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The construction of Brownian motion on the Sierpinski carpet

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ABSTRACT. — Let F_n denote the n-th stage in the construction of the Sierpinski carpet, and let W_t^n be Brownian motion on F_n, with normal reflection at the internal boundaries. We show that there exist constants α_n such that $X_t^n = W_{\alpha_n t}^n$ are tight. From this we obtain a continuous strong Markov process X on $F = \bigcap_n F_n$.

RÉSUMÉ. — Soit F_n l'ensemble obtenu à la n-ième étape de la construction du tapis de Sierpinski, et soit W_t^n le mouvement brownien sur F_n, avec réflection normale sur les parties intérieures de la frontière. Nous montrons l'existence de constantes α_n telles que la suite des lois de $X_t^n = W_{\alpha_n t}^n$ soit tendue. En passant à la limite, cela permet de construire un processus fortement markovien continu X sur $F = \bigcap_n F_n$.

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0. INTRODUCTION

The Sierpinski carpet (Sierpinski [15]) is the fractal subset of $\mathbb{R}^2$ defined as follows. Divide the unit square into nine identical squares, each with sides of length 1/3. Remove the central one. Divide each of the remaining eight squares into nine identical squares, each with sides of length 1/9 and remove the central one. See *Fig.* 1.

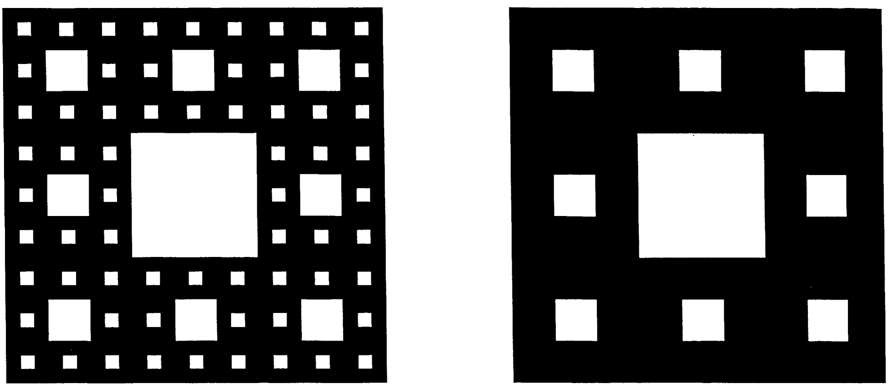

Fig. 1

Continue this process indefinitely. The carpet is what remains.

It is clear from its construction that the carpet is a closed, connected set with Lebesgue measure 0. It has Hausdorff dimension $\log 8/\log 3$.

The purpose of this paper is to construct a "Brownian motion" whose state space is the Sierpinski carpet. That is, we construct a strong Markov process with continuous paths which is nondegenerate, nondeterministic, whose state space is the Sierpinski carpet, and which is preserved under certain transformations of the state space. (*See* Section 6 for a more precise statement.)

In fact, we construct Brownian motions for a whole class of fractals that are formed in a manner similar to the Sierpinski carpet. [*See* the first two paragraphs of Section 1 and (1.1).]

The motivation of our problem comes from mathematical physics, which has an extensive literature concerning random walks and diffusions on fractals. See Rammal and Toulouse [14] for an introduction. There are also connections with Kesten's work on random walks on percolation clusters [8].

In Barlow-Perkins [1], Goldstein [7], and Kusuoka [10], a "Brownian motion" was constructed on another fractal, the Sierpinski gasket (*see* [11]; see *Fig.* 2).

