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On the sample path behavior of the first passage time process of a Brownian motion with drift

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ABSTRACT. — Consider the first passage time process $\{M(t), t \geqq 0\}$ of a Brownian motion $\{X(s), s \geqq 0\}$ with positive drift, *i.e.* $M(t) = \inf\{s \geqq 0 : X(s) \geqq t\}$. In this paper, we establish strong limit theorems on the behavior of the sample path modulus of continuity of $\{M(t), t \geqq 0\}$, characterized by the maximal and minimal increments $\Delta^{\pm}(T, h) = \pm \sup_{0 \leqq t \leqq T-h} \pm (M(t+h) - M(t))$ for $0 \leqq h \leqq T$. The case where $h = K_T = O(\log T)$ as $T \to \infty$ is of particular interest here. The results are deduced from their corresponding analogues for partial sums of inverse Gaussian random variables, which are developed first.

Key words : Strong limit theorems, first passage time process, Brownian motion with drift, partial sums of inverse Gaussian random variables, sample path behavior, maximal and minimal increments.

RÉSUMÉ. — Nous étudions le module de continuité du processus de premier temps de passage associé à un mouvement Brownien à dérive positive $\{X(s), s \geqq 0\}$. Plus précisément, si $M(t) = \inf\{s \geqq 0 : X(s) \geqq t\}$,

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nous étudions le comportement limite des incréments maximaux et minimaux $\Delta^{\pm}(T,h)=\pm \sup_{0\leqq t\leqq T-h} \pm(M(t+h)-M(t))$, où $0\leqq h\leqq T$ est une fonction de T. Le cas particulier où $h=K_T=O(\log T)$ lorsque $T\to\infty$ fait l'objet d'une étude approfondie. Nos résultats sont déduits de leurs analogues obtenus pour les incréments de sommes partielles de variables aléatoires indépendantes de même loi gaussienne inverse.

1. INTRODUCTION

Let $\{W(t), t\geqq 0\}$ be a standard Wiener process, and consider the Brownian motion with positive drift $\mu>0$ and variance $\sigma^2>0$ defined by

$$X(t)=\mu t+\sigma W(t) \qquad \text{for} \quad t\geqq 0. \tag{1.1}$$

We will be concerned with the *first passage time process* of $\{X(t), t\geqq 0\}$, defined by

$$M(t)=\inf\{s\geqq 0: X(s)\geqq t\} \qquad \text{for} \quad t\geqq 0. \tag{1.2}$$

The distribution of $M(t)$ was first discovered by Schrödinger (1915), and used later by Wald (1947) as a limiting form of the distribution of the sample size necessary to complete a sequential probability ratio test (SPRT). It is often called the *inverse Gaussian* or *Wald* distribution, whose density is given for $t>0$ by

$$f_t(x)=f_{\mu,\sigma;t}(x)=\frac{tx^{-3/2}}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\mu x-t)^2}{2\sigma^2 x}\right) \qquad \text{for} \quad x>0. \tag{1.3}$$

It is obvious that the process $\{M(t), t\geqq 0\}$ has independent and stationary increments so that (1.3) characterizes its finite dimensional distributions.

Aside of SPRT's, a large number of applications have been found for $\{M(t), t\geqq 0\}$ (*see* e.g. Tweedie (1957 *a, b*), Chhikara and Folks (1978), Jørgensen (1961), Lerche (1986) and the references therein.

The purpose of this paper is to investigate the sample path behavior of $\{M(t), t\geqq 0\}$. A simple application of the Komlós, Major and Tusnády (1976) strong invariance principle (*see* Lemma 5.1 in the sequel) shows that, without loss of generality, we can assume that $\{M(t), t\geqq 0\}$ is defined on the same probability space as a standard Wiener process

$\{\hat{W}(t), t \geqq 0\}$, such that, almost surely,

$$\sup_{0 \leqq t \leqq T} \left| M(t) - \frac{t}{\mu} - \sigma \mu^{-3/2} \hat{W}(t) \right| = O(\log T) \quad \text{as } T \to \infty. \tag{1.4}$$

It follows that the global behavior of $\{M(t), t \geqq 0\}$ is governed by the Brownian motion $\hat{B}(t) = (t/\mu) + \sigma \mu^{-3/2} \hat{W}(t)$. From there one can obtain by routine arguments all "classical" asymptotic results such as the CLT and the LIL. We omit details.

We will concentrate our interest in the local behavior of $\{M(t), t \geqq 0\}$, with emphasis upon the sample path modulus of continuity, characterized by the maximal and minimal increments:

$$\Delta^{\pm}(T, h) = \pm \sup_{0 \leqq t \leqq T-h} \pm (M(t+h) - M(t)) \quad \text{for } 0 \leqq h \leqq T. \tag{1.5}$$

In the sequel, we will investigate the limiting behavior of $\Delta_T^{\pm} = \Delta^{\pm}(T, K_T)$, where $\{K_T\}_{T>0}$ denotes a function such that $0 < K_T \leqq T$ for all $T > 0$.

Such statistics have been widely studied in the case of the Wiener process, starting with the pioneering work of Lévy (1937) [*see* e.g. Taylor (1974), Révész (1982)]. On the other hand, there does not seem to exist in the literature any serious attempt to study $\Delta_T^{\pm}$ in general. The aim of this paper is to fill this gap.

It is noteworthy that for $\mu = 0$, the process $\{M(t), t \geqq 0\}$ exhibits a specific behavior, being seen [*see* e.g. Basu and Wasan (1975)] a stable process with index 1/2. We will not consider this case here and limit ourselves to the situation where $\mu > 0$.

The rest of our paper is organized as follows. In Section 2, we give some general bounds and evaluations for the inverse Gaussian distributions which are required for the proofs of our theorems. Sections 3 and 4 are devoted to the study of increments of partial sums of inverse Gaussian random variables. In Section 3, we apply general result on this problem, while Section 4 is entirely new. Section 5 contains our main theorems. Our last Section 6 contains some further comments.

2. PROPERTIES OF THE INVERSE GAUSSIAN DISTRIBUTIONS

In the literature [Tweedie (1957 *a*, *b*), Johnson and Kotz (1970), Chhikara and Folks (1978)] it is usual to say that X follows an *inverse Gaussian* $I(\nu, \lambda)$ distribution if it has density, for $\nu > 0$ and $\lambda > 0$

$$g(x) = g_{\nu, \lambda}(x) = \frac{\lambda^{1/2}}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{\lambda (x-\nu)^2}{2\nu^2 x}\right) \quad \text{for } x > 0. \tag{2.1}$$

Note for further use that $M(t)$ follows an $I(t/\mu, t^2/\sigma^2)$ distribution.
In general, the first moments of X are

$$E(X)=\nu, \qquad \mathrm{Var}(X)=\nu^3/\lambda \qquad \text{and} \qquad E((X-\nu)^3)=3\nu^5/\lambda^2. \tag{2.2}$$

Let Φ denote the distribution of a standard normal $N(0,1)$ distribution. We have

$$G(x)=P(X\leqq x)=\Phi\left(\sqrt{\lambda/x}\left(\frac{x}{\nu}-1\right)\right) + e^{2\lambda/\nu}\Phi\left(-\sqrt{\lambda/x}\left(\frac{x}{\nu}+1\right)\right). \tag{2.3}$$

The moment-generating function of X is

$$\psi(s)=E(e^{sX})=\exp\left(\frac{\lambda}{\nu}\left(1-\left(1-\frac{2\nu^2 s}{\lambda}\right)^{1/2}\right)\right) \quad \text{for} \quad -\infty<s<\lambda/2\nu^2. \tag{2.4}$$

Our first lemma uses (2.3) to evaluate large deviation probabilities.

LEMMA 2.1. – *Let G be as in* (2.3). *Then:*
1° *Uniformly over* $x>0$, $\nu>0$ *and* $\lambda>0$ *such that*

$$\left(\sqrt{\lambda/x}\left(\frac{x}{\nu}-1\right)\right)^2 \Big/ \left(\frac{x}{\nu}+1\right) \to \infty \qquad \text{and} \qquad x>\nu. \tag{2.5}$$

We have

$$1-G(x)=(1+o(1))\frac{2\nu^2\sqrt{x/\lambda}}{(x^2-\nu^2)\sqrt{2\pi}}\exp\left(-\frac{\lambda}{2x}\left(\frac{x}{\nu}-1\right)^2\right). \tag{2.6}$$

2° *Uniformly over* $x>0$, $\nu>0$ *and* $\lambda>0$ *such that*

$$\left(\sqrt{\lambda/x}\left(\frac{x}{\nu}-1\right)\right)^2 \Big/ \left(\frac{x}{\nu}+1\right) \to \infty \qquad \text{and} \qquad x<\nu. \tag{2.7}$$

We have

$$G(x)=(1+o(1))\frac{2\nu^2\sqrt{x/\lambda}}{(\nu^2-x^2)\sqrt{2\pi}}\exp\left(-\frac{\lambda}{2x}\left(\frac{x}{\nu}-1\right)^2\right). \tag{2.8}$$

Proof. – We make use of the well-known expansion [*see* e.g. Feller (1968), p. 175].

$$\Phi(-z)=1-\Phi(z)=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{z}-\frac{1+o(1)}{z^3}\right)\exp\left(-\frac{1}{2}z^2\right) \quad \text{as } z\to\infty. \tag{2.9}$$

Thus, (2.9) and the remark that (2.5) implies that $\sqrt{\lambda/x}\left(\frac{x}{\nu}+1\right)\to\infty$ yield

$$1-G(x)=\frac{1+o(1)}{\sqrt{2\pi}}\sqrt{x/\lambda}\left(\left(\frac{x}{\nu}-1\right)^{-1}-\left(\frac{x}{\nu}+1\right)^{-1}\right)\exp\left(-\frac{\lambda}{2x}\left(\frac{x}{\nu}-1\right)^{2}\right),$$

which is equivalent to (2.6). The proof of (2.8) is similar and will be omitted. □

Our next lemma reformulates Lemma 2.1 in terms of M (*t*).

LEMMA 2.2. — 1° *Uniformly over* $t>0$ *and* $\alpha\geqq 1/\mu$ *such that*

$$(\sqrt{t/\alpha}(\mu\alpha-1))^2/(\mu\alpha+1)\to\infty. \tag{2.10}$$

We have

$$P(M(t)\geqq t\alpha)=\frac{2\sigma\sqrt{\alpha}(1+o(1))}{(\mu^2\alpha^2-1)\sqrt{2\pi t}}\exp\left(-\frac{t(\alpha\mu-1)^2}{2\sigma^2\alpha}\right). \tag{2.11}$$

2° *Uniformly over* $t>0$ *and* $0<\alpha<1/\mu$ *such that*

$$(\sqrt{t/\alpha}(\mu\alpha-1))^2/(\mu\alpha+1)\to\infty. \tag{2.12}$$

We have

$$P(M(t)\leqq t\alpha)=\frac{2\sigma\sqrt{\alpha}(1+o(1))}{(1-\mu^2\alpha^2)\sqrt{2\pi t}}\exp\left(-\frac{t(\alpha\mu-1)^2}{2\sigma^2\alpha}\right). \tag{2.13}$$

Proof. — Since M (*t*) follows an $I(t/\mu, t^2/\sigma^2)$ distribution, the formal changes $\nu=t/\mu$, $\lambda=t^2/\sigma^2$ and $x=t\alpha$ in (2.5)-(2.8) yield (2.10)-(2.13) as sought. □

We will now consider results related to the moment-generating function of M (*t*) which, by setting $\nu=t/\mu$ and $\lambda=t^2/\sigma^2$ in (2.4), is given by

$$\psi_t(s)=E\left(\exp(sM(t))\right)=\exp\left(\frac{\mu t}{\sigma^2}\left(1-\left(1-2\frac{\sigma^2}{\mu^2}s\right)^{1/2}\right)\right)$$
$$\text{for}\quad -\infty<s<s_0:=\mu^2/2\sigma^2. \tag{2.14}$$

Let $m_t(s)=\psi_t'(s)/\psi_t(s)$. We have

$$m_t(s)=\frac{t}{\mu}\left(1-2\frac{\sigma^2}{\mu^2}s\right)^{-1/2}\qquad\text{for}\quad -\infty<s<s_0. \tag{2.15}$$

By (2.15), it is obvious that, for any $0<a<\infty$, the equation $m_t(s)=a$ has a unique root $s_t(a)\in(-\infty, s_0)$ given by

$$s_t(a)=\frac{\mu^2}{2\sigma^2}\left(1-\frac{t^2}{\mu^2a^2}\right)\quad\Leftrightarrow\quad m_t(s_t(a))=a. \tag{2.16}$$

Denote by $\zeta_t(a)=\sup_s(as-\log\psi_t(s))=as_t(a)-\log\psi_t(s_t(a))$ for $0<a<\infty$.

