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Bin-packing problems for a renewal process

by

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ABSTRACT. — A sequence $X_1, X_2, \ldots$ of i.i.d. (0,1]-valued random variables is distributed into blocks successively as follows. A block starting with X_k ends with X_{k+j} iff $X_k + \ldots + X_{k+j} \leqq 1 < X_k + \ldots + X_{k+j+1}$. Analyzing this procedure, which is known as Next-Fit algorithm for bin packing, we ask, e.g., for the asymptotic behavior of the number of X_i's placed in the n-th block and of the sum of the variables $X_n, X_{n-1}, \ldots$ which are placed in the same block as X_n at the time X_n enters (both as $n \to \infty$).

Key words : Bin packing, renewal process, Markov chain with continuous state space.

RÉSUMÉ. — Une suite $X_1, X_2, \ldots$ de variables aléatoires indépendantes, équidistribuées et à valeurs dans (0, 1] est partagée successivement aux blocs. Le bloc qui commence avec X_k se termine avec X_{k+j} si, et seulement si, $X_k + \ldots + X_{k+j} \leqq 1 < X_k + \ldots + X_{k+j+1}$. Pour de grandes valeurs de n on étudie le comportement asymptotique du nombre des variables aléatoires placées dans le n-ième bloc et de la somme des variables $X_n, X_{n-1}, \ldots$ qui sont placées dans le même bloc que X_n, avant que X_{n+1} arrive. L'application au problème de « bin-packing » est exposée.

Mots clés : Processus de renouvellement, bin-packing, chaîne de Markov avec une espace réelle des états.

Classification A.M.S. : 60 K 05, 60 J 20.

1. INTRODUCTION

Let $X_1, X_2, \ldots \in (0, 1]$ be independent random variables with the common distribution function F. Let $X_0 \in [0, 1)$ be independent of all X_n, $n \geqq 1$. We consider the partitioning of the corresponding renewal process $S_n = X_0 + X_1 + \ldots + X_n$ into blocks of total length at most 1. Thus if

$$S_n \leqq 1 < S_{n+1}$$

and

$$X_{n+1} + \ldots + X_{n+k} \leqq 1 < X_{n+1} + \ldots + X_{n+k+1},$$

the first block has length S_n, the second one has length $X_{n+1} + \ldots + X_{n+k}$, and so on. The following questions will be treated:

(*a*) What is the asymptotic behavior of the number of X_i's placed in the n-th block, as $n \to \infty$?

(*b*) How large is the probability that a given initial value x of X_0 will eventually lead to an increase of the number of blocks (compared with $X_0 = 0$)?

(*c*) How can one describe the limiting behavior of the portion of the current block which is occupied just after the placement of X_n?

A more intuitive description of these problems can be given in terms of a storage procedure. A sequence of storage demands of random sizes $X_1, X_2, \ldots$ are to be satisfied by stores which are all of size 1. X_0 units of the first store are already occupied. The demands may not be broken into pieces. Hence, if $S_n \leqq 1$ and $1 < S_{n+1}$, the demands $X_1, \ldots, X_n$ are successively placed in the first store, X_{n+1} becomes the first amount to be placed in the second store, and so on. The language of "stores" and "demands" will be used throughout.

This storage procedure is closely related to the so-called bin-packing problem in which for a given set A of real numbers in (0, 1] and an infinite set of bins, each with capacity 1, one wants to "store" the members of A in a minimal number of bins [*see*, e. g., Johnson (1974), Johnson *et al.* (1974), and Coffman (1978)]. There is an abounding number of applications of the bin-packing problem to computer science and industry, e. g. stock cutting or the assignment of tracks in disks. These and many further examples are discussed in Johnson *et al.* (1974) and Coffman (1976). In the probabilistic analysis of bin packing the model explained above has been introduced by Coffman *et al.* (1980) and further studied by Ong *et al.* (1986) under the name "Next-Fit algorithm". In Coffman *et al.* (1980) the expected performance of this rule is bounded by the (unknown) expected optimal number of bins; Ong *et al.* (1986) show that the ratio of the expected number of bins to the number of stored demands converges to a constant. In these papers one can also find some explicit results in the cases of uniform or truncated exponential demand distributions.