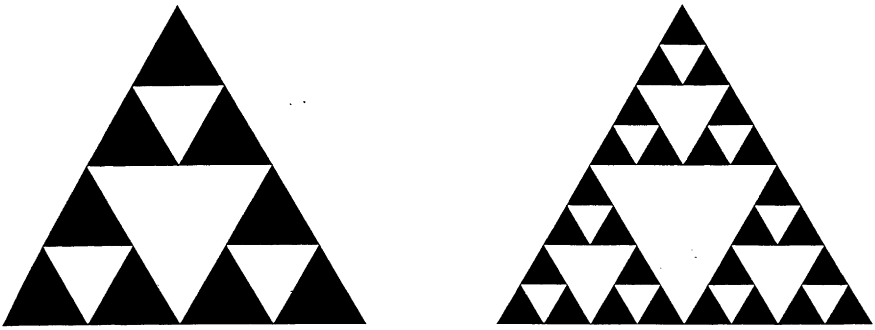

Fig. 2

This is the fractal formed by taking the equilateral triangle with sides of length of 1, dividing it into four equal equilateral triangles, removing the central one, dividing the remaining three triangles into four, and continuing. Interestingly, the Brownian motion on the Sierpinski gasket behaves quite differently from Brownian motion on $\mathbb{R}^d$. For example, instead of the ubiquitous $t^{1/2}$ scaling of ordinary Brownian motion, one has scaling by the factor $t^{\log 2/\log 5}$.

The proofs in [1], [7] and [10] relied very heavily on the fact that the Sierpinski gasket is a *finitely ramified* fractal: that is, it can be disconnected by removing finitely many points. Any continuous process X on the gasket must pass through the vertices of the triangles, and the discrete time process obtained by looking at X at the successive hits of these vertices is a simple random walk on a suitable graph. One can do exact computations with these random walks, e. g., the estimates on transition densities in [1].

By contrast, the Sierpinski carpet is an infinitely ramified fractal (*i. e.*, it is not finitely ramified). Consequently, we had to employ quite different techniques. And it should not be at all surprising that our results are much less precise than the corresponding results for the gasket.

Our procedure is to consider W_t^n, an ordinary Brownian motion with normal reflection on the boundaries of F_n, where F_n is the n-th stage in the construction of the fractal. A little thought suggests, and computer simulation confirms, that as $n \to \infty$, W_t^n escapes from neighborhoods of the starting point more and more slowly and eventually gets trapped at the starting point. We prove that there exist constants α_n such that if $X_t^n = W_{\alpha_n t}^n$, *i. e.*, X^n is W^n speeded up deterministically, then the laws of the X^n are tight yet the limit is nondegenerate.

Section 1 consists primarily of notation and definitions. In Section 2 we obtain some lower bounds for certain hitting probabilities. These are used in Section 3 to obtain a Harnack inequality, which is key to what follows. Section 4 uses the Harnack inequality to obtain bounds on expected lifetimes and Section 5 establishes tightness of the laws of the X^n. The limit process is constructed in Section 6. Section 7 contains estimates for the Green's function of the limiting Brownian motion.

Section 8 mentions some open problems. These include the construction for fractals imbedded in $\mathbb{R}^d$, $d \geqq 3$, the problem of general conditions for pointwise recurrence, the relationship of uniqueness and scale invariance and how to establish them, and the notion of Brownian motion on randomly generated fractals.

1. PRELIMINARIES

We begin by defining our state space. Let $F_0=[0, 1]^2$, the unit square with lower left corner at the origin. Let $k \geqq 3$ be fixed. Divide F_0 into k^2 identical squares with sides of length k^{-1} and vertices (ik^{-1}, jk^{-1}), $0 \leqq i, j \leqq k$. Let S_{ij} denote the closed square whose lower left corner is (ik^{-1}, jk^{-1}). Let $1 \leqq R \leqq k^2-1$. We form F_1' by removing from F_0 the closure of R of the S_{ij}, and we let $F_1=cl(F_1')$, where $cl(A)$ denotes the closure of A. We now repeat the process: for each S_{ij} that was not removed, we divide that S_{ij} into k^2 equal squares with sides of length k^{-2} and we remove the same pattern of little squares from S_{ij} as we did to form F_1 from F. Thus, if $S_{ij} \subseteqq F_1$, then

$$S_{ij} \cap F_2 = k^{-1} F_1 + (k^{-1} i, k^{-1} j).$$

Repeating this process we obtain F_3, F_4, . . ., a decreasing sequence of closed subsets of F_0 with $|F_n|=((k^2-R)/k^2)^n$, where $|A|$ denotes the Lebesgue measure of A. Let $F=\bigcap F_n$. The set F is the state space, and is a fractal with Hausdorff dimension $d=\log(k^2-R)/\log k$.

