-1}(A), A \in \mathscr{E}\}$. T and S are $\mathscr{F}^0$-measurable. The restriction of $\mathscr{F}$ to

$$E^{]s, t]} = T^{-1}(t) \cap S^{-1}(s) \qquad \text{is} \quad \sigma(X_u, s < u \leqq t).$$

Kolmogorov's theorem allows to construct a unique system of probabilities $\mathbb{P}_{x, s, t}$ on $E^{]s, t]}$, $\sigma(X_u, u \in]s, t])$ such that :

$$\mathbb{P}_{x, s, t}(\cap \{X_{u_i} \in A_i\}) = P_{u_i - s} 1_{A_1} \ldots P_{u_n - u_{n-1}} 1_{A_n}(x)$$

for $s < u_1 \leqq \ldots \leqq u_n \leqq t$ and $A_i \in \mathscr{E}$.

Using a generating semi algebra (*cf.* [N3], § I-6) of $\mathscr{F}$, composed with sets of the form $\{S < t_1\} \cap \{t_n \leqq T\} \cap \cap_i \{X_{t_i} \in A_i\} \cap \{X_T \in B\}$ with $t_1 < t_2 < \ldots < t_n$, A_i and B belonging to $\mathscr{E}$, and the monotone class theorem, it appears that :

(1) *For any* $A \in \mathscr{F}$, $P_{x, s, t}(A \cap E^{]s, t]})$ *is a measurable function of* (x, s, t).
For any parameter $\tau > 0$, we can define a probability $\mathbb{P}^\tau_{x, s}$ on $(F, \mathscr{F})$, carried by $\{S = s\}$, by setting

$$\mathbb{P}^\tau_{x, s}(A) = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} \mathbb{P}_{x, s, s+t}(A \cap E^{]s, s+t]}) dt.$$

For any $\omega \in F$ and $u \in]S(\omega), +\infty[(]-\infty, T(\omega)[)$, define $k_u(\omega)$, $(\theta_u \omega)$ to be the restriction of ω to $]S(\omega), T(\omega) \wedge u]$, $(S(\omega) \vee u, T(\omega)]$.

For $0 \leqq h < T(\omega) - S(\omega)$, let $e_h(\omega)$ be the restriction of ω to $]S(\omega), T(\omega) - h]$, *i. e.* $k_{T(\omega) - h} \omega$.

The following measurability properties are easy to check :

(2) *For any* $A \in \mathscr{F}$ *and* $u \in \mathbb{R}$, $k_u^{-1}(A)$ *and* $\theta_u^{-1}(A)$ *belong to* $\mathscr{F}$.

(3) *For any* $B \in \mathscr{F}^0$ *and* $h \in \mathscr{R}^+$, $e_h^{-1}(B)$ *belongs to* $\mathscr{F}^0$.

If ω and ω' belong to F and $S(\omega') = T(\omega)$, the path $\omega \omega'$ is defined on $]S(\omega), T(\omega')]$ by : $\omega \omega' = \omega$ on $]S(\omega), T(\omega)]$, $\omega \omega' = \omega'$ on $]S(\omega'), T(\omega')]$.

(4) *For any* $A \in \mathscr{F}$, $\{(\omega, \omega'), \omega \omega' \in \mathscr{A}\}$ *is* $\mathscr{F}_0 \otimes \mathscr{F}$ *measurable*.

The following properties can be easily checked.

P1 (Markov property).

For any A, $B \in \mathscr{F}$, $t > s \in \mathbb{R}$, $x \in E$,

$$\mathbb{P}^\tau_{x, s}(k_t^{-1}(A) \cap \theta_t^{-1}(B)) = \mathbb{E}^\tau_{x, s}(k_t^{-1}(A) \mathbb{P}^\tau_{X_{t, t}}(\theta_t^{-1}B))$$

P2 *For any* $A \in \mathscr{F}^0$ *and* $h > 0$, $\mathbb{P}^\tau_{x,s}(e_h^{-1}(A)) = e^{-h/\tau} \mathbb{P}^\tau_{x,s}(A)$

P3 *For any* $A \in \mathscr{F}$ *and* $\tau_1, \tau_2 > 0$,

$$\int \{\omega\omega' \in A\} \, \mathbb{P}^{\tau_1}_{x,s}(d\omega) \, \mathbb{P}^{\tau_2}_{X_T(\omega), \, T(\omega)}(d\omega') = \mathbb{P}^{\tau_1+\tau_2}_{x,s}(A).$$

2. BRANCHING PATHS

Let U be the space of empty or finite sequences of positive integers, *i. e.* "words". There is a natural composition law on U as well as a lexicographical order induced by the order on $\mathbb{N} - \{0\}$. Define the mapping π from $U - \{\varnothing\}$ onto U by $\pi(j_1 \ldots j_n) = j_1 \ldots j_{n-1}$. Set $|j_1 \ldots j_n| = n$, and $|\varnothing| = 0$, to define the lenght of a word.

Following Neveu [N1], we define trees as sets of words δ containing $\varnothing$ and such that if $uj \in \delta$, for $j \in \mathbb{N} - \{0\}$ and $u \in U$, then $u \in \delta$ and also $ui \in \delta$ for any $i \leq j$.

Define the height $|\delta|$ of a tree as the length of the longest word it contains. If δ is a tree, and $u \in \delta$, then $T_u \delta = \{v \in U, uv \in \delta\}$ is a tree.

