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Time-changes of self-similar Markov processes

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ABSTRACT. – Let X_t be a β-self-similar, $\beta>0$, transient Markov process on $(0, \infty)$. We show that if X_{T_t} (T_t is the right continuous inverse of a continuous additive functional A_t) is an α-self-similar Markov process, $\alpha>0$, then

$$A_t=k\int_0^t X_h^{1/\alpha-1/\beta}\,dh \quad \text{for some } k>0.$$

A result concerning time-changes of a transient Lévy process is also given.

Key words : Self-similar, Markov process, time-change, Lévy process.

RÉSUMÉ. – Soit X_t un processus β-self-similaire transient et de Markov sur $(0, \infty)$, $\beta>0$. Notons T_t l'inverse continu à droite du fonctionnelle additive A_t. Nous montrons que si X_{T_t} est un processus α-self-similaire et de Markov, $\alpha>0$, alors

$$A_t=k\int_0^t X_h^{1/\alpha-1/\beta}\,dh \quad \text{pour quelque } k>0.$$

Un résultat concernant le changement de temps d'un processus de Lévy transient est également donné.

0. INTRODUCTION

α-self-similar Markov processes (a-ssmp) on $(0, \infty)$ were introduced by J. Lamperti [5]. The process (X_t, P^x) with a state space $(0, \infty)$ is called α-ssmp, $\alpha>0$, if there exists a Borel semigroup $(P_t(\ ,\ ))_{t \geqq 0}$ on $(0, \infty) \times \mathscr{B}(0, \infty)$ satisfying

(i) $P_0(\ ,\ )=I$;

(ii) $P_t(x, A)=P_{at}(a^\alpha x, a^\alpha A)$ for all $t>0$, $a>0$, $x \in (0, \infty)$, $A \in \mathscr{B}(0, \infty)$, such that (X_t, P^x) is a time homogeneous Markov process with a transition function $(P_t(\ ,\ ))_{t \geqq 0}$ and with sample paths that are P^x-almost surely right-continuous with left limits for all $x \in (0, \infty)$.

It was proved in [6] that every α-ssmp with "nice paths" (*see* Notation) on $(0, \infty)$ has a weak dual with respect to the measure $x^{1/\alpha - 1}\, dx$. In this note we apply this result and characterize, the theory developed in [3] as a main tool, all the possible ways to time-change a transient self-similar process (in fact, an α-ssmp is transient iff it is cotransient; *see* Proposition) to another self-similar process. We also obtain a result concerning time-changes of a Lévy process. For simplicity, we assume $\alpha>0$, but the results can easily be generalized to negative α.

1. – NOTATION. DEFINITIONS

Ω notes the space of all functions ω from $[0, \infty) \rightarrow (0, \infty) \cup \{\Delta\}$ (Δ denotes the point used as a "graveyard"; we assume Δ is an isolated point), which satisfy

(*a*) $\omega(t)=\Delta$ for $t \geqq \zeta(\omega)=\inf\{t \geqq 0; \omega(t)=\Delta\}$;

(*b*) ω is right continuous and ω or $1/\omega$ has left limits on $[0, \infty)$ at every $t \in (0, \zeta(\omega)]$.

Such Ω is called the space of "nice paths".

DEFINITION. – Let $\alpha>0$ be given. A stochastic process (X_t, P^x) with a state space $(0, \infty) \cup \{\Delta\}$ is called α-ssmp on $(0, \infty)$ if the following is satisfied: there exists a Borel semigroup $(P_t(\ ,\ ))_{t \geqq 0}$ on $(0, \infty) \times \mathscr{B}(0, \infty)$ with the properties:

(i) $P_0(\ ,\ )=I$;

(ii) $P_t(x, A)=P_{at}(a^\alpha x, a^\alpha A)$ for all $t \geqq 0$, $a>0$, $x \in (0, \infty)$, $A \in \mathscr{B}(0, \infty)$, such that (X_t, P^x) is a time homogeneous Markov process with a transition function $(P_t(\ ,\ ))_{t \geqq 0}$ and $t \rightarrow X_t \in \Omega\ P^x$-a. s. for $x \in (0, \infty)$.