We have

$$\zeta_t(a)=\frac{1}{2\sigma^2 a}(\mu a-t)^2 \qquad \text{for} \quad a>0 \quad \text{and} \quad t>0. \tag{2.17}$$

It is noteworthy that $\zeta_t(.)$ is decreasing on $(0, t/\mu)$ and increasing on $(t/\mu, \infty)$. It follows that, for any $x>0$, there exist two distinct roots $0<a_t^-(x)<\frac{t}{\mu}<a_t^+(x)<\infty$, of the equation $\zeta_t(a)=x$, given by

$$a_t^\pm(x)=\frac{t}{\mu}+\frac{\sigma^2 x}{\mu^2}\left(1\pm\left(1+\frac{2t\mu}{\sigma^2 x}\right)^{1/2}\right)=ta_1^\pm(x/t). \tag{2.18}$$

3. PARTIAL SUMS OF INVERSE GAUSSIAN RANDOM VARIABLES – GENERAL RESULTS

Let $\delta>0$ be fixed. In this section, we will consider the sequence $\{X_n, n\geqq 1\}$ of independent random variables following a common $I(\delta/\mu, \delta^2/\sigma^2)$ distribution, defined by

$$X_n=M(n\delta)-M((n-1)\delta) \qquad \text{for} \quad n=1,2,\ldots \tag{3.1}$$

Let $S_0=0$ and $S_n=X_1+\ldots+X_n=M(n\delta)$. Consider the statistic

$$U^\pm(n,b)=\pm\max_{0\leqq k\leqq n-b}\pm(S_{k+b}-S_k), \tag{3.2}$$

where $0\leqq b\leqq n$ is integer.

Let $0\leqq b_n\leqq n$ be an integer sequence. In the sequel, we shall investigate the limiting behavior of $U_n^\pm=U^\pm(n, b_n)$ under various assumptions imposed on $\{b_n\}$. The motivation for this study comes from the following inequalities whose proof is a straightforward consequence of the fact that $M(.)$ is nondecreasing. We have, for $\delta\leqq K\leqq T$,

$$U^+\left(\left[\frac{T}{\delta}\right],\left[\frac{K}{\delta}\right]\right)\leqq\Delta^+(T,K)\leqq U^+\left(\left[\frac{T}{\delta}\right]+1,\left[\frac{K}{\delta}\right]+2\right), \tag{3.3}$$

where $[u]$ denotes the integer part of u. Similar inequalities hold for U^-.

By (3.3) we can translate without difficulty the results below in terms of $\Delta_T^\pm$. This will be made in the forthcoming sections.

Our first lemma uses the fact [*see* e. g. (2.14)] that the moment-generating function of X_1 is finite in a neighborhood of zero. By (2.2) and

Komlós, Major and Tusnády (1976) we have:

LEMMA 3.1. — *It is possible to define the sequence* $\{X_n, n \geqq 1\}$ *on a probability space which carries a Wiener process* $\{\hat{W}(t), t \geqq 0\}$ *such that*

$$\limsup_{T \to \infty} \left| S_{[T]} - \frac{\delta}{\mu} T - \delta^{1/2} \sigma \mu^{-3/2} \hat{W}(T) \right| \Big/ \log T < \infty \quad \text{a.s.} \quad \square \tag{3.4}$$

By combining (3.4) with the results of M. Csörgő and Révész (1979) and the classical Erdös-Rényi (1970) theorem, we obtain easily the following proposition (*see* e.g. M. Csörgő and Révész (1981), Theorems 2.4.3 and 3.1.1).

PROPOSITION 3.1. — *Assume that* b_n *is an integer sequence such that* $1 \leqq b_n \leqq n$ *and* $b_n \uparrow$. *Assume further that there exists a real-valued sequence* $\tilde{b}_n \uparrow$ *such that* $n^{-1} \tilde{b}_n \downarrow$, *jointly with*

$$b_n = \tilde{b}_n (1 + o(1)) \text{ as } n \to \infty. \tag{3.5}$$

Then

1° *If* $b_n / \log n \to \infty$, *joinly with* $(\log(n/b_n))/\log\log n \to \infty$, *we have*

$$\lim_{n \to \infty} \left(U_n^{\pm} - \frac{\delta}{\mu} b_n \right) / (\sigma \mu^{-3/2} (2 \delta b_n \log(n/b_n))^{1/2}) = \pm 1 \quad \text{a.s.} \tag{3.6}$$

2° *If* $b_n / \log n \to c \in (0, \infty)$, *we have*

$$\lim_{n \to \infty} \left(U_n^{\pm} - \frac{\delta}{\mu} b_n \right) \Big/ \left(\frac{\sigma^2}{\mu^2} (\log n) \left(1 \pm \left(1 + \frac{2\mu}{\sigma^2} \frac{\delta b_n}{\log n} \right)^{1/2} \right) \right) = 1 \quad \text{a.s.} \tag{3.7}$$

Proof. — We limit ourselves to the proof of (3.7) which follows from (2.18) and the Erdös-Rényi (1970) Theorem whose statement can be stated as follows. Whenever $b_n \sim c \log n$, we have

$$\lim_{n \to \infty} (U_n^{\pm} / (b_n a_\delta^{\pm}(1/c))) = 1 \quad \text{a.s.} \quad \square$$

The rate of convergence in (3.6) and (3.7) follows from the results of Deheuvels, Devroye and Lynch (1986) and Deheuvels and Steinebach (1987), used jointly with the results of Section 2. We have the following proposition.

PROPOSITION 3.2. — *Assume that* b_n *is an integer sequence such that* $1 \leqq b_n \leqq n$ *and* $b_n \uparrow$. *Assume further that there exists a real-valued sequence* $\tilde{b}_n \uparrow$ *such that*

$$b_n - \tilde{b}_n = O(\min(b_n / \log n, \log n))$$

and (3.8)

$$\tilde{b}_{n+1} - \tilde{b}_n = O(\tilde{b}_n / (n \log n)) \quad \text{as } n \to \infty.$$

Then, under either of the assumptions (i) *and* (*ii*) *below:*
(i) $b_n(\log\log n)^2/\log^3 n \to \infty$ *and* $(\log\log(n/b_n))/\log\log n \to 1$ *as* $n \to \infty$;
(ii) $b_n/\log n \to c \in (0, \infty]$ *and* $b_n/\log^p n \to 0$ *as* $n \to \infty$, *for some* $p > 1$.
We have

$$\limsup_{n\to\infty} \frac{t_n^\pm (\mathrm{U}_n^\pm - b_n a_n^\pm)}{\log\log n} = \frac{3}{2}$$

and (3.9)

$$\liminf_{n\to\infty} \frac{t_n^\pm (\mathrm{U}_n^\pm - b_n a_n^\pm)}{\log\log n} = \frac{1}{2} \quad \text{a.s.,}$$

where

$$a_n^\pm = \frac{\delta}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log(n/b_n)}{b_n}\right)\left(1 \pm \left(1 + \frac{2\mu}{\sigma^2}\left(\frac{\delta b_n}{\log(n/b_n)}\right)\right)^{1/2}\right) = a_\delta^\pm\left(\frac{\log(n/b_n)}{b_n}\right). \quad (3.10)$$

and

$$t_n^\pm = s_\delta(a_n^\pm) = \frac{\mu^2}{2\sigma^2}\left(1 - \frac{\delta^2}{\mu^2}(a_n^\pm)^{-2}\right). \quad (3.11)$$

We now concentrate on the case of *small increments*, *i.e.*, when the sequence b_n satisfies $b_n/\log n \to 0$ as $n \to \infty$. In this case, aside of the results of Huse and Steinebach (1984) which do not apply for the inverse Gaussian distributions, the only available results are due to Mason (1989). We cite his main theorem in the following proposition.

PROPOSITION 3.3. – *Let* $\xi_1, \xi_2, \ldots$ *be an i.i.d. sequence of random variables with partial sums* $\eta_0 = 0$ *and* $\eta_n = \xi_1 + \ldots + \xi_n$. *Set* $\Psi(s) = \mathrm{E}(\exp(s\xi_1))$, $\mathrm{G}(x) = \mathrm{P}(\xi_1 \leq x)$, *and assume that*
(i) $0 < \omega = \sup\{x : \mathrm{G}(x) < 1\} \leq \infty$;
(ii) $\mathrm{P}(\xi_1 = x) < 1$ *for all* x;
(iii) $\sup\{s : \Psi(s) < \infty\} > 0$;
(iv) if $\omega = \infty$, *then* $\lim_{x\uparrow\omega} \gamma(-\log(1 - \mathrm{G}(x)))/x = 1$,
where $\zeta(a) = \sup_s\{as - \log\Psi(s)\}$ *and* $\gamma(x) = \sup\{a : \zeta(a) \leq x\}$.

Then, for all integer sequences $b_n\uparrow$ *such that* $1 \leq b_n \leq n$ *and* $b_n/\log n \to 0$ *as* $n \to \infty$, *we have*

$$\lim_{n\to\infty}\left(\max_{0\leq k\leq n-b_n}(\eta_{k+b_n} - \eta_k)\right)\Big/\left(b_n\gamma\left(\frac{\log n}{b_n}\right)\right) = 1 \quad \text{a.s.} \quad (3.12)$$

Observe that $\xi_n = \mathrm{X}_n$ satisfies the assumptions of Proposition 3.5 with $\omega = \infty$, $\Psi(s) = \psi_\delta(s)$ given by (2.14), $\zeta(a) = \zeta_\delta(a)$ given by (2.17), and by

(2.18),

$$\gamma(x)=a_\delta^+(x)=\frac{\delta}{\mu}+\frac{\sigma^2 x}{\mu^2}\left(1+\left(1+\frac{2\delta\mu}{\sigma^2 x}\right)^{1/2}\right)\sim\frac{2\sigma^2 x}{\mu^2}\quad\text{as } x\to\infty. \tag{3.13}$$

Moreover, by taking $\nu=\delta/\mu$ and $\lambda=\delta^2/\sigma^2$ in (2.6), we see that

$$-\log(1-G(x))\sim\frac{\lambda}{2x}\left(\frac{x}{\nu}\right)^2=\frac{\mu^2 x}{2\sigma^2}\quad\text{as } x\to\infty. \tag{3.14}$$

It is obvious by (3.13) and (3.14) that Condition (iv) in (Proposition 3.3 is satisfied.

Likewise, we see that $\xi_n=1-X_n$ satisfies the assumptions of Proposition 3.3 with $\omega=1$, $\Psi(s)=e^s\psi_\delta(-s)$, $\zeta(a)=\zeta_\delta(1-a)$, and

$$\gamma(x)=1-a_\delta^-(x)=1-\frac{\delta}{\mu}-\frac{\sigma^2 x}{\mu^2}\left(1-\left(1+\frac{2\delta\mu}{\sigma^2 x}\right)^{1/2}\right)\to 1\quad\text{as } x\to\infty.$$

An application of Proposition 3.3 in either case shows that the following proposition holds.

PROPOSITION 3.4. – *Assume that b_n is an integer sequence such that $b_n\uparrow$, $1\leqq b_n\leqq n$, and $b_n/\log n\to 0$ as $n\to\infty$. Then we have*

$$\lim_{n\to\infty} U_n^+\Big/\left(\frac{2\sigma^2}{\mu^2}\log n\right)=1\quad\text{a.s.} \tag{3.16}$$

and

$$\lim_{n\to\infty} U_n^-/b_n=0\quad\text{a.s.} \tag{3.17}$$

It is obvious from Proposition 3.4 that the results so obtained are rather coarse with respect to what is known for $b_n/\log n\nrightarrow 0$. This motivates our next section where we investigate the rate of convergence of (3.16) and (3.17). Aside of the specific interest of such evaluations with respect to our study, it will become clear in the sequel that the methods we use may be generalized to more general sequences after simple modifications.

