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MIKLÓS CSÖRGÖ

LAJOS HORVÁTH

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On the distributions of L_p norms of weighted quantile processes

by

Miklós CSÖRGÖ (*)

Department of Mathematics and Statistics, Carleton
University, Ottawa K1S 5B6, Canada

and

Lajos HORVÁTH ()**

Bolyai Institute, Szeged University, H-6720 Szeged,
Aradi vértanúk tere 1, Hungary

and
Department of Mathematics, University of Utah
Salt Lake City, UT 84112, U.S.A.

ABSTRACT. — We prove limit theorems for the distributions of L_p norms of weighted quantile processes. The limiting random variables are represented as integrals of weighted Wiener and exponential partial sum processes.

RÉSUMÉ. — Nous démontrons des théorèmes sur les distributions limites des L_p -normes des processus quantiles pondérés. Les variables aléatoires limites sont représentées par des intégrales d'un processus gaussien pondéré et d'un processus d'une somme des variables aléatoires exponentielles.

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1. INTRODUCTION

Let X_1, X_2, \dots be independent identically distributed random variables with distribution function F . The quantile function Q is defined by

$$Q(t) = \inf \{x: F(x) \geq t\}, \quad 0 < t < 1.$$

Let $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ be the order statistics of X_1, \dots, X_n , and define the empirical quantile function Q_n by

$$Q_n(t) = X_{k,n}, \quad (k-1)/n < t \leq k/n \quad (k=1, 2, \dots, n). \quad (1.1)$$

In this paper we study the asymptotic behaviour of L_p norms of the process

$$\mu_n(t) = Q(t) - Q_n(t), \quad 0 < t < 1 \quad (1.2)$$

in weighted metrics. If F is absolutely continuous, one can introduce a normalized version of μ_n , the so-called quantile process

$$\rho_n(t) = n^{1/2} f(Q(t)) (Q(t) - Q_n(t)), \quad 0 < t < 1. \quad (1.3)$$

Here f is the density function of F , and it is assumed to be positive on the open support of F , (t_F, t^F) , where $t_F = \sup \{x: F(x) = 0\}$ and $t^F = \inf \{x: F(x) = 1\}$. We use the notation $t_F = Q(0)$ and $t^F = Q(1)$.

We can define U_i , $1 \leq i \leq n$, independent uniform-(0, 1) r. v.'s with their corresponding order statistics $U_{1,n} \leq U_{2,n} \leq \dots \leq U_{n,n}$ and uniform empirical quantile function

$$U_n(t) = U_{k,n}, \quad (k-1)/n < t \leq k/n \quad (k=1, \dots, n) \quad (1.4)$$

such that

$$Q(U_k) \stackrel{\mathcal{D}}{=} X_k \quad (k=1, 2, \dots).$$

The corresponding uniform quantile process is

$$u_n(t) = n^{1/2} (t - U_n(t)), \quad 0 < t < 1. \quad (1.5)$$

The process $\{\rho_n(t), 0 < t < 1\}$ was first studied by Bickel (1967), Pyke and Shorack (1968) and Shorack (1972). Strong approximations of u_n were given by M. Csörgö and Révész (1975). They also studied the uniform distance of u_n and ρ_n as well as their strong approximations in M. Csörgö and Révész (1978, 1981) [see also M. Csörgö (1983) and references therein]. Initiated by Chibisov (1964), the limiting behaviour of u_n and ρ_n in weighted metrics (the so-called $\|\cdot\|_q$ metrics) has been investigated by several authors, including Pyke and Shorack (1968), Shorack (1972, 1979, 1982), O'Reilly (1974), Dudley (1981), Dudley and Philipp (1983), Pollard (1984), Mason (1984), M. Csörgö, S. Csörgö, Horváth and Révész (1984), M. Csörgö and Mason (1985), M. Csörgö, S. Csörgö and Horváth (1986), M. Csörgö, S. Csörgö, Horváth and Mason (1986), and M. Csörgö and

Horváth (1986). M. Csörgö and Horváth (1989) and Horváth (1987) studied the limit distributions of $\sup |\rho_n(t)|/(t^\nu L(t))$ and $\sup |u_n(t)|/(t^\nu L(t))$, when L is a slowly varying function at zero and where the sup is taken over intervals of the form $[1/(n+1), 1/2]$, $[1/(n+1), k_n/n]$ and $[k_n/n, 1/2]$. Here $\{k_n, n \geq 1\}$ is a sequence of positive numbers such that

$$1 \leq k_n \leq n \quad \text{and} \quad k_n/n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (1.6)$$

A very complete description of the almost sure behaviour of weighted versions of the uniform quantile process u_n was given by Einmahl and Mason (1988), who in their Remark 3 note that the extension of their results to ρ_n under various regularity conditions is straightforward. Further details and references on these problems are given in Shorack and Wellner (1986, Chapters 3, 11, 16 and 18).

M. Csörgö and Horváth [CsH] (1988 *b*) study the asymptotic behaviour of $\int |u_n(t)|^p/(t^\nu L(t)) dt$, $1 \leq p < \infty$, where the limits of integration are the just mentioned intervals. In this paper we consider the corresponding functionals of ρ_n and μ_n over the same intervals. Corresponding results over similar subintervals of $[1/2, 1)$ can be formulated and proved in the same way.

Our L_p norms, $1 \leq p < \infty$, are motivated by traditional L_1 and L_2 norm investigations and by the fact that such functionals for ρ_n and μ_n are of statistical interest. The L_2 norm has been extensively studied in the literature. For example, the method of least squares (minimization of L_2 norm) enjoys well known optimality properties. On the other hand, the least absolute error is a recognized superior robust method specially well suited to longer tailed error distributions. However, the L_1 norm as a base for statistical data analysis, which may include estimation or model fitting, hypothesis testing, cluster analysis and nonparametric analysis, has not been studied very extensively and asymptotic results are scarce. In particular, according to the Fortet-Mourier theorem [*cf.*, e.g., Shorack and Wellner (1986, pp. 64-65)], the L_1 norm $\int_0^1 |Q(t) - Q_n(t)| dt$ is the Wasserstein distance of the distributions F and F_n , where F_n is the empirical distribution function of the r. v.'s X_i ($i=1, \dots, n$). Our results are believed to be new not only for L_p in general, but also for L_1 and L_2 as well.