Remark 1. – It was proved in [4] that every α-ssmp on $(0, \infty)$ automatically is strongly Markov.

A Markov process $(X_t, P^x, x \in (0, \infty))$ is said to be *in weak duality* with a Markov process $(\hat{X}_t, \hat{P}^x, x \in (0, \infty))$ with respect to a σ-finite measure

η, if for all bounded f, g in $\mathscr{B}(0, \infty)$

$$\int f(x) \mathrm{E}^x g(\mathrm{X}_t) \eta(dx) = \int \hat{\mathrm{E}}^x f(\hat{\mathrm{X}}_t) g(x) \eta(dx), \quad \text{for all } t>0.$$

Let $(\mathrm{X}_t, \mathrm{P}^x)$ be in weak duality with $(\hat{\mathrm{X}}_t, \hat{\mathrm{P}}^x)$ with respect to a measure η. $(\mathrm{X}_t, \mathrm{P}^x)$ is said to be *transient*, if

$$\mathrm{U}f(x) = \mathrm{E}^x\left\{\int_0^\infty f(\mathrm{X}_t)\, dt\right\} < \infty$$

for all x, all bounded, non-negative Borel functions f on $(0, \infty)$ with compact support (*see* alternative definitions for transience in [2]). If the dual process $(\hat{\mathrm{X}}_t, \hat{\mathrm{P}}^x)$ is transient, then $(\mathrm{X}_t, \mathrm{P}^x)$ is said to be *cotransient*.

Remark 2. — It was shown in [6] that an α-ssmp on $(0, \infty)$ has a weak dual, with respect to the measure $x^{1/\alpha - 1}\, dx$, and the dual process is also an α-ssmp.

2. — THEOREMS

We assume throughout this paper that $(\mathrm{X}_t, \mathrm{P}^x)$ is transient (as we shall *see* in Proposition, for self-similar processes the transience is equivalent to the cotransience). According to [6], $(\mathrm{X}_{\mathrm{T}_t}, \mathrm{P}^x)$ is an α-ssmp if $(\mathrm{X}_t, \mathrm{P}^x)$ is β-ssmp and T_t is the right continuous inverse of an additive functional $k\int_0^t \mathrm{X}_h^{1/\alpha - 1/\beta}\, dh$. We shall show that this is the only possible way to time-change $(\mathrm{X}_t, \mathrm{P}^x)$ to an α-ssmp.

PROPOSITION. — *Let* $(\mathrm{X}_t, \mathrm{P}^x)$ *be a* β*-ssmp on* $(0, \infty)$, $\beta>0$. *Then it is transient iff it is cotrasient.*

Proof. — According to [4] (Th. 2.3) and [5] (Th. 4.1), there is one to one correspondence between a β-ssmp X_t on $(0, \infty)$ and a Lévy process Z_t on $(-\infty, +\infty)$ (that is, Z_t is a strong Markov process which have stationary independent increments and right continuous paths with left limits) defined by $\mathrm{Z}_t = \log \mathrm{X}_{\mathrm{T}_t}$, where T_t is the right continuous inverse of an additive functional

$$\int_0^t \mathrm{X}_h^{-1/\beta}\, dh.$$

It is also easily seen that X_t is transient iff Z_t is transient. Now for Z_t there exists a weak dual $\hat{\mathrm{Z}}_t$ with respect to the Lebesgue measure such that also $\hat{\mathrm{Z}}_t$ is a Lévy process. As shown in [6], starting from $\hat{\mathrm{Z}}_t$ one can construct a β-ssmp $\hat{\mathrm{X}}_t$, which is a weak dual to X_t with respect to the measure $x^{1/B-1}\,dx$. Now $\hat{X}_t$ is transient iff $\hat{Z}_t$ is transient and so it suffices to show that $\hat{Z}_t$ is transient iff it is cotransient. If (Z_t, Q^z) is a Lévy process then Z_t under Q^z has the same distribution as $z+Z_t$ under Q^0 and thus easy calculations show that $\hat{Z}_t$ under $\hat{Q}^z$ has the same distribution as $z-Z_t$ under Q^0. This shows that Z_t is transient iff $\hat{Z}_t$ is transient and gives thus the assertion.