Set $\upsilon(\delta) = \sup(j \in \mathbb{N} - \{0\}, j \in \delta)$ if $\delta \neq \{\varnothing\}$, and $\upsilon(\{\varnothing\}) = 0$.

Let $\hat{F}$ be the space of branching paths $\alpha = (\alpha_0, \xi_\alpha)$ defined by a tree α_0 and a mapping ξ_α from α_0 into F, such that $T(\xi_\alpha(\pi(u))) = S(\xi_\alpha(u))$ for any $u \in \alpha_0 - \varnothing$.

F is imbedded in $\hat{F}$ in a canonical way. If $\omega \in F$, we identify it with the element ω' of $\hat{F}$ such that $\omega'_0 = \{\varnothing\}$ and $\xi_{\omega'}(\varnothing) = \omega$.

We set $\upsilon(\alpha) = \upsilon(\alpha_0)$, $|\alpha| = |\alpha_0|$.

If $u \in \alpha_0$, $T_u \alpha$ is defined by $(T_u \alpha)_0 = T_u(\alpha_0)$ and $\xi_{T_u \alpha}(v) = \xi_\alpha(uv)$.

Set $T_\alpha(u) = T(\xi_\alpha(u))$, $X_\alpha(u) = X_T(\xi_\alpha(u))$ and $S_\alpha(u) = S(\xi_\alpha(u))$ for any $u \in \alpha_0$. We define also $\zeta(\alpha) = \sup_{u \in \alpha_0} T_\alpha(u)$, $\sigma(\alpha) = S_\alpha(\varnothing)$.

Let $\hat{\mathscr{F}}$ be the smallest σ-algebra on $\hat{F}$ containing the sets

$$\Lambda_{u, A} = \{\alpha \in \hat{F}, u \in \pi(\alpha_0) \text{ and } X_\alpha(u) \in A\},$$

for all $u \in U$, $A \in \mathscr{E}$ and

$$\Gamma_{u, B} = \{\alpha \in \hat{F}, u \in \alpha_0 \text{ and } \xi_\alpha(u) \in B\},$$

for all $u \in U$, $B \in \mathscr{F}^0$.

(5) *If* $\Xi \in \hat{\mathscr{F}}$, $\Xi^{(i)} = \{\alpha, i \in \alpha_0 \text{ and } T_i \alpha \in \Xi\} \in \hat{\mathscr{F}}$.

(If $\Xi = \Lambda_{u, A}$, $\Xi^{(i)} = \Lambda_{iu, A}$ etc.)

Set $\hat{\mathscr{F}}_0 = \{\{\alpha, \xi_\alpha(\varnothing) \in A\} \, A \in \mathscr{F}^0\}$, and define by induction the σ-algebra $\hat{\mathscr{F}}_{(n)}$ generated by subsets of $\hat{F}$ of the form :

$$\left\{ \{\alpha, \upsilon(\alpha) = 0 \text{ and } \xi_\alpha(\varnothing) \in A_0\} \right.$$

$$\bigcup_{i=1}^{\infty} \bigcup \{\alpha, \upsilon(\alpha) = i, \xi_\alpha(\varnothing) \in A_i \text{ and } T_j\alpha \in \Xi_{i,j} \text{ for } j \leq i\},$$

$$\left. A_0 \in \mathscr{F}^0, \, A_i \in \mathscr{F}, \, \Xi_{i,j} \in \hat{\mathscr{F}}_{(n+1)} \right\}.$$

($\hat{\mathscr{F}}_{(n)}$ describes the history until the n-th generation).

By (5), $\hat{\mathscr{F}}_{(n)} \subseteq \hat{\mathscr{F}}$ for every n. Moreover $\Lambda_{u, A} \in \hat{\mathscr{F}}_{(|u|+1)}$ and $\Gamma_{u, B} \in \hat{\mathscr{F}}_{(|u|)}$. Hence

(6) $\hat{\mathscr{F}}$ is the σ-algebra generated by the algebra $\bigcup \hat{\mathscr{F}}_{(n)}$.

Let q be a probability on $\mathbb{N}$ with $\sum_0^\infty k\, q(k) < \infty$. We shall extend the probabilities $\mathbb{P}^\tau_{x,s}$ to $\hat{F}$ as follows :

PROPOSITION 1. — *There exist a unique system of probabilities* $\mathbb{P}^{\tau,q}_{x,s}$ *on* $(\hat{F}, \hat{\mathscr{F}})$ *such that* :
 (a) *For any* $\Xi \in \hat{\mathscr{F}}$, $\mathbb{P}^{\tau,q}_{x,s}(\Xi)$ *is measurable in* (x, s).
 (b) $\mathbb{P}^{\tau,q}_{x,s}$ *induces* $\mathbb{P}^\tau_{x,s}$ *on* $(F, \mathscr{F}^0)$.
 (c) *For any* $A \in \mathscr{F}^0$, $B \in \mathscr{E}$, $n \in \mathbb{N} - \{0\}$, $\Xi_1 \ldots \Xi_m \in \hat{\mathscr{F}}_{(n)}$,

$$\mathbb{P}^{\tau,q}_{x,s}(\{\alpha, \xi_\alpha(\varnothing) \in A, X_\alpha(\varnothing) \in B, \upsilon(\alpha) = m \text{ and } T_i\alpha \in \Xi_i \text{ for } i \leq m\})$$

$$= q(m) \, \mathbb{E}^\tau_{x,s}\left(1_A \, 1_{\{X_T \in B\}} \prod_{i=1}^m \mathbb{E}^{\tau,q}_{X_T, T}(\Xi_i) \right).$$

