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Capacity and energy for multiparameter Markov processes

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ABSTRACT. — We obtain upper and lower inequalities relating capacity and energy for multiparameter processes whose components are independent (one parameter) Markov processes. We apply our results to the resolution of a conjecture of Hendricks and Taylor on multiple points of Lévy processes.

Key words : Capacity, energy, additive functionals, Kuznetsov measures, multiparameter processes, Lévy processes, path intersections.

RÉSUMÉ. — Nous obtenons des inégalités supérieures et inférieures entre la capacité et l'énergie des processus à deux indices multiples, dont les composantes sont des processus de Markov indépendants (à un seul

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indice). Ces résultats servent à résoudre une conjecture de Hendricks et Taylor, concernant les points multiples des processus de Lévy.

1. INTRODUCTION

Part of the usefulness of probabilistic potential theory is that it allows one to compute probabilistic quantities in analytic ways. We are interested in generalizing a classical computation relating capacity and energy. Recall that for Brownian motion in $\mathbb{R}^d$, $d \geqq 3$, one defines the capacity $\Gamma(B)$ of a set B to be the probability of hitting B, "starting from infinity." More precisely, if m denotes Lebesgue measure, and if $U(x, dy) = u(x, y)\, m\,(dy)$ is the Brownian potential kernel, then one chooses measures ν_k such that $\nu_k U \uparrow m$, and one sets

$$\Gamma(B) = \lim_{k \to \infty} P^{\nu_k}(\text{hit}\, B). \tag{1.1}$$

This probabilistic quantity can be computed analytically as follows: set

$$e(\mu) = \iint u(x, y)\, \mu(dx)\, \mu(dy),$$

$$I(B) = \inf \{ e(\mu);\ \mu \text{ is a probability measure on } B \}.$$

Then

$$\Gamma(B) = 1/I(B). \tag{1.2}$$

One obstruction to the study of multiparameter Markov processes is that such exact computations seem to be impossible. Our object in this paper is to show that for a large class of multiparameter processes the functionals $\Gamma(.)$ and $1/I(.)$ are equivalent; that is, Γ lies within constant multiples of $1/I(.)$. Such a relation is good enough to determine whether a given set B is polar. We use this result to verify a conjecture of Hendricks and Taylor concerning multiple points of Lévy processes. There are several previous results along these lines (Dynkin [D81], Evans [E87 a]), to the effect that $\Gamma(B) > 0$ if $I(B) < \infty$.

More precisely, for a class of multiparameter processes (including those of the form

$$\mathbf{X}_\mathbf{t} = (X^1_{t^1}, X^2_{t^2}, \ldots, X^n_{t^n}), \qquad \mathbf{t} = (t^1, \ldots, t^n), \tag{1.3}$$

where the X^i are independent Borel right processes), we exhibit a pair of elementary inequalities relating "capacity" to the energy of additive functionals (more properly, homogeneous random measures). The capacity used here is defined in terms of Kuznetsov measures, and our arguments are adaptations of ones used for 1-parameter processes by K. M. Rao, P.-A. Meyer, and J. Azéma. For processes **X** of the form (1.3) the "Kuznetsov capacity" can be identified with a more classical capacity analogous to (1.1). Under additional hypotheses on the X^i we can identify the additive functional energy with the analogue of $e(\mu)$, μ being the Revuz measure of the additive functional in question. We thereby obtain the inequalities

$$2^{-n}/I(B) \leqq \Gamma(B) \leqq 2^{3n}/I(B). \tag{1.4}$$

We make use of a construction of E. B. Dynkin [D81], which associates an additive functional with each measure of finite energy. (As noted above, Dynkin's construction directly yields a weak form of the left hand inequality in (1.4).) For one-parameter results on capacity and energy, without symmetry assumptions, *see* [H79], [P-SR85], or the survey in [P-S87].

Our principal results are Theorem (2.12) (Kuznetsov capacity vs. additive functional energy), Corollary (3.5) (classical capacity vs. additive functional energy), and Theorem (4.16) (classical capacity vs. classical energy).

It is possible to give an analytic expression for additive functional energies under broader conditions than those imposed here. This gain in generality comes at considerable technical cost and requires a deeper development of multiparameter potential theory. Also, our arguments apply to other processes; for example, the Ornstein-Uhlenbeck sheet. We hope to address these issues in subsequent papers.

We would like to thank Ed Perkins for suggesting several applications of our methods.

2. THE BASIC INEQUALITIES

In this paper we are chiefly interested in processes of the form (1.3), but in this section we develop capacity/energy inequalities in a wider setting, with an eye to future applications.

We begin by describing a canonical setting for stationary multiparameter processes with random birth and death times. Let E be a Lusin metrizable space with Borel sets $\mathscr{E}$. Let W be the space of paths $\mathbf{w}: \mathbb{R}^n \to E \cup \{\Delta\}$ which are E-valued and right continuous on some nonempty open rectangle $]\boldsymbol{\alpha}(\mathbf{w}), \boldsymbol{\beta}(\mathbf{w})[$, taking the value Δ off this rectangle. Here $n \geqq 1$ is fixed throughout the paper, and $\Delta \notin E$ serves as cemetery. We let $]\mathbf{s}, \mathbf{t}[$ denote the set of $\mathbf{r} \in \mathbb{R}^n$ such that $s^i < r^i < t^i$, $\forall i$, with a similar convention for $\mathbf{s} < \mathbf{t}$.

The symbols $\mathbf{0},\mathbf{1},\boldsymbol{\infty}$ denote the vectors whose components are all $0,1,\infty$ respectively. Let $(\mathbf{Y}_\mathbf{t};\ \mathbf{t}\in\mathbb{R}^n)$ be the coordinate process on W and put $\mathscr{G}^0=\sigma\{\mathbf{Y}_\mathbf{t};\ \mathbf{t}\in\mathbb{R}^n\}$. Define a family of shift operators on W by

$$(\sigma_\mathbf{t}\mathbf{w})(\mathbf{s})=\mathbf{w}(\mathbf{t}+\mathbf{s}),\qquad \mathbf{s},\ \mathbf{t}\in\mathbb{R}^n.$$

Throughout this section we fix a σ-finite measure Q on $(W, \mathscr{G}^0)$ that is *stationary:* $\sigma_\mathbf{t}(Q)=Q$, $\forall\,\mathbf{t}\in\mathbb{R}^n$. Let $\mathscr{G}$ denote the Q-completion of $\mathscr{G}^0$, and $\mathscr{N}$ the class of Q-null sets in $\mathscr{G}$.

For an arbitrary σ-field $\mathscr{F}$, we will denote the set of positive (resp. bounded, resp. bounded and positive) $\mathscr{F}$-measurable random variables by $p\mathscr{F}$ (resp. $b\mathscr{F}$, resp. $bp\mathscr{F}$).

A $\mathscr{G}$-measurable random time $\mathbf{S}$: $W\to\bar{\mathbb{R}}^n=[-\infty,\infty]^n$ is *homogeneous* provided $\mathbf{t}+\mathbf{S}\circ\sigma_\mathbf{t}(\mathbf{w})=\mathbf{S}(\mathbf{w})$, $\forall\,\mathbf{t}\in\mathbb{R}^n$, $\forall\,\mathbf{w}\in W$. We assume that Q is *dissipative* in the sense of [Fi88 a]: there is a homogeneous time $\mathbf{S}$ such that $Q(\mathbf{S}\notin\mathbb{R}^n)=0$. Let $\mathscr{A}^0$ denote the class of $(\sigma_\mathbf{t})$-invariant events in $\mathscr{G}$, and put $\mathscr{A}=\mathscr{A}^0\vee\mathscr{N}$. As in [Fi88 a], if $\mathbf{S}$ and $\mathbf{T}$ are homogeneous times, $A\in\mathscr{A}$, and $B\in\mathscr{R}^n$ (the Borel sets in $\mathbb{R}^n$), then

$$Q(\mathbf{S}\in B,\ \mathbf{T}\in\mathbb{R}^n;\ A)=Q(\mathbf{S}\in\mathbb{R}^n,\ \mathbf{T}\in B;\ A). \tag{2.1}$$

This allows us to define a measure $\mathbb{P}$ on $(W, \mathscr{A})$ by

$$\mathbb{P}(A)=Q(A;\ \mathbf{S}\in]\mathbf{0},\mathbf{1}]),\qquad A\in\mathscr{A}, \tag{2.2}$$

where $\mathbf{S}$ is any homogeneous time such that $Q(\mathbf{S}\notin\mathbb{R}^n)=0$. Formula (2.1) makes it clear that the R.H.S. of (2.2) doesn't depend on the choice of $\mathbf{S}$. One can invert (2.2) as follows. Given $F\in p\mathscr{G}^0$ define $\bar F=\int_{\mathbb{R}^n}F\circ\sigma_\mathbf{t}\,d\mathbf{t}\in p\mathscr{A}$; then as in [Fi88 a],

$$Q(F)=\mathbb{P}(\bar F),\qquad F\in p\mathscr{G}^0. \tag{2.3}$$

In particular, $\mathbb{P}$ is σ-finite. Fubini's theorem and a completion argument show that if $F\in p\mathscr{G}$ then $\bar F$ is well-defined a. s. Q and (up to $\mathbb{P}$-null sets) determines a unique element of $\mathscr{A}$. Moreover, (2.3) remains valid for $F\in p\mathscr{G}$. This observation will be used frequently in the sequel without special mention. The following consequence of (2.3) will also be used later.

(2.4) LEMMA. — *There is a sequence* $(\mathbf{S}_k)$ *of homogeneous times such that, a. s.* Q, $\boldsymbol{\alpha}<\mathbf{S}_k<\boldsymbol{\beta}$ *on* $\{\mathbf{S}_k\in\mathbb{R}^n\}$, $\forall\,k\geqq 1$, *and* $\mathbf{S}_k\downarrow\boldsymbol{\alpha}$.

Proof. — Q being σ-finite we can choose $F\in bp\mathscr{G}^0$ with $\{F>0\}=\{\boldsymbol{\alpha}<\mathbf{0}<\boldsymbol{\beta}\}$ and $Q(F)<\infty$. By (2.3), $\bar F<\infty$ a. s. $\mathbb{P}$. But from (2.2) we deduce that $\mathbb{P}|_{\mathscr{A}}$ has the same null sets as $Q|_{\mathscr{A}}$, hence $\bar F<\infty$ a. s. Q. Thus

$$f_t=\int_{]-\infty,\,t]\times\mathbb{R}^{n-1}}F\circ\sigma_\mathbf{s}\,d\mathbf{s}\downarrow 0\quad\text{as } t\downarrow\alpha^1,\quad \text{a. s. Q},$$

since $\{s; F(\sigma_s w)>0\}=]\alpha(w), \beta(w)[$. Put $S_k^1=\inf\{t; f_t>k^{-1}\}$, and define S_k^i, $i=2, \ldots, n$ analogously. The homogeneous times $S_k=(S_k^1, S_k^2, \ldots, S_k^n)$, $k\geqq 1$, fit the bill. □

Motivated by Getoor and Steffens [GSt87], we define a capacity Γ by

$$\Gamma(B)=\mathbb{P}(\{Y \text{ hits } B\}), \qquad B\in\mathscr{E}.$$

Here $\{Y \text{ hits } B\}=\{Y_t\in B \text{ for some } t\in]\alpha, \beta[\}$. Set

$$I_{raw}(B)=\inf\{2^{-n}\,\mathbb{P}(Z^2);\ Z\in p\mathscr{A},\ \mathbb{P}(Z)=1,\ Z=0 \text{ off } \{Y \text{ hits } B\}\},$$

observing the usual convention, $\inf\varnothing=\infty$.

(2.5) LEMMA. — $\Gamma(B)=2^{-n}/I_{raw}(B)$, $\forall\, B\in\mathscr{E}$.