4. PARTIAL SUMS OF INVERSE GAUSSIAN RANDOM VARIABLES – SMALL INCREMENTS

Throughout this section, we assume that $1\leqq b_n\leqq n$ is an integer sequence such that $b_n/\log n\to 0$ as $n\to\infty$. In the first place, we will obtain the rate of convergence in (3.17). A close look to the results of Mason (1989) (*see* e.g. Proposition 3.3) hints that (3.12) should hold for $\omega=0$ when applied to the sequence $\xi_n=-X_n$, $n=1,2,\ldots$ Here, we have $\Psi(s)=\psi_\delta(-s)$,

$\zeta(a)=\zeta_\delta(-a)$, and

$$\gamma(x)=-a_\delta^-(x)=-\frac{\delta}{\mu}-\frac{\sigma^2 x}{\mu^2}\times\left(1-\left(1+\frac{2\delta\mu}{\sigma^2 x}\right)^{1/2}\right)=-(1+o(1))\frac{\delta^2}{2\sigma^2 x}\quad\text{as } x\to\infty. \tag{4.1}$$

Thus (3.12) may be rewritten as

$$\lim_{n\to\infty} U_n^- \Big/ \left(\frac{\delta^2}{2\sigma^2}\left(\frac{b_n^2}{\log n}\right)\right)=1 \quad \text{a.s.} \tag{4.2}$$

The extension of Mason's Proposition 3.3 for $\omega=0$ is an open problem. A reasonable guess is that one would need in this case to replace the condition (iv) of this proposition by $\lim_{x\uparrow\omega}\gamma(-\log P(\xi_1>x))/x=1$ for $\omega=0$. It is noteworthy that this last requirement is satisfied for inverse Gaussian distributions [*i.e.* by (2.8) with $\nu=\delta/\mu$ and $\lambda=\delta^2/\mu^2$, we have $-\log P(\xi_1>x)=-\log P(X_1<-x)\sim-\delta^2/\{2\sigma^2 x\}$ as $x\uparrow\omega=0$, while by (4.1) we get $\gamma(u)\sim-\delta^2/\{2\sigma^2 u\}$ as $u\to\infty$]. Limiting ourselves to inverse Gaussian distributions, we will now show that (4.2) is correct under this assumption.

PROPOSITION 4.1. – *Assume that b_n is an integer sequence such that $b_n\uparrow$, $1\leqq b_n\leqq n$, and $b_n/\log n\to 0$ as $n\to\infty$. Then we have*

$$\lim_{n\to\infty} U_n^- \Big/ \left(\frac{\delta^2}{2\sigma^2}\left(\frac{b_n^2}{\log n}\right)\right)=1 \quad \text{a.s.} \tag{4.3}$$

Proof. – In the first place, we have the obvious inequality

$$P(U_n^-\leqq x)\leqq n\,P(M(\delta b_n)\leqq x). \tag{4.4}$$

By (4.4) taken with $x=(1-\varepsilon)\left(\frac{\delta^2}{2\sigma^2}\right)\left(\frac{b_n^2}{\log n}\right)$, used jointly with (2.8) taken with $\nu=\delta b_n/\mu$ and $\lambda=(\delta b_n)^2/\sigma^2$, we obtain as $n\to\infty$

$$P\left(U_n^-\leqq(1-\varepsilon)\left(\frac{\delta^2}{2\sigma^2}\right)\left(\frac{b_n^2}{\log n}\right)\right)=\frac{(1+o(1))\,n\sqrt{1-\varepsilon}}{\sqrt{\pi\log n}}\exp\left(-\frac{\log n}{1-\varepsilon}\left(1-\frac{\delta\mu}{2\sigma^2}\left(\frac{b_n(1-\varepsilon)}{\log n}\right)\right)^2\right),$$

which for all fixed $0<\varepsilon<1$ is ultimately less than or equal to $n^{-(1/4)\varepsilon}$ as $n\to\infty$.

Next, we use the assumption that b_n is nondecreasing. Denote by $n_1 < n_2 < \ldots$ the sequence defined by

$$n_1 = 1, \quad n_j = \min(\inf\{[a^l] > n_{j-1} : l = 1, 2, \ldots\},\\ \inf\{m > n_{j-1} : b_m < b_{m+1}\}), \quad j = 2, 3, \ldots, \tag{4.5}$$

where $a > 1$ is a constant precised below.

It is noteworthy that, for $n_{j-1} < n \leqq n_j$, we have $U^-_{n_{j-1}+1} \geqq U^-_n \geqq U_{n_j}$ and $b_n = b_{n_j}$. Moreover, if $x_{n,\varepsilon} = (1-\varepsilon)\left(\frac{\delta^2}{2\sigma^2}\right)\left(\frac{b_n^2}{\log n}\right)$, we have $x_{n_{j-1}+1,\varepsilon} \geqq x_{n,\varepsilon} \geqq x_{n_j,\varepsilon}$. Also, (4.5) implies that

$$\limsup_{j \to \infty} (\log n_{j+1}) / \log n_j \leqq \log a. \tag{4.6}$$

Therefore, the choice of $a = \exp\left(\left(1 - \frac{1}{2}\varepsilon\right) \Big/ (1-\varepsilon)\right)$ implies that, for all j sufficiently large

$$\bigcup_{n_{j-1} < n \leqq n_j} \{U^-_n \leqq x_{n,\varepsilon}\} \subset \{U^-_{n_j} \leqq x_{n_j, 1/4\varepsilon}\} \tag{4.7}$$

Since $P(U_n \leqq x_{n, 1/4\varepsilon})$ is ultimately less than or equal to $n^{-\varepsilon/16}$, we see by Borel-Cantelli that $P(U^-_n \leqq x_{n,\varepsilon} \text{ i.o.}) = 0$ whenever $\sum_j n_j^{-\varepsilon/16} < \infty$. Next, we see that for $\theta > 0$

$$\sum_j n_j^{-\theta} \leqq \sum_j [a^j]^{-\theta} + \sum_m m^{-\theta}(b_{m+1} - b_m)\\ = O(1) + \sum_m b_m((m-1)^{-\theta} - m^{-\theta}) < \infty. \tag{4.8}$$

An application of this result for $\theta = \varepsilon/16$ and for an arbitrary choice of $0 < \varepsilon < 1$ suffices to prove that

$$\liminf_{n \to \infty} U^-_n \Big/ \left(\frac{\delta^2}{2\sigma^2}\left(\frac{b_n^2}{\log n}\right)\right) \geqq 1 \quad \text{a.s.} \tag{4.9}$$

For the other half of our proof, we make use of the inequality $U^-_n \leqq V^-_n$, where $V^-_n = \min_{1 \leqq i \leqq N(n)} (S_{ib_n} - S_{(i-1)b_n})$, and $N(n) = [n/b_n]$. Moreover, as in (4.4), we have

$$P(V^-_n > x_{n,\varepsilon}) = [1 - P(M(\delta b_n) \leqq x_{n,\varepsilon})]^{N(n)}\\ \leqq \exp(-N(n) P(M(\delta b_n) \leqq x_{n,\varepsilon}))\\ = \exp\left(-\frac{(1+o(1))\, n\sqrt{1-\varepsilon}}{b_n\sqrt{\pi \log n}}\right.\\ \left. \times \exp\left(-\frac{\log n}{1-\varepsilon}\left(1 - \frac{\delta\mu}{2\sigma^2}\left(\frac{b_n(1-\varepsilon)}{\log n}\right)\right)^2\right)\right), \tag{4.10}$$

which for $\varepsilon<0$ is ultimately less than $\exp(-n^{-(1/2)\varepsilon/(1-\varepsilon)})$ as $n\to\infty$. Since this expression is summable in n, the Borel-Cantelli lemma implies that, for all $\varepsilon<0$, $P(V_n^- > x_{n,\varepsilon}\text{ i.o.})=0$.

This, jointly with the inequality $U_n^- \leqq V_n^-$, implies that

$$\limsup_{n\to\infty} U_n^- \Big/ \left(\frac{\delta^2}{2\sigma^2}\left(\frac{b_n^2}{\log n}\right)\right) \leqq 1 \quad \text{a.s.} \tag{4.11}$$

The proof of Proposition 4.1 follows directly from (4.9) and (4.11). □

We will obtain rates of convergence for the limit in (4.3) at the end of this section. In the first place, we will investigate what happens for (3.16). Our main result concerning this problem is stated below.

PROPOSITION 4.2. – *Assume that b_n is an integer sequence such that $b_n\uparrow$, $1\leqq b_n\leqq n$ and $b_n/\log n\to 0$ as $n\to\infty$. Let A_n^+ be defined by*

$$A_n^+ = \frac{\delta}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log(nb_n)}{b_n}\right)\left(1+\left(1+\frac{2\mu}{\sigma^2}\left(\frac{\delta b_n}{\log(nb_n)}\right)\right)^{1/2}\right) = a_\delta^+\left(\frac{\log(nb_n)}{b_n}\right). \tag{4.12}$$

Then we have

$$\limsup_{n\to\infty}\frac{U_n^+ - b_n A_n^+}{(2\sigma^2/\mu^2)\log\log n} = -\frac{1}{2}$$

$$\textit{and}\quad \liminf_{n\to\infty}\frac{U_n^+ - b_n A_n^+}{(2\sigma^2/\mu^2)\log\log n} = -\frac{3}{2} \quad \text{a.s.} \tag{4.13}$$

Remark 4.1. – Using the well-known expansion

$$(1+u)^{1/2} = 1+\frac{u}{2}-\frac{u^2}{8}+O(u^3)$$

valid for $u\geqq 0$, we obtain the following expansion of A_n^+ as given in (4.12)

$$b_n A_n^+ = \frac{2\sigma^2}{\mu^2}\log(nb_n) + \frac{2\delta}{\mu}b_n - \frac{\delta^2 b_n^2}{2\sigma^2\log(nb_n)} + O(b_n^3/(\log(nb_n))^2).$$

Hence, whenever $b_n = o((\log n)^{1/2}(\log\log n)^{1/2})$ as $n\to\infty$, we can replace in (4.13) $b_n A_n^+$ by the simple expression

$$\frac{2\sigma^2}{\mu^2}\log(nb_n) + \frac{2\delta}{\mu}b_n.$$

A similar expansion of a_n^+ as given in (3.10) yields

$$b_n a_n^+ = \frac{2\sigma^2}{\mu^2}\log(n/b_n) + \frac{2\delta}{\mu}b_n - \frac{\delta^2 b_n^2}{2\sigma^2\log(n/b_n)} + O(b_n^3/(\log(n/b_n))^2).$$

Thus, if we assume that $b_n = o(\log n)$ (recall that $b_n \geqq 1$), we obtain

$$\frac{b_n A_n^+ - b_n a_n^+}{\log\log n} = \frac{4\sigma^2}{\mu^2}\left(\frac{\log b_n}{\log\log n}\right) \times \left(1 + O\left(\frac{b_n^2}{\log^2 n}\right)\right) = \frac{4\sigma^2}{\mu^2}\left(\frac{\log b_n}{\log\log n}\right) + o(1). \quad (4.14)$$

In the same range [*i.e.* $b_n = o(\log n)$ as $n \to \infty$], it can also be seen that if t_n^+ is as in (3.11), we have $t_n^+ = s_\delta(a_n^+) \sim s_\delta(A_n^+) \sim \mu^2/(2\sigma^2)$ as $n \to \infty$. This in combination with (4.14) shows that *the conclusion* (3.9) *of Proposition* 3.2 *coincides with the conclusion* (4.13) *of Proposition* 4.2 *in the range where* $(\log b_n)/\log\log n \to 1$ *as* $n \to \infty$, *whereas these statements differ when* $(\log b_n)/\log\log n \to c < 1$ *as* $n \to \infty$.

This result is interesting in itself, since it brings a negative answer to the conjecture that (3.9) could be true in general (assuming weak additional assumptions on the distribution of X_1) outside the ranges covered by the theorems of Deheuvels, Devroye and Lynch (1986) and of Deheuvels and Steinebach (1987), which correspond to

$$(\log b_n)/\log\log n \geqq 1 + O(1/\log\log n) \text{ as } n \to \infty. \quad \square$$

Proof of Proposition 4.2. – The proof is captured in the following sequence of lemmas.