In the proof of the following theorem actually cotransience (and not transience) is used.

THEOREM 2.1. — *Let* (X_t, P^x) *be a transient* β*-ssmp on* $(0, \infty)$, $\beta>0$, *and let* A_t *be a continuous additive functional of* X_t *with* T_t *as the right continuous inverse*, i. e.

$$T_t=\inf\{s\geqq 0;\ A_s>t\}.$$

If the process (X_{T_t}, P^x) *is* α*-ssmp, then there exists* $k>0$ *such that*

$$A_t(\omega)=k\int_0^t X_h^{1/\alpha-1/\beta}(\omega)\,dh,\quad \textit{for all}\ t<\zeta(\omega).$$

Proof. — As mentioned in Remark 2, (X_t, P^x) has a weak dual $(\hat{X}_t, \hat{P}^x)$ with respect to the measure $x^{1/\beta-1}\,dx$ such that also $(\hat{X}_t, \hat{P}^x)$ is β-ssmp. Let A_t be a continuous additive functional of X_t and let (X_{T_t}, P^x), T_t is the right continuous inverse of A_t, be an α-ssmp. Let further ν_A be the Revuz measure corresponding to A_t. According to the result of Getoor and Sharpe [3]

$$\int \nu_A(dx)f(x)\,\hat{U}(x, dy)=E^y\left\{\int_0^\infty f(X_t)\,dA_t\right\}y^{1/\beta-1}\,dy \tag{2.1}$$

for any bounded, non-negative Borel function f with compact support.

The right side of (2.1) is equal to $E^y\left\{\int_0^\infty f(X_{T_t})\,dt\right\}y^{1/\beta-1}\,dy$, which, because of the α-self-similarity of (X_{T_t}, P^x), is equal to

$$\begin{aligned}
&E^{a^\alpha y}\left\{\int_0^\infty f(a^{-\alpha}X_{T_{at}})\,dt\right\}y^{1/\beta-1}\,dy\\
&\qquad =a^{-1}\,E^{a^\alpha y}\left\{\int_0^\infty f(a^{-\alpha}X_{T_t})\,dt\right\}y^{1/\beta-1}\,dy\\
&\qquad\qquad =a^{-1}\,E^{a^\alpha y}\left\{\int_0^\infty f(a^{-\alpha}X_t)\,dA_t\right\}y^{1/\beta-1}\,dy \quad (2.2)
\end{aligned}$$

From (2.1) and (2.2) we obtain, by substituting $z=a^\alpha y$,

$$\int \nu_A(dx) f(x) \hat{U}(x, d(a^{-\alpha} z))$$
$$= a^{-\alpha/\beta - 1} E^z \left\{ \int_0^\infty f(a^{-\alpha} X_t)\, dA_t \right\} z^{1/\beta - 1}\, dz$$
$$= a^{-\alpha/\beta - 1} \int \nu_A(d(a^\alpha x)) f(x) \hat{U}(a^\alpha x, dz) \quad (2.3)$$

The β-self-similarity of $(\hat{X}_t, \hat{P}^x)$ implies

$$\hat{U}(x, d(a^{-\alpha} z)) = \int_0^\infty \hat{P}^x \{ \hat{X}_t \in d(a^{-\alpha} z) \}\, dt$$
$$= \int_0^\infty \hat{P}^{a^\alpha x} \{ \hat{X}_{a^{\alpha/\beta} t} \in dz \}\, dt = a^{-\alpha/\beta} \hat{U}(a^\alpha x, dx)$$

This, together with (2.3), gives

$$\nu_A(dx)\, \hat{U}(a^\alpha x, dz) = a^{-1} \nu_A(d(a^\alpha x))\, \hat{U}(a^\alpha x, dz) \quad (2.4)$$

Because X_t is cotransient we have $\hat{U} f(x) < +\infty$, $\forall x$. Thus

$$\nu_A(d(ax)) = a^{1/\alpha} \nu_A(dx), \qquad \forall a > 0. \quad (2.5)$$

Applying the well-known uniqueness result for a Haar measure we obtain

$$\nu_A(dx) = k x^{1/\alpha - 1}\, dx = k (x^{1/\alpha - 1/\beta})\, x^{1/\beta - 1}\, dx, \quad \text{for some } k > 0,$$

which gives

$$A_t = k \int_0^t X_h^{1/\alpha - 1/\beta}\, dh \text{ for some } k > 0.$$

Remark. – The special case of Theorem 1 is $\alpha = \beta$, which gives $A_t = kt$. This means that the only possible way to time-change a transient β-ssmp to another β-ssmp is a linear time-change.