Our analysis of the storage process is based on the investigation of two Markov chains involved. First consider the store in which the n-th demand X_n is placed. Let U_n be the amount stored there immediately before the $(n+1)$-th demand arrives. Taking into account the recursion

$$U_{n+1}=(U_n+X_{n+1})\,1_{\{U_n+X_{n+1}\leqq 1\}}+X_{n+1}\,1_{\{U_n+X_{n+1}>1\}} \tag{1.1}$$

$$U_0=X_0 \tag{1.2}$$

(1_A denotes the indicator function of the event A) it is seen that $(U_n)_{n\geqq 0}$ is a homogeneous Markov chain. Further, let τ_n be the number of demands placed in the n-th store, and define $V_1=U_0$, $V_{n+1}=X_{\tau_1+\ldots+\tau_n+1}$, $n\geqq 1$. V_n is the size of the first demand placed in the n-th store. Since obviously

$$P(V_{n+1}\in B \mid V_n=v_n, \ldots, V_1=v_1) = \sum_{k=1}^{\infty} P(S_k\leqq 1-v_n<S_{k+1}, X_{k+1}\in B), \qquad n\geqq 1 \tag{1.3}$$

the sequence $(V_n)_{n\geqq 1}$ also forms a homogeneous Markov chain.

In section 2 we give a detailed analysis of $(U_n)_{n\geqq 0}$. It turns out that it is an aperiodic Harris chain whose behavior is independent of the initial value in a very strong sense. The limiting distribution function $M(x)$ of $(U_n)_{n\geqq 0}$ is characterized by an integral equation. Asymptotically, M is the distribution function of the load an arriving demand will find in the current store. Coffman *et al.* (1980) study the ergodic behavior of the related Markov chain $(L_n)_{n\geqq 1}$, where L_n is the sum of all demands placed in the n-th store.

In section 3 we determine the probability $p(u)$ that a single demand of size u is responsible for the need of an additional store. $p(u)$ can be considered as a measure of the load caused by a demand u which is more informative than its mere size. In fact, if $u<u'$, but $p(u)$ and $p(u')$ are close to each other [even $p(u)=p(u')$ is possible], the sequences of demands starting with u or u', respectively, need the same number of stores with high probability. Therefore the function $p(u)$ in a sense provides a more reasonable basis for assessing the costs due to the storage of a demand.

Finally, section 4 deals with the calculation of the mean number of demands placed together in a store. $E(\tau_n)$ obviously is a measure for the "real" capacity of the n-th store with respect to the underlying demand process. An asymptotic formula for $E(\tau_n)$ is derived using martingale arguments. In the case of the uniform distribution some results on the sequence $(\tau_n)_{n\geqq 1}$ can be found in Coffman *et al.* (1980).

We note that it may happen that with probability 1 every store is charged with the same number of demands, although the X_i are not constant. The situation is cleared in the following lemma. $[x]$ denotes the integer part of $x \geqq 0$. Let W be the set of points of increase of the distribution function F and $\alpha = \inf W$, $\beta = \sup W$.

LEMMA. — *Assume* $X_0 = 0$. τ_1 *is almost surely constant iff one of the following two conditions holds:*

(i) $[\alpha^{-1}] = [\beta^{-1}]$.

(ii) $\alpha^{-1} = [\alpha^{-1}] = \beta^{-1} + 1$ *and* $P(X_1 = \alpha) = 0$.

Proof. — We can restrict our attention to the case $0 < \alpha < \beta$. Clearly, $[\alpha^{-1}]$ is an upper bound and $[\beta^{-1}]$ is a lower bound for the number of demands placed in a store, and we always have $P(\tau_1 = [\beta^{-1}]) > 0$. If $P(X_1 = \alpha) > 0$, it follows that $P(\tau_1 = [\alpha^{-1}]) > 0$. Thus in this case condition (i) is necessary and sufficient. Now let $P(X_1 = \alpha) = 0$. If α^{-1} is not an integer, we again have $P(\tau_1 = [\alpha^{-1}]) > 0$. But if α^{-1} is an integer, $P(\tau_1 \leqq \alpha^{-1} - 1) = 1$ and $P(\tau_1 = \alpha^{-1} - 1) > 0$. Thus $\beta^{-1} = \alpha^{-1} - 1$ is necessary and sufficient for $\tau_1 = \text{Const.}$ almost surely.