LEMMA 4.1. – *Let* $\varepsilon_n \to 0$ *be a sequence of positive numbers. Then, uniformly over all sequences* $\{y_n\}$ *such that* $|y_n| \leqq \varepsilon_n b_n A_n^+$, *we have*

$$n P(M(\delta b_n) \geqq b_n A_n^+ + y_n) = (1 + o(1)) \frac{\delta\mu}{2\sigma^2\sqrt{\pi}} (\log n)^{-3/2} \exp\left(-\frac{\mu^2}{2\sigma^2}(1 + o(1)) y_n\right). \quad (4.15)$$

Proof. – In view of (2.6) taken with $\nu = \delta b_n/\mu$, $\lambda = \delta^2 b_n^2/\sigma^2$ and $x = b_n A_n^+ + y_n$, and of (4.12), which ensures that $x \sim \frac{2\sigma^2}{\mu^2}\log n$, we see that, if $x_+ = b_n A_n^+$,

$$\frac{2\nu^2\sqrt{x/\lambda}}{(x^2 - \nu^2)\sqrt{2\pi}} \sim \frac{\delta\mu}{2\sigma^2\sqrt{\pi}} (\log n)^{-3/2} b_n \quad \text{as } n \to \infty \quad (4.16)$$

and

$$\begin{aligned}\frac{\lambda}{2x}\left(\frac{x}{\nu} - 1\right)^2 &= \frac{\mu^2 x}{2\sigma^2} - \frac{\delta\mu}{\sigma^2} b_n + \frac{\delta^2}{2\sigma^2 x} b_n^2 \\ &= \frac{\lambda}{2x_+}\left(\frac{x_+}{\nu} - 1\right)^2 + \frac{\mu^2}{2\sigma^2} y_n + \frac{\delta^2 b_n^2}{2\sigma^2}\left(\frac{1}{x} - \frac{1}{x_+}\right) \\ &= \zeta_{\delta b_n}(b_n A_n^+) + \frac{\mu^2}{2\sigma^2} y_n - (1 + o(1)) \frac{\delta^2}{2\sigma^2} y_n/(A_n^+)^2, \quad (4.17)\end{aligned}$$

where $\zeta_t(a)$ is as in (2.17). Observe now that $b_n A_n^+ = a_{\delta b_n}^+ (\log (nb_n))$, so that by (2.18), $\zeta_{\delta b_n}(b_n A_n^+) = \log (nb_n)$. This, jointly with (2.6), (4.16), (4.17) and the observation that $A_n^+ \sim \frac{2\sigma^2}{\mu^2}\left(\frac{\log n}{b_n}\right) \to \infty$, completes the proof of (4.15). □

LEMMA 4.2. — *Assume that b_n is an integer sequence such that $b_n \uparrow$, $1 \leqq b_n \leqq n$, and $b_n/\log n \to 0$ as $n \to \infty$. Let A_n^+ be as in* (4.12). *Then*

$$\limsup_{n \to \infty} \frac{U_n^+ - b_n A_n^+}{(2\sigma^2/\mu^2) \log\log n} \leqq -\frac{1}{2} \quad \text{a.s.} \tag{4.18}$$

Proof. — Let $\{n_j\}$ be as in (4.5). Obviously, if $n_{j-1} < n \leqq n_j$, $b_n = b_{n_j}$ and $U_n^+ \leqq U_{n_j}^+$. Since $\limsup_{n \to \infty} (n_j/n_{j-1}) \leqq a$, we see by (4.12), that, uniformly over $n_{j-1} < n \leqq n_j$,

$$b_n A_n^+ - b_{n_j} A_{n_j}^+ = b_{n_j}(A_n^+ - A_{n_j}^+) = 2(1+o(1)) \frac{\sigma^2}{\mu^2} \log (n/n_j) = O(1), \tag{4.19}$$

and

$$(\log\log n)/\log\log n_j \to 1 \quad \text{as } j \to \infty. \tag{4.20}$$

Therefore, in order to prove (4.18), it is enough to show that, for any $\varepsilon > 0$, we have $P\left(U_{n_j}^+ \geqq b_{n_j} A_{n_j}^+ + \left(\varepsilon - \frac{1}{2}\right)\left(\frac{2\sigma^2}{\mu^2} \log\log n_j\right) \text{i.o.}\right) = 0$, which, in view of (4.15) and by Borel-Cantelli, reduces to show that for any $\varepsilon > 0$

$$\begin{aligned} \sum_j (\log n_j)^{-\varepsilon-1} &< \sum_j (\log [a^j])^{-\varepsilon-1} + \sum_m (\log m)^{-\varepsilon-1} (b_{m+1} - b_m) \\ &= O(1) + \sum_m b_m ((\log (m-1))^{-\varepsilon-1} - (\log m)^{-\varepsilon-1}) < \infty. \end{aligned} \tag{4.21}$$

This completes the proof of (4.18). □

LEMMA 4.3. — *Assume that b_n is an integer sequence such that $1 \leqq b_n \leqq n$, and $b_n/\log n \to 0$ as $n \to \infty$. Let A_n^+ be as in* (4.12). *Then*

$$\liminf_{n \to \infty} \frac{U_n^+ - b_n A_n^+}{(2\sigma^2/\mu^2) \log\log n} \leqq -\frac{3}{2} \quad \text{a.s.} \tag{4.22}$$

Proof. — By using the fact that, if $\{E_n\}$ is a sequence of events, $P(E_n) \to 1 \Rightarrow P(E_n \text{ i.o.}) = 1$, we see that (4.22) follows from the statement that, for all $\varepsilon > 0$,

$$P\left(U_n^+ \geqq b_n A_n^+ + \left(\varepsilon - \frac{3}{2}\right)\left(\frac{2\sigma^2}{\mu^2} \log\log n\right)\right) \to 0 \quad \text{as } n \to \infty. \tag{4.23}$$

Since (4.15) implies that the probability above is less than or equal to $O(1)(\log n)^{-\varepsilon/2}$ for all n sufficiently large, we have (4.22) as sought. □

Lemmas 4.1 and 4.2 capture the easy halves of the statements in (4.13). In order to complete the proof of Proposition 4.2, we shall need the following two technical lemmas.

LEMMA 4.4 [Chung and Erdös (1952)]. – *For arbitrary events* $E_1, \ldots, E_N$, *we have*

$$P\left(\bigcup_{i=1}^{N} E_i\right) \geqq \left(\sum_{i=1}^{N} P(E_i)\right)^2 \Bigg/ \left(\sum_{i=1}^{N} P(E_i) + \sum_{1 \leqq i \neq j \leqq N} P(E_i \cap E_j)\right). \quad (4.24)$$

LEMMA 4.5. – *Let* $1 \leqq l < b$, *and set*

$$S_l = X_1 + \ldots + X_l,$$

$$S'_{b-l} = X_{l+1} + \ldots + X_b \quad \text{and} \quad S''_l = X_{b+1} + \ldots + X_{b+l}.$$

Then, for any $0 < y < x$ *such that*

$$\frac{\mu(x-y) - \delta l}{(\mu(x-y) + \delta l)^{1/2} (x-y)^{1/4}} \to \infty \quad \text{and} \quad \frac{\mu y - \delta(b-l)}{(\mu y + \delta(b-l))^{1/2} y^{1/4}} \to \infty,$$

we have ultimately

$$\begin{aligned} P(S_l + S'_{b-l} \geqq x, S'_{b-l} + S''_l \geqq x) &= P(S_b \geqq x, S_{b+l} - S_l \geqq x) \\ &\leqq \left(\frac{4\delta\sigma}{\mu^2\sqrt{2\pi}}\right)\left\{ y^{-3/2}(b-l)\exp\left(-\frac{1}{2\sigma^2 y}(\mu y - \delta(b-l))^2\right) \right. \\ &\left. + (x-y)^{-3/2}\, l \exp\left(-\frac{1}{2\sigma^2(x-y)}(\mu(x-y) - \delta l)^2\right) P(S_b \geqq x) \right\}. \quad (4.25) \end{aligned}$$

Proof. – Note (*see* Deheuvels, Devroye and Lynch (1986), p. 215) that we have the inequality

$$P(S_l + S'_{b-l} \geqq x, S'_{b-l} + S''_l \geqq x) \leqq P(S_{b-l} \geqq y) + P(S_l \geqq x-y)\, P(S_b \geqq x).$$

Next, we observe that

$$P(S_{b-l} \geqq y) = P(M(\delta(b-l)) \geqq y), \qquad P(S_l \geqq x-y) = P(M(\delta l) \geqq x-y),$$

and apply Lemma 2.2. The conclusion now follows from (2.11) and the observation that our assumptions imply that $(x-y)/\delta l \to \infty$, and $y/(\delta(b-l)) \to \infty$.

LEMMA 4.6. – *Assume that* b_n *is an integer sequence such that* $1 \leqq b_n \leqq n$, *and* $b_n/\log n \to 0$ *as* $n \to \infty$. *Let* A_n^+ *be as in* (4.12). *Then*

$$\limsup_{n \to \infty} \frac{U_n^+ - b_n A_n^+}{(2\sigma^2/\mu^2)\log\log n} \geqq -\frac{1}{2} \quad \text{a.s.} \quad (4.26)$$

Proof. – Let $m_j=[a^j]$, $j=1,2,\ldots$, for some fixed $a>1$, and set for $j\geqq j_0$ large enough

$$R_j^+=\max_{m_{j-1}\leqq k\leqq m_j-b_{m_j}}(S_{k+b_{m_j}}-S_k).$$

Since $m_j-m_{j-1}\sim(1-a^{-1})m_j$ and $b_{m_j}/m_j\to 0$ as $j\to\infty$, this definition is possible and $R_{j_0}^+, R_{j_0+1}^+, \ldots$ are independent. By Borel-Cantelli and the inequality $R_j^+\leqq U_{m_j}^+$, we are done if we prove that, for any $\varepsilon>0$,

$$\sum_j P\left(R_j^+\geqq b_{m_j}A_{m_j}^+ +\left(-\varepsilon-\frac{1}{2}\right)\left(\frac{2\sigma^2}{\mu^2}\log\log m_j\right)\right)=:\sum_j P_j^+=\infty. \quad (4.27)$$

Now $P_j^+=P\left(\bigcup_{i=1}^{N}E_i\right)$, where $E_i=\{S_{i+b}-S_i\geqq x\}$, $b=b_{m_j}$,

$$N=m_j-m_{j-1}-b\sim(1-a^{-1})m_j,$$

and

$$x=b_{m_j}A_{m_j}^+ +\left(-\varepsilon-\frac{1}{2}\right)\left(\frac{2\sigma^2}{\mu^2}\log\log m_j\right).$$

By Lemma 4.4 used jointly with the remark that E_i and E_j are independent for $|i-j|>b$ we have

$$P_j^+\geqq(NP(E_0))^2/(NP(E_0)+(NP(E_0))^2+2N\sum_{l=1}^{b-1}P(E_0\cap E_l)). \quad (4.28)$$

Lemma 4.1 implies the existence of a $j_1\geqq j_0$ such that, for all $j\geqq j_1$,

$$j^{-1+(7/8)\varepsilon}\leqq NP(E_0)\leqq j^{-1+(9/8)\varepsilon}. \quad (4.29)$$

Next we use Lemma 4.5 to evaluate

$$P(E_0\cap E_l)=P(S_b\geqq x, S_{b+l}-S_l\geqq x).$$

We choose x as above, and let $y=x-la$ in (4.25). Observe by (4.12) that $b/x\to 0$, so that we have always $0<x-ba<x-la=y$ for all $1\leqq l<b$, and hence

$$(\mu y-\delta(b-l))^2/y\sim\mu^2 x\to\infty.$$

Likewise,

$$(\mu(x-y)-\delta l)^2/(x-y)=l(\mu a-\delta)^2/a\to\infty \text{ as } l\to\infty \text{ for } a\neq\delta/\mu.$$

By all this, we may apply (4.25). Note that in the right-hand-side of (4.25), $y\sim x\to\infty$, $(b-l)y^{-3/2}=O(bx^{-3/2})\to 0$, and $l(x-y)^{-3/2}=O(l^{-1/2})\to 0$ as

$l \to \infty$. Moreover

$$\frac{1}{2\sigma^2 y}(\mu y-\delta(b-l))^2=\frac{1}{2\sigma^2 x}(\mu x-\delta b)^2-\frac{\mu l}{2\sigma^2}(a\mu-2\delta)$$
$$+\frac{\delta^2}{2\sigma^2 y}(b-l)^2-\frac{\delta^2}{2\sigma^2 x}b^2\leqq\frac{1}{2\sigma^2 x}(\mu x-\delta b)^2-\frac{\mu l}{2\sigma^2}(a\mu-2\delta)$$
$$+\frac{\delta^2 b^2 la}{2\sigma^2 xy}=\frac{1}{2\sigma^2 x}(\mu x-\delta b)^2-\frac{\mu l}{2\sigma^2}(a\mu-2\delta+o(1)).$$

Likewise,

$$\frac{1}{2\sigma^2(x-y)}(\mu(x-y)-\delta l)^2=l\frac{(\mu a-\delta)^2}{2\sigma^2 a}.$$

Up to now, the choice of a had remained open. We choose $a>1$ in such a way that $\frac{\mu}{2\sigma^2}(a\mu-2\delta)\geqq 1+\log 2$ and $\frac{(\mu a-\delta)^2}{2\sigma^2 a}\geqq\log 2$. By (4.25), we have ultimately in $j\to\infty$ and $l\to\infty$

$$P(E_0\cap E_l)\leqq\left(\frac{8\delta\sigma}{\mu^2\sqrt{2\pi}}\right)$$
$$\times\left\{bx^{-3/2}\exp\left(-\frac{1}{2\sigma^2 x}(\mu x-\delta b)^2\right)+P(S_b\geqq x)\right\}2^{-l}$$
$$=2^{-l}O(P(E_0)),\quad (4.30)$$

where we have used again (2.6). It follows from (4.30) and (4.29) that

$$N\sum_{l=1}^{b-1}P(E_0\cap E_l)\leqq j^{(1/4)\varepsilon}NP(E_0)+\sum_{l=[j^{(1/4)\varepsilon}]}^{b-1}P(E_0\cap E_l)$$
$$\leqq j^{(11/8)\varepsilon-1}+2^{-[j^{(1/4)\varepsilon}]}j^{(9/8)\varepsilon-1}O(1).$$