In [5] J. Lamperti introduced a continuous additive functional

$$A_t(\omega) = \int_0^t X_h^{-1/\beta}(\omega)\, dh,$$

where (X_t, P^x) is a β-ssmp on $(0, \infty)$. He showed that if T_t is the right continuous inverse of A_t, then the time-changed process (X_{T_t}, P^x) is a strong Markov process on $(0, \infty)$ such that it is *multiplicatively invariant*, *i.e.*

$$Q_t(x, A) = Q_t(ax, aA), \quad \text{for all } t > 0,\ a > 0,\ X \in (0, \infty),\ A \in \mathscr{B}(0, \infty),$$

where $Q_t(\ ,\ )$ is a transition function for (X_{T_t}, P^x).

The following theorem says that this, possibly multiplied by a constant, is the only way to time-change (X_t, P^x) to a multiplicatively invariant process:

THEOREM 2.2. — *Let* (X_t, P^x) *be a transient* β*-ssmp on* $(0, \infty)$ *and let* A_t *be a continuous additive functional of* (X_t, P^x) *and* T_t *the right continuous inverse of* A_t. *If* (X_{T_t}, P^x) *is multiplicatively invariant, then there exists* $k>0$ *such that*

$$A_t = k \int_0^t X_t^{-1/\beta}\, dh \quad \textit{for all } t<\xi.$$

The proof is similar to that of Theorem 2.1 and will therefore be omitted.

Finally, we shall present a result concerning time-changes of a Lévy process. It was already remarked by Lamperti [5] that (Y_t) is a multiplicatively invariant strong Markov process with "nice paths" (*see* Notation) on $(0, \infty)$ iff $(Z_t) = (\log Y_t)$ is a Lévy process on $(-\infty, -\infty)$. We will show

THEOREM 2.3. — *The only possible way to time-change a transient Lévy process on* $(-\infty, +\infty)$ *to another Lévy process is a linear time-change* $t \to kt$, $k>0$.

We need the following Lemma:

LEMMA. — *Let* (Y_t, P^x) *be a transient, multiplicatively invariant strong Markov process with "nice paths"* (see *Notation*) *on* $(0, \infty)$. *Then the only possible way to time-change* (Y_t, P^x) *to another process of the same type is a linear time-change* $t \to kt$, $k>0$.

Proof. — What we have to show is, that if A_t is a continuous additive functional of (Y_t) with T_t as the right continuous inverse and (Y_{T_t}, P^x) is multiplicatively invariant, then there exists $k>0$ such that $A_t = kt$ for all $t<\xi$. According to [6], (Y_t, P^x) has a weak dual with respect to the measure $x^{-1}\,dx$. We can now show, by the same way as in Theorem 2.1, that A_t has a Revuz measure

$$\nu_A(dx) = kx^{-1}\,dx, \quad \text{for some } k>0,$$

which gives the assertion.

Proof of Theorem 2.3. — Let (Z_t, Q^z) be a transient Lévy process on $(-\infty, +\infty)$. Then, as remarked in the proof of Proposition, the weak dual $(\hat{Z}_t, \hat{Q}^z)$, which also is a Lévy process, is transient. Now $(\exp Z_t, Q^{\log x})$, $x>0$, is multiplicatively invariant and has, as shown in [6], a weak dual $(\exp \hat{Z}_t, \hat{Q}^{\log x})$ with respect to the measure $x^{-1}\,dx$. It is easily seen that (Z_t) is transient iff $(\exp Z_t)$ is transient and so, according to Lemma, $(\exp Z_t)$ cannot be time-changed to another multiplicatively invariant process with "nice paths" otherwise than by the linear time change $t \to kt$, $k>0$. Thus one to one correspondence between the class of Lévy processes and the class of multiplicatively invariant processes with "nice paths" gives the assertion.

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