In what follows we shall always assume that τ_1 is *not* almost surely constant, *i.e.*, that neither (i) nor (ii) of the lemma holds.

2. THE MARKOV CHAIN $(U_n)_{n \geqq 0}$

We retain the notation of section 1 and start by studying the homogeneous Markov chain $(U_n)_{n \geqq 0}$ defined by (1.1) and the initial condition $U_0 = u$. P^{U_n} denotes the distribution of U_n. For background on Markov chains with continuous state space see Asmussen (1987, chapter VI.3), Laslett *et al.* (1987), and Tweedie (1975).

THEOREM 1. — $(U_n)_{n \geqq 0}$ is an aperiodic Harris chain. P^{U_n} converges in total variation to the probability measure whose distribution function M is uniquely determined by the equation

$$M(x) = \int_0^x F(x-u)\, dM(u) + \int_{1-x}^1 [F(x) - F(1-u)]\, dM(u). \tag{2.1}$$

Proof. — (1) $(U_n)_{n \geqq 0}$ is a Harris chain. In order to show this, we must find a probability measure λ on $[0, 1]$, an $\varepsilon > 0$ and a recurrent "regeneration set" $R \subset [0, 1]$ such that

$$P(U_1 \in B \mid U_0 = u) \geqq \varepsilon \lambda(B) \tag{2.2}$$

for all $u \in R$ and all Borel subsets B of $(0, 1]$. First we assume that $\alpha > 0$. Then define $\varepsilon = P(X_1 > \alpha)$ and

$$\lambda(B) = P(X_1 \in B,\, X_1 > \alpha)/P(X_1 > \alpha), \qquad R = [1 - \alpha, 1].$$

It is easily checked that, for $u \in R$,

$$P(U_1 \in B \mid U_0 = u) \geqq P(X_1 > \alpha,\, X_1 \in B) = \varepsilon \lambda(B).$$

Further we have to show that R is recurrent, *i.e.* $P(\tau(R) < \infty \mid U_0 = u) = 1$ for all $u \in [0, 1)$, where $\tau(R)$ is the first entrance time for R. This follows immediately from the easily verified fact that there exists a $\gamma > 0$ with the following property: For every $u \in [0, 1]$ there is a $m \in \{1, \ldots, [\alpha^{-1}]\}$ such that

$$P(U_{n+m} \in R \mid U_n = u) \geqq \gamma.$$

Now let $\alpha = 0$. In this case every interval in $[0, 1]$ is recurrent. Choose a $\tilde{\alpha} > 0$ such that $P(X_1 > \tilde{\alpha}) > 0$ and define $\tilde{R}$, $\tilde{\varepsilon}$ and $\tilde{\lambda}$ as R, ε and λ, but with α replaced by $\tilde{\alpha}$. Then again $P(U_1 \in B \mid U_0 = u) \geqq \tilde{\varepsilon}\tilde{\lambda}(B)$ for all $u \in \tilde{R}$ and all Borel subsets B of $[0, 1]$. (2.2) is proved.

(2) $(U_n)_{n \geqq 0}$ is P^{X_1}-irreducible (according to the definition in Tweedie (1975)), *i.e.*, for every $u \in [0, 1]$ and every Borel set $B \subset [0, 1]$ satisfying $P(X_1 \in B) > 0$ there exists a positive integer n such that

$$P(U_n \in B \mid U_0 = u) > 0.$$

The probability in question is not smaller than

$$P(u + X_1 + \ldots + X_{n-1} \leqq 1 < u + X_1 + \ldots + X_n,\ X_n \in B).$$

If $\alpha = \inf W > 0$, this latter probability is positive for $n = [\alpha^{-1}(1-u) + 1]$. If $\alpha = 0$, it is positive for all sufficiently large n.