Proof. — The stated equality is trivial if $\Gamma(B)=0$. If $0<\Gamma(B)<\infty$ put $H=1_{\{Y \text{ hits } B\}}\in p\mathscr{A}$ and note that

$$2^n I_{raw}(B)\leqq \mathbb{P}([H/\Gamma(B)]^2)=1/\Gamma(B). \tag{2.6}$$

If $\Gamma(B)=\infty$ choose indicators $H_k\uparrow H$ with $0<\mathbb{P}(H_k)<\infty$, replace $H/\Gamma(B)$ by $H_k/\mathbb{P}(H_k)$ in (2.6) and let $k\to\infty$ to reach the same conclusion.

The reverse inequality was suggested by an argument of Rao [Ra87]. Fix $Z\in p\mathscr{A}$ with $\mathbb{P}(Z)=1$ and $Z=0$ off $\{Y \text{ hits } B\}$. Using the Cauchy-Schwarz inequality,

$$\mathbb{P}(Z^2)=\mathbb{P}(Z^2, Y \text{ hits } B)\geqq \frac{\mathbb{P}(Z, Y \text{ hits } B)^2}{\mathbb{P}(Y \text{ hits } B)}=\frac{1}{\Gamma(B)},$$

hence $I_{raw}(B)\geqq 2^{-n}/\Gamma(B)$. □

Relation (2.5) yields useful inequalities for a modification of I_{raw} which we now describe. Define σ-fields

$$\mathscr{G}_t^{i,0}=\sigma\{Y_s;\ s^i\leqq t^i\}, \qquad \hat{\mathscr{G}}_t^{i,0}=\sigma\{Y_s;\ s^i\geqq t^i\},$$
$$\mathscr{G}_t^i=\bigcap_{s>t}(\mathscr{G}_s^{i,0}\vee\mathscr{N}), \qquad \hat{\mathscr{G}}_t^i=\bigcap_{s<t}(\hat{\mathscr{G}}_s^{i,0}\vee\mathscr{N}),$$

so that $(\mathscr{G}_t^i)$ is a filtration when $\mathbb{R}^n$ carries its natural partial order, and $(\hat{\mathscr{G}}_t^i)$ is a "reverse filtration" in the obvious sense. Let $\mathscr{I}$ denote the class of Q-evanescent subsets of $W\times\mathbb{R}^n$, and put $\mathscr{M}=(\mathscr{G}\otimes\mathscr{R}^n)\vee\mathscr{I}$, $\mathscr{M}^0=\mathscr{G}^0\otimes\mathscr{R}^n$. Associated with $(\mathscr{G}_t^i)$ [resp. $(\hat{\mathscr{G}}_t^i)$] is the i-optional (resp. i-copredictable) σ-field $\mathscr{O}^i$ (resp. $\hat{\mathscr{P}}^i$) on $W\times\mathbb{R}$. For instance, $\mathscr{O}^1$ (resp. $\hat{\mathscr{P}}^1$) is spanned by $\mathscr{I}$ and the class of processes of the form $V_t(w)=A_{t^1}(w)\,B(t^2, \ldots, t^n)$, where $A_{t^1}\in bp\mathscr{G}_t^1$ (resp. $bp\hat{\mathscr{G}}_t^1$), $B\in bp\mathscr{R}^{n-1}$, and $t^1\mapsto A_{t^1}(w)$ is right continuous on $\mathbb{R}$, $\forall\, w\in W$. For example, if $f\in p\mathscr{E}$ then $f\circ Y\in\mathscr{O}^i$ for all i. (As usual, $f(\Delta)=0$.) However $f\circ Y$ need not i-copredictable; *see* (2.8) below concerning these matters.

A *random measure* (RM) is a kernel $\kappa=\kappa(w, B)$ from $(W, \mathscr{G})$ to $(\mathbb{R}^n, \mathscr{R}^n)$. An RM κ is i-optional (resp. i-copredictable) if there is a strictly positive

V in $\mathcal{O}^i$ (resp. $\hat{\mathcal{P}}^i$) such that the process

$$Z_t = \int 1_{]-\infty, t^i]}(s^i)\, V_s\, \kappa(ds) \left(\text{resp. } \int 1_{[t^i, \infty[}(s^i)\, V_s\, \kappa(ds) \right)$$

is finite a.s. Q and in $\mathcal{O}^i$ (resp. $\hat{\mathcal{P}}^i$). If κ is an RM and $Q\left(\int V_t\, \kappa(dt) \right) < \infty$ for some strictly positive $V \in \mathcal{O}^i$ then we can define the i-optional dual projection of κ: this is the unique i-optional RM $o^i \kappa$ such that

$$Q\left(\int V_s\, o^i \kappa(ds) \right) = Q\left(\int V_s\, \kappa(ds) \right), \qquad \forall V \in \mathcal{O}^i.$$

The i-optional projection of a process $V \in p\mathcal{M}$ is then defined as the unique process $o^i V \in \mathcal{O}^i$ such that

$$Q\left(\int o^i V_s\, \kappa(ds) \right) = Q\left(\int V_s\, o^i \kappa(ds) \right), \qquad \forall V \in \mathcal{O}^i$$

for every RM κ such that $Q(\kappa(\mathbb{R}^n))$ is finite. The i-copredictable projection (and dual projection) operator $\hat{p}^i$ is defined similarly; it produces predictable (dual) projections with respect to the reverse filtration $(\hat{\mathcal{G}}^i_t)$. If $V \in p\mathcal{M}$ and $Z \in p\mathcal{O}^i$ (resp. $p\hat{\mathcal{P}}^i$) then $o^i(ZV) = Z . o^i V$ (resp. $Z . \hat{p}^i V$) up to Q-evanescence, and the dual results are equally valid. Implicit in the above discussion are the section theorems: if V is a process in $p\mathcal{M}$ (resp. $p\mathcal{O}^i$, resp. $p\hat{\mathcal{P}}^i$) such that $Q\left(\int V_t\, \kappa(dt) \right) = 0$ for all RM's (resp. i-optional RM's, resp. i-copredictable RM's) κ with $Q(\kappa(\mathbb{R}^n)) < \infty$, then $V \in \mathcal{I}$. The details of the construction of the operators o^i and $\hat{p}^i$ are the same as those of the optional and predictable projections found in [B81], [MZ80] and especially [M78]. (Trivial modifications are needed to cope with $\mathbb{R}^n$ as parameter set instead of $\mathbb{R}^n_+$, while the σ-finiteness of Q can be handled by using the techniques of [D77].) Our operators differ from those of [D77] and [Fi87] in that these sources impose additional structure on the behavior of processes and RM's off the set $]\boldsymbol{\alpha}, \boldsymbol{\beta}[$.

We will assume the following regularity hypothesis (and in paragraph 4 will impose conditions that guarantee that it holds)

$$\hat{p}^i (1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[}) = 1 \quad \text{on }]\boldsymbol{\alpha}, \boldsymbol{\beta}[, \quad \forall i. \tag{2.7}$$

(The reader will note that we are suppressing "Q-a.s." statements when no confusion can arise.) Note that $1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[} \in \mathcal{O}^i$ for all i, so (2.7) always holds if $\hat{p}^i$ is replaced by o^i. The most important consequence of (2.7) is contained in the following

(2.8) LEMMA. — *If $f \in p\mathscr{E}$ then $f \circ \mathbf{Y} \in \mathcal{O}^i$ for all i. If, in addition,* (2.7) *holds then*

$$\hat{p}^i (f \circ \mathbf{Y}) = f \circ \mathbf{Y} \quad \text{on }]\boldsymbol{\alpha}, \boldsymbol{\beta}[, \quad \forall i.$$

Proof. — It is enough to prove this for $i=1$ and f bounded, positive, and continuous. To see that $f \circ \mathbf{Y} \in \mathcal{O}^1$ let $Z_\mathbf{t}^k = 1_{\{t^1 \geq S_k^1\}} f \circ \mathbf{Y}_\mathbf{t}$, where S_k^1 is as in the proof of Lemma (2.4), and define, for $l \in \mathbb{N}$ and $\mathbf{j} = (j^2, \ldots, j^n) \in \mathbb{Z}^{n-1}$,

$$B(l, \mathbf{j}) = \{(t^2, \ldots, t^n) \in \mathbb{R}^{n-1};\, j^i 2^{-l} \leqq t^i < (j^i + 1) 2^{-l},\, i = 2, \ldots, n\},$$
$$\mathbf{b}(l, \mathbf{j}) = ((j^2+1) 2^{-l}, \ldots, (j^n+1) 2^{-l}),$$
$$C(t^1, l, \mathbf{j}) = \{\mathbf{w} \in W;\, t^1 \geqq S_k^1(\mathbf{w}),\, \mathbf{b}(l, \mathbf{j}) > (\alpha^2(\mathbf{w}) + l^{-1}, \ldots, \alpha^n(\mathbf{w}) + l^{-1})\}.$$

Note that $C(t^1, l, \mathbf{j}) \in \mathscr{G}_\mathbf{t}^1$ since $S_k^1 > \alpha^1$. Clearly

$$Z_\mathbf{t}^k(\mathbf{w}) = \lim_{l \to \infty} \sum_{\mathbf{j} \in \mathbb{Z}^{n-1}} 1_{C(t^1, l, \mathbf{j})} f(\mathbf{Y}_{(t^1, \mathbf{b}(l, \mathbf{j}))}) 1_{B(l, \mathbf{j})}(t^2, \ldots, t^n),$$

and so $Z^k \in \mathcal{O}^1$. Since $Z_\mathbf{t}^k \uparrow f \circ \mathbf{Y}_\mathbf{t}$, $\forall \mathbf{t}$ a. s. Q, we have $f \circ \mathbf{Y} \in \mathcal{O}^1$.

To prove the second assertion note that by the section theorem it suffices to show that

$$Q\left(\int \hat{p}^1 (f \circ \mathbf{Y})\, d\kappa\right) = Q\left(\int f \circ \mathbf{Y}\, d\kappa\right)$$

for every RM κ carried by $]\boldsymbol{\alpha}, \boldsymbol{\beta}[$ and such that $Q(\kappa(\mathbb{R}^n)) < \infty$. Define

$$Z_\mathbf{t} = \liminf_{s^1 \to t^1,\, s^1 > t^1,\, s^1 \text{ rational}} f \circ \mathbf{Y}_{(s^1, t^2, \ldots, t^n)}.$$

Arguing as in the 1-optional case we see that $Z \in \hat{\mathscr{P}}^1$, and, since f is continuous, $1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[} Z = f \circ \mathbf{Y}$. Thus, using (2.7) for the first equality,

$$Q\left(\int f \circ \mathbf{Y}\, d\kappa\right) = Q\left(\int Z (\hat{p}^1 1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[})\, d\kappa\right) = Q\left(\int \hat{p}^1 (Z 1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[})\, d\kappa\right)$$
$$= Q\left(\int Z 1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[}\, d(\hat{p}^1 \kappa)\right) = Q\left(\int f \circ \mathbf{Y}\, d(\hat{p}^1 \kappa)\right)$$
$$= Q\left(\int \hat{p}^1 (f \circ \mathbf{Y})\, d\kappa\right),$$

and the lemma is proved. □

A *raw homogeneous random measure* (RHRM) is a kernel $\kappa = \kappa(\mathbf{w}, B)$ from $(W, \mathscr{G})$ to $(\mathbb{R}^n, \mathscr{R}^n)$ such that

(2.9) (i) $\kappa(\mathbf{w}, .)$ is carried by $]\boldsymbol{\alpha}(\mathbf{w}), \boldsymbol{\beta}(\mathbf{w})[$, for Q a.e. $\mathbf{w} \in W$;

(ii) there is an $f \in \mathscr{E}$, $f > 0$ with

$$Q\left(\int_{]0, 1]} f(\mathbf{Y}_\mathbf{t})\, \kappa(d\mathbf{t})\right) < \infty;$$

(iii) $\kappa(\sigma_\mathbf{t} \mathbf{w}, B) = \kappa(\mathbf{w}, B + \mathbf{t})$, $\forall B \in \mathscr{R}^n$, for Q a. e. $\mathbf{w}$, $\forall \mathbf{t} \in \mathbb{R}^n$.