Some other topological notation: let $G_n(\varepsilon)=F_n \cap [0, 1-\varepsilon]^2$, $G(\varepsilon)=F \cap [0, 1-\varepsilon]^2$, ∂A be the boundary of A in $\mathbb{R}^2$, int(A) the interior of A, $\partial_u F_n=[0, 1]^2-[0, 1)^2$, $[x, y]$ the line segment connecting x and y, $B_\varepsilon(x)$ the open ball in $\mathbb{R}^2$ with center x and radius ε. Let $\mathscr{S}_r$ be the set of squares of side k^{-r} with lower hand corners (ik^{-r}, jk^{-r}), $i, j \in \mathbb{Z}$. We sometimes write $x \in \mathbb{R}^2$ as $(x^{(1)}, x^{(2)})$.

Let μ_n be Lebesgue measure on F_n normalized to be 1 : $\mu_n(A)=|A \cap F_n|/|F_n|$. Then μ_n converges weakly to μ, the $x^{\log(k^2-R)/\log k}$ Hausdorff measure on F.

We now make some assumptions on F_1.

(1.1)

(i) None of S_{0j}, $0 \leqq j \leqq k-1$, are removed from F_0.

(ii) int(F_1) is connected.

(iii) If S_{ij} and $S_{i+1, j+1}$ are removed, so is at least one of $S_{i+1, j}$, $S_{i, j+1}$. Similarly, if S_{ij} and $S_{i-1, j+1}$ are removed, so is at least one of $S_{i-1, j}$, $S_{i, j+1}$.

(iv) If S_{ij} is removed, so are $S_{k-1-i,\,j}$, $S_{i,\,k-1-j}$, $S_{k-1-i,\,k-1-j}$ and $S_{j,\,i}$.

Assumption (iii) prevents the occurrence in any block of four squares of only two diagonal squares remaining. Assumption (iv), that of symmetry, is crucial in what follows.

If $x=(x^{(1)}, x^{(2)})\in F_n$, let i and j be such that $ik^{-n} \leqq x^{(1)} < (i+1)\,k^{-n}$, $jk^{-n} \leqq x^{(2)} < (j+1)\,k^{-n}$. Let $S\in\mathcal{S}_n$ be the square with lower left corner at (ik^{-n}, jk^{-n}). We now form a 2×2 block of squares $D_n(x)$ as follows. If $x^{(1)} < ik^{-n}+k^{-n}/2$, adjoin the square to the left of S; otherwise adjoin the square to the right of S. If $x^{(2)} < jk^{-n}+k^{-n}/2$, adjoin the square below S; otherwise adjoin the square above S. Let $D_n(x)$ consist of S, the two adjoined squares, and a fourth square that makes $D_n(x)$ into a $2k^{-n}\times 2k^{-n}$ square. Thus $D_n(x)$ is the 2×2 block of squares which has x closest to the center. Note $|x-y| \geqq \frac{1}{2}k^{-n}$ for all $y\in\partial D_n(x)$.

Finally, we can talk about random processes. Let W_t^n be Brownian notion with absorption on $\partial_u F_n$ and normal reflection on $\partial F_n - \partial_u F_n$. It is clear that W_t^n inherits many of the path properties of W^0, and in particular,

$$P^x(W_t^n=y \text{ for some } t>0)=0 \quad \text{for all } x, y\in F_n. \tag{1.2}$$