Proof. $\mathbb{P}^{\tau,q}_{x,s}$ can be defined by induction on each $\hat{\mathscr{F}}_{(n)}$ [by (c)]. The σ-additivity of the projective limit follows from the existence of a compact class (*cf.* [N3], § I-6) generated by sets of the form

$$\{u \in \pi(\alpha_0)\} \cap \{X_\alpha(u) \in K\},$$

$$\{u \in \alpha_0\} \cap \{a \leq S_\alpha(u) \leq b\} \cap \{c \leq T_\alpha(u) \leq d\} \cap \bigcap_{i=1}^n \{X_{t_i}(\xi_\alpha(u)) \in K_i\}$$

with $K, K_i \in K(E)$; where $K(E)$ denotes the class of compact subsets of E.

N. B. — Set, for any $n \in \mathbb{N}$, $z_n = \{u \in \alpha_0, |u| = n\}$.
 z_n is a Galton Watson process of reproduction law q. Hence, if $\sum k\, q(k) \leq 1$, α_0 is almost surely finite for any (x, s, τ).

Similary set $|Z_t| = \{ u \in \alpha_0, S_\alpha(u) \leqq t < T_\alpha(u) \}$. Under $\mathbb{P}_{x,s}^{\tau,q}, |Z_{t+s}|$ is a continuous time Galton Watson process with reproduction law q and time constant τ.

We extend k_t to $\{ \alpha, \sigma(\alpha) < t \}$ as follows : $(k_t \alpha)_0 = \{ u \in \alpha_0, S_\alpha(u) < t \}$ and $\xi_{k_t \alpha} = k_t \xi_\alpha$.

To extend θ_t, set $\hat{Z}_t(\alpha) = \{ v \in \alpha_0, t \in [S_\alpha(v), T_\alpha(v)[\}$.

If $u \in \hat{Z}_t(\alpha)$, define $\theta_{t,u}(\alpha)$ as follows :

$(\theta_{t,u}\alpha)_0 = T_u \alpha_0$ and, if $\beta = \theta_{t,u}\alpha$, we define $\xi_\beta(\varnothing) = \theta_t \xi_\alpha(u)$ and $\xi_\xi(v) = \xi_\alpha(uv)$ for any $v \in T_u \alpha_0 - \varnothing$.

Measurability properties

(7) *For any* $\Xi \in \hat{\mathscr{F}}$, $k_t^{-1}(\Xi)$ *is* $\hat{\mathscr{F}}$ *measurable.*
Indeed : $k_t^{-1} \Lambda_{u,A} = \Lambda_{u,A} \cap \{ T_\alpha(u) < t \}$ [since $u \in \pi(k_t \alpha)_0$ if $T_\alpha(u) < t$]

$$k_t^{-1}(\Gamma_{u,B}) = \Gamma_{u, k_t^{-1}(B)} \in \hat{\mathscr{F}}.$$

Set $\hat{\mathscr{F}}_t = \{ k_t^{-1}\Xi, \Xi \in \hat{\mathscr{F}} \}$.
(8) *For any* $u \in U$ *and* $\Xi \in \hat{\mathscr{F}}$, $\{ \alpha \in \hat{\mathscr{F}}, u \in \hat{Z}_t(\alpha) \text{ and } \theta_{t,u}(\alpha) \in \Xi \}$ *belongs to* $\hat{\mathscr{F}}$.

(Indeed for example $\{ u \in \hat{Z}_t \} \cap \theta_{t,u}^{-1}(\Gamma_{v,B}) = \{ u \in \hat{Z}_t \} \cap \Gamma_{uv,B}$ if $v \neq \varnothing$, and $\{ u \in \hat{Z}_t \} \cap \theta_{t,u}^{-1} \Gamma_{\varnothing,B} = \{ u \in \hat{Z}_t \} \cap \Gamma_{u, \theta_t^{-1}(B)}$.)

For $u \in \hat{Z}_t$, set $X(t,u) = X(\pi(u))$ if $t = S(u)$ and $X(t,u) = X_t(\xi(u))$ if $t > S(u)$. Then, we have

PROPOSITION 2. — *For any* $s < t, u_1, u_2 \ldots u_n$ *and* $\Xi_1, \ldots, \Xi_n \in \hat{\mathscr{F}}$,

$$\mathbb{E}_{x,s}^{\tau,q}\left(\bigcap_{i=1}^n \{ u_i \in \hat{Z}_t \} \cap \theta_{t,u_i}^{-1}(\Xi_i) \,\Big|\, \hat{\mathscr{F}}_t \right) = \prod_{i=1}^n 1_{\{ u_i \in \hat{Z}_t \}} \mathbb{E}_{X(t,u_i),t}^{\tau,q}(\Xi_i).$$

Sketch of proof. First one shows that for any $A \in \mathscr{F}$ and $\Xi \in \hat{\mathscr{F}}$,

$\mathbb{P}_{x,s}^{\tau,q}(\{ \xi(\varnothing) \in k_t^{-1}(A) \} \cap \theta_{t,\varnothing}^{-1}(\Xi) \cap \{ t < T(\varnothing) \})$
$$= \mathbb{E}_{x,s}^\tau(k_t^{-1}(A) \cap \{ t < T \} \, \mathbb{E}_{X_t,t}^{\tau,q}(\Xi))$$

by using the Markov property P1.