In fact, our κ's will usually satisfy a condition stronger than (ii), namely $\mathbb{P}(\kappa(\mathbb{R}^n))<\infty$. We drop the pejorative "raw" and speak of a homogeneous random measure (HRM) if κ satisfies (2.9) (i)-(iii) and

(iv) $\quad o^i\kappa=\hat{p}^i\kappa=\kappa$, up to Q-indistinguishability, $\forall\, i=1, \ldots, n$.

Note that the existence of these projections is guaranteed by (ii) above. In the sequel π is a generic symbol representing any of the $2n$ projections o^i, $\hat{p}^i$, $i=1, \ldots, n$.

(2.10) LEMMA. – *Let* κ *be an* RHRM. *Then*

(*a*) $\kappa(\mathbb{R}^n)\in p\mathscr{A}$;

(*b*) $\pi\kappa$ *is an* RHRM *and* $\mathbb{P}(\kappa(\mathbb{R}^n))=\mathbb{P}(\pi\kappa(\mathbb{R}^n))$;

(*c*) $\mathbb{P}([\pi\kappa(\mathbb{R}^n)]^2)\leqq 4\,\mathbb{P}([\kappa(\mathbb{R}^n)]^2)$.

Proof. – Point (*a*) follows from a "perfection" result (A.5) found in the appendix, and guarantees that $\mathbb{P}(\kappa(\mathbb{R}^n))$ is well defined.

By virtue of (2.7), $\pi(1_{]\alpha,\beta[})=1$ on $]\alpha,\beta[$. Thus if $V\in p\,\mathcal{O}^i$ or $p\,\hat{\mathscr{P}}^i$ according as $\pi=o^i$ or $\hat{p}^i$,

$$Q\left(\int V_t\, 1_{]\alpha,\beta[}(t)\,\pi\kappa(dt)\right)=Q\left(\int V_t\,\pi(1_{]\alpha,\beta[})(t)\,\kappa(dt)\right)$$
$$=Q\left(\int V_t\,\kappa(dt)\right)=Q\left(\int V_t\,\pi\kappa(dt)\right),$$

since κ is carried by $]\alpha,\beta[$ a. s. Q. It follows that $\pi\kappa$ satisfies (2.9) (i). Next, using the invariance of Q and the fact that $\mathscr{R}^n$ is countably generated, we see that $\pi\kappa$ inherits property (2.9) (iii) from κ. Finally, note that if $f\in bp\mathscr{E}$, then $\pi(f\circ Y\,d\kappa)=f\circ Y\,d(\pi\kappa)$, because of (2.8). By (2.3) and the homogeneity of κ and $\pi\kappa$,

$$\mathbb{P}\left(\int f\circ Y_t\,\pi\kappa(dt)\right)=Q\left(\int 1_{]0,1]}(t)\,f\circ Y_t\,\pi\kappa(dt)\right)$$
$$=Q\left(\int 1_{]0,1]}(t)\,f\circ Y_t\,\kappa(dt)\right)=\mathbb{P}\left(\int f\circ Y_t\,\kappa(dt)\right).$$

This yields the second assertion in (*b*) (take $f=1_E$) as well as the fact that $\pi\kappa$ satisfies (2.9) (ii). Point (*b*) is therefore proved.

Point (*c*) is Meyer's energy inequality adapted to the present context (*see* [M78] or VI (95.3) of [DM80]). We adapt his argument and, since some care is required, give a complete proof. First note that it suffices to prove (*c*) under the auxiliary hypothesis $\mathbb{P}(\kappa(\mathbb{R}^n))<\infty$. For if f is as in (ii) of (2.9), then the RHRM $\kappa_k(dt)=(kf\wedge 1)(Y_t)\,\kappa(dt)$ satisfies $\mathbb{P}(\kappa_k(\mathbb{R}^n))<\infty$, $\pi\kappa_k(dt)=(fk\wedge 1)(Y_t)\,\pi\kappa(dt)$ [because of (2.8)]. Once (*c*) is proved for each κ_k we can let $k\uparrow\infty$ to obtain the desired result. Our proof requires the following lemma which will also prove useful later. It is an immediate consequence of (2.3).

(2.11) LEMMA. — *Let* κ *and* γ *be* RHRM's. *Given* $\varphi \in p(\mathscr{R}^n \otimes \mathscr{R}^n)$ *define* $g(\mathbf{b}) = \int_{\mathbb{R}^n} \varphi(\mathbf{a}, \mathbf{a}+\mathbf{b})\, d\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$. *Then*

$$\mathbb{P}\left(\iint g(\mathbf{t}-\mathbf{s})\,\gamma(d\mathbf{s})\,\kappa(d\mathbf{t})\right) = \mathrm{Q}\left(\iint \varphi(\mathbf{s}, \mathbf{t})\,\gamma(d\mathbf{s})\,\kappa(d\mathbf{t})\right).$$

Proceeding with the proof of (2.10) (*c*), let κ be an RHRM and set $\tilde{\kappa} = \pi\kappa$. We consider only the case $\pi = o^1$, the other cases being similar. Note that if $\mathbf{s}$ and $\mathbf{t}$ in $\mathbb{R}^n$ have the same first coordinate, then $\mathscr{G}^1_{\mathbf{s}} = \mathscr{G}^1_{\mathbf{t}}$. Thus the σ-field $\mathscr{F}(t^1) = \mathscr{G}^1_{\mathbf{t}}$, $\mathbf{t} = (t^1, t^1, \ldots, t^1)$, is well-defined and $(\mathscr{F}(t);\ t \in \mathbb{R})$ is a filtration. Put $\mathrm{H}(t) = \{\mathbf{s} \in \mathbb{R}^n;\ s^1 \leq t\}$. Evidently the increasing process $\mathrm{B}_t := \kappa(\mathrm{H}(t))$ has $(\mathscr{F}(.), \mathrm{Q})$-dual optional projection $\mathrm{A}_t = \tilde{\kappa}(\mathrm{H}(t))$. This implies that the processes $(\mathrm{B}_\infty - \mathrm{B}_t) + (\mathrm{B}_\infty - \mathrm{B}_{t-})$ and $(\mathrm{A}_\infty - \mathrm{A}_t) + (\mathrm{A}_\infty - \mathrm{A}_{t-})$ have the same $(\mathscr{F}(.), \mathrm{Q})$-optional projection. Using (2.11) with $\varphi(\mathbf{s}, \mathbf{t}) = 1_{\{s^1 < t^1\}}\psi(\mathbf{s}) + 1_{\{s^1 \geq t^1\}}\psi(\mathbf{t})$, where $\psi \in p\mathscr{R}^n$ with $\int_{\mathbb{R}^n} \psi(\mathbf{t})\, d\mathbf{t} = 1$, we compute

$$\begin{aligned}
\mathbb{P}(\tilde{\kappa}(\mathbb{R}^n)^2) &= \mathrm{Q}\left(\int_{\mathbb{R}^n} \psi(\mathbf{t})\,[(\mathrm{A}_\infty - \mathrm{A}_{t^1}) + (\mathrm{A}_\infty - \mathrm{A}_{t^1-})]\,\tilde{\kappa}(d\mathbf{t})\right) \\
&= \mathrm{Q}\left(\int_{\mathbb{R}^n} \psi(\mathbf{t})\,[(\mathrm{B}_\infty - \mathrm{B}_{t^1}) + (\mathrm{B}_\infty - \mathrm{B}_{t^1-})]\,\tilde{\kappa}(d\mathbf{t})\right) \\
&\leq 2\,\mathrm{Q}\left(\int_{\mathbb{R}^n} \psi(\mathbf{t})\,\kappa(\mathbb{R}^n)\,\tilde{\kappa}(d\mathbf{t})\right) \\
&= 2\,\mathbb{P}(\kappa(\mathbb{R}^n)\,\tilde{\kappa}(\mathbb{R}^n))
\end{aligned}$$

where we have used (2.3) for the last equality. Invoking the Cauchy-Schwarz inequality we obtain

$$\mathbb{P}(\tilde{\kappa}(\mathbb{R}^n)^2) \leq 2\,\mathbb{P}(\kappa(\mathbb{R}^n)^2)^{1/2}\,\mathbb{P}(\tilde{\kappa}(\mathbb{R}^n)^2)^{1/2}$$

whence the desired inequality [assuming $\mathbb{P}(\tilde{\kappa}(\mathbb{R}^n)^2) < \infty$]. If $\mathbb{P}(\tilde{\kappa}(\mathbb{R}^n)^2) = \infty$, replace A_t by $\mathrm{A}_t \wedge k$ in the above to obtain

$$\begin{aligned}
\mathbb{P}([\tilde{\kappa}(\mathbb{R}^n) \wedge k]^2) &\leq 2\,\mathbb{P}(\kappa(\mathbb{R}^n)\,[\tilde{\kappa}(\mathbb{R}^n) \wedge k]) \\
&\leq 2k\,\mathbb{P}(\kappa(\mathbb{R}^n)) < \infty.
\end{aligned}$$

Once more Cauchy-Schwarz yields

$$\mathbb{P}([\tilde{\kappa}(\mathbb{R}^n) \wedge k]^2) \leq 4\,\mathbb{P}(\kappa(\mathbb{R}^n)^2)$$

and the result obtains upon letting $k \uparrow \infty$. $\square$

We say that two projections π_1, π_2 *commute on* $]\alpha, \beta[$ if

$$1_{]\alpha,\beta[}\pi_1\pi_2 Z = 1_{]\alpha,\beta[}\pi_2\pi_1 Z, \qquad \forall Z \in p\mathcal{M}.$$

We say that Q is *π-Markov* if the $2n$ projections o^i, $\hat{p}^i$, $i=1, \ldots, n$ commute on $]\alpha, \beta[$.
In this case the dual projections commute when applied to RHRM's. We modify I_{raw} by defining

$$I_{HRM}(B) = \inf\{2^{-n}\mathbb{P}(\kappa(\mathbb{R}^n)^2);\ \kappa \text{ is an HRM carried by } \{t;\ Y_t \in B\}, \text{ with } \mathbb{P}(\kappa(\mathbb{R}^n)) = 1.\}$$

(2.12) THEOREM. — *Assume that Q is dissipative, π-Markov, and satisfies* (2.7). *Then*

$$2^{-n}/I_{HRM}(B) \leqq \Gamma(B) \leqq 2^{3n}/I_{HRM}(B), \qquad \forall B \in \mathcal{E}.$$

Proof. — The lower bound follows from Lemma (2.5) and part (*a*) of (2.10), and the upper bound is trivial if $\Gamma(B)=0$. To prove the upper bound in case $\Gamma(B)>0$, we construct a suitable HRM, adapting a device of Azéma [A72]. First we construct a homogeneous time $\mathbf{T}$ such that $\mathbf{T} \in]\alpha, \beta[$ on $\{\mathbf{T} \in \mathbb{R}^n\}$ and

$$\text{(2.13)} \qquad Q(\mathbf{Y_T} \notin B,\ \mathbf{T} \in \mathbb{R}^n) = 0;$$

$$\text{(2.14)} \qquad Q(\mathbf{Y} \text{ hits } B,\ \mathbf{T} \notin \mathbb{R}^n) = 0.$$

To this end recall the sequence $(\mathbf{S}_k)$ of Lemma (2.4). By a standard selection argument there are times

$$\mathbf{R}_k \geqq \mathbf{0}, \qquad \mathbf{R}_k\big|_{\{\alpha<\mathbf{S}_k<\beta\}} \in [\sigma\{\mathbf{Y}_{\mathbf{S}_k+\mathbf{t}};\ \mathbf{t} \geqq \mathbf{0}\} \vee \mathcal{N}] \cap \{\alpha < \mathbf{S}_k < \beta\}$$

such that (2.13) holds with $\mathbf{T}$ replaced by $\mathbf{T}_k := \mathbf{S}_k + \mathbf{R}_k \circ \sigma_{\mathbf{S}_k}$, and such that

$$Q(\mathbf{T}_k \notin \mathbb{R}^n,\ \mathbf{Y_t} \in B \text{ for some } \mathbf{t} \geqq \mathbf{S}_k) = 0.$$