For any process X in F or F_n or $\mathbb{R}^2$, we let

$$\begin{aligned} \tau(X) &= \inf\{t: X_t\in\partial_u F\},\\ \sigma_0^n(X) &= \inf\{t: X_t\in \bigcup_{S\in\mathcal{S}_n}\partial S\}, \\ \sigma_1^n(X) &= \inf\{t\geqq 0: X_t\in\partial D_n(X_0)\}, \end{aligned} \tag{1.3}$$

and

$$\sigma_{r+1}^n(X)=\inf\{t\geqq \sigma_r^n(X): X_t\in\partial D_n(X_{\sigma_r^n(X)})\}.$$

In Sect. 4 we will use

LEMMA 1.1. — *Let* X, $Y_1, \ldots, Y_n$ *be non-negative random variables satisfying*

(*a*) $$X \geqq \sum_{i=1}^n Y_i$$

(*b*) $P(Y_i \leqq x \mid \sigma(Y_j, j\leqq i-1)) \leqq p+bx, \qquad i=1, \ldots, n, \quad x\geqq 0,$

where $0<p<1$, $b>0$. *Then*

$$P(X\leqq x) \leqq \exp\left(2\left(\frac{bnx}{p}\right)^{1/2} - n\log\frac{1}{p}\right), \qquad x\geqq 0.$$

Proof. — Let Z be a random variable with distribution function G(*x*), where

$$G(x)=p+bx, \qquad 0\leqq x\leqq (1-p)/b, \qquad G(0-)=0.$$

Then $E(e^{-uY_i} \mid \sigma(Y_j, j \leqq i-1)) \leqq E(e^{-uZ})$. Now, writing $q=1-p$,

$$\begin{aligned} E e^{-uZ} &= p + \int_0^{q/b} e^{-ux}\, b\, dx \\ &= p + bu^{-1}(1-e^{-uq/b}). \end{aligned}$$

Thus

$$\begin{aligned} P(X \leqq x) &= P(e^{-uX} \geqq e^{-ux}) \\ &\leqq e^{ux} E e^{-uX} \\ &\leqq e^{ux}(p+bu^{-1}(1-e^{-uq/b}))^n \\ &\leqq e^{ux}(p+bu^{-1})^n \\ &\leqq p^n \exp\left(ux + \frac{bn}{pu}\right). \end{aligned}$$

The result now follows by setting $u=(bn/px)^{1/2}$. □

2. KNIGHT'S MOVES

Let F_n and W^n be as described in the previous section. We now wish to obtain estimates on the probability of certain kinds of behaviour for W^n. As we will ultimately be letting $n \to \infty$ (in Section 6) these bounds must be uniform in n.

We will adopt the convention that any constant (δ say) mentioned in the statement of a theorem or lemma is independent of n, unless its dependence is indicated in some obvious way (such as writing δ_n).

Let $n \geqq 0$ be fixed, and let $m \leqq n$. F_n consists of $(k^2-R)^m$ identical sets, each contained in a square in $\mathscr{S}_m$, and each symmetric under the rotations and reflections which preserve the square.

By this symmetry the behaviour of W^n between times $\sigma_i^m = \sigma_i^m(W^n)$ and σ_{i+1}^m can be reduced to the following (see *Fig.* 3).

$W^n(\sigma_i^m(W^n))$ lies in the bold cross in the center, and after time σ_i^m the process W^n runs in the 4 squares shown [which is the set $D_m(W^n(\sigma_i^m))$] until it hits the outer boundary, at time σ_{i+1}^m.

We wish to obtain bounds on the probability $W^n(\sigma_{i+1}^m)$ lies in various parts of the boundary; to do this we will describe two "moves" the process can make from $W^n(\sigma_i^m)$, and will obtain lower bounds on their success.

$D_m(W^n(\sigma_i^m))$ is contained in four squares in $\mathscr{S}_m$ and one or more of these squares may be in F_n^c: to describe the situation, we first reduce to a canonical set-up.