Then, one proves the Proposition by induction on the "degree" $|u_1| + |u_2| + \ldots + |u_n|$, using the branching property c in proposition 1 to reduce the degree after choosing test set, in $\hat{\mathscr{F}}_t$ of the form $k_t^{-1}(\Lambda)$, with $\Lambda \in \hat{\mathscr{F}}_{(n)}$ for some n.

N. B. A stronger version of this proposition has been proved in [C], when P_t is the heat semigroup.

3. GENERATING FUNCTIONAL

For any positive bounded measure μ on E and $f \in \mathcal{E}^+$, $0 < f(x) \leq 1$ for every x, set :

$$f^\mu = \exp\left(\operatorname{Log} f \, d\mu\right).$$

Define $Z_t(\alpha) = (\sum_{u \in \hat{Z}_t(\alpha)} \varepsilon_{X(t,\,u)})$.

The law of Z_t is determined by the generating functional

$$\Phi^\tau_{t-s}(f)(x) = E^{\tau,\,q}_{x,\,s}(f^{Z_t}) \qquad (t > s).$$

Applying proposition 1 (c), we have

$$(9) \qquad \Phi^\tau_t(f) = e^{-t/\tau} P_t f + \frac{1}{\tau} \int_0^t e^{-s/\tau} P_s \hat{q}(\Phi^\tau_{t-s}(f)) ds$$

$\hat{q}$ being the generating function of q.
(9) can be written in the equivalent form

$$(10) \qquad \Phi^\tau_t(f) = P_t f + \frac{1}{\tau} \int_0^t P_s [\hat{q}(\Phi^\tau_{t-s}(f)) - \Phi^\tau_{t-s}(f)] \, ds$$

Indeed, assuming (9), (10) is equivalent to the identity

$$(1 - e^{-t/\tau}) P_t f + \frac{1}{\tau} \int_0^t (1 - e^{-s/\tau}) P_s \hat{q}(\Phi^\tau_{t-s}(f)) \, ds = \frac{1}{\tau} \int_0^t (P_{t-s}(\Phi^\tau_s(f)) \, ds$$

Replacing $\Phi^\tau_s(f)$ by its expression in (9) in the second member, we get the first one by performing an integration.

The semigroup property

$$(11) \qquad \Phi^\tau_{t+s} = \Phi^\tau_t \circ \Phi^\tau_s$$

follows from the branching markov property (proposition 2).

For any f, $0 < f \leq 1$, equation (10) has a unique solution $\Phi^\tau_t(f)$ verifying

$$\Phi^\tau_0(f) = f, \qquad 0 \leq \Phi^\tau_t(f) \leq 1.$$

Set $a_t = \Phi^{(1)}_t(f) - \Phi^{(2)}_t(f)$. Then

$$a_t = \int_0^t (P_s(\hat{q}(\Phi^{(1)}_{t-s}) - \hat{q}(\Phi^{(2)}_{t-s})) \, ds$$

and

$$\|a_t\|_\infty \leq \frac{1}{\tau} \int_0^t \|a_s\|_\infty (1 + \sum k\, q(k))\, ds,$$

hence $a_t = 0$ $\Big]$.

The semigroup property follows also from the uniqueness.

(Set $\Phi_t^{(s)} = \Phi_t$ for $t \leq s$ and $\Phi_t^{(s)} = \Phi_{t-s} \circ \Phi_s$ for $t \geq s$, and check $\Phi_t^{(s)}$ verifies the same integral equation as Φ_t.)

Note also that for any bounded $g \in \mathscr{E}^+$, and $t > s$,

$$\mathbb{E}_{x,s}^{\tau,q}\left(\int g\, dZ_t\right) = -\frac{d}{d\varepsilon}\Phi_{t-s}^\tau(e^{-\varepsilon g})\big|_{\varepsilon=0}.$$

Differentiating (10) and solving the resulting linear integral equation yields

(12) $$\mathbb{E}_{x,s}^{\tau,q}\left(\int g\, dZ_t\right) = \exp\left(\frac{t-s}{\tau}\hat{q}'(1) - \frac{t-s}{\tau}\right)P_t g.$$

Note also that $\mathbb{P}_{x,s}^{\tau,q}(\zeta < t) = \Phi_{t-s}^\tau(0+)$.

In the following we shall be particulary interested in the laws of reproduction q_β, $\beta \in]0, 1]$ defined by the generating functions

$$\hat{q}_\beta(u) = \frac{1}{1+\beta}(1-u)^{1+\beta} + u, \qquad u \in [0, 1] \quad (cf.\, [Z], [H]).$$

Then equation (10) can be solved for $f = u$ (const.) to yield

$$\Phi_t^\tau(u) = 1 - \frac{1-u}{(1 + t\tau^{-1}(1-u)^\beta)^{1/\beta}}.$$

In particular $\Phi_t^\tau(0+) = 1 - \left(\dfrac{\tau}{\tau+t}\right)^{1/\beta}$.

4. ERASURE

e_h extends to $\{\zeta - \sigma > h\}$ as follows :

$$(e_h\alpha)_0 = \{u \in \alpha_0, \zeta(\alpha) - h > S_\alpha(u)\}$$

and $\xi_{e_h\alpha}(u) = k_{\zeta(\alpha)-h}\xi_\alpha(u)$ for any $u \in (e_h\alpha)_0$.

(13) *For any* $\Xi \in \hat{\mathscr{F}}$, $e_h^{-1}(\Xi) \in \hat{\mathscr{F}}$.