As further applications of our results, we note also that all the theorems of this exposition remain true with $p=1$ and the absolute value signs removed. In this case we end up with asymptotic results for functionals

like, for example,

$$\int_{1/(n+1)}^{n/(n+1)} g(t) (Q(t) - Q_n(t)) dt$$

and

(1.7)

$$\int_{1/(n+1)}^{k_n/(n+1)} g(t) (Q(t) - Q_n(t)) dt,$$

where g stands for any of the appropriate weight functions of our theorems. The asymptotic representations of these random variables are the corresponding earlier ones with $p=1$ and no absolute value signs. The thus obtained asymptotics for integrals like those of (1.7) translate to results for sums like

$$\sum_{i=1}^n c_{i,n} X_{i,n} \quad \text{and} \quad \sum_{i=1}^{k_n} c_{i,n} X_{i,n}, \quad (1.8)$$

where, typically, $c_{i,n} = \int_{i/n}^{(i+1)/n} g(t) dt$. The first sum in (1.8) has the form of the much investigated sums of linear combinations of order statistics [cf., e. g., Chernoff, Gastwirth and Johns (1967), Stigler (1969), Mason (1981), as well as Serfling (1980, Chapter 8) and Shorack and Wellner (1986, Chapter 18)]. According to our theorems, the limiting behaviour of $\sum_{i=1}^n c_{i,n} X_{i,n}$ being normal or non-normal will be determined by that of the linear combination of the smallest, respectively the largest order statistics. For further results on sums of order statistics we refer to S. Csörgö, Horváth and Mason (1986), S. Csörgö and Mason (1985, 1986), S. Csörgö, Deheuvels and Mason (1985), S. Csörgö, Haeusler and Mason (1988), Deheuvels and Mason (1988), CsH (1988a), and Lo (1989). Recently Mason and Shorack (1988) announced necessary and sufficient conditions for asymptotic normality of linear combinations of order statistics.

2. MAIN RESULTS

We say that q is a positive function on $(0, 1/2]$, if $\inf_{\delta \leq t \leq 1/2} q(t) > 0$ for each $0 < \delta < 1/2$. The process $\{B(t); 0 \leq t \leq 1\}$ denotes a Brownian bridge throughout, and $\{W(t); 0 \leq t < \infty\}$ stands for the standard Wiener process.

THEOREM 2.1. — *Let q be a positive function on $(0, 1/2]$, $1 \leq p < \infty$, and assume*

- (i) $f(Q(t))$ is positive and continuous on $(0, 1/2]$,

(ii) $\int_0^{1/2} t^{p/2}/q(t) dt < \infty,$

(iii) $1/f(Q(t)) \leq w(t)$, where w is a monotone function on $(0, 1/2]$,

(iv) $\int_0^{1/2} (f(Q(t)) w(t/\lambda))^p t^{p/2}/q(t) dt < \infty$ for all $\lambda > 1$ if w is non-increasing,

and for all $0 < \lambda < 1$ if w is non-decreasing on $(0, 1/2]$.

Then with $\{k_n\}$ as in (1.6) we have, as $n \rightarrow \infty$,

$$\int_{1/(n+1)}^{1/2} |\rho_n(t)|^{p/q}(t) dt \xrightarrow{\mathcal{D}} \int_0^{1/2} |\mathbf{B}(t)|^{p/q}(t) dt, \tag{2.1}$$

$$\int_{k_n/n}^{1/2} |\rho_n(t)|^{p/q}(t) dt \xrightarrow{\mathcal{D}} \int_0^{1/2} |\mathbf{B}(t)|^{p/q}(t) dt, \tag{2.2}$$

and

$$\int_{1/(n+1)}^{k_n/n} |\rho_n(t)|^{p/q}(t) dt \xrightarrow{P} 0. \tag{2.3}$$

Remark 2.1. — If $w(t) = t^\alpha$, $-\infty < \alpha < \infty$, and $p=2$, then (iv) of Theorem 2.1 is equivalent to

$$\int_0^{1/2} f^2(Q(t)) t^{1+2\alpha}/q(t) dt < \infty.$$

Thus our Theorem 2.1 can be considered as an extension of Theorem 3 of Mason (1984).

A look at (2.3) immediately suggests that on multiplying by an appropriate sequence of constants tending to ∞ , we should be able to get a non-degenerate limit distribution. In order to do so, we use weight functions of the form $t^\nu L(t)$ ($-\infty < \nu < \infty$), where $L(t)$ is positive on $(0, 1/2]$, Lebesgue measurable and

$$\lim_{t \rightarrow 0} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for all } \lambda > 0. \tag{2.4}$$

Also, we need stronger conditions than those of Theorem 2.1. Namely we assume

C:

(i) f is twice differentiable on its open support

(ii) $f(Q(t)) > 0$, $0 < t < 1$,

(iii) $\sup_{0 < t < 1} t(1-t) \frac{|f'(Q(t))|}{f^2(Q(t))} < \infty$,

which also figure in M. Csörgö and Révész (1978).

THEOREM 2.2. — *We assume that condition C holds and $1 \leq p < \infty$. Let L be a slowly varying function at zero and k_n as in (1.6). If*

$-\infty < \nu < 1 + p/2$, then, as $n \rightarrow \infty$,

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) \int_{1/(n+1)}^{k_n/n} |\rho_n(t)|^p / (t^\nu L(t)) dt \xrightarrow{\mathcal{D}} \int_0^1 t^{-\nu} |W(t)|^p dt.$$

The case of $\nu = 1 + p/2$ is separating the light weights from the heavy ones. In this case we have the following central limit theorems.

THEOREM 2.3. — *We assume that condition C holds and $1 \leq p < \infty$. Let $\{k_n\}$ be as in (1.6). Then, as $n \rightarrow \infty$,*

$$(2D \log n)^{-1/2} \left\{ \int_{1/(n+1)}^{1/2} \rho_n^2(t) / (t(1-t))^2 dt - m \log n \right\} \xrightarrow{\mathcal{D}} N(0, 1), \quad (2.5)$$

$$(2D \log k_n)^{-1/2} \left\{ \int_{1/(n+1)}^{k_n/n} \rho_n^2(t) / (t(1-t))^2 dt - m \log k_n \right\} \xrightarrow{\mathcal{D}} N(0, 1), \quad (2.6)$$

and

$$(2D \log(n/k_n))^{-1/2} \left\{ \int_{k_n/n}^{1/2} \rho_n^2(t) / (t(1-t))^2 dt - m \log(n/k_n) \right\} \xrightarrow{\mathcal{D}} N(0, 1), \quad (2.7)$$

where $D = D(p)$ is a positive constant, $m = m(p) = E|N(0, 1)|^p$, and $N(0, 1)$ stands for the standard normal r. v.

Now we turn to dealing with heavy weights.