(3) $(U_n)_{n \geqq 0}$ is aperiodic. This will follow from a much stronger property which we shall prove next. Let $U_n(u)$ and $U_n(u')$ be the Markov chains generated by the same sequence $X_1, X_2, \ldots$ via the recurrence relation (1.1), but with different initial values u and u'. Then

$$\lim_{n \to \infty} P(U_n(u) = U_n(u')) = 1. \tag{2.3}$$

Note that $U_k(u) = U_k(u')$ for some k implies $U_n(u) = U_n(u')$ for all $n \geqq k$. (2.3) follows if we can show that there exists an $\eta > 0$ and a $k \in \mathbb{N}$ such that

$$P(U_k(u) \neq U_k(u')) \leqq 1 - \eta \quad \text{for all } u, u' \in [0, 1]. \tag{2.4}$$

Since $U_n(u) \neq U_n(u')$ for all n implies $U_{ik}(u) \neq U_{ik}(u')$ for all $i \in \mathbb{N}$, (2.4) entails [for the moment using the abbreviation $U_i = U_{ik}(u)$, $U'_i = U_{ik}(u')$]

$$\begin{aligned} P(U_n(u) \neq U_n(u') \text{ for all } n) &\leqq P(U_j \neq U'_j, \ldots, U_1 \neq U'_1) \\ &\leqq \iint_{u_1 \neq u'_1} \cdots \iint_{u_{j-1} \neq u'_{j-1}} P(U_j \neq U'_j \mid U_{j-1} = u_{j-1}, U'_{j-1} = u'_{j-1}) \\ &\qquad \times P(U_{j-1} \in du_{j-1}, U'_{j-1} \in du'_{j-1} \mid U_{j-2} = u_{j-2}, U'_{j-2} = u'_{j-2}) \\ &\qquad\qquad \times \ldots P(U_1 \in du_1, U'_1 \in du'_1) \\ &\leqq (1-\eta) P(U_{j-1} \neq U'_{j-1}, \ldots, U_1 \neq U'_1) \leqq \ldots \leqq (1-\eta)^j \end{aligned}$$

for all $j \in \mathbb{N}$, and hence $P(U_n(u) \neq U_n(u')$ for all $n) = 0$ proving (2.3). It remains to show how η and k can be found such that (2.4) holds.

For this purpose we choose γ, $\delta \in [\alpha, \beta]$ with the following properties:

(i) $0 < \gamma < \delta$.

(ii) $[\gamma^{-1}] \neq [\delta^{-1}]$.

(iii) γ and δ are points of increase of F.

We call a demand "small", if its size falls into the interval $(\gamma-\varepsilon, \gamma+\varepsilon)$, and "large", if its size falls into $(\delta-\varepsilon, \delta+\varepsilon)$. Here $\varepsilon > 0$ is taken small enough to ensure that i small and j large demands can be placed in the same store iff $i\gamma+j\delta < 1$ (in the case when γ and δ are atoms of P^{X_1}, only demands of size exactly equal to γ or δ will be called small or large, respectively). This choice of ε makes it possible to treat small and large demands as if they were constantly equal to γ and δ, respectively. By (iii), small and large demands occur with positive probability.

We consider two storage processes in parallel, the first one with initial value u and the second one with initial value u'. Let $u < u'$. We start by a sequence of small demands of length m, where we choose m large enough to guarantee that in both storage processes at least the second store is maximally filled. Thus in both processes there is a store containing the maximal number $N=[\gamma^{-1}]$ of small demands. Looking at the last stores of both processes there are two possibilities: Either they contain the same number of small demands; then $U_m(u)=U_m(u')$. Or the last store of the second process takes a lead of say l small demands. In this second case we have $l \in \{1, \ldots, N-1\}$. Now we shall show how to diminish l to $l-1$ by continuing with small and large demands.