Indeed

$$e_h^{-1} \Lambda_{u, A} = \Lambda_{u, A} \cap \{ \zeta - h > T_\alpha(u) \}$$

$$e_h^{-1} \Gamma_{u, B} = \{ \zeta - h > S_u(u) \} \cap \Gamma_{u, k\zeta^{-1}_h B},$$

where $\Gamma_{u, k\zeta^{-1}_h B} = \{ \zeta - h \geq t \} \cap \Gamma_{u, B} \in \hat{\mathscr{F}}$.

PROPOSITION 3. — *For any* $\Xi \in \hat{\mathscr{F}}$,

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\Xi)) = \lambda_h \mathbb{P}_{x, s}^{\tau, q^{(h)}}(\Xi)$$

with

$$\lambda_h = \mathbb{P}_{x, s}^{\tau}(\zeta > s + h) = 1 - \Phi_h^{\tau}(0+)$$

and

$$q^{(h)}(k) = \frac{1}{1 - \lambda_h} \left(\sum_{l \geq k} q(l) \binom{l}{k} (1 - \lambda_h)^k \lambda_h^{l-k} - \lambda_h \delta_{0k} \right).$$

The proof given by Neveu in [N2], p. 106-107, can be easily generalized to this context, using the caracterization of $\mathbb{P}_{x, 0}^{\tau, q}$ given in proposition 1 and property P2 to prove the identity for $\Xi \in \hat{\mathscr{F}}_{(n)}$, by induction on n.

To start the recurrence, we need to compute $\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1} \{ \xi \emptyset) \in A \})$ for $A \in \mathscr{F}_0$.

We show that for $k > 0$,

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{ \xi(\emptyset) \in A \} \cap \{ \upsilon = k \})) = \sum_{l \leq k} q(l) \binom{l}{k} (1 - \lambda_h)^{l-k} \lambda_h^l \mathbb{P}_{x, s}^{\tau}(A).$$

Then, if μ denotes the law of ζ-σ under $\mathbb{P}_{., .}^{\tau, q}$,

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{ \xi_\emptyset \in A \} \cap \{ \upsilon = 0 \}))$$

$$= \sum q(l) \int_0^h d_\mu(z_1) \dots \int_0^h d_\mu(z_l) \mathbb{E}_{x, s}^{\tau}(e_{h - max z_i}^{-1}(A))$$

$$= \text{some constant}) \mathbb{P}_{x, s}^{\tau}(A) \quad \text{by } P_2.$$

It follows that

$$\mathbb{P}_{x, s}^{\tau, q}(e_n^{-1} \{ \xi(\emptyset) \in A \}) = \lambda_h \mathbb{P}_{x, s}^{\tau, q}(A)$$

and

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{ \xi(\emptyset) \in A \} \cap \{ \upsilon = 0 \})) = \lambda_h q^{(h)}(0) \mathbb{P}_{x, s}^{\tau}(A).$$

Then one proves the recurrence on events of the form

$$\{ \xi(\emptyset) \in A \} \cap \{ \upsilon = k \} \cap \bigcap_{i=1}^k T_i^{-1}(\Xi_i)$$

with $k > 0$, $A \in \mathscr{F}_0$ and $\Xi_i \in \hat{\mathscr{F}}_{(n)}$.

In particular, if $q = q_\beta$,

$$1 - \lambda_h = \left(\frac{\tau}{\tau + h} \right)^{1/\beta} \quad \text{and} \quad \hat{q}_\beta^{(h)}(u) = \frac{\tau}{\tau + h} \frac{(1-u)^{1+\beta}}{1+\beta} + u.$$

Set $Z_{t,s} = Z_s \circ e_{t-s}$ for $t > s$. Then

$$Z_{t,s}(\alpha) = \sum_{u \in Z_s(\alpha)} \varepsilon_{X(s,u)} 1_{\{\zeta(\theta_{s,u}\alpha) \geq t\}}.$$

Hence for any $t_1 < t_2 < \ldots < t_n, f_1, \ldots, f_n \in \mathscr{E}^+$,

$$\prod_{i=1}^{n} f_i^{Z_{t_n, t_i}} = \prod_{u \in Z_{t_1}} \left[f_1(X(t_1, u)) \theta_{t_1, u}^{-1} \left(1_{\{\zeta \geq t_2\}} \prod_{i=2}^{n} f_i^{Z_{t_n, t_i}} \right) + \theta_{t_1, u}^{-1} (1_{\{\zeta \leq t_2\}}) \right].$$

By the branching Markov property, it follows that for $t_0 < t_1$

$$\mathbb{E}_{x, t_0}^{\tau, q} \left[\prod_{i=1}^{n} (f_i^{Z_{t_n, t_i}}) \right] = \Phi_{t_1 - t_0} R_{t_2 - t_1}^{f_1} \cdots R_{t_n - t_{n-1}}^{f_{n-1}} (f_n)(x)$$

with $R_s^f(g) = f(\Phi_s(g) - \Phi_s(0+) + \Phi_s(0+))$.