Note that each $\mathbf{T}_k$ is a homogeneous time; hence so is the time $\mathbf{T}$ defined by

$$\mathbf{T} = \begin{cases} \mathbf{T}_K, & \text{on } \{\mathbf{Y} \text{ hits } B\}, \\ \infty, & \text{otherwise,} \end{cases}$$

where $K = \inf\{k;\ \mathbf{S}_k \in]\alpha, \beta[,\ \mathbf{Y_t} \in B \text{ for some } \mathbf{t} \geqq \mathbf{S}_k\}$. Evidently $\mathbf{T}$ satisfies (2.13) and (2.14). If $\Gamma(B) < \infty$ then $\kappa_0 := 1_{\{\mathbf{T} \in]\alpha, \beta[\}}\varepsilon_{\mathbf{T}}$ is an RHRM and $\kappa := (o^1 o^2 \ldots o^n \hat{p}^1 \ldots \hat{p}^n)\kappa_0$ is an HRM [(iv) of (2.9) holds because of the π-Markov property of Q]. Moreover,

$$\mathbb{P}(\kappa(\mathbb{R}^n)) = \Gamma(B)$$
$$\mathbb{P}(\kappa(\mathbb{R}^n)^2) \leqq 4^{2n}\mathbb{P}(\kappa_0(\mathbb{R}^n)^2) = 4^{2n}\Gamma(B)$$

[*see* Lemma (2.10)]. In view of (2.8), κ is carried by $\{\mathbf{t};\ \mathbf{Y_t}\in \mathrm{B}\}$ since κ_0 is so carried. Thus, provided $0<\Gamma(\mathrm{B})<\infty$,

$$\mathrm{I_{HRM}}(\mathrm{B})\leqq 2^{-n}\,\mathbb{P}([\kappa(\mathbb{R}^n)/\Gamma(\mathrm{B})]^2)\leqq 2^{3n}/\Gamma(\mathrm{B})$$

as required. If $\Gamma(\mathrm{B})=\infty$ then by the σ-finiteness of $\mathbb{P}$ we can choose $(\mathrm{F}_k)\subset p\mathscr{A}^0$ with $0<\mathrm{F}_k\uparrow 1$ and $\mathbb{P}(\mathrm{F}_k)<\infty$. Then $\kappa_k:=\mathrm{F}_k.\kappa_0$ is an RHRM with $\mathbb{P}(\kappa_k(\mathbb{R}^n))<\infty$ and $\kappa_k\uparrow\kappa_0$. By the previous argument, since $\kappa_k(\mathbb{R}^n)^2\leqq\kappa_k(\mathbb{R}^n)\leqq 1$,

$$\mathrm{I_{HRM}}(\mathrm{B})\leqq\frac{2^{3n}\,\mathbb{P}(\kappa_k(\mathbb{R}^n)^2)}{\mathbb{P}(\kappa_k(\mathbb{R}^n))^2}\leqq 2^{3n}/\mathbb{P}(\kappa_k(\mathbb{R}^n))\to 0$$

as $k\to\infty$. $\square$

Remark. – In Dynkin [D81], $o^1\ldots o^n\hat{p}^1\ldots\hat{p}^n$ is called the *central projection.*

3. KUZNETSOV MEASURES AND CAPACITY

In this section we describe the specific measure Q to which Theorem (2.12) will be applied, and we obtain a classical expression for $\Gamma(\mathrm{B})$.

For $i=1,\ \ldots,\ n$, let $(\mathrm{X}^i_t,\ \mathrm{P}^{i,\,x})$ be Borel right Markov processes with Lusin state spaces $(\mathrm{E}^i,\ \mathscr{E}^i)$, semigroups (P^i_t), and resolvents $(\mathrm{U}^{i,\,q})$. We assume that the X^i are *transient* in the sense that there exist $f^i\in\mathscr{E}^i$, $f^i>0$ such that $\mathrm{U}^i f^i<\infty$. On the product of the underlying probability spaces define

$$\mathbf{X_t}=(\mathrm{X}^1_{t^1},\ \ldots,\ \mathrm{X}^n_{t^n}),\qquad \mathbf{t}=(t^1,\ \ldots,\ t^n)\in\mathbb{R}^n_+$$
$$\mathrm{P}^{\mathbf{x}}=\mathrm{P}^{1,\,x^1}\otimes\ldots\otimes\mathrm{P}^{n,\,x^n},\qquad \mathbf{x}=(x^1,\ \ldots,\ x^n)\in\mathrm{E}=\mathrm{E}^1\times\ldots\times\mathrm{E}^n,$$

so that the X^i are independent processes under each $\mathrm{P}^{\mathbf{x}}$. The semigroup and resolvent of $\mathbf{X}$ are

$$\mathrm{P_t}(\mathbf{x},\ d\mathbf{y})=\mathrm{P}^1_{t^1}(x^1,\ dy^1)\otimes\ldots\otimes\mathrm{P}^n_{t^n}(x^n,\ dy^n),$$
$$\mathrm{U^q}(\mathbf{x},\ d\mathbf{y})=\mathrm{U}^{1,\,q^1}(x^1,\ dy^1)\otimes\ldots\otimes\mathrm{U}^{n,\,q^n}(x^n,\ dy^n),$$

where $\mathbf{t},\ \mathbf{q}\in\mathbb{R}^n_+$.

Given (P^i_t)-excessive measures m^i, form the product $m=m^1\otimes\ldots\otimes m^n$, and note that m is an excessive measure for $(\mathrm{P_t})$. By a theorem of Kuznetsov [K74], for each i there is a stationary measure Q^i on W^i (the space of paths $w:\mathbb{R}\to\mathrm{E}^i\cup\{\Delta\}$ as in paragraph 2 with $n=1$) such that

(i) $\mathrm{Q}^i(\mathrm{Y}^i_t\in dx^i)=m^i(dx^i),\ \forall\, t\in\mathbb{R}$;

(3.1) (ii) $(\mathrm{Y}^i_t;\ t\in\mathbb{R})$ is Markov under Q^i with semigroup P^i_t, and random "life interval" $]\alpha^i,\ \beta^i[$.

The product measure $\mathrm{Q}^1\otimes\ldots\otimes\mathrm{Q}^n$ on $\mathrm{W}^1\times\ldots\times\mathrm{W}^n$ can be carried to the space W of paragraph 2 via the natural map $\mathrm{W}^1\times\ldots\times\mathrm{W}^n\to\mathrm{W}$.

Note the identification of $]\boldsymbol{\alpha}(\mathbf{w}), \boldsymbol{\beta}(\mathbf{w})[$ with $\times_{i=1}^{n}]\alpha^i(\omega^i), \beta^i(\omega^i)[$. Since each X^i is transient, the Q^i are dissipative (*see* [Fi88a]), hence Q is dissipative. Owing to the strong Markov property of X^i, the projections o^i and $\hat{p}^i$ commute on $]\boldsymbol{\alpha}, \boldsymbol{\beta}[$ for $i=1, \ldots, n$ (The one-parameter version of this result (with slightly different definitions) is proved in [D77], and in (4.12) of [Fi87]. The n-parameter version, for processes of the form $V_{t_1}(w^1) V_{t_2}(w^2) \ldots V_{t^n}(w^n)$, then follows by independence, and the general case in turn by monotone class arguments.) As noted by Dynkin [D81] the independence of Y^i and $Y^j (i \neq j)$ implies that o^i and $\hat{p}^j$ commute, as do o^i, o^j and $\hat{p}^i$, $\hat{p}^j$. Thus Q is π-Markov.

Define

$$\tilde{\Gamma}(B) = \sup \{ P^{\nu}(X_t \in B \text{ for some } t > 0); \\ \nu = \nu^1 \otimes \ldots \otimes \nu^n, \text{ each } \nu^i U^i \leqq m^i \}.$$

(3.2) LEMMA. — *(a)* $\Gamma = \tilde{\Gamma}$ *on* $\mathscr{E}$.

(b) If $\nu_k^i U^i \uparrow m^i$ *as* $k \to \infty$, *for each i, then*

$$P^{\nu_k}(X_t \in B \textit{ for some } t > 0) \uparrow \Gamma(B), \qquad \forall B \in \mathscr{E},$$

where $\nu_k = \nu_k^1 \otimes \ldots \otimes \nu_k^n$.

Remark. — It is *not* true that for general ν (not necessarily a product measure), $\nu U \leqq m \Rightarrow P^{\nu}(X_t \in B \text{ for some } t > 0) \leqq \tilde{\Gamma}(B)$.

Proof. — We prove that $\tilde{\Gamma} \leqq \Gamma$ using Skorokhod embedding. Let $\mathscr{H}^i = \sigma\{Y_t^i;\ t \in \mathbb{R}\}$ so that $\mathscr{H}^i \cap \Omega^i = \sigma\{X_t^i;\ t > 0\}$. Here

$$\Omega^i := \{ w \in W^i;\ \alpha^i(w) = 0,\ Y_{0+}^i(w) \text{ exists in } E \}$$

is taken to be the sample space of X^i. Let τ_t^i be the birthed shift operator on W^i:

$$\tau_t^i(w)(s) = \begin{cases} w(s+t), & s > 0, \\ \Delta, & s \leqq 0. \end{cases}$$

It follows from Theorem 3.1 of [Fi88b] that if $\nu^i U^i \leqq m^i$ then there exists a family $\{T^i(u),\ 0 \leqq u \leqq 1\}$ of homogeneous $(\mathscr{G}_t^i)$ stopping times such that $\alpha^i \leqq T^i(u) < \beta^i$ if $T^i(u) < \infty$ and

$$\nu^i U^i f = \int_0^1 Q^i(f \circ Y_0^i;\ T^i(u) < 0)\, du.$$

Applying [FM86, (2.4)] we obtain, for $s > 0$,

$$\nu^i P_s^i f = \int_0^1 Q^i(f \circ Y_{T^i(u)+s}^i;\ 0 < T^i(u) \leqq 1)\, du.$$

A comparison of finite dimensional distributions now reveals that

$$P^{i,\,\nu^i}(1_{\Omega^i} F) = \int_0^1 Q^i((1_{\Omega^i} F) \circ \tau^i_{T^i(u)};\ 0 < T^i(u) \leqq 1)\,du,$$

for $F \in p\mathscr{H}^i$. Let $\tau_{\mathbf{t}}$ be the operator on W analogous to the $\tau^i_{t^i}$, and consider the homogeneous time $\mathbf{T}(\mathbf{u}) = (T^1(u^1), \ldots, T^n(u^n))$. By (2.1), if $\Omega = \times_{i=1}^{n} \Omega^i$ then

$$\begin{aligned} P^{\nu}(1_{\Omega} F) &= \int_{]0,1]} Q((1_{\Omega} F) \circ \tau_{\mathbf{T}(\mathbf{u})};\ \mathbf{T}(\mathbf{u}) \in]\mathbf{0}, \mathbf{1}])\, d\mathbf{u} \\ &= \int_{]0,\ 1]} \mathbb{P}((1_{\Omega} F) \circ \tau_{\mathbf{T}(\mathbf{u})};\ \mathbf{T}(\mathbf{u}) \in \mathbb{R}^n)\, d\mathbf{u}, \end{aligned}$$

provided $F \in p\mathscr{G}^0$. It follows from the last display that if $F \in p\mathscr{G}^*$, where $\mathscr{G}^*$ denotes the universal completion of $\mathscr{G}^0$, then $1_{\Omega} F$ is measurable over the P^{ν}-completion of $\bigotimes_{i=1}^{n} (\mathscr{H}^i \cap \Omega^i)$ and the asserted equalities hold. Taking for F the indicator of $\{\mathbf{Y}_{\mathbf{t}} \in B \text{ for some } \mathbf{t} \in \mathbb{R}^n\} \in \mathscr{G}^*$ we have $1_{\Omega} F = 1_{\{\mathbf{X}_{\mathbf{t}} \in B \text{ for some } \mathbf{t} > \mathbf{0}\}}$ and $(1_{\Omega} F) \circ \tau_{\mathbf{s}} \leqq F$ whenever $\boldsymbol{\alpha} \leqq \mathbf{s} < \boldsymbol{\beta}$. Thus $P^{\nu}(\mathbf{X}_{\mathbf{t}} \in B \text{ for some } \mathbf{t} > \mathbf{0}) \leqq \Gamma(B)$, hence $\tilde{\Gamma}(B) \leqq \Gamma(B)$. On the other hand, by (4.6) of [FM86] we can find homogeneous times S^i_k on W^i, $S^i_k \downarrow\downarrow \alpha^i$, and a sequence μ^i_k such that $\mu^i_k U^i \uparrow m^i$ and

$$P^{i,\,\mu^i_k}(1_{\Omega^i} F) = Q^i((1_{\Omega^i} F) \circ \tau^i_{S^i_k};\ 0 < S^i_k \leqq 1)$$

for $F \in p\mathscr{H}^i$. Put $\mu_k = \mu^1_k \otimes \ldots \otimes \mu^n_k$, $\mathbf{S}_k = (S^1_k, \ldots, S^n_k)$. Then

$$\begin{aligned} P^{\mu_k}(1_{\Omega} F) &= Q((1_{\Omega} F) \circ \tau_{\mathbf{S}_k};\ \mathbf{S}_k \in]\mathbf{0}, \mathbf{1}]) \\ &= \mathbb{P}((1_{\Omega} F) \circ \tau_{\mathbf{S}_k};\ \mathbf{S}_k \in \mathbb{R}^n), \qquad F \in \mathscr{G}^0. \end{aligned}$$