First, we move the square $D_m(W^n(\sigma_i^m))$ so that its center is at the origin, and we then expand it by k^m. We then rotate it so that $W^n(\sigma_i^m)$ lies on

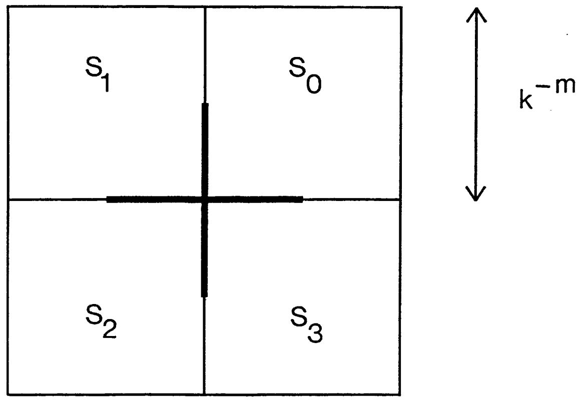

Fig. 3

the positive x-axis. Let φ be the linear map on $\mathbb{R}^2$ we have described, and let

$$X_t = \varphi(W^n(\sigma_i^m + t)), \qquad t \geqq 0.$$

X is a time changed reflecting Brownian motion on $\varphi(F_n) \subseteqq \mathbb{R}^2$: we write $P^{(x)}(\cdot)$ for $P(\cdot \mid X_0 = x)$. Let $S_0, \ldots, S_3$ be the 4 squares in $\mathcal{S}_0$ which make up $\varphi(D_m(W^n(\sigma_i^m)))$. We say S_i is *filled* if $\operatorname{int}(S_i) \subseteqq \varphi(F_n)^c$, and *unfilled* otherwise: the process X can only move in the unfilled squares. As W^n has reached $W^n(\sigma_i^m)$, at least one of S_0, S_3 must be unfilled. Assume it is S_0; if not, we can modify φ to include a reflection in the x-axis.

Let $B = \varphi(D_m(W^n(\sigma_i^m)))$. There are now 7 possible arrangements of filled and unfilled squares in B:

(2.1) (*a*) all unfilled

(b_j) S_j filled, the rest unfilled ($1 \leqq j \leqq 3$)

(c_1) S_1 and S_2 filled, the rest unfilled

(c_2) S_2 and S_3 filled, the rest unfilled

(*d*) S_1, S_2, S_3 filled, the rest unfilled.

[The case when S_1 and S_3 only are filled has been ruled out by assumption (1.1) (iii)].

We now describe the first of our two moves, "the corner". Set

$$T = \inf\{t \geqq 0 : X_t \in (S_0 \cup S_3)^c\}.$$

Thus $X_T \in \partial(S_0 \cup S_3)$: we divide this boundary into 12 sections, each of length 1/2, which we label counterclockwise $L_1, \ldots, L_{12}$, starting with $L_1 = [(1, 0), (1, 1/2)]$. Note that $T = \sigma_1^0(X)$ if and only if $X_T \notin \bigcup_{j=5}^{8} L_j$. Set

$$p_i(x_0) = P(X_T \in L_i \mid X_0 = x_0).$$

THEOREM 2.1. — *(a) If* S_3 *is filled then* $p_6(x) \geqq 1/6$ *for all* $x \in [(0, 0), (0, 1/2)]$.

(b) If S_3 *is unfilled then* $p_6(x) \geqq 1/12$ *for all* $x \in [(0, 0), (0, 1/2)]$.

Proof. — By (1.2) $P(X_T \in L_i \cap L_{i+1}) = 0$. So $\sum_{i=1}^{12} p_i(x) = 1$.

The proof will use the reflection symmetry of X in B. Indeed, from this symmetry it is clear that, if S_3 unfilled then $p_i(x) = p_{13-i}(x)$ for $1 \leqq i \leqq 6$. Thus, it is sufficient to consider case *(a)*.

We use two kinds of reflection: (simple) first entry reflection, and "last exist reflection"-which we describe in more detail below.