Hence, by (4.28) and (4.29), we have ultimately in $j\to\infty$

$$P_j^+\geqq(j^{(14/8)\varepsilon-2})/(4j^{(11/8)\varepsilon-1})=\frac{1}{4}j^{-1+(3/8)\varepsilon},$$

which is not summable in j. This, jointly with (4.27) completes the proof of Lemma 4.6. □

LEMMA 4.7. – *Assume that b_n is an integer sequence such that $b_n\uparrow$, $1\leqq b_n\leqq n$, and $b_n/\log n\to 0$ as $n\to\infty$. Let A_n^+ be as in* (4.12). *Then*

$$\liminf_{n\to\infty}\frac{U_n^+-b_nA_n^+}{(2\sigma^2/\mu^2)\log\log n}\geqq-\frac{3}{2}\quad \text{a.s.}\qquad (4.31)$$

Proof. – Let $\{n_j\}$ be as in (4.5). By the same arguments as in the proof of Lemma 4.2, we see that we are done if we can show that for any $\varepsilon>0$

$$\sum_j P\left(U_{n_j}^+ < b_{n_j} A_{n_j}^+ + \left(-\varepsilon-\frac{3}{2}\right)\left(\frac{2\sigma^2}{\mu^2}\log\log n_j\right)\right)=0. \quad (4.32)$$

For the proof of (4.32), we shall need an upper bound for

$$P_n=P\left(U_n^+ < b_n A_n^+ + \left(-\varepsilon-\frac{3}{2}\right)\left(\frac{2\sigma^2}{\mu^2}\log\log n\right)\right).$$

Fix $\Lambda>0$ and set $j=[\Lambda\log n]$. Denote by I_n the set of all integers of the form $r[j^{\varepsilon/4}]$, $r=0, 1, \ldots$ For integer l and fixed n, define, as in Lemma 7 of Deheuvels, Devroye and Lynch (1986),

$$Q_l = \max_{2lj\leqq i<(2l+1)j,\, i\in I_n} (S_{i+b_n}-S_i).$$

Observe that $b_n=o(\log n)=o(j)$ as $n\to\infty$. Therefore, for all n sufficiently large, the random variables $Q_0, Q_1, \ldots$ are independent. Moreover,

$$U_n^+ \geqq \max_{0\leqq l\leqq L} Q_l,$$

where $L=L_n$ is the largest integer such that $(2L+1)j-1\leqq n-b_n$, *i.e.*

$$L=\left[\frac{1}{2}\left(\frac{n-b_n+1}{[\Lambda\log n]}\right)-1\right]\sim\frac{n}{2\Lambda\log n}\sim\frac{1}{2}nj^{-1} \quad \text{as } n\to\infty. \quad (4.33)$$

Let $x=b_n A_n^+ + \left(-\varepsilon-\frac{3}{2}\right)\left(\frac{2\sigma^2}{\mu^2}\log\log n\right)$. Using the independence of the Q_l's for n sufficiently large, we have

$$P_n=P(U_n^+<x)\leqq P(\max_{0\leqq l\leqq L} Q_l<x)$$
$$=\prod_{l=0}^{L} P(Q_l<x)\leqq\exp\left(-\sum_{l=0}^{L} P(Q_l\geqq x)\right). \quad (4.34)$$

Let N_l be the number of indices in $I_n\cap\{2lj, 2lj+1, \ldots, (2l+1)j-1\}$. We see that, uniformly in $l\geqq 0$, $N_l\sim j^{1-(\varepsilon/4)}$ as $n\to\infty$. Now, making use of Lemma 4.4, we have

$$P(Q_l\geqq x)$$
$$\geqq (N_l P(E_0))^2/\left(N_l P(E_0)+(N_l P(E_0))^2+2N_l\sum_{k=1}^{b_n-1} P(E_0\cap E_k)\right), \quad (4.35)$$

where $E_k=\{S_{k+b_n}-S_k\geqq x\}$.

By Lemma 4.1, we see, by the same arguments as in (4.29), that there exists a j_0 such that $j>j_0$ implies

$$n^{-1}j^{1-(\varepsilon/4)+(7/8)\varepsilon}\leqq N_l P(E_0)\leqq n^{-1}j^{1-(\varepsilon/4)+(9/8)\varepsilon}. \quad (4.36)$$

Now, by exactly the same arguments as used in the proof of Lemma 4.6, the denominator in (4.35) can be, for a suitable choice of the constant $\Lambda > 0$, bounded from above by

$$N_l P(E_0)(1+o(1)) + 2 N_l P(E_0) j^{(1/4)\varepsilon}(1+o(1)) \leqq 4 N_l P(E_0) j^{(1/4)\varepsilon},$$

for all n sufficiently large. This, jointly with (4.35) and (4.36), implies that

$$P(Q_l \geqq x) \geqq (n^{-2} j^{2+(5/4)\varepsilon})/(4 n^{-1} j^{1+(9/8)\varepsilon}) = \frac{1}{4} n^{-1} j^{1+(1/8)\varepsilon}.$$

Hence, by (4.33) and (4.34), we have, for all n sufficiently large,

$$P_n \leqq \exp\left(-\frac{1}{4} L n^{-1} j^{1+(1/8)\varepsilon}\right)$$
$$= \exp\left(-\frac{1}{8}(1+o(1)) j^{(1/8)\varepsilon}\right) \leqq n^{-\varepsilon/16}. \quad (4.37)$$

By (4.5) and (4.37), the proof of (4.32) reduces to

$$\sum_j [a^j]^{-\varepsilon/16} + \sum_m m^{-\varepsilon/16}(b_{m+1} - b_m) = O(1) + \sum_m b_m m^{-1-\varepsilon/16} O(1) < \infty.$$

This completes the proof of Lemma 4.7. □

The proof of Proposition 4.2 now follows by combining (4.18), (4.22), (4.26) and (4.31). □

Remark 4.2. – Notice that the assumptions of Proposition 4.2 do not require any regularity assumptions (such as (3.8)) imposed upon the growth of b_n. A close look to the proof of Theorem 2 in Deheuvels and Steinebach (1987) [*see* also Theorem 5 and Remark 5 in Deheuvels, Devroye and Lynch (1986)], shows that we can partially relax the conditions (3.8) in Proposition 3.2. For this, observe that Lemmas 12, 13 and 14 in Deheuvels and Steinebach (1987) do not assume (3.8) which is only required for their Lemma 11, *i.e.* for showing that, under the assumptions that $b_n/\log n \to \infty$ and $b_n = O(\log^p n)$ for some $p > 1$ as $n \to \infty$, we have, under (3.8) and using the notation of (3.9),

$$\limsup_{n\to\infty} \frac{t_n^+(U_n^+ - b_n a_n^+)}{\log\log n} \leqq \frac{3}{2}$$

and (4.38)

$$\liminf_{n\to\infty} \frac{-t_n^-(U_n^- - b_n a_n^-)}{\log\log n} \geqq -\frac{3}{2} \quad \text{a.s.}$$

The proof of the above-mentioned Lemma 11 makes use of the subsequence $m_j = [a^j]$, where $a > 1$ is fixed. By repeating verbatim this proof with the formal replacement of m_j by n_j as defined in (4.5), we obtain that

(4.38) holds whenever for any $\varepsilon>0$, $\sum_j (\log n_j)^{-1-(1/4)\varepsilon}<\infty$. This, in turn, is satisfied whenever

$$\sum_j (\log [a^j])^{-1-(1/4)\varepsilon}+\sum_m (b_m-b_{m-1})(\log m)^{-1-(1/4)\varepsilon}$$
$$=O(1)+O(1)\sum_m b_m m^{-1}(\log m)^{-2-(1/4)\varepsilon}<\infty. \quad (4.39)$$

Interestingly, (4.39) does not hold for $b_n=[(\log n)^r]$ and $r\geqq 1$. On the other hand, (4.39) is satisfied if

$$\lim_{n\to\infty} (\log b_n)/\log\log n=1. \quad (4.40)$$

The same arguments [observe also by Remark 4.1 that (4.40) implies the equivalence of (3.9) and (4.13)] used for the case when $b_n/\log n \to c\in(0,\infty)$ as $n\to\infty$ [*see* Remark 6 in Deheuvels and Steinebach (1987)] show that the conclusion of Proposition 3.2 holds when (3.8) is replaced by (4.40). We omit the details.

We shall make use of this remark in Theorem 5.1 (1°) for $c>0$.

We conclude this section by an analogue of Proposition 4.2 for U_n^-. At times in the sequel, b_n will be a sequence of possibly non integer numbers, and we shall let then $U_n^-=\min_{0\leq i\leq n-b_n} (M(\delta(i+b_n))-M(\delta i))$.

PROPOSITION 4.3. – *Assume that b_n is a sequence of integers such that $b_n\uparrow$, $1\leqq b_n\leqq n$, and $b_n/\log n\to 0$ as $n\to\infty$. Let A_n^- be defined by*

$$A_n^-=\frac{\delta}{\mu}+\frac{\sigma^2}{\mu^2}\left(\frac{\log n}{b_n}\right)\left(1-\left(1+\frac{2\mu}{\sigma^2}\left(\frac{\delta b_n}{\log n}\right)\right)^{1/2}\right)=a_\delta^-\left(\frac{\log n}{b_n}\right). \quad (4.41)$$

Then we have

$$\limsup_{n\to\infty}\frac{U_n^- - b_n A_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n}=\frac{1}{2}$$

and (4.42)

$$\liminf_{n\to\infty}\frac{U_n^- - b_n A_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n}=-\frac{1}{2} \quad \text{a.s.}$$

Remark 4.3. – An expansion of A_n^- as given in (4.41) yields

$$b_n A_n^-=\frac{1}{2\sigma^2}\frac{\delta^2 b_n^2}{\log n}+O(b_n^3/\log^2 n). \quad (4.43)$$

Hence, whenever $b_n=o(\log\log n)$, we may replace in (4.42) $b_n A_n^-$ by $\frac{1}{2\sigma^2}\frac{\delta^2 b_n^2}{\log n}$.

Let t_n^- and a_n^- be defined as in (3.10) and (3.11). We see that, for $b_n/\log n \to 0$, $t_n^- \sim \frac{-\delta^2}{2\sigma^2}(a_n^-)^{-2} \sim -\frac{2\sigma^2}{\delta^2}\frac{\log^2 n}{b_n^2}$. Moreover, by (2.18) and (4.43)

$$t_n^- b_n (A_n^- - a_n^-) \sim \left(-\frac{2\sigma^2}{\delta^2}\frac{\log^2 n}{b_n^2}\right) b_n \times \left(\frac{-\delta^2 b_n \log b_n}{2\sigma^2 \log n}\right) = \log b_n. \quad (4.44)$$

In view of (4.44), a comparison of (3.9) and (4.42) shows that *the statements of Propositions* 3.2 *and* 4.3 *coincide in the range where* $(\log b_n)/\log\log n \to 1$ *but differ otherwise.* We have here the same observation for U_n^- as that given in Remark 4.1 for U_n^+.

Proof of Proposition 4.3. – The proof is captured in the following sequence of lemmas.