Taking $F = 1_{\{\mathbf{Y} \text{ hits } B\}}$ we have

$$(1_{\Omega} F) \circ \tau_{\mathbf{S}_k} = 1_{\{\mathbf{S}_k \neq \infty\}} 1_{\{\mathbf{Y}_{\mathbf{t}} \in B \text{ for some } \mathbf{t} > \mathbf{S}_k\}} \uparrow 1_{\{\mathbf{Y} \text{ hits } B\}}$$

since $\mathbf{S}_k \downarrow\downarrow \boldsymbol{\alpha}$. Consequently $P^{\mu_k}(\mathbf{X} \text{ hits } B) \uparrow \Gamma(B)$, so $\tilde{\Gamma}(B) \geqq \Gamma(B)$, and (*a*) is proved.

To show (*b*), observe that we have just shown the conclusion for the particular sequence μ_k. Let ν_k be as in the statement of the lemma. The function $f(\mathbf{x}) = P^{\mathbf{x}}(\mathbf{X}_{\mathbf{t}} \in B \text{ for some } \mathbf{t} > \mathbf{0})$ is easily seen to be separately excessive in each component x^i of $\mathbf{x}$. If we fix i and x^j, $j = 1, \ldots, n$, $j \neq i$, then

$$\int f(\mathbf{x})\, \nu^i_k(dx^i) \quad \text{is increasing in } k, \tag{3.3}$$

(3.4) $$\uparrow \lim \int f(\mathbf{x})\, \nu_k^i(dx^i) = \uparrow \lim \int f(\mathbf{x})\, \mu_k^i(dx^i).$$

(These assertions follow immediately if $x^i \mapsto f(\mathbf{x})$ takes the form $U^i g$, and because of our transience hypothesis any excessive function (of X^i) is the increasing limit of such functions: *see* [DM87, XII.T17].) By (3.3), $\nu_{k^1}^1 \otimes \ldots \otimes \nu_{k^n}^n(f)$ is increasing in each of $k^1, \ldots, k^n$, and therefore its limit does not depend on how we take the k^i to infinity. If we let them $\uparrow \infty$ in turn, then (3.4) shows that each $\nu_{k^i}^i$ may be replaced by $\mu_{k^i}^i$ without affecting the limit. Thus, if we let $k^1 = \ldots = k^n = k \to \infty$ then

$$\uparrow \lim \nu_k(f) = \uparrow \lim \mu_k(f) = \Gamma(B)$$

showing (*b*). □

Finally, by Theorem (2.12) we have the following

(3.5) COROLLARY. – *Let the* X^i *be transient Borel right processes, and assume* (2.7). *Then*

$$2^{-n}/I_{\mathrm{HRM}}(B) \leqq \tilde{\Gamma}(B) \leqq 2^{3n}/I_{\mathrm{HRM}}(B), \qquad \forall\, B \in \mathscr{E}.$$

(In the next section we shall impose additional hypotheses ensuring that condition (2.7) is met.)

4. CLASSICAL ENERGY

We maintain the hypotheses and notation of paragraph 3, so that the X^i are independent transient Borel right processes.

Given an RHRM κ define a measure μ_κ on $(E, \mathscr{E})$ by

(4.1) $$\mu_\kappa(f) = \mathbb{P}\left(\int_{\mathbb{R}^n} f \circ \mathbf{Y}_{\mathbf{t}}\, \kappa(d\mathbf{t})\right), \qquad f \in p\mathscr{E}.$$

Applying (2.3) with $F = \int_{\mathbb{R}^n} g(\mathbf{t}, \mathbf{Y}_{\mathbf{t}})\, \kappa(d\mathbf{t})$ we find that for $g \in p(\mathscr{R}^n \otimes \mathscr{E})$,

(4.2) $$Q\left(\int_{\mathbb{R}^n} g(\mathbf{t}, \mathbf{Y}_{\mathbf{t}})\, \kappa(d\mathbf{t})\right) = \int_{\mathbb{R}^n} d\mathbf{t} \int_E \mu_\kappa(d\mathbf{x})\, g(\mathbf{t}, \mathbf{x}).$$

We call μ_κ the *Revuz measure* of κ (Dynkin [D81] calls it the "characteristic measure" of κ). Note that μ_κ is σ-finite by (ii) of (2.9), and if (2.7) holds then $\mu_{\pi\kappa} = \mu_\kappa$ for any RHRM.

Our goal in this section is to express the "energy" $2^{-n}\,\mathbb{P}(\kappa(\mathbb{R}^n)^2)$ of an HRM κ as a classical energy integral of μ_κ relative to the appropriate kernel function. To do this with minimal preliminary work we assume fairly stringent conditions on the X^i.

We assume that, relative to m^i, X^i has a transient Borel right strong dual process $\hat{X}^i$ (with semigroup $\hat{P}^i_t$ and resolvent $\hat{U}^{i,q}(x, dy)$). Thus m^i is assumed to be a reference measure, and

$$m^i(dx)\, P^i_t(x, dy) = m^i(dy)\, \hat{P}^i_t(y, dx).$$

In particular there are kernel functions $u^{i,q}$ such that $u^{i,q}(\,.\,, y)$ is q-excessive (for (P^i_t)), $u^{i,q}(x,\,.\,)$ is q-coexcessive (*i. e.*, q-excessive for $(\hat{P}^i_t)$), and

$$U^{i,q}(x, dy) = u^{i,q}(x, y)\, m^i(dy).$$
$$\hat{U}^{i,q}(y, dx) = u^{i,q}(x, y)\, m^i(dx).$$

See Chapter VI of [BG68]. Because X^i and $\hat{X}^i$ are transient, the above discussion is valid even when $q=0$, and we write u^i for $u^{i,0}$.

The *symmetrized potential density* u^S is defined on $E \times E$ by

$$(4.3) \qquad u^S(\mathbf{x}, \mathbf{y}) = 2^{-n} \sum_J \prod_{i=1}^{n} u^i(x^i_J, y^i_J)$$

where the sum extends over all sets $J \subset \{1, 2, \ldots, n\}$, and $(x^i_J, y^i_J) = (y^i, x^i)$ if $i \in J$, $= (x^i, y^i)$ if $i \notin J$. For any positive measure μ on $(E, \mathscr{E})$ we have the energy integral

$$e(\mu) = \iint \mu(d\mathbf{x})\, u^S(\mathbf{x}, \mathbf{y})\, \mu(d\mathbf{y}).$$

(Note that if $n=1$ then $e(\mu)$ is unchanged if u^S is replaced by u; if $n>1$ then this is true for all μ only if u is symmetric.)

Our final hypothesis on the processes X^i is of a technical nature. We assume that X^i and $\hat{X}^i$ are *special standard* processes, $i=1, \ldots, n$. The reader is referred to paragraph 16 of [GSh84] for a precise definition of this term; roughly speaking, a special standard process is a standard process that is quasi-left-continuous in its Ray topology. In particular, X^i and $\hat{X}^i$ have left limits up to (but not necessarily at) their lifetimes. For example (because of strong duality), if X^i is symmetric (*i. e.*, $\hat{X}^i = X^i$) then hypothesis (H) holds (semipolars are polar) in which case results from paragraph 16 of [GSh84] imply that X^i is special standard. This hypothesis has two important consequences, which we collect in the following

(4.4) LEMMA. — *(a)* (2.7) *holds.*

(b) If κ *is an* HRM *then* κ *is carried by* $\{\mathbf{t} \in]\boldsymbol{\alpha}, \boldsymbol{\beta}[;\ \mathbf{Y}_\mathbf{t} = \mathbf{Y}_{\mathbf{t}-}\}$.

Proof. — To preserve the usual order of time, we argue first with predictable projections. Let ρ_i be the Q^i-predictable projection on $W^i \times \mathbb{R}$. Without loss of generality, we take $i=1$. By Theorem 4 of [WM71], we see that the Q^1-predictable projection of $f \circ Y^1_t$ is $\bar{P}^1_0 f \circ Y^1_{t-}$. Here the left limit Y^1_{t-} is taken in the Ray topology, and $\bar{P}^1_t$ is the extension of the semigroup of X^1 to a Ray-Knight compactification of $E \cup \{\Delta\}$. By (16.18) of [GSh84], $Y^1_{t-} \in E$ for all $t \in]\alpha^1, \beta^1[$, a. s. Q^1. Taking $f = 1_E$ and noting that $\bar{P}_0^1 1_E(x) = P_0 1_E(x) = 1$ if $x \in E$, we obtain $\rho^1(1_{]\alpha^1, \beta^1[}) = 1$ on $]\alpha^1, \beta^1[$. The same argument works in reversed time as well, so $\hat{\rho}^1(1_{]\alpha^1, \beta^1[}) = 1$ on $]\alpha^1, \beta^1[$ also. By independence,

$$\begin{aligned}\hat{p}^1(1_{]\boldsymbol{\alpha}, \boldsymbol{\beta}[}) &= \hat{\rho}^1(1_{]\alpha^1, \beta^1[}) \prod_{i=2}^{n} 1_{]\alpha^i \beta^i[} \\ &= 1 \quad \text{on }]\boldsymbol{\alpha}, \boldsymbol{\beta}[,\end{aligned}$$

showing (a).