First note that if a store contains the maximal number N of small demands, $r=[(1-N\delta)/(\delta-\gamma)]$ is the maximal number of small demands in it which can be replaced by large ones. Now continue both processes until in the second process the current store and an additional one is filled. Two subcases must be distinguished:

1. *subcase.* $r > 0$. Replace $2r$ small demands by large ones. For this purpose choose the r small demands which are placed as the first ones in the last store of the second process, and the r small demands which are the last ones of the preceding store. By the construction, this replacement does not change the number of demands in the stores of the second process, but it does in the *first* process. The lead of the second process is reduced from 1 to $l-1$.

2. *subcase.* $r=0$. Replace the small demand which is the last one in the last maximally filled store of the first process by a large demand. This large demand then has to be placed as the first demand in the next store of the first process so that again the lead of the second process is reduced.

These considerations complete the proof of (2.4).

Now let $d \in \mathbb{N}$ be the period of the Markov chain. Then there is a partition $[0, 1]=E_1 \cup \ldots \cup E_d \cup N$ of the state space into a transient set N and non-empty sets $E_1, \ldots, E_d$ for which $P(U_1 \in E_{i+1} \mid U_0 \in E_i)=1$ for

$i=1, 2, \ldots$ (where we identify E_{d+1} with E_1 and so on). Assume $d \geqq 2$. Choose $u \in E_1$, $u' \in E_2$. By (2.3),

$$P(U_i(u) \in E_1) - P(U_i(u') \in E_1) \to 0, \quad \text{as } n \to \infty. \tag{2.5}$$

On the other hand, by the choice of u and u',

$$\begin{aligned} P(U_i(u) \in E_1) = 1 \quad &\text{iff } i \equiv 0 \pmod d \\ P(U_i(u') \in E_1) = 1 \quad &\text{iff } i \equiv d-1 \pmod d. \end{aligned}$$

But this contradicts (2.5). It follows that $d=1$. $(U_n)_{n \geqq 0}$ is aperiodic.

(4) $(U_n)_{n \geqq 0}$ is ergodic. We only have to verify Doeblin's condition [Tweedie (1975), p. 386]: There exists a probability measure θ on $[0, 1]$, a positive integer n and a $\delta > 0$ such that, whenever $\theta(B) \leqq \delta$,

$$P(U_n \in B \mid U_0 = u) \leqq 1 - \delta \quad \text{for all } u \in [0, 1).$$

Note that

$$\begin{aligned} P(U_n \in B \mid U_0 = u) &\leqq P(X_1 + \ldots + X_n \leqq 1 - u) \\ &\quad + \sum_{i=1}^{n} P(\tau_1 + \ldots + \tau_j = i - 1 \text{ for some } j,\ X_i + \ldots + X_n \in B) \\ &\leqq r + \tilde{\theta}(B), \end{aligned} \tag{2.6}$$

where

$$r = P(X_1 + \ldots + X_n \leqq 1) < 1 \quad \text{for all sufficiently large } n$$

and

$$\tilde{\theta}(B) = \sum_{i=1}^{n} P(X_1 + \ldots + X_{n-i+1} \in B).$$

Set $\theta(B) = \tilde{\theta}(B)/\tilde{\theta}([0, 1])$, $\delta = (1-r)/(n+1)$. Then, by (2.6), if $\theta(B) \leqq \delta$,

$$P(U_n \in B \mid U_0 = u) \leqq r + \tilde{\theta}([0, 1])\,\theta(B) \leqq r + n\,\theta(B) \leqq r + n\delta = 1 - \delta,$$

as claimed.

(5) It now follows from the ergodic theorem for Markov chains that there is a uniquely determined invariant probability measure which is also the limit in total variation of P^{U_n}. Let M be its distribution function. The invariance is characterized by the property that if M is the distribution

function of U_0, then also

$$M(x)=P(U_1 \leqq x)=\int P(U_1 \leqq x \mid U_0=u)\, dM(u)$$

$$=\int [P(u+X_1 \leqq x,\ u+X_1 \leqq 1)+P(X_1 \leqq x,\ u+X_1 > x)]\, dM(u)$$

$$=\int_0^x F(x-u)\, dM(u)+\int_{1-x}^1 [F(x)-F(1-u)]\, dM(u).$$

The Theorem is completely proved.