LEMMA 4.8. – *Let* $\varepsilon_n \to 0$ *be a sequence of positive numbers. Then, uniformly over all sequences* $\{y_n\}$ *such that* $|y_n| \leqq \varepsilon_n b_n A_n^-$, *we have*

$$n P(M(\delta b_n) \leqq b_n A_n^- - y_n) = (1+o(1))\pi^{-1/2}(\log n)^{-1/2} \exp\left(-2\sigma^2(1+o(1))\frac{\log^2 n}{\delta^2 b_n^2} y_n\right). \quad (4.45)$$

Proof. – By (2.8) taken with $\nu = \delta b_n/\mu$, $\lambda = \delta^2 b_n^2/\sigma^2$ and $x = b_n A_n^- - y_n$, and by (4.43) which ensures that $x \sim \frac{\delta^2}{2\sigma^2}(b_n^2/\log n)$, we see that if $x_- = b_n A_n^-$,

$$\frac{2\nu^2\sqrt{x/\lambda}}{(\nu^2 - x^2)\sqrt{2\pi}} \sim \pi^{-1/2}(\log n)^{-1/2}, \quad (4.46)$$

and

$$\frac{\lambda}{2x}\left(\frac{x}{\nu}-1\right)^2 = \frac{\lambda}{2x_-}\left(\frac{x_-}{\nu}-1\right)^2 - \frac{\mu^2}{2\sigma^2}y_n + \frac{\delta^2 b_n^2}{2\sigma^2}\left(\frac{1}{x}-\frac{1}{x_-}\right)$$
$$= \zeta_{\delta b_n}(b_n A_n^-) - \frac{\mu^2}{2\sigma^2}y_n + (1+o(1))\, 2\sigma^2 \frac{\log^2 n}{\delta^2 b_n^2} y_n$$
$$= \zeta_{\delta b_n}(b_n A_n^-) + (1+o(1))\, 2\sigma^2 \frac{\log^2 n}{\delta^2 b_n^2} y_n. \quad (4.47)$$

where $\zeta_t(a)$ is as in (2.17). By (2.18), we see that $b_n A_n^- = a_{\delta b_n}^-(\log n)$, so that $\zeta_{\delta b_n}(b_n A_n^-) = \log n$. This, jointly with (2.8), (4.46) and (4.47), completes the proof of (4.45). □

LEMMA 4.9. — *Assume that* b_n *is an integer sequence such that* $b_n \uparrow$, $1 \leqq b_n \leqq n$, *and* $b_n/\log n \to 0$ *as* $n \to \infty$. *Let* A_n^- *be as in* (4.41). *Then*

$$\liminf_{n \to \infty} \frac{U_n^- - b_n A_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n} \geqq -\frac{1}{2} \quad \text{a.s.} \tag{4.48}$$

Proof. — Let $\{n_j\}$ be as in (4.5). Recall that $b_n = b_{n_j}$ for $n_{j-1} < n \leqq n_j$. It follows that $U_n^- \geqq U_{n_j}$ for $n_{j-1} < n \leqq n_j$. Next, we see that, uniformly over all $n_{j-1} < n \leqq n_j$, we have as $j \to \infty$

$$b_n^2 \frac{\log\log n}{\log^2 n} = (1 + o(1))\, b_{n_j}^2 \frac{\log\log n_j}{\log^2 n_j},$$

and

$$b_n A_n^- = b_{n_j} A_{n_j}^- + O\left(\frac{b_{n_j}^2}{\log^2 n_j}\right) = b_{n_j} A_{n_j}^- + o\left(b_{n_j}^2 \frac{\log\log n_j}{\log^2 n_j}\right).$$

Hence, using Borel-Cantelli, all we need is to prove that for any $\varepsilon > 0$,

$$\sum_j P\left(U_{n_j}^- \leqq b_{n_j} A_{n_j} - \left(\frac{1}{2} + \varepsilon\right)\left(\frac{\delta^2 b_{n_j}^2}{2\sigma^2}\right)\frac{\log\log n_j}{\log^2 n_j}\right) =: \sum_j R_j < \infty.$$

By Lemma 4.8, we see that, for all j sufficiently large,

$$R_j \leqq 2\pi^{-1/2} (\log n_j)^{-1/2} \exp\left(-\left(\frac{1}{2} + \frac{1}{2}\varepsilon\right)\log\log n_j\right) \leqq (\log n_j)^{-1-(1/4)\varepsilon}.$$

The conclusion now follows by the same arguments as in (4.39). □

LEMMA 4.10. — *Let* b_n *be as in Lemma* 4.3, *but possibly non integer. We have*

$$\limsup_{n \to \infty} \frac{U_n^- - b_n A_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n} \geqq \frac{1}{2} \quad \text{a.s.} \tag{4.49}$$

Proof. — The arguments are similar to those used in the proof of Lemma 4.3, with the replacement of Lemma 4.1 by Lemma 4.8. We omit details. □

LEMMA 4.11. — *Let* b_n *be as in Lemma* 4.3, *but possibly non integer. We have*

$$\liminf_{n \to \infty} \frac{U_n^- - b_n A_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n} \leqq -\frac{1}{2} \quad \text{a.s.} \tag{4.50}$$

Proof. — The arguments are along the lines of proof for Lemma 4.6, but certain modifications are necessary. Let $m_j = [a^j]$ $(a > 1)$ and set

$$R_j^- = \min_{m_{j-1} < k \leqq m_j - b_{m_j}} (S_{k+b_{m_j}} - S_k).$$

It is enough (*see* the proof of Lemma 4.6) to show that for all $\varepsilon>0$

$$\sum_j \mathrm{P}\left(\mathrm{R}_j^- \leqq b_{m_j} \mathrm{A}_{m_j}^- - \left(\left(\frac{1}{2}-\varepsilon\right)\left(\frac{\delta^2}{2\sigma^2}\right) b_{m_j}^2 \frac{\log\log m_j}{\log^2 m_j}\right)\right) =: \sum_j \mathrm{P}_j^- = \infty.$$

Now $\mathrm{P}_j^- = \mathrm{P}\left(\bigcup_{i=1}^{\mathrm{N}} \mathrm{E}_i\right)$, where

$$\mathrm{E}_i = \{\mathrm{M}(\delta(i+b)) - \mathrm{M}(\delta i) \leqq x\},$$
$$b = b_{m_j}, \qquad \mathrm{N} = m_j - m_{j-1} - b \sim (1-a^{-1}) m_j,$$

and

$$x = b_{m_j} \mathrm{A}_{m_j}^- - \left(\left(\frac{1}{2}-\varepsilon\right)\left(\frac{\delta^2}{2\sigma^2}\right) b_{m_j} \frac{\log\log m}{\log^2 m_j}\right)$$
$$\sim \frac{\delta^2}{2\sigma^2} b_{m_j}^2/\log m_j \sim \frac{\delta^2}{2\sigma^2 \log a}(b^2/j) \quad \text{as } j\to\infty.$$

By Lemmas 4.4 and 4.8, we have

$$\mathrm{P}_j^- \geqq (\mathrm{NP}(\mathrm{E}_0))^2 \Big/ \left(\mathrm{NP}(\mathrm{E}_0) + (\mathrm{NP}(\mathrm{E}_0))^2 + 2\mathrm{N} \sum_{l=1}^{b-1} \mathrm{P}(\mathrm{E}_0 \cap \mathrm{E}_l)\right),$$

and there exists a j_0 such that, for all $j \geqq j_0$,

$$j^{-1+(7/8)\varepsilon} \leqq \mathrm{NP}(\mathrm{E}_0) \leqq j^{-1+(9/8)\varepsilon}.$$

To proceed as in the proof of Lemma 4.6, it is enough to show that

$$\sum_{l=[j^{(1/4)\varepsilon}]}^{b-1} \mathrm{P}(\mathrm{E}_0 \rangle \mathrm{E}_l) = O(\mathrm{P}(\mathrm{E}_0)) \quad \text{as } j\to\infty. \tag{4.51}$$

By Lemma 4.5, we have

$$\mathrm{P}(\mathrm{E}_0 \cap \mathrm{E}_l) \leqq \mathrm{P}(\mathrm{M}(\delta(b-l)) \leqq y) + \mathrm{P}(\mathrm{M}(\delta l) \leqq x-y)\,\mathrm{P}(\mathrm{E}_0), \tag{4.52}$$

where we choose y in such a way that $(b-l)^2/y = b^2/x$, *i.e.* by setting

$$y = x\left(1-\frac{l}{b}\right)^2 = x - \frac{lx}{b}\left(2-\frac{l}{b}\right) := x - l\Gamma.$$

Observe that, uniformly over $1 \leqq l \leqq b$,

$$(1+o(1))\left(\frac{\delta^2}{2\sigma^2} b/\log m_j\right) \leqq \Gamma \leqq (1+o(1))\left(\frac{\delta^2}{\sigma^2} b/\log m_j\right)$$
$$= o(1) \quad \text{as } j\to\infty.$$

Moreover,

$$\frac{1}{2\sigma^2 y}(\mu y-\delta(b-l))^2=\frac{1}{2\sigma^2}\left(\frac{\delta^2(b-l)^2}{y}-2\mu b\delta+2\mu l\delta+\mu^2(y+l\Gamma-l\Gamma)\right)$$
$$=\frac{1}{2\sigma^2}\left(\frac{(\mu x-\delta b)^2}{x}+l(2\delta\mu-\mu^2\Gamma)\right)$$
$$=\frac{1}{2\sigma^2}\left(\frac{(\mu x-\delta b)^2}{x}+2l\mu\delta(1+o(1))\right).$$

By (2.7) taken with $\nu=\delta(b-l)/\mu$ and $\lambda=\delta^2(b-l)^2/\sigma^2$, we see that

$$\sqrt{\lambda/y}\left(\frac{y}{\nu}-1\right)=\frac{\delta b}{\sigma\sqrt{x}}\left(\frac{\mu x(b-l)}{\delta b^2}-1\right)\to-\infty$$

and (4.53)

$$\frac{y}{\nu}=\frac{\mu x(b-l)}{\delta b^2}\to 0 \quad \text{as } j\to\infty.$$

Here, we have used the facts that $b/\sqrt{x}\sim\sqrt{j}(2\sigma^2\log a)^{1/2}/\delta\to\infty$ and that

$$\mu x(b-l)/b^2\leqq\mu x/b=O(b_{m_j}/\log m_j)\to 0 \quad \text{as } j\to\infty.$$

By (4.53) and (2.8), it follows that, uniformly over $1\leqq l\leqq b$,

$$P(M(\delta(b-l))\leqq y)$$
$$=O\left\{\frac{\sqrt{x}}{b}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{(\mu x-\delta b)^2}{x}+2l\mu\delta(1+o(1))\right)\right)\right\}$$
$$=O\left\{P(E_0)\exp\left(-\frac{l\mu\delta}{2\sigma^2}\right)\right\},$$

so that

$$\sum_{l=1}^{b-1}P(M(\delta(b-l))\leqq y)=O(P(E_0)) \quad \text{as } j\to\infty. \tag{4.54}$$

Likewise, by (2.7) taken with $\nu=\delta l/\mu$ and $\lambda=\delta^2 l^2/\sigma^2$, we see that, for $l\geqq[j^{1/4}]$,

$$\sqrt{\lambda/(x-y)}\left(\frac{x-y}{\nu}-1\right)=\frac{\delta l}{\sigma\sqrt{l\Gamma}}\left(\frac{\Gamma\mu}{\delta}-1\right)\to-\infty$$

and

$$\frac{x-y}{\nu}=\frac{\Gamma\mu}{\delta}\to 0 \quad \text{as } j\to\infty.$$

Hence, by (2.8), we have uniformly over $[j^{1/4}] \leqq l \leqq b$

$$\mathrm{P}(\mathrm{M}(\delta l) \leqq x-y) = O\left\{\frac{\sqrt{x-y}}{l}\exp\left(-\frac{1}{2\sigma^2(x-y)}(\mu(x-y)-\delta l)^2\right)\right\}$$
$$= O\left\{\sqrt{\Gamma/l}\exp\left(-\frac{\delta^2 l}{2\sigma^2\Gamma}(1+o(1))\right)\right\} = o\{\exp(-l)\} \quad \text{as } j\to\infty,$$

so that

$$\sum_{l=[j^{1/4}]}^{b-1} \mathrm{P}(\mathrm{M}(\delta l) \leqq x-y) = o(1) \quad \text{as } j\to\infty. \tag{4.55}$$

By (4.53), (4.54) and (4.55), we have (4.51), which completes the proof of our lemma. □

LEMMA 4.12. — *Let b_n be as in Lemma 4.7, but possibly non integer, We have*

$$\limsup_{n\to\infty} \frac{\mathrm{U}_n^- - b_n \mathrm{A}_n^-}{(\delta^2 b_n^2/2\sigma^2)(\log\log n)/\log^2 n} \leqq \frac{1}{2} \quad \text{a.s.} \tag{4.56}$$

Proof. — In the first place, we show that for any $\varepsilon>0$, we have

$$\mathrm{P}_n := \mathrm{P}\left(\mathrm{U}_n^- \geqq b_n \mathrm{A}_n^- - \left(-\frac{1}{2}-\varepsilon\right)\left(\frac{\delta^2 b_n^2}{2\sigma^2}\right)\frac{\log\log n}{\log^2 n}\right) \leqq n^{-\varepsilon/16} \tag{4.57}$$

for all n sufficiently large. The arguments to prove (4.57) are along the lines of Lemma 4.7 with similar modifications as in the previous proof. So details may be omitted here.