THEOREM 2.4. — *We assume that condition C holds and $1 \leq p < \infty$. Let L be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $p/2 + 1 < \nu < \infty$, then, as $n \rightarrow \infty$,*

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) \int_{k_n/n}^{1/2} |\rho_n(t)|^p / (t^\nu L(t)) dt \xrightarrow{\mathcal{D}} \int_0^1 t^{\nu-(p+2)} |W(t)|^p dt. \quad (2.8)$$

When integrating from $1/(n+1)$ to k_n/n , the limit is determined only by the smallest order statistics. The asymptotic behaviour of extreme order statistics has a considerable literature. Gnedenko (1943) gives necessary and sufficient conditions on F under which there exist sequences $a(n) > 0$ and $b(n)$ such that the normalized minimum $(X_{1,n} - b(n))/a(n)$ has a non-degenerate limiting distribution. The limiting extreme value distribution is

one of the three functions:

$$\Phi_\gamma(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 1 - \exp(-|x|^{-\gamma}), & \text{if } x < 0, \end{cases}$$

$$\Psi_\gamma(x) = \begin{cases} 1 - \exp(-x^\gamma), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and

$$\Lambda(x) = \exp(-\exp(-x)), \quad -\infty < x < \infty,$$

where γ is a positive constant. The distribution function F is said to belong to the domain of attraction of Φ_λ , Ψ_λ or Λ [$F \in \mathcal{D}(\Phi_\lambda)$, $F \in \mathcal{D}(\Psi_\lambda)$ or $F \in \mathcal{D}(\Lambda)$] if there exist sequences $a(n) > 0$ and $b(n)$ such that $\lim_{n \rightarrow \infty} P\{(X_{1,n} - b(n))/a(n) \leq x\} =$ one of these distribution functions for all

x . De Haan (1975) and Galambos (1978) give detailed accounts of characterizations of the domain of attractions of Φ_λ , Ψ and Λ .

Throughout this paper $\{S_i; i \geq 1\}$ stands for partial sums of i.i.d. exponential r. v.'s with expectation one. Let also

$$[x] = \begin{cases} [x], & \text{if } x \text{ is not an integer} \\ [x] - 1, & \text{if } x \text{ is an integer,} \end{cases}$$

where $[x]$ is integer part of x .

THEOREM 2.5. — *We assume that $1 \leq p < \infty$ and $F \in \mathcal{D}(\Phi_\gamma)$. Let H be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $1 - p/\gamma < \theta < \infty$, then, as $n \rightarrow \infty$,*

$$n^{1-\theta} (H(1/n) / |Q(1/n)|^p) \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt$$

$$\xrightarrow{\mathcal{D}} \int_1^\infty u^{-(\theta+p/\gamma)} \left| \left(\frac{S_{[u]+1}}{u} \right)^{-1/\gamma} - 1 \right|^p du.$$

Assuming also that the density function f of F exists, we can extend Theorem 2.5 to the case of $1/2 - p/\gamma < \theta < \infty$. Namely we can use von Mises' sufficient conditions for the domain of attraction of Φ_γ :

C1: $f = F'$ exists and is positive on $(-\infty, x_0]$ for some x_0 and

$$\lim_{x \rightarrow -\infty} \frac{|x|f(x)}{F(x)} = \gamma > 0. \tag{2.9}$$

We note that if f is non-decreasing on $(-\infty, x_0]$ and $F \in \mathcal{D}(\Phi_\gamma)$, then (2.9) holds true [cf. Theorem 2.7.1 in De Haan (1975)].

THEOREM 2.6. — *We assume that condition C1 holds and $1 \leq p < \infty$. Let H be a positive, slowly varying function at zero and $\{k_n\}$ as in (1.6). If*

$1/2 - p/\gamma < \theta < \infty$, then, as $n \rightarrow \infty$,

$$n^{1-\theta} (\mathbf{H}(1/n)/\mathbf{Q}(1/n))^p \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta \mathbf{H}(t)) dt \\ \xrightarrow{\mathcal{D}} \int_1^\infty u^{-(\theta+p/\gamma)} \left| \left(\frac{\mathbf{S}_{\lfloor [u] \rfloor + 1}}{u} \right)^{-1/\gamma} - 1 \right|^p du.$$

From Theorem 2.6 we immediately obtain a result for ρ_n .

THEOREM 2.7. — *We assume that condition C1 holds and $1 \leq p < \infty$. Let \mathbf{L} be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $p/2 + 1 < \nu < \infty$, then, as $n \rightarrow \infty$,*

$$n^{p/2+1-\nu} \mathbf{L}(1/n) \int_{1/(n+1)}^{k_n/n} |\rho_n(t)|^p / (t^\nu \mathbf{L}(t)) dt \\ \rightarrow \gamma^p \int_1^\infty u^{p-\nu} \left| \left(\frac{\mathbf{S}_{\lfloor [u] \rfloor + 1}}{u} \right)^{-1/\gamma} - 1 \right|^p du.$$

Next we consider the case of $\mathbf{F} \in \mathcal{D}(\Psi_\gamma)$. In this case $t_{\mathbf{F}} = \mathbf{Q}(0) > -\infty$.

THEOREM 2.8. — *We assume that $\mathbf{F} \in \mathcal{D}(\Psi_\gamma)$ and $1 \leq p < \infty$. Let \mathbf{H} be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $1 + p/\gamma < \theta < \infty$, then, as $n \rightarrow \infty$,*

$$n^{1-\theta} (\mathbf{H}(1/n)/(\mathbf{Q}(1/n) - \mathbf{Q}(0)))^p \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta \mathbf{H}(t)) dt \\ \xrightarrow{\mathcal{D}} \int_1^\infty \mu^{p/\gamma-\theta} \left| \left(\frac{\mathbf{S}_{\lfloor [u] \rfloor + 1}}{u} \right)^{1/\gamma} - 1 \right|^p du.$$

Von Mises gives the following sufficient condition for $\mathbf{F} \in \mathcal{D}(\Psi_\gamma)$.

C2: $-\infty < t_{\mathbf{F}} = \mathbf{Q}(0)$, $f = \mathbf{F}'$ exists, and is positive on $(t_{\mathbf{F}}, x_0]$ for some x_0 ,

$$\lim_{x \rightarrow t_{\mathbf{F}}} \frac{(x - t_{\mathbf{F}})f(x)}{\mathbf{F}(x)} = \gamma > 0. \quad (2.10)$$

Here we should note that if f is non-decreasing on $(t_{\mathbf{F}}, x_0]$, then (2.10) is a necessary and sufficient condition for $\mathbf{F} \in \mathcal{D}(\Psi_\gamma)$ [cf. Theorem 2.7.2 in De Haan (1975)].