3. THE PROBABILITY OF A SINGLE DEMAND TO INCREASE THE NUMBER OF STORES

We compare the cases $U_0=0$ and $U_0=u\in[0, 1)$. If $U_0=u$, does this necessitate the use of additional stores? Let $p_m(u)$ be the probability that in the case $U_0=u$ more stores are needed than in the case $U_0=0$, if m demands must be satisfied. Obviously $p_m(u)$ is non-decreasing with respect to u, non-increasing with respect to m and $p_m(0)=0$, $p_m(1)=1$, $p_1(u)=1-F(1-u)$. Let $p(u)=\lim_{m\to\infty} p_m(u)$.

LEMMA. — *If* $U_0=u$, *at most one more store is needed than in the case* $U_0=0$.

Proof. — For the moment do not use the first store at all, but start storing $X_1, \ldots, X_m$ in the second store. Suppose that one ends up with the $(n+1)$-th store. Now try to transpose the first X_1 units from the second to the first store. If this is not possible (*i.e.*, $u+X_1 > 1$), one needs $n+1$ stores if $U_0=u$. Otherwise proceed by trying to shift the next X_2 units from the second to the first store, and so on. Clearly this procedure leads to the placement of the m demands generated if $U_0=u$, which, consequently, needs at most $n+1$ stores. But given $U_0=0$, exactly n stores are needed.

THEOREM 2. — $p(u)$ satisfies the integral equation

$$p(u)=1-F(1-u)+\int_0^{1-u} [p(u+x)-p(x)]\, dF(x) \tag{3.1}$$

Proof. — Conditioning on X_1 we obtain

$$p_m(u)=1-F(1-u)+\int_0^{1-u} [p_{m-1}(u+x)-p_{m-1}(x)]\, dF(x). \tag{3.2}$$

For if $X_1 > 1-u$, the storage process really starts with the second store so that one store (the first one) is needed additionally. But if $X_1 = x \in (0, 1-u]$, X_1 is placed in the first store, and the probability that one extra store is needed (relative to $X_2, \ldots, X_m$) is equal to the probability that one extra store is necessary under the condition that $u+x$ units in the first store are occupied minus the corresponding probability given that x units of the first store are occupied. This proves (3.2). Since $p(u)$ is the monotone limit of $p_m(u)$, as $m \to \infty$, (3.1) follows from (3.2) by Lebesgue's dominated convergence theorem.

The recursion (3.2) also provides a convenient method for the approximate determination of $p(u)$, since an exact solution of (3.1) usually does not seem possible.

4. THE MEAN NUMBER OF DEMANDS IN A STORE

τ_n has been defined to be the number of demands placed in the n-th store. In this section we shall derive the limit of the expected value $E(\tau_n)$.

THEOREM 3:

$$\lim_{n \to \infty} E(\tau_n) = \left(\int_0^1 p(x)\, dF(x) \right)^{-1}. \tag{4.1}$$

Example. — Let us consider the uniform case $F(x) = x$, $0 \leqq x \leqq 1$. Using Theorem 2 it is easily checked that $p(u) = u(2-u)$. A straightforward calculation then yields $\lim_{n \to \infty} E(\tau_n) = 3/2$.

This limit has already been derived by Coffman *et al.* (1980) by a different method.

Proof of Theorem 3. — Let T_n be the number of the store in which the n-th demand placed. Let $\mu = \int_0^1 p(x)\, dF(x)$ and denote by $\mathscr{A}_n$ the σ-field generated by $X_0, \ldots, X_n, T_0, \ldots, T_n$. We shall show that