Next, we introduce a sequence of integers ν_j defined as follows. Fix an $a>1$, and define

$$\nu_1 = 1$$

and

$$\nu_j = \min\{\inf([a^i] > \nu_{j-1} : i=1,2,\ldots),\ \inf(n>\nu_{j-1} : b_n > b_{\nu_{j-1}} + \theta(\nu_{j-1}))\} \tag{4.58}$$
$$\text{for } j=2,3,\ldots, \quad \text{where } \theta(\nu) = b_\nu/\log\nu.$$

Observe by (4.58) that we have

$$\limsup_{j\to\infty}(\nu_j/\nu_{j-1}) \leqq a \quad \text{and} \quad \lim_{j\to\infty}((\log\nu_j)/(\log\nu_{j-1})) = 1. \tag{4.59}$$

Moreover, uniformly over all $\nu_{j-1} \leqq n < \nu_j$, we have

$$b_n/b_{\nu_{j-1}} \to 1 \quad \text{as } j\to\infty. \tag{4.60}$$

Let $H(K,T)=Ka_1^-\left(\frac{\log T}{K}\right)$. We see that if $K/\log T \to 0$, then

$$\frac{\partial}{\partial K} H(K,T) \sim \frac{1}{\sigma^2}(K_T/\log T). \tag{4.61}$$

This, jointly with the observation that $b_n A_n^- = H(\delta b_n, n)$, shows that, uniformly over all $\nu_{j-1} \leqq n < \nu_j$, we have

$$b_n A_n^- - b_{\nu_{j-1}} A_{\nu_{j-1}}^- = O(\theta(\nu_{j-1}) b_{\nu_j}/\log \nu_j)$$
$$= O(b_{\nu_j}^2/\log^2 \nu_j) \quad \text{as } j \to \infty. \tag{4.62}$$

By (4.60) and (4.62), we see that (4.56) reduces to show that, for all $\varepsilon > 0$,

$$\sum_j P_{\nu_j} < \infty. \tag{4.63}$$

Let $m_j = [a^j]$. If $m_{j-1} \leqq \nu_{l-1} < \nu_l \leqq m_j$, then

$$b_{\nu_l} - b_{\nu_{l-1}} > b_{\nu_{l-1}}/\log \nu_{l-1} \geqq b_{m_{j-1}}/\log m_j$$

(here, we have used the assumption that $b_n \uparrow$). Therefore, we have the inequality

$$(b_{m_{j-1}}/\log m_j)(\#\{\nu_l : m_{j-1} \leqq \nu_l < m_j\} - 1) \leqq b_{m_j} - b_{m_{j-1}},$$

which in turn (recall that $b_n \geqq 1$) implies that

$$\#\{\nu_l : m_{j-1} \leqq \nu_l < m_j\} \leqq 1 + (b_{m_j} - b_{m_{j-1}}) \log m_j = O(\log^2 m_j) = o(j^2)$$

as $j \to \infty$ (here, we have used the fact that $b_{m_j} = O(\log m_j)$ as $j \to \infty$). Hence, by (4.57), (4.63) reduces to (notice that $m_j/m_{j-1} \leqq a^2/(a-1)$ for $j \geqq 1$)

$$\sum_j \nu_j^{-\varepsilon/16} \leqq \left(\frac{a^2}{a-1}\right)^{\varepsilon/16} \sum_j m_j^{-\varepsilon/16} \#\{\nu_l : m_{j-1} \leqq \nu_l \leqq m_j\}$$
$$= O\Big(\sum_j j^2 a^{-\varepsilon j/16}\Big) < \infty, \tag{4.64}$$

which completes the proof of Lemma 4.12. □

Proof of Proposition 4.3. – It follows directly from Lemmas 4.9-4.12. □

5. INCREMENTS OF THE INVERSE GAUSSIAN PROCESS

Our main theorem concerning $\Delta_T^+ = \Delta^+(T, K_T)$ as defined in (1.5) is stated below.

THEOREM 5.1. – *Let* K_T *be a function such that* $K_T\uparrow$ *and* $0<K_T\leqq T$. *Define*

$$\alpha_T^+ = \frac{1}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log(TK_T)}{K_T}\right)\left(1+\left(1+\frac{2\mu}{\sigma^2}\left(\frac{K_T}{\log(TK_T)}\right)\right)^{1/2}\right) = a_1^+\left(\frac{\log(TK_T)}{K_T}\right), \quad (5.1)$$

$$\beta_T^+ = \frac{1}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log(T/K_T)}{K_T}\right)\left(1+\left(1+\frac{2\mu}{\sigma^2}\left(\frac{K_T}{\log(T/K_T)}\right)\right)^{1/2}\right) = a_1^+\left(\frac{\log(T/K_T)}{K_T}\right), \quad (5.2)$$

and set $t_T^+ = s_1(\beta_T^+)$, *where* $s_1(.)$ *is as in* (2.16).

1° *If*

$$\lim_{T\to\infty} (K_T/\log T) = c\in[0,\infty), \quad (5.3)$$

we have

$$\limsup_{T\to\infty} \frac{t_T^+(\Delta_T^+ - K_T\alpha_T^+)}{\log\log T} = -\frac{1}{2}$$

and (5.4)

$$\liminf_{T\to\infty} \frac{t_T^+(\Delta_T^+ - K_T\alpha_T^+)}{\log\log T} = -\frac{3}{2} \quad \text{a.s.}$$

2° *If* K_T *has first derivative* K_T' *such that* $K_T'/K_T = O(1/(T\log T))$ *as* $T\to\infty$, *and if*

$$\liminf_{T\to\infty} (K_T/\log T) > 0, \quad (5.5)$$

we have

$$\limsup_{T\to\infty} \frac{t_T^+(\Delta_T^+ - K_T\beta_T^+)}{\log\log T} = \frac{3}{2}$$

and (5.6)

$$\liminf_{T\to\infty} \frac{t_T^+(\Delta_T^+ - K_T\beta_T^+)}{\log\log T} = \frac{1}{2} \quad \text{a.s.}$$

Proof. – First, notice that the assumption that $K_T'/K_T = O(1/(T\log T))$ implies (after integration) that $K_T = O((\log T)^p)$ as $T\to\infty$, for some $p>0$. This in combination with (5.5) implies that $(\log\log(T/K_T))/\log\log T \to 1$ as $T\to\infty$. Next, let $\delta=1$, $n=[T]+m'$ and $b_n=[K_T]+m''$, where m' and m'' are fixed integers. By (3.3), it suffices to prove that (5.4) [resp. (5.6)] holds under (5.3) [resp. (5.5)] with U_n^+ replacing Δ_T^+, and independently of m' and m''. For this, we make use of Proposition 4.2 for (5.4) when $c=0$, of Proposition 3.2 for (5.4) when $0<c<\infty$ [note here that we need not impose (3.8) in this case because of Remark 4.2. Otherwise, one would need to assume regularity conditions like the existence of K'_T and $K'_T/K_T = O(1/(T\log T))$ as $T\to\infty$], and for (5.6). In all three cases we have $t_n^+ \sim t_T^+$ as $T\to\infty$, the definition of t_n^+ given in the theorem being in agreement with (3.11). Finally, the proof boils down to show that both $b_n a_n^+ - K_T\beta_T^+$ and $b_n A_n^+ - K_T\alpha_T^+$ are $o((\log\log T)/t_T^+)$ as $T\to\infty$. For this, observe that by (5.1) and (4.12) we have, under (5.3),

$$b_n A_n^+ - K_T\alpha_T^+ = b_n a_1^+(b_n^{-1}\log(nb_n)) - K_T a_1^+(K_T^{-1}\log(TK_T))$$
$$= O(\log(nb_n/TK_T)) + O(b_n - K_T) = O(1),$$

which is more than enough. A similar argument holds for $b_n A_n^+ - K_T\beta_T^+$ under (5.5).

We now present the analogue of Theorem 5.1 for $\Delta_T^- = \Delta^-(T, K_T)$.

THEOREM 5.2. – *Let K_T be a function such that $K_T\uparrow$ and $0<K_T\leqq T$. Assume that K_T has first derivative K'_T such that $K'_T/K_T = O(1/(T\log T))$ as $T\to\infty$. Define*

$$\alpha_T^- = \frac{1}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log T}{K_T}\right)\left(1-\left(1+\frac{2\mu}{\sigma^2}\left(\frac{K_T}{\log T}\right)\right)\right)^{1/2} = a_1^-\left(\frac{\log T}{K_T}\right), \quad (5.7)$$

$$\beta_T^- = \frac{1}{\mu} + \frac{\sigma^2}{\mu^2}\left(\frac{\log(T/K_T)}{K_T}\right)\left(1-\left(1+\frac{2\mu}{\sigma^2}\left(\frac{K_T}{\log(T/K_T)}\right)\right)\right)^{1/2}$$
$$= a_1^-\left(\frac{\log(T/K_T)}{K_T}\right), \quad (5.8)$$

and set $t_T^- = s_1(\beta_T^-)$, where $s_1(.)$ is as in (2.16).

1° *If* $\liminf_{T\to\infty}(K_T/\log T)>0$, *then*

$$\limsup_{T\to\infty}\frac{t_T^-(\Delta_T^- - K_T\beta_T^-)}{\log\log T} = \frac{3}{2}$$

and (5.9)

$$\liminf_{T\to\infty}\frac{t_T^-(\Delta_T^- - K_T\beta_T^-)}{\log\log T} = \frac{1}{2}\quad \text{a.s.}$$

2° *If* $K_T/\log T\to 0$ *as* $T\to\infty$, *then almost surely*

$$\frac{1}{2}\leqq\limsup_{T\to\infty}\frac{t_T^-(\Delta_T^- - K_T\beta_T^-)}{\log\log T}\leqq\frac{3}{2},$$
$$-\frac{1}{2}\leqq\liminf_{T\to\infty}\frac{t_T^-(\Delta_T^- - K_T\beta_T^-)}{\log\log T}\leqq\frac{1}{2}, \quad (5.10)$$

$$\begin{aligned}\frac{1}{2} &\leqq \limsup_{T\to\infty} \frac{t_T^-(\Delta_T^- - K_T\alpha_T^-)}{\log\log T} \leqq \frac{3}{2},\\ -\frac{1}{2} &\leqq \liminf_{T\to\infty} \frac{t_T^-(\Delta_T^- - K_T\alpha_T^-)}{\log\log T} \leqq \frac{1}{2},\end{aligned} \tag{5.11}$$

while, as $T \to \infty$,

$$\frac{t_T^-(\Delta_T^- - K_T\beta_T^-)}{\log\log T} - \frac{t_T^-(\Delta_T^- - K_T\alpha_T^-)}{\log\log T} = (1+o(1))\frac{\log K_T}{\log\log T}. \tag{5.12}$$

Proof. – The proof of Part 1 follows the lines of the proof of Theorem 5.1 with the formal replacement of Δ_T^+ by Δ_T^-. Therefore, we will omit details. Assume from now on that $K_T/\log T \to 0$ as $T \to \infty$, and consider the proof of Part 2. The following facts will be used in the sequel.

Fact 1. – We have $(\log\log T)/t_T^- \sim \dfrac{-1}{2\sigma^2}(K_T^2/\log^2 T)\log\log T$ as $T \to \infty$.

Fact 2. – We have $K_T\alpha_T^- - K_T\beta_T^- \sim \dfrac{-1}{2\sigma^2}(K_T^2/\log^2 T)\log K_T$ as $T \to \infty$.

Fact 3. – Let $H(K, T) = K a_1^-\left(\dfrac{\log T}{K}\right)$. Then, as $K/\log T \to 0$,

$$\frac{\partial}{\partial K} H(K, T) \sim \frac{1}{\sigma^2}\{K/\log T)$$

and (5.13)

$$\frac{\partial}{\partial T} H(K, T) \sim \frac{-1}{2\sigma^2 T}(K/\log T)^2.$$

Fact 4. – For any $\delta > 0$, we have

$$U^-\left(\left[\frac{T}{\delta}\right]+1, \frac{1}{\delta}K_T - 1\right) \leqq \Delta^-(T, K_T) \leqq U^-\left(\left[\frac{T}{\delta}\right], \frac{1}{\delta}K_T\right), \tag{5.14}$$

where we use the notation $U^-(n, b) = \min_{0 \leqq i \leqq [n-b]} (M(\delta(i+b)) - M(\delta i))$.