THEOREM 2.9. — *We assume that condition C2 holds and $1 \leq p < \infty$. Let \mathbf{H} be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $1/2 + p/\gamma < \theta < \infty$, then, as $n \rightarrow \infty$,*

$$n^{1-\theta} (\mathbf{H}(1/n)/(\mathbf{Q}(1/n) - \mathbf{Q}(0)))^p \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta \mathbf{H}(t)) dt \\ \xrightarrow{\mathcal{D}} \int_1^\infty u^{p/\gamma-\theta} \left| \left(\frac{\mathbf{S}_{\lfloor [u] \rfloor + 1}}{u} \right)^{1/\gamma} - 1 \right|^p du.$$

THEOREM 2.10. — We assume that condition C2 holds and $1 \leq p < \infty$. Let L be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $p/2 + 1 < \nu < \infty$, then, as $n \rightarrow \infty$,

$$n^{p/2+1-\nu} L(1/n) \int_{1/(n+1)}^{k_n/n} |\rho_n(t)|^p / (t^\nu L(t)) dt \xrightarrow{\mathcal{D}} \gamma^p \int_1^\infty u^{p-\nu} \left| \left(\frac{S_{\lfloor [u] \rfloor + 1}}{u} \right)^{1/\gamma} - 1 \right|^p du.$$

Next we consider the third domain of attraction.

THEOREM 2.11. — We assume that $F \in \mathcal{D}(\Lambda)$ and $1 \leq p < \infty$. Let H be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $1 < \theta < \infty$, then, as $n \rightarrow \infty$,

$$n^{1-\theta} \frac{H(1/n)}{(Q(e/n) - Q(1/n))^p} \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt \xrightarrow{\mathcal{D}} \int_1^\infty u^{-\theta} \left| \log \frac{S_{\lfloor [u] \rfloor + 1}}{u} \right|^p du.$$

Now we state the von Mises sufficiency condition for $F \in \mathcal{D}(\Lambda)$.

C3: $f = F'$ exists, and positive on $(t_F, X_0]$ for some x_0 ,

$$\lim_{x \rightarrow t_F} \frac{f(x) \int_{t_F}^x F(u) du}{F^2(x)} = 1. \quad (2.11)$$

De Haan (1975) proves in his Theorem 2.7.1 that if f is non-decreasing on $(t_F, x_0]$, then (2.11) is also necessary for $F \in \mathcal{D}(\Lambda)$.

THEOREM 2.12. — We assume that condition C3 holds and $1 \leq p < \infty$. Let H be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $p/2 < \theta < \infty$, then, as $n \rightarrow \infty$,

$$n^{1-\theta} \frac{H(1/n)}{(Q(e/n) - Q(1/n))^p} \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt \xrightarrow{\mathcal{D}} \int_1^\infty u^{-\theta} \left| \log \frac{S_{\lfloor [u] \rfloor + 1}}{u} \right|^p du.$$

THEOREM 2.13. — We assume that condition C3 holds and $1 \leq p < \infty$. Let L be a slowly varying function at zero and $\{k_n\}$ as in (1.6). If $p/2 + 1 < \nu < \infty$, then, as $n \rightarrow \infty$,

$$n^{p/2+1-\nu} L(1/n) \int_{1/(n+1)}^{k_n/n} |\rho_n(t)|^p / (t^\nu L(t)) dt \xrightarrow{\mathcal{D}} \int_1^\infty u^{p-\nu} \left| \log \frac{S_{\lfloor [u] \rfloor + 1}}{u} \right|^p du.$$

3. PROOFS

First we state two lemmas which are frequently used in the sequel.

LEMMA 3.1 [Wellner (1978)]. — We have, as $n \rightarrow \infty$,

$$\sup_{1/(n+1) \leq t \leq n/(n+1)} \left(\frac{U_n(t)}{t} \vee \frac{t}{U_n(t)} \right) = O_P(1). \quad (3.1)$$

LEMMA 3.2. — Let L be a slowly varying function at zero and $\{k_n\}$ as in (1.6). Then

$$\lim_{t \rightarrow 0} \sup_{a \leq \lambda \leq b} \left| \frac{L(t)}{L(\lambda t)} - 1 \right| = 0, \quad 0 < a \leq b < \infty, \quad (3.2)$$

and for all $\varepsilon > 0$ we have, as $n \rightarrow \infty$,

$$\sup_{1/(n+1) \leq t \leq k_n/n} (nt)^{-\varepsilon} \left| \frac{L(1/n)}{L(t)} - 1 \right| = o(1), \quad (3.3)$$

$$k_n^{-\varepsilon} \sup_{1/(n+1) \leq t \leq k_n/n} \frac{L(k_n/n)}{L(t)} = O(1), \quad (3.4)$$

and

$$\sup_{1/(n+1) \leq t \leq k_n/n} (nt)^{-\varepsilon} \frac{L(1/n)}{L(t)} = O(1). \quad (3.5)$$

Lemma 3.2 follows from well-known properties of slowly varying functions. For details we refer to De Haan (1975), Seneta (1975) or CsH (1988 b, 1989).

For later use we note that if $p \geq 1$, then we have

$$\left| |a|^p - |b|^p \right| \leq p 2^{p-1} |a-b|^p + p 2^{p-1} |b|^{p-1} |a-b| \quad (3.6)$$

and

$$|a+b|^p \leq 2^p (|a|^p + |b|^p). \quad (3.7)$$

Proof of Theorem 2.1. — We give details only for the case when w of condition (iii) is non-increasing. The non-decreasing case is proved similarly.