To show (b), first "perfect" κ as in Proposition (A.5) of the appendix. Again take $i=1$. Then for a. e. fixed $w^2, \ldots, w^n$,

$$\kappa^1(w^1, dt^1) = \kappa(w^1, \ldots, w^n, dt^1 \times \mathbb{R}^{n-1})$$

is an RHRM on W^1. Let $a_{t^1}(w^1) \in p(\sigma\{Y_t^1; t \in \mathbb{R}\} \otimes \mathscr{R})$ and put $b = \hat{\rho}^1(a)$. For $C \in p\sigma\{Y^2, \ldots, Y^n\}$ set

$$A_{\mathbf{t}}(\mathbf{w}) = C(\mathbf{w})\, a_{t^1}(w^1), \quad B_{\mathbf{t}}(\mathbf{w}) = C(\mathbf{w})\, b_{t^1}(w^1).$$

Then $\hat{p}^1 A = B$, so

$$\begin{aligned}Q\left(C \int_{\mathbb{R}} a_t \kappa^1(dt)\right) &= Q\left(\int_{\mathbb{R}^n} A_{\mathbf{t}} \kappa(d\mathbf{t})\right) \\ &= Q\left(\int_{\mathbb{R}^n} B_{\mathbf{t}} \kappa(d\mathbf{t})\right) \\ &= Q\left(C \int_{\mathbb{R}} b_t \kappa^1(dt)\right),\end{aligned}$$

hence $Q^1\left(\int_{\mathbb{R}} a_t \kappa^1(dt)\right) = Q^1\left(\int_{\mathbb{R}} b_t \kappa^1(dt)\right)$ for a. e. $w^2, \ldots, w^n$. Letting a range through a countable sequence generating $\sigma\{Y_t^1; t \in \mathbb{R}\} \otimes \mathscr{R}$ we see that κ^1 is Q^1-copredictable for a. e. $w^2, \ldots, w^n$. Similarly κ^1 is Q^1-optional. Thus by (5.3) of [GSh84], κ^1 is a. s. carried by $\{t \in]\alpha^1, \beta^1[; Y_{t-}^1 = Y_t^1\}$, so that κ is a. s. carried by $\{\mathbf{t} \in]\boldsymbol{\alpha}, \boldsymbol{\beta}[; Y_{t^1}^1 = Y_{t^1 -}^1\}$. Since this is true for $i \neq 1$ as well, (b) must hold. $\square$

A measure on $\mathbb{R}^n$ is *projectively continuous* if each of its 1-dimensional marginals is diffuse. Following Dynkin [D81] we say that an HRM is *continuous* if $\kappa(\mathbf{w}, .)$ is projectively continuous for Q-a. e. $\mathbf{w}$. The following evaluation is the key result of this section.

(4.5) Theorem. — *Let* κ *be an* HRM *with Revuz measure* μ. *Then*

(i) $2^{-n} \mathbb{P}(\kappa(\mathbb{R}^n)^2) \geqq e(\mu)$;

(ii) $2^{-n} \mathbb{P}(\kappa(\mathbb{R}^n)^2) = e(\mu)$ *if and only if* κ *is continuous.*

The proof of (4.5) requires a number of preliminary lemmas. To begin let $D = \{(\mathbf{s}, \mathbf{t}) \in (\mathbb{R}^n)^2; s^i = t^i \text{ for some } i\}$. Given an HRM κ let $\kappa^2(\mathbf{w}, .)$ be the measure $\kappa(\mathbf{w}, .) \otimes \kappa(\mathbf{w}, .)$ on $(\mathbb{R}^n)^2$. Clearly κ is continuous if and only if $\kappa^2(\mathbf{w}, D) = 0$ for Q-a. e. $\mathbf{w}$. For $J \subset \{1, 2, \ldots, n\}$ define a partial order $<_J$ on $\mathbb{R}^n$ by $\mathbf{s}<_J\mathbf{t}$ if and only if $s^i<t^i$, $\forall i\in J^c$; $s^i>t^i$, $\forall i\in J$. Let $R_J=\{\mathbf{t}\in\mathbb{R}^n;\ \mathbf{0}<_J\mathbf{t}\}$ denote the positive orthant of $\mathbb{R}^n$ in the order $<_J$. If κ is any HRM then

$$\mathbb{P}(\kappa(\mathbb{R}^n)^2)=\mathbb{P}(\kappa^2(D))+\sum_J \mathbb{P}\left(\iint 1_{R_J}(\mathbf{t}-\mathbf{s})\,\kappa(d\mathbf{s})\,\kappa(d\mathbf{t})\right).$$

Thus to prove (4.5) we must check that for each $J\subset\{1,\ldots,n\}$,

$$\text{(4.6)}\qquad \mathbb{P}\left(\iint 1_{R_J}(\mathbf{t}-\mathbf{s})\,\kappa(d\mathbf{s})\,\kappa(d\mathbf{t})\right)=\int \mu_\kappa(d\mathbf{x})\,u_J(\mathbf{x},\mathbf{y})\,\mu_\kappa(d\mathbf{y}),$$

where

$$u_J(\mathbf{x},\mathbf{y})=\prod_{i=1}^n u^i(x_J^i,y_J^i),\qquad \mathbf{x}=(x^1,\ldots,x^n),\quad \mathbf{y}=(y^1,\ldots,y^n).$$

In fact we prove (4.6) only for $J=\varnothing$ and $J=\{1,\ldots,n\}$. Of course, (4.6) for J is equivalent to (4.6) for J^c, but giving both arguments allows us to keep track of the roles of right limits ($J=\varnothing$) and left limits ($J=\{1,\ldots,n\}$). The general case falls to the same methods, but a mixture of right limits and left limits would be required, leading to an even more intricate notation.

The following simple consequence of (2.11) is valid for arbitrary J.

(4.7) LEMMA. — *Let* $\varphi\in p(\mathscr{R}^n\otimes\mathscr{R}^n)$ *satisfy* $\int_{\mathbb{R}^n}\varphi(\mathbf{a},\mathbf{a}+\mathbf{b})\,d\mathbf{a}=1$, $\forall\,\mathbf{b}$. *Then for any* $J\subset\{1,\ldots,n\}$,

$$\mathbb{P}\left(\iint 1_{R_J}(\mathbf{t}-\mathbf{s})\,\kappa(d\mathbf{t})\right)=Q\left(\iint \varphi(\mathbf{s},\mathbf{t})\,1_{R_J}(\mathbf{t}-\mathbf{s})\,\kappa(d\mathbf{s})\,\kappa(d\mathbf{t})\right).$$

To evaluate the Q-expectation in (4.7) explicity we need an expression for the "optional projection" of $\mathbf{t}\mapsto\kappa(R_J+\mathbf{t})$. To this end it is convenient to transfer the laws $P^{\mathbf{x}}$ and $\hat{P}^{\mathbf{x}}=\hat{P}^{1,x^1}\otimes\ldots\otimes\hat{P}^{n,x^n}$ to subsets of W. Define

$$\Omega=\{\mathbf{w}\in W;\ \boldsymbol{\alpha}(\mathbf{w})=\mathbf{0},\ \mathbf{Y}_{\mathbf{0}+}(\mathbf{w})\text{ exists in E}\}$$
$$\hat{\Omega}=\{\mathbf{w}\in W;\ \boldsymbol{\beta}(\mathbf{w})=\mathbf{0},\ \mathbf{Y}_{\mathbf{0}-}(\mathbf{w})\text{ exists in E}\}.$$

Let $\mathscr{F}^0$ and $\hat{\mathscr{F}}^0$ denote the respective traces of $\mathscr{G}^0$ on Ω and $\hat{\Omega}$. Without changing notation we carry $P^{\mathbf{x}}$ and $\hat{P}^{\mathbf{x}}$ to $(\Omega,\mathscr{F}^0)$ and $(\hat{\Omega},\hat{\mathscr{F}}^0)$ in such a way that the coordinate processes

$$(\mathbf{w}_{\mathbf{t}+};\ \mathbf{t}\geqq\mathbf{0})\qquad(\mathbf{w}\in\Omega);$$
$$(\mathbf{w}_{(-\mathbf{t})-};\ \mathbf{t}\geqq\mathbf{0})\qquad(\mathbf{w}\in\hat{\Omega}),$$

under P^x and $\hat{P}^x$ are realizations of X and $(\hat{X}_t) : =(\hat{X}^1_{t^1}, \hat{X}^2_{t^2}, \ldots, \hat{X}^n_{t^n})$ respectively. Let $o=o^1 \ldots o^n$ and $\hat{o}=\hat{o}^1 \ldots \hat{o}^n$, where the $\hat{o}^i$ are now co-optional projections (*i. e.* optional projections relative to the reverse filtrations $\hat{\mathcal{G}}^i_t$). Since the X^i are independent, the next result follows by standard monotone class arguments (*cf.* Dynkin [D81]). We omit the proof.

(4.8) LEMMA. — *Let* $H \in \hat{\mathcal{G}}^0_0$, $\hat{H} \in \mathcal{G}^0_0$, *and let* $\mathbf{r} \geqq \mathbf{0}$. Then

$$o(H \circ \sigma_{.+\mathbf{r}})_{\mathbf{t}} = P^{Y_t}(P^{Y_{r+}}(H)), \qquad \forall\, \mathbf{t} \in]\boldsymbol{\alpha}, \boldsymbol{\beta}[,$$
$$\hat{o}(\hat{H} \circ \sigma_{.-\mathbf{r}})_{\mathbf{t}} = \hat{P}^{Y_{t-}}(\hat{P}^{Y_{(-r)-}}(\hat{H})), \qquad \forall\, \mathbf{t} \in]\boldsymbol{\alpha}, \boldsymbol{\beta}[,$$

up to Q-evanescence.

Given an HRM κ, let H_t be the modification of $\kappa(R_\varnothing + \mathbf{t})$ constructed in (A.6). Let $\hat{H}_t$ be the dual object, corresponding to $\kappa(R_{\{1, \ldots, n\}} + \mathbf{t})$. Define

$$v_\kappa(x) = P^x(H_0), \qquad \hat{v}_\kappa(x) = \hat{P}^x(\hat{H}_0).$$

Note that if κ has the special form $\kappa(dt) = f(Y_t)\, dt$ then $v_\kappa = U f$, $\hat{v}_\kappa = \hat{U} f$, and $\mu_\kappa = f.m$.

(4.9) LEMMA. — *Let* κ *and* γ *be* HRM's *with* v_κ *and* $\hat{v}_\gamma$ *as described above. Then*

$$\mathbb{P}\left(\iint 1_{R_\varnothing}(\mathbf{t}-\mathbf{s})\, \gamma(d\mathbf{s})\, \kappa(d\mathbf{t})\right) = \mu_\kappa(\hat{v}_\gamma) = \mu_\gamma(v_\kappa). \tag{4.10}$$

Proof. — Apply Lemma (2.11) with $\varphi(\mathbf{s}, \mathbf{t}) = 1_{R_\varnothing}(\mathbf{t}-\mathbf{s})\, \psi(\mathbf{s})$, where $\psi \in p\mathcal{R}^n$ with $\int_{\mathbb{R}^n} \psi(\mathbf{s})\, d\mathbf{s} = 1$. Then

$$\begin{aligned}
\mathbb{P}\left(\iint 1_{R_\varnothing}(\mathbf{t}-\mathbf{s})\, \gamma(d\mathbf{s})\, \kappa(d\mathbf{t})\right) &= Q\left(\int \psi(\mathbf{s})\, \gamma(d\mathbf{s})\, \kappa(R_\varnothing + \mathbf{s})\right) \\
&= Q\left(\int \psi(\mathbf{s})\, \gamma(d\mathbf{s})\, H_0 \circ \sigma_{\mathbf{s}}\right) \\
&= Q\left(\int \psi(\mathbf{s})\, \gamma(d\mathbf{s})\, v_\kappa(Y_{\mathbf{s}})\right) \\
&= \mu_\gamma(v_\kappa).
\end{aligned}$$

We have used the "perfect homogeneity" given by (A.6) for the second equality, (4.8) (with $\mathbf{r}=\mathbf{0}$) and the "optionality" of γ for the third equality, and (4.2) for the fourth. This establishes the second equality in (4.10). Using the "co-optionality" of κ, a similar argument shows that the L.H.S. of (4.10) is equal to

$$Q\left(\int \psi(\mathbf{t})\, \kappa(d\mathbf{t})\, \hat{v}_\gamma(Y_{\mathbf{t}-})\right). \tag{4.11}$$

But (4.4) allows another appeal to (4.2), hence (4.11) equals $\mu_\kappa(\hat{v}_\gamma)$ as required. □

Let $U_J\mu(\mathbf{x}) = \int u_J(\mathbf{x}, \mathbf{y})\,\mu(d\mathbf{y})$.