Hence, it is easy to check that, for $h > 0$

$$\mathbb{E}_{x, 0}^{\tau, q} (f^{Z_t} \mid \sigma(Z_{t, s}, s < t - h)) = \left[\frac{\Phi_h(f) - \Phi_h(0+)}{1 - \Phi_h(0+)} \right]^{Z_{t, t-h}}$$

This formula says that the conditional law of Z_t given $Z_{t,s} = \mu$ is identical to the conditional law of Z_{t-s} given $Z_0 = \mu$ and that each point of μ survives until $t - s$. In particular

$$(14) \qquad \mathbb{E}_{x, 0}^{\tau, q} \left(\int g \, dZ_t \mid \sigma(Z_{t, s}, s < t - h) \right) = \frac{1}{1 - \Phi_h^{\tau}(0+)} \int P_h g \, dZ_{t, t-h}$$

5. REDUCTION

We define a reduction operation R on trees, by induction on the height. It amounts to extending the lifetime of particles whose offspring is only one particle. R(δ) will be defined together with a mapping N_δ from R(δ)

into $\mathbb{N}$, as follows :

- $R(\{\varnothing\})=(\{\varnothing\})$, $N_{\{\varnothing\}}(\varnothing)=0$;
- if $v \in R(\delta)$, $N_\delta(v)=inf(m, v\,1^m \in \delta$ and $\upsilon(T_{v_1m}\delta)\neq 1)$ [1] and $vi \in R(\delta)$ if and only if $N(v)<\infty$ and $i\leq\upsilon(T_{v_1\,N_\delta(v)}\delta)$.

In the following, *we assume* $q(1)\neq 1$ so that $N(v)<\infty$ a. s. when $v \in R(\alpha_0)$.

Set

$$\hat{F}_0=\{\alpha\in\hat{\mathscr{F}}, N(v)<\infty \text{ for any } v\in R(\alpha_0)\}.$$

$\hat{F}_0\in\hat{\mathscr{F}}$, is stable under k_t, $\theta_{t,u}$ etc. and has full measure under all $P_{x,s}^{\tau,q}$. Hence all the results we have obtained are still valid on $\hat{F}_0$. From now on, we shall work on $\hat{F}_0$ only.

R can be extended to $\hat{F}_0$ as follows : if $\alpha\in\hat{F}_0$, $(R\alpha)_0=R\alpha_0$ and

$$\xi_{R\alpha}(v)=\prod_{i=0}^{N(v)}\xi_\alpha(v\,1^i) \text{ for any } v\in R\alpha_0 \text{ (where the product refers to the conca-}$$

tenation of paths).

PROPOSITION 4. — $R\,\mathbb{P}_{x,s}^{\tau,q}=\mathbb{P}_{x,s}^{\tau_R,q_R}$ *with*

$$\tau_R=\frac{\tau}{1-q(1)} \quad and \quad q_R(k)=\frac{1}{1-q(1)}(q(k)-q(1)\delta_1(k)).$$

Sketch of proof. Note first that for any $A\in\mathscr{F}$,

$$\mathbb{E}_{x,s}^{\tau,q}\left(\prod_{i=0}^{N(\varnothing)}\xi(1^i)\in A\right)=\mathbb{P}_{x,s}^{\tau_R}(A) \quad \text{by P3.}$$

Then prove that $\mathbb{P}_{x,s}^{\tau,q}(R^{-1}(\Xi))=\mathbb{P}_{x,s}^{T_R,q_R}(\Xi)$ for Ξ in $\hat{\mathscr{F}}_{(n)}$ by induction on n.

COROLLARY. — *For any* $\Xi\in\hat{\mathscr{F}}$,

$$\mathbb{P}_{x,s}^{\tau,q_\beta}(R^{-1}\circ e_h^{-1}(\Xi))=\left(\frac{\tau}{\tau+h}\right)^{1/\beta}\mathbb{P}_{x,s}^{h+\tau,q_\beta}(\Xi).$$

[An easy computation shows $(q_\beta^{(h)})_R=q_\beta$.]

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(1) 1^m means 1 written m times.

6. SUPERPROCESSES

Let μ be a probability on $(E, \mathscr{E})$, ψ and ε be positive parameters. Set $\mathbb{P}_\mu^\varepsilon = \int \mu(dx) \, \mathbb{P}_{x, 0}^{\varepsilon; \, q_\beta}$ (β will be fixed in the following). Let π_μ^ε be a Poisson point process in $\hat{F}_0$, with intensity $\psi \varepsilon^{-1/\beta} \mathbb{P}_\mu^\varepsilon$. For any $F \in \hat{\mathscr{F}}^+$,

$$(15) \qquad \mathrm{Log}\,(E\,(e^{-\int F(\alpha)\pi^\varepsilon\mu(d\alpha)})) = \psi \varepsilon^{-1/\beta} \, \mathbb{E}_\mu^\varepsilon(e^{-F} - 1).$$

π_μ^ε describes the evolution of a population starting at time 0 with a Poisson distribution of intensity $\psi \varepsilon^{-1/\beta}\mu$, and with reproduction law q_β. Each particle has an exponential lifetime of average ε and moves independently of the others according to the law determined by the semigroup P_t.

From the corollary to proposition 4, it follows that $R \circ e_h \pi_\mu^\varepsilon$ and $\pi_\mu^{\varepsilon + h}$ have the same distribution.

Taking a sequence $\varepsilon_n \downarrow 0$ and applying the Ionescu-Tulcea theorem, it appears one can define the processes π_μ^ε simultaneously, so that

$$(16) \qquad R \circ e_h \pi_\mu^\varepsilon = \pi_\mu^{\varepsilon + h} \quad \text{for any } \varepsilon, h \in \mathbb{R}^+ - \{0\}.$$

Set $\tilde{Z}_t^{\varepsilon, \, \mu} = \int Z_t(\alpha) \, \pi_\mu^\varepsilon(d\alpha)$. (For fixed ε, it describes the position of the population living at time t.)