By using the construction of M. Csörgö, S. Csörgö, Horváth and Mason (1986), we can assume without loss of generality that there exists a sequence of Brownian bridges $\{B_n(t), 0 \leq t \leq 1\}$ such that, as $n \rightarrow \infty$,

$$n^\alpha \sup_{1/(n+1) \leq t \leq n/(n+1)} |u_n(t) - B_n(t)| / (t(1-t))^{(1/2)-\alpha} = O_P(1) \quad (3.8i)$$

for all $0 \leq \alpha < 1/2$, and

$$\sup_{0 \leq t \leq 1} |u_n(t) - B_n(t)| = o_P(1). \quad (3.8ii)$$

Let $\varepsilon > 0$. Then by the mean value theorem

$$\int_{\varepsilon}^{1/2} |\rho_n(t)|^{p/q}(t) dt = \int_{\varepsilon}^{1/2} \left(\frac{f(Q(t))}{f(Q(\theta_n(t)))} \right)^p \frac{|u_n(t)|^p}{q(t)} dt, \quad (3.9)$$

where

$$t \wedge U_n(t) \leq \theta_n(t) \leq U_n(t) \vee t.$$

By condition (i), (3.8 ii), (3.9) and (3.10) we obtain for all $\varepsilon > 0$

$$\int_{\varepsilon}^{1/2} |\rho_n(t)|^{p/q}(t) dt \xrightarrow{\mathcal{D}} \int_{\varepsilon}^{1/2} |B(t)|^{p/q}(t) dt.$$

Next by condition (ii) [cf. Shepp (1966) or (4.2) in CsH (1988 b)] we have for all $\delta > 0$

$$\lim_{\varepsilon \rightarrow 0} P \left\{ \int_0^{\varepsilon} |B(t)|^{p/q}(t) dt > \delta \right\} = 0, \quad (3.11)$$

and we now show

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \int_{1/(n+1)}^{\varepsilon} |\rho_n(t)|^{p/q}(t) dt > \delta \right\} = 0. \quad (3.12)$$

By condition (iii) and (3.1) it is enough to prove that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \int_{1/(n+1)}^{\varepsilon} (f(Q(t)) w(t/\lambda) |u_n(t)|)^{p/q}(t) dt > \delta \right\} = 0 \quad (3.13)$$

for all $\delta > 0$ and $\lambda > 1$. Using now (3.7), (3.8 i) with $\alpha = 0$ and condition (iv), we get

$$\begin{aligned} \int_{1/(n+1)}^{\varepsilon} (f(Q(s)) w(t/\lambda) |u_n(t) - B_n(t)|)^{p/q}(t) dt \\ = O_P(1) \int_0^{\varepsilon} (f(Q(t)) w(t/\lambda))^p t^{p/2}/q(t) dt. \end{aligned} \quad (3.14)$$

It is easy to see that

$$\begin{aligned} E \int_{1/(n+1)}^{\varepsilon} (f(Q(t)) w(t/\lambda) |B(t)|)^{p/q}(t) dt \\ \leq m(p) \int_0^{\varepsilon} (f(Q(t)) w(t/\lambda))^p t^{p/2}/q(t) dt, \end{aligned} \quad (3.15)$$

where $m(p) = E |N(0, 1)|^p$ ($N(0, 1)$ stands for a standard normal random variable), and hence Markov's inequality and (3.14) imply (3.13). This completes the proof of (2.1).

Now (2.2) and (2.3) also follow from (3.11) and (3.13).

Proof of Theorem 2.2. — We first prove that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) \int_{1/(n+1)}^{k_n/n} \left| \rho_n(t) \right|^p - \left| u_n(t) \right|^p / (t^\nu L(t)) dt = o_p(1), \quad (3.16)$$

and then the result follows from Theorem 2.2 in CsH (1988 *b*). We follow the proof of Theorem 3.1 in CsH (1989). By a two term Taylor expansion we get

$$\rho_n(s) = u_n(s) - (1/2)n^{-1/2} \frac{u_n^2(s)}{s(1-s)} \left\{ \frac{s(1-s)}{\theta_n(s)(1-\theta_n(s))} \right\} \times \left\{ \frac{f'(Q(\theta_n(s)))}{f^2(Q(\theta_n(s)))} \right\} \left\{ \frac{f(Q(s))}{f(Q(\theta_n(s)))} \right\},$$

where

$$U_n(s) \wedge s < \theta_n(s) < U_n(s) \vee s, \quad 0 < s < 1.$$

By Lemma 3.1,

$$\sup_{1/(n+1) \leq s \leq n/(n+1)} \frac{s(1-s)}{\theta_n(s)(1-\theta_n(s))} = O_p(1), \quad n \rightarrow \infty,$$

and by C.1 (iii) and Lemma 4.5.2 of M. Csörgö and Révész (1981)

$$\sup_{1/(n+1) \leq s \leq n/(n+1)} \frac{f(Q(s))}{f(Q(\theta_n(s)))} = O_p(1), \quad n \rightarrow \infty.$$

Using again C.1 (iii) we get

$$\left| \rho_n(t) - u_n(t) \right| = O_p(1) n^{-1/2} u_n^2(t)/t, \quad 1/(n+1) \leq t \leq 1/2, \quad (3.17)$$

where $O_p(1)$ does not depend on t . Using (3.6) and (3.17) we get

$$\begin{aligned} & \left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) \int_{1/(n+1)}^{k_n/n} \left| \rho_n(t) \right|^p - \left| u_n(t) \right|^p / (t^\nu L(t)) dt \\ &= O_p(1) \left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) n^{-p/2} \int_{1/(n+1)}^{k_n/n} u_n^{2p}(t) / (t^{\nu+p} L(t)) dt \\ &+ O_p(1) \left(\frac{k_n}{n}\right)^{\nu-(p/2)+1} L(k_n/n) n^{-1/2} \int_{1/(n+1)}^{k_n/n} \left| u_n(t) \right|^{p+1} / (t^{\nu+1} L(t)) dt. \end{aligned}$$

We show that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) n^{-p/2} \int_{1/(n+1)}^{k_n/n} u_n^{2p}(t) / (t^{\nu+p} L(t)) dt = o_p(1). \quad (3.18)$$

Whenever $-\infty < \nu < 1$, (3.18) immediately follows from Theorem 2.2 in CsH (1988 *b*). Let now $\nu = 1$. Then by (3.4) and Corollary 2.1 in CsH

(1988 *b*) we have

$$\begin{aligned} k_n^{-p/2} L(k_n/n) \int_{1/(n+1)}^{k_n/n} u_n^{2p}(t)/(t^{p+1} L(t)) dt \\ = O(1) k_n^{\varepsilon-p/2} \int_{1/(n+1)}^{k_n/n} u_n^{2p}(t)/t^{p+1} dt = O_P(k_n^{\varepsilon-p/2} (\log k_n)^{1/2}) \end{aligned}$$

for all $\varepsilon > 0$, which gives again (3.18). Let $1 < v < p/2 + 1$. By Theorem 2.5 in CsH (1988 *b*) and (3.3)

$$\begin{aligned} \left(\frac{k_n}{n}\right)^{v-(p/2+1)} L(k_n/n) n^{-p/2} \int_{1/(n+1)}^{k_n/n} u_n^{2p}(t)/(t^{p+v} L(t)) dt \\ = O_P(1) k_n^{v-(p/2+1)} L(k_n/n)/L(1/n) = O_P(1) k_n^{v-(p/2+1)+\varepsilon} \end{aligned}$$

for all $\varepsilon > 0$. On choosing ε small enough, the proof of (3.18) is complete. A similar argument shows that we have also

$$\left(\frac{k_n}{n}\right)^{v-(p/2+1)} L(k_n/n) n^{-1/2} \int_{1/(n+1)}^{k_n/n} |u_n(t)|^{p+1}/(t^{v+1} L(t)) dt = o_P(1)$$

and this in turn implies (3.16).