(4.12) COROLLARY. — *Let* κ *be an* HRM *with Revuz measure* μ_κ. *Then*

$$v_\kappa = U_\varnothing \mu_\kappa \quad m-\text{a. e.},$$
$$\hat{v}_\kappa = U_{\{1,\ldots,n\}} \mu_\kappa \quad m-\text{a. e.}$$

Proof. — Take γ in (4.9) to have the form $g(\mathbf{Y}_t)\,dt$. Varying g gives the conclusion. □

(4.13) LEMMA. — *Let* κ, γ *be as before. Then*

$$\mu_\gamma(v_\kappa) = \mu_\gamma(U_\varnothing \mu_\kappa),$$
$$\mu_\kappa(\hat{v}_\gamma) = \mu_\kappa(U_{\{1,\ldots,n\}} \mu_\gamma).$$

Proof. — $U_\varnothing \mu_\kappa$ is excessive, in the sense that

$$q^1 \ldots q^n U^{\mathbf{q}}(U_\varnothing \mu_\kappa) \uparrow U_\varnothing \mu_\kappa \quad \text{as each } q^i \uparrow \infty.$$

Since $U^{\mathbf{q}}(\mathbf{x}, d\mathbf{y}) \ll m(d\mathbf{y})$, we conclude from Corollary (4.12) that $q^1 \ldots q^n U^{\mathbf{q}}(v_\kappa) \uparrow U_\varnothing \mu_\kappa$ as each $q^i \uparrow \infty$. Thus, by the argument of (4.9),

$$\begin{aligned}
\mu_\gamma(U_\varnothing \mu_\kappa) &= \uparrow \lim q^1 \ldots q^n \int_{R_\varnothing} e^{-\mathbf{q}\cdot\mathbf{r}} Q\Big(\int \psi(\mathbf{s})\,\gamma(d\mathbf{s})\, v_\kappa(\mathbf{Y}_{\mathbf{s}+\mathbf{r}})\Big)\,d\mathbf{r} \\
&= \uparrow \lim q^1 \ldots q^n \int_{R_\varnothing} e^{-\mathbf{q}\cdot\mathbf{r}} Q\Big(\int \psi(\mathbf{s})\,\gamma(d\mathbf{s})\, P^{\mathbf{Y}_\mathbf{s}}(P^{\mathbf{Y}_{\mathbf{r}+}}(H_0))\Big)\,d\mathbf{r} \\
&= \uparrow \lim q^1 \ldots q^n \int_{R_\varnothing} e^{-\mathbf{q}\cdot\mathbf{r}} Q\Big(\int \psi(\mathbf{s})\,\gamma(d\mathbf{s})\, H_0 \circ \sigma_{\mathbf{r}+\mathbf{s}}\Big)\,d\mathbf{r} \\
&= \uparrow \lim q^1 \ldots q^n \int_{R_\varnothing} e^{-\mathbf{q}\cdot\mathbf{r}} Q\Big(\int \psi(\mathbf{s})\,\gamma(d\mathbf{s})\, \kappa(R_\varnothing + \mathbf{r} + \mathbf{s})\Big)\,d\mathbf{r} \\
&= Q\Big(\int \psi(\mathbf{s})\,\gamma(d\mathbf{s})\, \kappa(R_\varnothing + \mathbf{s})\Big).
\end{aligned}$$

In the proof of Lemma (4.9), we showed that this equaled $\mu_\gamma(v_\kappa)$. The second equality follows similarly. □

(4.14) COROLLARY. — *Formula* (4.6) *holds if* $J = \varnothing$ *or* $\{1, \ldots, n\}$.

Proof. — Take $\kappa = \gamma$ in (4.9) and use (4.13). □

As noted previously, entirely analogous arguments yield (4.6) for arbitrary $J \subset \{1, \ldots, n\}$. Thus the proof of Theorem (4.5) is complete. □

See Dynkin [D 81] for the following result. (Dynkin assumes $P_t(\mathbf{x}, .) \ll m$, but this is implied by our hypotheses (Fukushima's theorem); *see* [Fu 76, Thm. 4.3.4] or [FG 88].)

(4.15) PROPOSITION. — *In addition to the other hypotheses of this section, assume that the* X^i *are symmetric* (i. e., $\hat{X}^i = X^i$). *If* μ *is a* σ-*finite measure on* $(E, \mathscr{E})$ *with* $e(\mu) < \infty$, *then there exists a unique (up to* Q-*indistinguishability) continuous* HRM κ *with Revuz measure* μ. *In particular* $e(\mu) = 2^{-n} \mathbb{P}(\kappa(\mathbb{R}^n)^2)$.

Now define for $B \in \mathscr{E}$,

$$I(B) = \inf \{ e(\mu) ; \ \mu \text{ is a probability measure on } (E, \mathscr{E}) \text{ with } \mu(E \setminus B) = 0 \}.$$

Combining (4.5) with (4.15) and (3.5) we arrive at our principal result:

(4.16) THEOREM. — *Let the* X^i *be as above (transient and special standard with transient special standard strong duals). Then*

(i) $\Gamma(B) \leqq 2^{3n}/I(B)$, $\forall B \in \mathscr{E}$;

(ii) *if each* X^i *is symmetric,* $2^{-n}/I(B) \leqq \Gamma(B)$, $\forall B \in \mathscr{E}$.

We conjecture that (4.16) (ii) is valid in much greater generality; it seems likely that Dynkin's result (4.15) is valid without symmetry provided each X^i satisfies Hunt's hypothesis (H) (semipolars are polar), and has lower semi-continuous excessive functions. In the symmetric case we have a different argument [not based on (2.12)] yielding (4.16) (ii) with 2^{-n} replaced by 2^{1-n}. Simple examples show that an inequality of the type (4.16) (ii) cannot be valid in the general (*i. e.*, nonsymmetric) case, unless the constant is at most 2^{-n}. We do not believe that 2^{1-n} is best possible in the symmetric case of (4.16) (ii) (except when $n=1$). We have a somewhat fanciful conjecture that in this case the best constant when $n=2$ is $\cong .8043$.

5. APPLICATIONS

We content ourselves with giving the converses to two results of S. Evans [E 87 a]. In both cases, Evans supplied a sufficient condition that we show to be necessary. His first result improved a result of Tongring [To 84] (Tongring obtained the same improvement in [To 88] together with a weaker necessary condition), and his second refined a result of LeGall, Rosen and Shieh [LRS 89]. Theorem (5.4) below resolves a conjecture of Hendricks and Taylor [HT 76] concerning necessary and sufficient conditions for a class of Lévy processes to have n-multiple points. See paragraph 7 of Taylor [Ta 86] and also Hawkes [H 78]. Let

$$I_n(B) = \inf \left\{ \int\int (\log^+ (1/|x-y|))^n \mu(dx) \mu(dy); \ \mu \text{ is a probability on } B \right\}.$$

(5.1) THEOREM. — *A necessary and sufficient condition for a Borel subset* B *of* $\mathbb{R}^2$ *to contain n-multiple points of* 2-*dimensional Brownian motion is that* $I_n(B) < \infty$.

Proof. – Let (Z_t) be a two dimensional Brownian motion started from the origin. Let $S_1, \ldots, S_n, T_1, \ldots, T_n$ be exponential random variables of mean 1, independent of Z and of each other. Set

$$Z_s^1 = \begin{cases} Z_{S_1+s}, & s \in]0, T_1[, \\ \Delta, & \text{otherwise}; \end{cases}$$

$$Z_s^{i+1} = \begin{cases} Z_{S_1+T_1+\ldots+S_i+T_i+S_{i+1}+s}, & s \in]0, T_{i+1}[, \\ \Delta, & \text{otherwise}; \end{cases}$$

$$\tilde{\mathbf{Z}}_s = \begin{cases} (Z_{s^1}^1, \ldots, Z_{s^n}^n), & \text{if each } Z_{s^i}^i \neq \Delta, \\ \Delta, & \text{otherwise.} \end{cases}$$

Write D for the diagonal $\{(x, \ldots, x); x \in \mathbb{R}^2\} \subset (\mathbb{R}^2)^n$. By a simple reduction, it will be enough to show that if diameter$(B) \leqq 1/2$ then $P(\tilde{\mathbf{Z}}$ hits $B^n \cap D) > 0$ if and only if $I_n(B) < \infty$.

Let X^i be independent Brownian motions killed at rate 1, m^i = Lebesgue measure on $\mathbb{R}^2$, $i = 1, \ldots, n$. Let Q be the Kuznetsov measure associated to $(X^1, \ldots, X^n)$ and $m^1 \otimes \ldots \otimes m^n$. Explicitly, under Q the Y^i are independent Brownian motions, killed at rate 1, but started homogeneously in time and space: if P_0^x denotes the common law of the X^i (started at x), then

$$Q(\boldsymbol{\alpha} \in d\mathbf{t}, \mathbf{Y}_{\boldsymbol{\alpha}+} \in d\mathbf{x}, Y^1_{\alpha^1+.} \in d\omega^1, \ldots, Y^n_{\alpha^n+.} \in d\omega^n) = d\mathbf{t}\, d\mathbf{x}\, P_0^{x^1}(d\omega^1) \ldots P_0^{x^n}(d\omega^n).$$

(If the killing were at rate q, then the R. H. S. above would need an additional factor of q.) It is easily seen that the law of $\tilde{\mathbf{Z}}$ has the same null sets as the Q-law of $Y \circ \sigma_{\boldsymbol{\alpha}}$. Thus by Theorem (4.16),

$$P(\tilde{\mathbf{Z}} \text{ hits } B^n \cap D) > 0 \;\Leftrightarrow\; Q(\mathbf{Y} \text{ hits } B^n \cap D) > 0 \;\Leftrightarrow\; I(B^n \cap D) < \infty.$$

Let $u^1(x, x')$ be the symmetric resolvent density for X^1. Then there are constants c, C such that

$$c \log^+ (|x - x'|^{-1}) \leqq u^1(x, x') \leqq C \log^+ (|x - x'|^{-1})$$

for $|x - x'| \leqq 1/2$, from which it is clear that if diameter $(B) \leqq 1/2$,

$$I(B^n \cap D) < \infty \;\Leftrightarrow\; I_n(B) < \infty. \quad \square$$

We now turn to the Hendricks-Taylor conjecture concerning n-multiple points for Lévy processes. Let Z_t be a Lévy process taking values in $\mathbb{R}^d$. Assume that it has q-resolvent densities, for $q > 0$. Let $u^1(x, y) = v^1(y - x)$ be the 1-resolvent density. Let B be the unit ball in $\mathbb{R}^d$. We will use the following conditions:

(5.2) $$v^1(0) > 0$$

(5.3) $$\int_B (v^1(x))^n \, dx < \infty.$$

(5.4) THEOREM. — *Let* (Z_t) *be a Lévy process with q-resolvent densities,* $q>0$. *Assume* (5.2). *Then in order that* Z *has n-multiple points it is necessary and sufficient that* (5.3) *holds.*

Remark. — Rogers [Ro89] also discusses the "sufficient" direction, and gives a number of equivalents to (5.3).

Proof. — We have not shown (4.15) in enough generality to give the sufficiency, so for that direction we appeal to Evans [E87b]. Thus we will concern ourselves with necessity. It turns out that (5.2) is not needed for this (in proving sufficiency, it is used to rule out examples such as subordinators).

Let $D=\{(x, \ldots, x); x\in\mathbb{R}^d\}$ be the diagonal in $(\mathbb{R}^d)^n$, and let X_s^i be independent copies of Z, killed at rate one. Let m^i be Lebesgue measure on $\mathbb{R}^d$, $i=1, \ldots, n$, and let Q be the Kuznetsov measure associated to $(X_{s^1}^1, \ldots, X_{s^n}^n)$ and $m^1\otimes\ldots\otimes m^n$. By assumption, the X^i are transient, special standard, and in strong duality with respect to m^i and $\hat{X}_t^i:=-X_t^i$. Thus Theorem (4.16) applies.