$$Y_n = T_n + p(U_n) - n\mu \tag{4.2}$$

is a martingale relative to $(\mathscr{A}_n)_{n \geqq 0}$. Indeed, we have

$$\begin{aligned}
E(Y_{n+1} \mid \mathscr{A}_n) &= \int_0^{1-U_n} [T_n + p(U_n + u)]\, dF(u) \\
&\quad + \int_{1-U_n}^{1} [T_n + 1 + p(u)]\, dF(u) - (n+1)\mu \\
&= T_n + 1 - F(1-U_n) + \int_0^{1-U_n} [p(U_n+u) - p(u)]\, dF(u) \\
&\quad + \int_0^1 p(u)\, dF(u) - (n+1)\mu = T_n + p(U_n) - n\mu,
\end{aligned}$$

where we have used Theorem 2 for the last equation. Next we must consider V_n, the size of the first demand placed in the n-th store. The Markov chain $(V_n)_{n \geqq 1}$ is P^{X_1}-irreducible (this is seen similarly as for $(U_n)_{n \geqq 0}$) and aperiodic (since $V_n = U_{N_n}$, where N_n is the number of the first demand placed in the n-th store, and $N_n \to \infty$, we even have $\lim_{n \to \infty} P(V_n(v) = V_n(v')) = 1$ for all $v, v' \in [0, 1]$). Again Doeblin's condition is satisfied, because

$$\begin{aligned}
P(V_2 \in B \mid V_1 = v) &\leqq P(X_1 + \ldots + X_n \leqq 1 - v) \\
&+ \sum_{j=0}^{n-1} P(X_1 + \ldots + X_j \leqq 1 - v < X_1 + \ldots + X_{j+1},\ X_{j+1} \in B) \leqq r + n\theta(B),
\end{aligned}$$

where $r = P(X_1 + \ldots + X_n \leqq 1) < 1$ for sufficiently large n and $\theta(B) = P(X_1 \in B)$. Setting $\delta = (1-r)/(n+1)$ we obtain

$$P(V_2 \in B \mid V_1 = v) \leqq r + n\delta = 1 - \delta, \qquad \text{if } \theta(B) \leqq \delta.$$

Thus P^{V_n} converges in total variation to a probability measure π.

Let $\tilde{X}_1, \tilde{X}_2, \ldots$ be independent random variables with distribution function F which are also independent of $X_1, X_2, \ldots$ Then τ_n has the same distribution as $\inf\{k \geqq 1 \mid V_n + \tilde{X}_1 + \ldots + \tilde{X}_k > 1\}$. It follows easily that τ_n converges in distribution to $\tau = \inf\{k \geqq 1 \mid V + \tilde{X}_1 + \ldots + \tilde{X}_k > 1\}$ where $P^V = \pi$ and V ist independent of the $\tilde{X}_i$. Since τ_n is stochastically smaller than $\inf\{k \geqq 1 \mid X_1 + \ldots + X_k > 1\}$, we further have $P(\tau_n > j) \leqq C\rho^j$ for some constants $C > 0$ and $\rho \in (0, 1)$ (*see* Shiryayev (1984), p. 601). This implies that $(\tau_n)_{n \geqq 1}$ is uniformly integrable. Consequently

$$E(\tau) = \lim_{n \to \infty} E(\tau_n).$$

It remains to determine $E(\tau)$.

Let $\sigma = \inf\{k \geqq 1 \mid X_0 + \ldots + X_k > 1\}$ and assume that X_0 has the distribution π. Then $E(\tau) = E(\sigma)$. We have

$$E(Y_\sigma) = E(Y_0), \tag{4.3}$$

because the optional stopping theorem can be applied to $(Y_n)_{n \geqq 0}$ and σ; this follows from a well-known result of martingale theory [Shiryayev (1984), p. 459], since $E(\sigma) < \infty$ and

$$|Y_{n+1} - Y_n| \leqq 1 + |p(U_{n+1}) - p(U_n)| \leqq 2.$$

(4.3) is tantamount to

$$E(T_\sigma) + E(p(U_\sigma)) - \mu E(\sigma) = E(T_0) + E(p(U_0)). \tag{4.4}$$

As $T_0 = 1$, $T_\sigma = 2$ and U_σ has the same distribution as U_1 (note that $U_\sigma = V_2$ and $U_0 = V_1$), (4.4) yields $E(\sigma) = \mu^{-1}$. The Theorem is proved.

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