Proof of Theorem 2.3. — By (3.17), (3.6) and Theorems 2.4 and 2.5 in CsH (1988 *b*)

$$\begin{aligned} \int_{1/(n+1)}^{1/2} \left| |\rho_n(t)|^p - |u_n(t)|^p \right| / t^{p/2+1} dt \\ = O_P(1) n^{-p/2} \int_{1/(n+1)}^{1/2} u_n^{2p}(t)/t^{3p/2+1} dt \\ + O_P(1) n^{-1/2} \int_{1/(n+1)}^{1/2} |u_n(t)|^{p+1}/t^{p/2+2} dt = O_P(1). \end{aligned}$$

Therefore, by Corollary 2.1 in CsH (1988 *b*) the result is immediate.

Proof of Theorem 2.3. — By (3.17) and Theorems 2.4 and 2.5 in CsH (1988 *b*)

$$\begin{aligned} \int_{1/(n+1)}^{1/2} |\rho_n(t) - u_n(t)|^p / t^{p/2+1} dt \\ = O_P(1) n^{-p/2} \int_{1/(n+1)}^{1/2} u_n^{2p}(t)/t^{3p/2+1} dt = O_P(1). \end{aligned}$$

Therefore, by Corollary 2.1 in CsH (1988 *b*) the result is immediate.

Proof of Theorem 2.4. — We prove

$$\left(\frac{k_n}{n}\right)^{v-(p/2+1)} L(k_n/n) \int_{k_n/n}^{1/2} \left| |\rho_n(t)|^p - |u_n(t)|^p \right| / (t^v L(t)) dt = o_P(1), \quad (3.19)$$

and then the result follows from Theorem 2.4 in CsH (1988 *b*). Using Theorem 2.4 in CsH (1988 *b*) we set

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) n^{-p/2} \int_{k_n/n}^{1/2} u_n^{2p}(t)/(t^{p+\nu} L(t)) dt = O_p(k_n^{-p/2}).$$

Applying again Theorem 2.4 in CsH (1988 *b*) we obtain

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L(k_n/n) n^{-1/2} \int_{k_n/n}^{1/2} |u_n(t)|^{p+1}/(t^{1+\nu} L(t)) dt = O_p(k_n^{-1/2}).$$

The last two statements imply (3.19) via (3.6) and (3.17).

The following lemma will be frequently used in the rest of this section.

LEMMA 3.3. — *Let $1-p/2 < \alpha < \infty$, $\beta \neq 0$ and $\{k_n\}$ be as in (1.6). Then, as $n \rightarrow \infty$,*

$$n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{U_n(t)}{t} \right)^\beta - 1 \right|^p dt \xrightarrow{\mathcal{D}} \int_1^\infty u^{-\alpha} \left| \left(\frac{S_{\lfloor u \rfloor + 1}}{u} \right)^\beta - 1 \right|^p du, \quad (3.20)$$

and

$$n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} |\log(U_n(t)/t)|^p dt \xrightarrow{\mathcal{D}} \int_1^\infty u^{-\alpha} \left| \log \frac{S_{\lfloor u \rfloor + 1}}{u} \right|^p du. \quad (3.21)$$

Proof. — It is well known that

$$\{U_{k,n}; 1 \leq k \leq n\} \stackrel{\mathcal{D}}{=} \{S_k/S_{n+1}; 1 \leq k \leq n\}. \quad (3.22)$$

Hence

$$n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{U_n(t)}{t} \right)^\beta - 1 \right|^p dt \stackrel{\mathcal{D}}{=} n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{S_{\lfloor nt \rfloor + 1}}{nt} \right)^\beta \left(\frac{n}{S_{n+1}} \right)^\beta - 1 \right|^p dt, \quad (3.23)$$

and

$$n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} |\log(U_n(t)/t)|^p dt \stackrel{\mathcal{D}}{=} n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \log \frac{S_{\lfloor u \rfloor + 1}}{nt} + \log \frac{n}{S_{n+1}} \right|^p dt.$$

Applying now (3.6) we obtain

$$\begin{aligned}
 n^{1-\alpha} & \left| \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{S_{[nt]+1}}{nt} \right)^\beta \left(\frac{n}{S_{n+1}} \right)^\beta \right. \right. \\
 & \quad \left. \left. - 1 \right|^p dt - \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{S_{[nt]+1}}{nt} \right)^\beta - 1 \right|^p dt \right| \\
 & \leq p 2^{p-1} n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} \left(\frac{S_{[nt]+1}}{nt} \right)^{\beta p} \left| \left(\frac{n}{S_{n+1}} \right)^\beta - 1 \right|^p t^{-\alpha} dt \\
 & \quad + p 2^{p-1} n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} \left(\frac{S_{[nt]+1}}{nt} \right)^\beta \left| \left(\frac{n}{S_{n+1}} \right)^\beta - 1 \right| t^{-\alpha} \\
 & \quad \quad \quad \times \left| \left(\frac{S_{[nt]+1}}{nt} \right)^\beta - 1 \right|^{p-1} dt.
 \end{aligned}$$

Using laws of large numbers and central limit theorem, under the conditions of Lemma 3.3 we obtain

$$\begin{aligned}
 n^{1-\alpha} & \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left(\frac{S_{[nt]+1}}{nt} \right)^{\beta p} \left| \left(\frac{n}{S_{n+1}} \right)^\beta - 1 \right|^p dt \\
 & = O_P(1) \left| \frac{n}{S_{n+1}} - 1 \right|^p n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} t^{-\alpha} dt \\
 & = \begin{cases} O_P(1) n^{1-\alpha-p/2} \left(\frac{k_n}{n} \right)^{1-\alpha}, & \text{if } 1-p/2 < \alpha < 1, \\ O_P(1) n^{-p/2} \log k_n, & \text{if } \alpha = 1, \\ O_P(1) n^{-p/2}, & \text{if } 1 < \alpha < \infty. \end{cases} \tag{3.24}
 \end{aligned}$$