Similary, set $\tilde{Z}_{t, \, s}^{\varepsilon, \, \mu} = \int Z_{t, \, s}(\alpha) \, \pi_\mu^\varepsilon(d\alpha)$. (It describes the population at times s whose offspring shall be alive at time t.)

By (16), we have for any $t > h > 0$,

$$(17) \qquad\qquad\qquad \tilde{Z}_{t, \, t-h}^{\varepsilon, \, \mu} = \tilde{Z}_{t-h}^{\varepsilon + h, \, \mu}.$$

Hence, from identity (14), for any bounded function f, and $0 < \eta < \varepsilon$, $E\,(<\tilde{Z}_{t-\eta}^{\eta, \, \mu}, P_\eta f> \,\|\, \sigma\,(\tilde{Z}_{t-\gamma}^{\gamma, \, \mu}, \gamma \geqq \varepsilon))$ can be computed and is equal to

$$(1 - \Phi_{\varepsilon - \eta}^\eta(0+))^{-1} <\tilde{Z}_{t-\varepsilon}^{\varepsilon, \, \mu}, P_\varepsilon f>.$$

$$\left(\text{If } \mu \text{ is a measure and } f \text{ a function, } \langle \mu, f \rangle = \int f \, d\mu. \right)$$

$$\left[\text{One uses the fact that for any } \sigma\text{-field } \mathscr{B} \subseteq \hat{\mathscr{F}}, \text{ and bounded measurable} \right.$$

$$\text{function F, } E\left(\int F \, d\pi^\varepsilon \,\Big|\, \sigma\,(\pi^\varepsilon(\Xi), \Xi \in \mathscr{B}) \right) = \pi^\varepsilon\,(\mathbb{E}_\mu^\varepsilon(F \,|\, \mathscr{B})). \right]$$

Moreover $(1 - \Phi_{\varepsilon - \eta}^\eta(0+)) = \left(\dfrac{\eta}{\varepsilon} \right)^{1/\beta}$. Hence, we have, for f positive,

(18)   *For any sequence* $\varepsilon_n \downarrow 0$, $Y_t^{\varepsilon_n, \mu}(f) = \varepsilon_n^{1/\beta} \psi^{-1} <\tilde{Z}_{t-\varepsilon_n}^{\varepsilon_n, \mu}, P_{\varepsilon_n} f>$ *is a positive martingale which has to converge a. s. for all f towards a limit* $Y_t^{\mu}(f)$, *clearly independent of the sequence* ε_n.

Using a dense countable $\mathbb{Q}$-vector subspace V of $C(E)$, we define a random measure μ_t such that

$$<\mu_t, f> = Y_t^{\mu}(f) \quad \text{a. s. for any } f \text{ in } C(E).$$

Since $|\mathbb{E}(\mu_t(f))| \leq \mu P_t(|f|)$, the a. s. equality extends to any bounded measurable function f.

THEOREM. $-$ μ_t *is a Markov process whose law is determined by the semigroup* Q_t *verifying :*

$$- \text{Log} \int Q_t(\mu, d\upsilon) \exp\left(- \int f d\upsilon\right) = \int U_t(f) d\mu$$

for any positive bounded measurable function f, *where* $U_t(f)$ *is uniquely determined by the equation*

$$U_t f = P_t f - \frac{\psi^{-\beta}}{1+\beta} \int_0^t P_s(U_{t-s}(f)^{1+\beta}) ds.$$

Proof . For any bounded f in $\mathscr{E}^+$, set

$$U_t(\mu, f) = - \text{Log } E(\exp(- Y_t^{\mu}(f)))$$
$$U_t^{\varepsilon}(\mu, f) = - \text{Log } E(\exp(- Y_t^{\varepsilon, \mu}(f))).$$

By (15), we have, $U_t^{\varepsilon}(\mu, f) = -\frac{1}{\varepsilon'} \mathbb{E}_{\mu}^{\varepsilon}(e^{-\varepsilon' Z_{t-\varepsilon} P_{\varepsilon} f} - 1)$, where $\varepsilon' = \varepsilon^{1/\beta} \psi^{-1}$.

Hence $U_t^{\varepsilon}(\mu, f) \leq \|f\|_{\infty}$ for all t, ε and more generally

$$|U_t^{\varepsilon}(f) - U_t^{\varepsilon}(g)| \leq \|f - g\|_{\infty}.$$

Also, if we set $U_t^{\varepsilon}(f)(x) = U_t^{\varepsilon}(\varepsilon_x, f)$, we have

$$U_t^{\varepsilon}(\mu, f) = \int \mu(dx) U_t^{\varepsilon}(f)(x),$$

and

(19) $$U_t^{\varepsilon}(f) = \frac{1}{\varepsilon'}(1 - \Phi_{t-\varepsilon}^{\varepsilon}(e^{-\varepsilon' P_{\varepsilon} f})).$$

By (18) since e^{-x} is convex, $U_t^{\varepsilon}(\mu, f)$ decreases with ε. By dominated convergence its limit has to be $U_t(\mu, f)$.

From (19) and (10) applied to $\Phi^\varepsilon_{t-\varepsilon}$ and $q = q_\beta$,

$$U^\varepsilon_t(f) = \frac{1}{\varepsilon'} P_{t-\varepsilon}(1 - e^{-\varepsilon' P_\varepsilon f}) - \frac{\psi^{-\beta}}{1+\beta} \int_0^{t-\varepsilon} P_s(U^\varepsilon_{t-s}(f)^{1+\beta}) \, ds.$$

Set $U_t(f)(x) = U_t(\varepsilon_x, f)$.