Choose S_i, T_i, Z^i, $\tilde{Z}$ as in the proof of (5.1). Assume that Z has n-multiple points with positive probability, so that $P(\tilde{Z}\text{ hits } D)>0$. By assumption, $Z_{0+}^{i+1}-Z_{T_i-}^i$ has a density with respect to Lebesgue measure (namely v^1) so that once more the law of $\tilde{Z}$ is absolutely continuous with respect to the Q-law of $\mathbf{Y}\circ\sigma_\alpha$. [The equivalence of these laws would follow from (5.2).] Thus $Q(\mathbf{Y}\text{ hits } D)>0$, and so $\Gamma(B)>0$. Let ν be the unique measure on D, all of whose marginals are Lebesgue measure. By (4.16) (i), there is a probability μ on D with $e(\mu)<\infty$. Theorem (8.1) of [E87a] contains an argument that in this case $e(\nu|_{B^n})<\infty$. In fact, we can see this directly from the proof of (2.12); that proof produced an HRM κ from a homogeneous time $\mathbf{T}$. Let $(\eta_\mathbf{x}\mathbf{w})(\mathbf{t})=\mathbf{w}(\mathbf{t})+\mathbf{x}$. We can easily construct $\mathbf{T}$ such that $\mathbf{Y_T}\in D$ on $\{\mathbf{T}\neq\infty\}$ and $\mathbf{T}\circ\eta_\mathbf{x}=\mathbf{T}$, $\forall\,\mathbf{x}\in D$. Then the Revuz measure of κ will be invariant under translations by $\mathbf{x}\in D$, and hence will equal $\lambda\nu$ for some $\lambda\in]0, \infty[$. The homogeneous time

$$\mathbf{T}_B=\begin{cases}\mathbf{T}, & \mathbf{Y_T}\in B^n,\\ \infty, & \text{otherwise,}\end{cases}$$

will in turn give rise to an HRM κ_B having Revuz measure $\lambda\nu|_{B^n}$, and satisfying

$$\mathbb{P}(\kappa_B(\mathbb{R}^n)^2)<\infty.$$

By (4.5), we therefore have $e(\nu|_{B^n})<\infty$, from which it is a simple matter to obtain that $\int_B (v^1(x))^n\,dx<\infty$ (note that u^S dominates $2^{-n}u_\varnothing$). □

Finally, Jay Rosen has pointed out to us that when the X^i are planar Brownian motions (killed at rate $q>0$, with each m^i being Lebesgue measure), then the HRM of (5.4) is a multiple of the "intersection local time" of Geman, Horowitz, and Rosen [GHR84]. (In [GHR84] this local time was obtained as a derivative of occupation time.) To see this note that an HRM is characterized by its Revuz measure, and the intersection local time of [GHR84] has Revuz measure ν while our HRM has Revuz measure $\lambda\nu$ for some $\lambda>0$. In fact, Brownian scaling shows that $\lambda=cq^{n/2}$ which $\to 0$ as $q\to 0$. This reflects the fact that when $q=0$, Q is conservative and hence admits no homogeneous times that are finite a. s.

It would be interesting to know if some form of "central projection" could be used to construct the renormalized self intersection local time of Varadhan [V69] and Le Gall [L85].

APPENDIX

This section contains several elementary perfection results concerning homogeneous processes and random measures. We use these results in section 2 [point (*a*) of Lemma (2.10)], and in section 4 [Lemmas (4.4), (4.9), and (4.13)]. In the latter case we must have perfection, both to allow us to fix the values of the w^i, and also to permit an explicit computation of optional and cooptional projections. In the former, we need perfection only because we chose (for simplicity) to use a "perfect" form of the invariant σ-field $\mathscr{A}$.

The setting and notation is that of sections 2 and 3. We say that a process $Z\in p\mathscr{M}$ is *perfectly homogeneous* if $Z_{\mathbf{t}}(\sigma_{\mathbf{s}}\mathbf{w})=Z_{\mathbf{t}+\mathbf{s}}(\mathbf{w})$ identically in $\mathbf{t}$, $\mathbf{s}\in\mathbb{R}^n$ and $\mathbf{w}\in W$.

(A.1) PROPOSITION. — *Let $Z\in p\mathscr{M}$ be right continuous and homogeneous ($Z_{\mathbf{t}}\circ\sigma_{\mathbf{s}}=Z_{\mathbf{t}+\mathbf{s}}$, $\forall\,\mathbf{t}\in\mathbb{R}^n$, Q-a. s., $\forall\,\mathbf{s}\in\mathbb{R}^n$). Then there exists a $\bar{Z}\in p\mathscr{M}^0$ that is perfectly homogeneous and Q-indistinguishable from Z.*

Proof. — We assume without loss of generality that $Z\in bp\mathscr{M}^0$. Define

$$G_0=\{\mathbf{w}\in W;\ Z_{\mathbf{t}+\mathbf{s}}(\mathbf{w})=Z_{\mathbf{t}}(\sigma_{\mathbf{s}}\mathbf{w}),\ \forall\,\mathbf{t},\ \text{for a. e. }\mathbf{s}\in\mathbb{R}^n\},$$

$$G=\{\mathbf{w}\in W;\ \sigma_{\mathbf{r}}\mathbf{w}\in G_0\ \text{for a. e. }\mathbf{r}\in\mathbb{R}^n\}. \tag{A.2}$$

Then $G_0\in\mathscr{G}^0$, $G\in\mathscr{A}^0$, and a Fubini argument shows that $Q(W\setminus G_0)=Q(W\setminus G)=0$.

Let $\Phi=\{\varphi\in p\mathscr{R}^n;\ \int_{\mathbb{R}^n}\varphi(\mathbf{s})\,d\mathbf{s}=1\}$ and define

$$Z^{\varphi}_{\mathbf{t}}(\mathbf{w})=\int\varphi(\mathbf{s})\,Z_{\mathbf{t}-\mathbf{s}}(\sigma_{\mathbf{s}}\mathbf{w})\,d\mathbf{s},\qquad \varphi\in\Phi.$$

Note that Z^{φ} is Q-indistinguishable from Z. Clearly $Z\in p\mathscr{M}^0$ and it is easy to check that for all $\mathbf{w}\in G$, $\varphi\in\Phi$,

$$Z^{\varphi_{\mathbf{r}}}_{\mathbf{t}}(\mathbf{w})=Z^{\varphi}_{\mathbf{t}}(\mathbf{w}),\quad\forall\,\mathbf{t},\qquad\text{for a. e.}\quad \mathbf{r}\in\mathbb{R}^n, \tag{A.3}$$

where $\varphi_{\mathbf{r}} = \varphi(\,.\, - \mathbf{r})$. Integrating (A.3) against $\psi(\mathbf{r})\,d\mathbf{r}$ we find that

(A.4) $$Z^{\psi}_{.}(\mathbf{w}) = Z^{\varphi}_{.}(\mathbf{w}), \qquad \forall\, \mathbf{w} \in G, \quad \forall\, \varphi,\ \psi \in \Phi.$$

Set $\bar{Z} = 1_G Z^{\varphi}$ where $\varphi \in \Phi$ is arbitrary. Since $Z^{\varphi}_{\mathbf{t}}(\sigma_{\mathbf{s}} \mathbf{w}) = Z^{\varphi}_{\mathbf{t}+\mathbf{s}}(\mathbf{w})$ identically, it follows from (A.4) and the invariance of G that $\bar{Z}$ has the required properties. □

The following variant of (A.1) was used in section 4.

(A.5) PROPOSITION. — *Let* κ *be an* RHRM. *There exists a kernel* κ_0 *from* $(W, \mathscr{G}^0)$ *to* $(\mathbb{R}^n, \mathscr{R}^n)$ *such that* $\kappa_0(\mathbf{w}, .) = \kappa(\mathbf{w}, .)$ *for* Q-*a. e.* $\mathbf{w}$, *and* $\mathbf{t} \mapsto \kappa_0(\mathbf{w}, B + \mathbf{t})$ *is perfectly homogeneous for each* $B \in \mathscr{R}^n$.

Proof. — By a standard result on the regularization of kernels [G75] we can assume that κ itself is a kernel from $(W, \mathscr{G}^0)$ to $(\mathbb{R}^n, \mathscr{R}^n)$. Define

$$G_0 = \{\mathbf{w} \in W;\ \kappa(\sigma_{\mathbf{t}} \mathbf{w}, B - \mathbf{t}) = \kappa(\mathbf{w}, B),\ \forall\, B \in \mathscr{R}^n, \text{ for a. e. } \mathbf{t} \in \mathbb{R}^n\}$$

and let G be as in (A.2). The argument used in the proof of (A.1) shows that $\kappa_0 := 1_G \kappa^{\varphi}$ is as required, where

$$\kappa^{\varphi}(\mathbf{w}, B) = \int \varphi(\mathbf{s})\, \kappa(\sigma_{\mathbf{s}} \mathbf{w}, B - \mathbf{s})\, ds,$$

and $\varphi \in \Phi$ is arbitrary. □

Write $\hat{p} = \hat{p}^1 \hat{p}^2 \ldots \hat{p}^n$.

(A.6) PROPOSITION. — *Let* κ *be an* RHRM *such that* $\hat{p}\kappa = \kappa$ *up to* Q-*evanescence. Let* $\hat{\mathscr{G}}^0_{\mathbf{t}+} = \sigma\{Y_{\mathbf{s}};\ \mathbf{t} < \mathbf{s}\}$. *Then there exists a process* $H \in p\mathscr{M}^0$, Q-*indistinguishable from* $\mathbf{t} \mapsto \kappa(R_{\varnothing} + \mathbf{t})$, *and a* Q-*full set* $G \in \mathscr{A}^0$ *such that*

(*a*) $\mathbf{t} \mapsto H_{\mathbf{t}}(\mathbf{w})$ *is decreasing and right continuous on* $\mathbb{R}^n$, $\forall\, \mathbf{w} \in W$;

(*b*) $1_G H$ *is perfectly homogeneous;*

(*c*) $H_{\mathbf{t}} \in \hat{\mathscr{G}}^0_{\mathbf{t}+}$, $\forall\, \mathbf{t} \in \mathbb{R}^n$.

Proof. — Let $K_{\mathbf{t}} = \kappa(R_{\varnothing} + \mathbf{t})$, so that $K \in p\mathscr{M}$ and $\mathbf{t} \mapsto K_{\mathbf{t}}$ is decreasing and right continuous on $\mathbb{R}^n$. Since $\hat{p}\kappa = \kappa$, $K_{\mathbf{t}}$ is $\hat{\mathscr{G}}_{\mathbf{t}}$-measurable for each $\mathbf{t} \in \mathbb{R}^n$. Thus there exist random variables $K^0_{\mathbf{t}} \in \hat{\mathscr{G}}^0_{\mathbf{t}}$ and Q-full events $B_{\mathbf{t}} \in \hat{\mathscr{G}}^0_{\mathbf{t}}$ such that $B_{\mathbf{t}} \subset \{K_{\mathbf{t}} = K^0_{\mathbf{t}}\}$, $\forall\, \mathbf{t} \in \mathbb{R}^n$. Define

$$Z^0_{\mathbf{t}}(\mathbf{w}) = \begin{cases} \sup\limits_{\substack{\mathbf{r} > \mathbf{t} \\ \mathbf{r} \in Q^n}} K^0_{\mathbf{r}}(\mathbf{w}), & \text{if } w \in \bigcap\limits_{\substack{\mathbf{r} > \mathbf{t} \\ \mathbf{r} \in Q^n}} B_{\mathbf{r}}, \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$Z_{\mathbf{t}} = \sup_{\substack{\mathbf{r} > \mathbf{t} \\ \mathbf{r} \in Q^n}} Z^0_{\mathbf{r}}.$$

Evidently Z is Q-indistinguishable from K, $\mathbf{t} \mapsto Z_{\mathbf{t}}(\mathbf{w})$ is decreasing and right continuous, and $Z_{\mathbf{t}} \in \hat{\mathscr{G}}^0_{\mathbf{t}+}$, $\forall\, \mathbf{t} \in \mathbb{R}^n$. If we now let Z^{φ} and G be as in the proof of (A.1), then $H = Z^{\varphi}$ and G have the desired properties. □

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