Similarly to (3.24) we get

$$n^{1-\alpha} \int_{1/(n+1)}^{k_n/n} \left(\frac{S_{[nt]+1}}{nt} \right)^\beta \left| \left(\frac{n}{S_{n+1}} \right)^\beta - 1 \right| t^{-\alpha} \left| \left(\frac{S_{[nt]+1}}{nt} \right)^\beta - 1 \right|^{p-1} dt = o_P(1).$$

Also,

$$\begin{aligned}
 n^{1-\alpha} & \int_{1/(n+1)}^{k_n/n} t^{-\alpha} \left| \left(\frac{S_{[nt]+1}}{nt} \right)^\beta - 1 \right|^p dt \\
 & = \int_{n/(n+1)}^{k_n} u^{-\alpha} \left| \left(\frac{S_{[u]+1}}{u} \right)^\beta - 1 \right|^p dt, \tag{3.25}
 \end{aligned}$$

and by laws of iterated logarithm

$$\int_{k_n}^\infty u^{-\alpha} \left| \left(\frac{S_{[u]+1}}{u} \right)^\beta - 1 \right|^p du = o_P(1).$$

This completes the proof of (3.20).

The proof of (3.21) is similar to that of (3.20). Hence the details are omitted.

Proof of Theorem 2.5. — The function F belongs to $\mathcal{D}(\Phi_\lambda)$ if and only if (see e. g. De Haan (1970), Theorem 2.3.1 and Corollary 1.2.15)

$$Q(t) = t^{-1/\gamma} G(t), \quad (3.26)$$

where G is a slowly varying function at zero. Using the latter characterization of the domain of attraction of Φ_γ we have

$$\begin{aligned} n^{1-\theta} (H(1/n)/|Q(1/n)|^p) & \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt \\ & = n^{1-(\theta+p/\gamma)} \frac{H(1/n)}{|G(1/n)|^p} \\ & \quad \times \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} \frac{G(U_n(t))}{G(t)} - 1 \right|^p \frac{|G(t)|^p}{t^{\theta+p/\gamma} H(t)} dt. \end{aligned}$$

On applying (3.6), (3.1), (3.3) and (3.4) with $\varepsilon = \frac{1}{4}(\theta + p/\gamma - 1)$, and (3.20) with $\beta = -p/\gamma$ and $\alpha = \theta + p/\gamma - \varepsilon$, we obtain

$$\begin{aligned} & n^{1-(\theta+p/\gamma)} \\ & \quad \times \int_{1/(n+1)}^{k_n/n} t^{-(\theta+p/\gamma)} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} \frac{G(U_n(t))}{G(t)} - 1 \right|^p \frac{H(1/n)|G(t)|^p}{H(t)|G(1/n)|^p} dt \\ & \quad - \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p t^{-(\theta+p/\gamma)} dt \leq n^{1-(\theta+p/\gamma)} \\ & \quad \times \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p \left| \frac{H(1/n)|G(t)|^p}{H(t)|G(1/n)|^p} - 1 \right| t^{-(\theta+p/\gamma)} dt \\ & + n^{1-(\theta+p/\gamma)} p 2^{p-1} \int_{1/(n+1)}^{k_n/n} \left(\frac{U_n(t)}{t} \right)^{-p/\gamma} \left| \frac{G(U_n(t))}{G(t)} - 1 \right|^p \frac{H(1/n)|G(t)|^p}{H(t)|G(1/n)|^p} t^{-(\theta+p/\gamma)} dt \\ & + n^{1-(\theta+p/\gamma)} p 2^{p-1} \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^{p-1} \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} \left| \frac{G(U_n(t))}{G(t)} - 1 \right| \\ & \quad \times \frac{H(1/n)|G(t)|^p}{H(t)|G(1/n)|^p} t^{-(\theta+p/\gamma)} dt \\ & = o(1) n^{1-(\theta+p/\gamma-\varepsilon)} \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p t^{-(\theta+p/\gamma-\varepsilon)} dt \\ & \quad + o_P(1) n^{1-(\theta+p/\gamma)} \int_{1/(n+1)}^{k_n/n} (nt)^{2\varepsilon} t^{-(\theta+p/\gamma)} dt \\ & + o_P(1) n^{1-(\theta+p/\gamma-\varepsilon)} \int_{1/(n+1)}^{k_n/n} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^{p-1} t^{-(\theta+p/\gamma-\varepsilon)} dt \\ & = o_P(1). \end{aligned}$$

Now (3.20) of Lemma 3.3 yields Theorem 2.5.

Proof of Theorem 2.6. — Let $D(t) = Q(t^{-\gamma})$. Under condition C1 we have [cf., e.g., Horváth (1987)]

$$Q(t) = t^{-1/\gamma} G(t), \tag{3.27}$$

$$f(Q(t)) = t^{1/\gamma+1} K(t), \tag{3.28}$$

$$\lim_{t \rightarrow 0} |G(t) K(t)| = \gamma, \tag{3.29}$$

and

$$D'(t) = -\frac{\gamma}{K(t^{-\gamma})}, \tag{3.30}$$

where G and K are slowly varying functions at zero. Then, by mean value theorem, we obtain

$$\begin{aligned} (n^{1-\theta} H(1/n) / |Q(1/n)|^p) & \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt \\ & = (n^{1-\theta} H(1/n) / |Q(1/n)|^p) \\ & \times \int_{1/(n+1)}^{k_n/n} |D((U_n(t))^{-1/\gamma}) - D(t^{-1/\gamma})|^p / (t^\theta H(t)) dt \\ & = (n^{1-\theta} H(1/n) / |Q(1/n)|^p) \\ & \times \int_{1/(n+1)}^{k_n/n} \left(\frac{\gamma}{K(\xi_n(t))} \left| (U_n(t))^{-1/\gamma} - t^{-1/\gamma} \right| \right)^p / (t^\theta H(t)) dt, \end{aligned}$$

where $\xi_n(t)$ satisfies (3.10). By (3.27) and (3.29) now, it suffices to consider

$$\begin{aligned} n^{1-(\theta+p/\gamma)} H(1/n) (K(1/n))^p & \int_{1/(n+1)}^{k_n/n} \left(\frac{1}{K(\xi_n(t))} \right)^p \\ & \times t^{-(\theta+p/\gamma)} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p / H(t) dt. \end{aligned}$$

Using (3.1), (3.2), and (3.3) with $0 < \varepsilon < \theta + 1/\gamma - 1/2$, in combination with Lemma 3.3 we have

$$\begin{aligned} n^{1-(\theta+p/\gamma)} & \int_{1/(n+1)}^{k_n/n} t^{-(\theta+p/\gamma)} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p \\ & \times \left| \frac{H(1/n) (K(1/n))^p}{H(t) (K(\xi_n(t)))^p} - 1 \right| dt \leq \sup_{1/(n+1) \leq t \leq k_n/n} (nt)^{-\varepsilon} \left| \frac{H(1/n) (K(1/n))^p}{H(t) (K(\xi_n(t)))^p} - 1 \right| \\ & \times n^{1-(\theta+p/\gamma-\varepsilon)} \int_{1/(n+1)}^{k_n/n} t^{-(\theta+p/\gamma-\varepsilon)} \left| \left(\frac{U_n(t)}{t} \right)^{-1/\gamma} - 1 \right|^p dt = o_p(1). \end{aligned}$$

Thus theorem follows by (3.20).