Since, for any ε_0 and $\varepsilon < \varepsilon_0$,

$$\int_0^{t-\varepsilon_0} P_s(U^\varepsilon_{t-s}(f)^{1+\beta}) \, ds + \|f\|_\infty^{1+\beta} \varepsilon_0$$

$$\geq \int_0^{t-\varepsilon} P_s(U^\varepsilon_{t-s}(f)^{1+\beta}) \, ds \geq \int_0^{t-\varepsilon_0} P_s(U^\varepsilon_{t-s}(f)^{1+\beta}) \, ds$$

we obtain $U_t(f) = P_t f - \dfrac{\psi^{-\beta}}{1+\beta} \displaystyle\int_0^t P_s(U_{t-s}(f)^{1+\beta}) \, ds$ by dominated convergence.

Uniqueness can be shown in the same way as for equation (10).

We also have the semigroup property $U_{t+s} = U_t \circ U_s$.

We still have to show that μ_t is a Markov process, which is equivalent to prove that

$$- \mathrm{Log}\, \mathrm{E}\left(\exp\left(-\sum_{i=1}^n Y_{t_i}(f_i)\right)\right) = U_{t_1}(f_1 + U_{t_2 - t_1}(f_2 + \ldots))$$

for $0 < t_1 < \ldots < t_n$ and bounded $f_i \in \mathscr{E}^+$.

The first member equals

$$- \lim_{\varepsilon \downarrow 0} \mathrm{Log}\, \mathrm{E}\left(\exp\left(-\varepsilon'\int\sum\langle Z_{t_i-\varepsilon}(\alpha), P_\varepsilon f_i\rangle \pi^\varepsilon_\mu(d\alpha)\right)\right)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon'}(1 - \Phi^\varepsilon_{t-\varepsilon}(e^{-\varepsilon' P_\varepsilon f_1} \Phi^\varepsilon_{t_2 - t_1}(e^{-\varepsilon' P_\varepsilon f_2} \ldots)))$$

$$= \lim_{\varepsilon \downarrow 0} \tilde{U}^\varepsilon_{t_1 - \varepsilon}(P_\varepsilon f_1 + V^\varepsilon_{t_2 - t_1}(P_\varepsilon f_2 + \ldots))) = (*)$$

with

$$V^\varepsilon_t(f) = -\frac{1}{\varepsilon'}\log(\Phi^\varepsilon_t(e^{-\varepsilon' f}))$$

and

$$\tilde{U}^\varepsilon_t(f) = \frac{1}{\varepsilon'} \mathbb{E}^\varepsilon(1 - e^{-\varepsilon' Z_t(f)}) = \frac{1}{\varepsilon'}(1 - \Phi^\varepsilon_t(e^{-\varepsilon' f}))$$

(note that $U_t^\varepsilon = \tilde{U}_{t-\varepsilon}^\varepsilon \circ P_\varepsilon$). We can replace V_t^ε by $\tilde{U}_t^\varepsilon$. Indeed,

$$\| V_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(f) \|_\infty \le \frac{1}{2\,\varepsilon'} \| 1 - \Phi_t^\varepsilon(e^{-\varepsilon'\,f}) \|_\infty^2 \le \frac{\varepsilon' \|f\|_\infty^2}{2}$$

when $\varepsilon' \|f\|_\infty < \dfrac{1}{2}$.

Moreover, $|\tilde{U}_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(g)| \le \|f - g\|$. Hence,

$$(*) = \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon (P_\varepsilon f_1 + \tilde{U}_{t_2-t_1}^\varepsilon (P_\varepsilon f_2 + \ldots)).$$

Finally since

$$|\tilde{U}_\varepsilon^\varepsilon(f) - P_\varepsilon f| \le \frac{1}{\varepsilon'\,\varepsilon\,(1+\beta)} \int_0^\varepsilon P_s (1 - \Phi_{t-s}^\varepsilon(e^{-\varepsilon'\,f}))^{1+\beta}\,ds$$

$$+ P_\varepsilon \left| \frac{1 - e^{-\varepsilon'f}}{\varepsilon'} - f \right| \le \frac{1}{\varepsilon\varepsilon'\,(1+\beta)} \left| \int_0^\varepsilon P_s (\varepsilon'\,\tilde{U}_{t-s}^\varepsilon(f))^{1+\beta}\,ds \right| + \frac{\varepsilon' \|f\|^2}{2}$$

$$\le \frac{\varepsilon \psi^{-\beta}}{1+\beta} \|f\|^{1+\beta} + \frac{\varepsilon' \|f\|^2}{2} \qquad and \qquad \tilde{U}_{t_1-t_2}^\varepsilon = \tilde{U}_\varepsilon^\varepsilon \tilde{U}_{t_1-t_2-\varepsilon}^\varepsilon,$$

$$(*) = \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon (P_\varepsilon (f_1 + \tilde{U}_{t_2-t_1-\varepsilon}^\varepsilon (P_\varepsilon (f_2 + \ldots)))$$
$$= \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1}^\varepsilon (f_1 + U_{t_2-t_1}^\varepsilon (f_2 + \ldots))$$
$$= U_{t_1} (f_1 + U_{t_2-t_1} (f_2 + \ldots))).$$

Q.E.D.

ACKNOWLEDGEMENTS

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