Let $a \geqq 1$ be fixed, and consider the sequence $m_j = [a^j]$, $j = 1, 2, \ldots$ Let $C > 0$ be a constant, and set $\delta = \delta_j = C j^{-1}(\log j) K_{m_j}$ for $m_j \leqq T < m_{j+1}$, $j = 1, 2, \ldots$ By Fact 4, we have the inequality

$$\Delta_T^- \geqq \min_{1 \leqq i \leqq M_j} V_{ij} =: D_j \qquad \text{for} \quad m_j \leqq T < m_{j+1}, \tag{5.15}$$

where $V_{ij} = M(\delta_j(i-2) + K_{m_j}) - M(\delta_j(i-1))$ for $1 \leqq i \leqq M_j$, and $M_j = \left[\dfrac{1}{\delta_j} m_{j+1}\right] + 1$. Here, we have used the assumption that $K_T \uparrow$ which implies that $K_{m_j} \geqq K_T$ for $T \geqq m_j$. Next, by the same arguments as used in the proof of Lemma 4.8, we see that, for any fixed r,

$$
\begin{aligned}
P_j(r) := P\left(D_j \leqq (K_{m_j} - \delta_j)\, a_1^- \left(\frac{1}{K_{m_j} - \delta_j} \log(m_j/K_{m_j})\right) + y_j\right)\Bigg) \\
= (1 + o(1))\, a m_j \delta_j^{-1} \pi^{-1/2} \\
\times (\log m_j)^{-1/2} \exp((1 + o(1))\, K_{m_j}^{-1} (\log m_j)^2 (2\sigma^2 y_j)) \\
= j^{r + (1/2) + o(1)} \quad \text{as } j \to \infty, \quad (5.16)
\end{aligned}
$$

where $y_j = \dfrac{r}{2\sigma^2} K_{m_j}^2 \dfrac{\log\log m_j}{\log^2 m_j}$. Consider now the event (for a fixed s)

$$
E_T(s) = \left\{ \Delta_T^- \leqq K_T a_1^- (K_T^{-1} \log(T/K_T)) + \frac{s}{2\sigma^2} K_T^2 \frac{\log\log T}{\log^2 T} \right\}.
$$

Observe by Fact 3 that, uniformly over $m_j \leqq T < m_{j+1}$, we have as $j \to \infty$

$$
\begin{aligned}
H(K_T, T/K_T) - H(K_{m_j}, m_j/K_{m_j}) \\
= O\left(m_j K'_{m_j} \frac{1}{\log m_j} K_{m_j}\right) + O((K_{m_j}/\log m_j)^2) \\
+ O(m_j K'_{m_j}/\log^2 m_j) = O((K_{m_j}/\log m_j)^2). \quad (5.17)
\end{aligned}
$$

Here, we have used the assumption that $K'_T/K_T = O(1/(T \log T))$ as $T \to \infty$.

Likewise, we have uniformly over $m_j \leqq T < m_{j+1}$ and for all j sufficiently large,

$$
\left| K_T^2 \frac{\log\log T}{\log T} - K_{m_j}^2 \frac{\log\log m_j}{\log m_j} \right| \leqq L K_{m_j}^2 \frac{\log\log m_j}{\log m_j} (a - 1), \quad (5.18)
$$

where $0 < L < \infty$ is a constant independent of $a > 1$. Hence, by (5.15)-(5.18), if the event $E_T(s)$ holds for an unbounded set of T's, then we have infinitely often in j

$$
\begin{aligned}
D_j \leqq K_{m_j} a_1^- (K_{m_j}^{-1} \log(m_j/K_{m_j})) \\
+ \frac{1}{2\sigma^2} (s + (1 + |s|)\, L(a - 1))\, K_{m_j}^2 \frac{\log\log m_j}{\log^2 m_j}. \quad (5.19)
\end{aligned}
$$

Another application of Fact 3 shows that, for all j sufficiently large,

$$
\begin{aligned}
|H(K_{m_j}, m_j/K_{m_j}) - H(K_{m_j} - \delta_j, m_j/(K_{m_j} - \delta_j))| \\
= O(\delta_j K_{m_j}/\log m_j) \leqq \frac{L'C}{2\sigma^2} K_{m_j}^2 \frac{\log\log m_j}{\log m_j}, \quad (5.20)
\end{aligned}
$$

where $0 < L' < \infty$ is a constant independent of $C > 0$. Thus, the probability that the event in (5.19) holds is by (5.16) for all j sufficiently large less than or equal to

$$
j^{s + (1 + |s|)\, L(a - 1) + CL' + (1/2) + \varepsilon}, \quad (5.21)
$$

where $\varepsilon > 0$ is another arbitrary constant.

Fix now $s < -3/2$ and observe that we can choose $C > 0$, $a - 1 > 0$ and $\varepsilon > 0$ so small that (5.21) is the general term of a convergent series. By Borel-Cantelli, it follows that

$$\liminf_{T \to \infty} (\Delta_T^- - K_T \beta_T^-) \Big/ \left(\frac{K_T^2}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \geqq -\frac{3}{2} \quad \text{a. s.}$$

Likewise, it follows from the preceding arguments that $P(E_T(s)) \to 0$ for all $s < -1/2$, so that

$$\limsup_{T \to \infty} (\Delta_T^- - K_T \beta_T^-) \Big/ \left(\frac{K_T^2}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \geqq -\frac{1}{2} \quad \text{a. s.} \qquad (5.23)$$

In a second step of our proof, we choose $\delta = 1$, $n = [T]$ and $b_n = K_T$ in Fact 4, so that (5.14) yields the inequality

$$\Delta_T^- = \Delta^-(T, K_T) \leqq U_n^- = U^-(n, b_n).$$

By Lemmas 4.11 and 4.12, it follows that, almost surely

$$\liminf_{T \to \infty} (\Delta_T^- - K_T A_n^-) \Big/ \left(\frac{K_T}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \leqq -\frac{1}{2}$$

and (5.24)

$$\limsup_{T \to \infty} (\Delta_T^- - K_T A_n^-) \Big/ \left(\frac{K_T^2}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \leqq \frac{1}{2},$$

where $A_n^- = a_1^- \left(\frac{\log [T]}{K_T} \right)$ and $n = [T]$. By Fact 3, we have evidently

$$K_T (A_n^- - \alpha_T^-) = H(K_T, [T]) - H(K_T, T)$$
$$= O(T^{-1} (K_T/\log T)^2) = o\left(K_T^2 \frac{\log\log T}{\log T} \right) \quad \text{as } T \to \infty. \qquad (5.25)$$

Hence, a direct consequence of (5.24) and (5.25) is that

$$\liminf_{T \to \infty} (\Delta_T^- - K_T \alpha_T^-) \Big/ \left(\frac{K_T^2}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \leqq -\frac{1}{2} \quad \text{a. s.,} \qquad (5.26)$$

and

$$\limsup_{T \to \infty} (\Delta_T^- - K_T \alpha_T^-) \Big/ \left(\frac{K_T^2}{2\sigma^2} \frac{\log\log T}{\log^2 T} \right) \leqq \frac{1}{2} \quad \text{a. s.}$$

An application of (5.22), (5.23), (5.26) and (5.27) in combination with Facts 1 and 2 completes the proof of Theorem 5.2. □

Remark 5.1. — 1° Whenever $(\log K_T)/\log\log T \to 1$, we see that (5.9) coincides with (5.10)-(5.11). Thus, in this range, the evaluations given in Theorem 5.2 are sharp and show that the limiting strong behaviors of

$\Delta^-(T, K_T)$ and of $U^-([T], K_T)$ coincide (up to the order of approximation considered).

2° On the other hand, if $\limsup_{T \to \infty} (\log K_T)/\log\log T < 1$, the results of Theorem 5.2 and Proposition 4.3 hint that the limiting strong behaviors of $\Delta^-(T, K_T)$ and $U^-([T], K_T)$, may be distinct. This question is open at present.

3° By taking $K_T = 1$ in (3.16), we obtain that

$$\limsup_{t \to \infty} (M(t) - M([t]))/\log t = \frac{2\sigma^2}{\mu^2} \quad \text{a.s.} \tag{5.28}$$

This brings evidence that the distance between $M(t)$ and its discrete approximant $M([t])$ may be important.

By (5.28), used jointly with Lemma 3.1, we obtain the following strong approximation result.

LEMMA 5.1. — *It is possible to define the process* $\{M(t), t \geq 0\}$ *on a probability space which carries a Wiener process* $\{\hat{W}(t), t \geq 0\}$ *such that*

$$\limsup_{t \to \infty} \left| M(t) - \frac{t}{\mu} - \sigma\mu^{-3/2}\hat{W}(t) \right| / \log T < \infty \quad \text{a.s.} \tag{5.29}$$

Combining Lemma 5.1 and Proposition 3.1 results in the following theorem which describes the large increment behavior of $M(.)$.

THEOREM 5.3. — *Assume that* K_T *is a function such that* $K_T \uparrow$, $0 < K_T \leq T$, $K_T/T \downarrow 0$, $K_T/\log T \to \infty$, *and* $(\log(T/K_T))/\log\log T \to \infty$ *as* $T \to \infty$, *then*

$$\lim_{T \to \infty} \left(\Delta_T^{\pm} - \frac{K_T}{\mu} \right) / (\sigma\mu^{-3/2}(2K_T \log(T/K_T))^{1/2}) = \pm 1 \quad \text{a.s.} \tag{5.30}$$

6. COMMENTS

It is interesting to discuss the results of the preceding sections in terms of the Lévy representation of $M(t)$ (*see* e.g. Itô and McKean (1965), p. 31). We can prove the following proposition.

PROPOSITION 6.1. — $\{M(t), t \geq 0\}$ *has the representation*

$$M(t) = \int_0^\infty x\, P([0, t] \times dx), \tag{6.1}$$

where $P(dt \times dx)$ *denotes the Poisson process on* $[0, \infty]^2$ *with mean measure*

$$\frac{x^{-3/2}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\mu^2 x}{2\sigma^2} \right) dt\, dx.$$

Proof. – We have (*see* Itô and McKean (1965) p. 31) a mean measure equal to $dt\,p(dx)$, where

$$\psi_t(s) = \mathrm{E}(\exp(s\,\mathrm{M}(t))) = \exp\left(t\int_0^\infty (e^{sx}-1)\,p(dx)\right).$$

By (2.14),

$$\begin{aligned}\psi_1(s) &= \lim_{m\to\infty} \exp(m(\psi_{1/m}(s)-1))\\ &= \exp\left(\lim_{m\to\infty}\int_0^\infty (e^{sx}-1)\frac{x^{-3/2}}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(m\mu x-1)^2}{2\sigma^2 m^2 x}\right)dx\right)\\ &= \exp\left(\left(\int_0^\infty (e^{sx}-1)\frac{x^{-3/2}}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\mu^2 x}{2\sigma^2}\right)dx\right)\right),\end{aligned}$$

as sought. □

Note that if $h(t)$ is a positive function,

$$\int_a^\infty\left(\int_{h(t)}^\infty \frac{x^{-3/2}}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\mu^2 x}{2\sigma^2}\right)dx\right)dt$$

stands for the mean number of times that $\mathrm{M}(t)$ has a jump at t which exceeds $h(t)$ for $a \leqq t < \infty$. This mean is finite if and only if the corresponding number is finite almost surely.

Under the assumption that $h(t)\to\infty$, this occurs if and only if

$$\int_a^\infty h(t)^{-3/2}\exp\left(-\frac{\mu^2}{2\sigma^2}h(t)\right)dt < \infty. \tag{6.2}$$

By taking in (6.2) $h(t) = (2\sigma^2/\mu^2)\log(\mathrm{T}/(\log \mathrm{T})^{(1/2)+\varepsilon})$, we obtain easily the following result.

PROPOSITION 6.2. – *Let* $\Delta^+_{\mathrm{T},0} = \lim_{h\downarrow 0}\Delta^+(\mathrm{T}, h)$. *Then*

$$\limsup_{\mathrm{T}\to\infty}\left(\frac{\Delta^+_{\mathrm{T},0} - (2\sigma^2/\mu^2)\log \mathrm{T}}{\log\log \mathrm{T}}\right) = -\frac{1}{2} \quad \text{a.s.} \tag{6.3}$$

It is noteworthy that (6.3) is in agreement with (5.4) in the case where $\mathrm{K_T}$ is constant (note that Theorem 5.1 assumes that $\mathrm{K_T}>0$ and $\mathrm{K_T}\uparrow$). Interestingly, (6.3) shows that (5.4) becomes invalid in general for those sequences $\mathrm{K_T}$ which tend to zero as $\mathrm{T}\to\infty$.

Aside of the specific interest of the inverse Gaussian process $\mathrm{M}(t)$, due to its intimate relationship with the Wiener process, our results have the interest of providing a rather complete description of the increments of this process in all possible ranges. The most interesting case at present corresponds to when $\mathrm{K_T}/\log \mathrm{T}\to 0$, and our theorems give explicit expansions in this situation. These should provide useful guide-lines for the study of increments of partial sums of independent random variables with arbitrary distributions. This is a virtually open problem [*see* e. g. Mason (1989)] for such small increments.

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