Proof of Theorem 2.7. — Using (3.27), (3.28), (3.29), and Theorem 2.6 with $\theta = \nu - 1 - p/\gamma$, $H(t) = L(t)/(K(t))^p$, the result is immediate.

Proof of Theorem 2.8. — By Theorem 2.3.2 and Corollary 1.2.1.5 in De Haan (1975) we know that $F \in \mathcal{D}(\Psi_\gamma)$ if and only if $-\infty < Q(0) = t_F$ and $Q(t) - Q(0)$ is $1/\gamma$ -varying at zero. We can assume without loss of generality that $Q(0) = 0$. Now this theorem is proved along the lines of the proof of Theorem 2.5. The details are omitted.

Proof of Theorem 2.9. — We can assume again that $Q(0) = 0$. Let now $D(t) = Q(t^\gamma)$. Condition C2 implies [see e.g. Horváth (1987)]

$$Q(t) = t^{1/\gamma} G(t), \quad (3.31)$$

$$f(Q(t)) = t^{1-1/\gamma} K(t), \quad (3.32)$$

$$\lim_{t \rightarrow 0} G(t) K(t) = \gamma, \quad (3.33)$$

and

$$D'(t) = \gamma/K(t^\gamma), \quad (3.34)$$

where G and K are slowly varying functions at zero. Using (3.31)-(3.34), proof of this theorem goes like that of Theorem 2.6.

Proof of Theorem 2.10. — This result follows immediately by (3.31)-(3.34) combined with Theorem 2.9.

Proof of Theorem 2.11. — Theorem 2.4.1 of De Haan (1975) implies that, if $F \in \mathcal{D}(\Lambda)$, then

$$\lim_{t \rightarrow 0} \sup_{a \leq x \leq b} \left| \frac{Q(tx) - Q(t)}{Q(et) - Q(t)} - \log x \right| = 0 \quad (3.35)$$

for all $0 < a \leq b < \infty$, and $Q(et) - Q(t)$ is a slowly varying function at zero. Using (3.6) we get

$$\begin{aligned} n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} \right|^p - \left| \log \frac{U_n(t)}{t} \right|^p dt \\ \leq p 2^{p-1} n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} - \log \frac{U_n(t)}{t} \right|^p dt \\ + p 2^{p-1} n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} - \log \frac{U_n(t)}{t} \right| \\ \times \left| \log \frac{U_n(t)}{t} \right|^{p-1} dt. \end{aligned} \quad (3.36)$$

By (3.1) and (3.35) we have for all $\theta > 1$ that

$$\begin{aligned} n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} - \log \frac{U_n(t)}{t} \right|^p dt \\ = O(1) \sup_{1/(n+1) \leq t \leq k_n/n} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} - \log \frac{U_n(t)}{t} \right|^p = o_p(1). \end{aligned} \quad (3.37)$$

Similarly we have also

$$n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \frac{Q(U_n(t)) - Q(t)}{Q(et) - Q(t)} - \log \frac{U_n(t)}{t} \right| \times \left| \log \frac{U_n(t)}{t} \right|^{p-1} dt = o_p(1). \quad (3.38)$$

From (3.3) with $0 < \varepsilon < \theta - 1$, (3.36)-(3.38), and (3.21) with $\alpha = \theta - \varepsilon$, we obtain

$$n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left(\frac{|\mu_n(t)|}{Q(et) - Q(t)} \right)^p \left| \left(\frac{Q(et) - Q(t)}{Q(e/n) - Q(1/n)} \right)^p \frac{H(1/n)}{H(t)} - 1 \right| dt = o_p(1) n^{1-\theta+\varepsilon} \int_{1/(n+1)}^{k_n/n} t^{-\theta+\varepsilon} \left(\frac{|\mu_n(t)|}{Q(et) - Q(t)} \right)^p dt = o_p(1). \quad (3.39)$$

Now result follows from (3.21).

Proof of Theorem 2.12. - Let $D(t) = Q(e^t)$, $-\infty < t < 0$. Condition C3 implies [see e.g. Horváth (1987)]

$$f(Q(t)) = tK(t), \quad (3.40)$$

$$D'(t) = 1/K(e^t), \quad (3.41)$$

and

$$\lim_{t \rightarrow 0} K(t)(Q(et) - Q(t)) = 1. \quad (3.42)$$

By (3.42) we can replace $1/(Q(e/n) - Q(1/n))$ by $K(1/n)$. Using now (3.41) we have

$$n^{1-\theta} H(1/n) (K(1/n))^p \int_{1/(n+1)}^{k_n/n} |\mu_n(t)|^p / (t^\theta H(t)) dt = n^{1-\theta} H(1/n) (K(1/n))^p \times \int_{1/(n+1)}^{k_n/n} \frac{1}{(K(\xi_n(t)))^p H(t)} \left| \log(U_n(t)/t) \right|^p / t^\theta dt,$$

where $\xi_n(t)$ is as in (3.10). From (3.1), (3.3) with $0 < \varepsilon < \theta - 1/2$, and (3.21) with $\alpha = \theta - \varepsilon$, we get

$$n^{1-\theta} \int_{1/(n+1)}^{k_n/n} t^{-\theta} \left| \log \frac{U_n(t)}{t} \right|^p \times \left| \left(\frac{K(1/n)}{K(\xi_n(t))} \right)^p \frac{H(1/n)}{H(t)} - 1 \right| dt = o_p(1) n^{1-\theta+\varepsilon} \int_{1/(n+1)}^{k_n/n} t^{-\theta+\varepsilon} \left| \log \frac{U_n(t)}{t} \right|^p dt = o_p(1).$$

The latter and (3.21) imply the theorem.

Proof of Theorem 2.13. - Let $\theta = \nu - 1$, $H(t) = L(t)/(K(t))^p$ in Theorem 2.12. Then by (3.40) and (3.42) we obtain the result.

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