Let $S = \inf\{t \geq 0 : X_t \in [(1/2, 0), (1/2, 1)]\}$. Since $S_0 \cap \varphi(F_n)$ is symmetric about the line $x^{(1)} = 1/2$, we have

$$\begin{aligned} p_1(x_0) &= P^{x_0}(X_T \in L_1, S < T) \\ &= E^{x_0} 1_{(S < T)} P^{X_S}(X_T \in L_1) \\ &= E^{x_0} 1_{(S < T)} P^{X_S}(X_T \in L_6) \quad \text{(symmetry)} \\ &= P^{x_0}(X_T \in L_6, S < T) \\ &\leqq p_6(x_0). \end{aligned}$$

Similarly we deduce that $p_2(x_0) \leqq p_5(x_0)$, $p_3(x_0) \leqq p_4(x_0)$.

We now consider reflection in the diagonal line $A_d = [(0, 0), (1, 1)]$. This is not so simple, as there is a reflecting boundary along the $x^{(1)}$-axis, while the $x^{(2)}$-axis is absorbing (for the process $X_{t \wedge T}$).

Write $A_x = [(0, 0), (1, 0)]$, and set

$$\begin{aligned} &\eta_0 = 0 \\ &\eta_r = \inf\{t \geqq \xi_{r-1} : X_t \in A_d\}, \qquad r \geqq 1 \\ &\xi_r = \inf\{t \geqq \eta_r : X_t \in A_x\}, \qquad r \geqq 0. \end{aligned}$$

As P^{x_0} (X_t hits 0) = 0, the stopping times ξ_r, η_r do not accumulate. So,

$$\begin{aligned} (2.2) \qquad p_4(x_0) &= \sum_{r=1}^{\infty} P^{x_0}(X_T \in L_4, \eta_r < T < \eta_{r+1}) \\ &= \sum_{r=1}^{\infty} E^{x_0} 1_{(T > \eta_r)} P^{X(\eta_r)}(X_T \in L_4, T < \eta_1). \end{aligned}$$

For $x \in A_d$, using the symmetry of X stopped at ξ_0,

$$\begin{aligned} P^x(X_T \in L_4, T < \eta_1) &= P^x(X_{\xi_0} \in L_4) \\ &= P^x(X_{\xi_0} \in L_1) \\ &\leqq P^x(X_T \in L_1, T < \eta_1). \end{aligned}$$

Hence, substituting in (2.2),

$$p_4(x_0) \leqq \sum_{r=1}^{\infty} P^{x_0}(X_T \in L_1, \eta_r < T < \eta_{r+1})$$
$$\leqq p_1(x_0).$$

An identical argument proves that $p_3(x_0) \leqq p_2(x_0)$. We will call this "last exit reflection" in the line A_d. A similar argument, using "last exit reflection" in the line [(0, 1/2), (1, 1/2)] gives $p_5(x_0) \leqq p_6(x_0)$, $p_2(x_0) \leqq p_1(x_0)$.

Putting these inequalities together yields $p_4(x_0) \leqq p_1(x_0) \leqq p_6(x_0)$, and $p_3(x_0) \leqq p_2(x_0) \leqq p_5(x_0) \leqq p_6(x_0)$. Hence, as the sum of the $p_i(x_0)$ is 1, $p_6(x_0) \geqq 1/6$. □

Remarks. — 1. If $x_0=0$, then $p_6(x_0)=1$: thus, in case (*a*) p_6 is the only one of the p_i for which a uniform lower bound is possible.

2. We suspect this bound is the best possible. Let k be large, and suppose F_1 is formed by removing all the squares except a very thin corridor around the edge of F_0, and a thicker, but still thin, corridor around the lines [(1/2, 0), (1/2, 1)], [(0, 1/2), (1, 1/2)]. Then, with high probability, if $x_0=(1/2, 0)$, X will fail to find the entrance to the first corridor, and will therefore eventually move close to the center of the square. And from there it is approximately equally likely to hit any of the L_i.

We now consider the basic square B. ∂B has length 8: we divide it into 16 sections, each of length 1/2, which we label counterclockwise $L_1, \ldots, L_{16}$, with L_1 as before. A move from $x \in [(0, 0), (1/2, 0)]$ to L_1 we call a "knight's move", and we wish to find a lower bound on its success, that is on $P^x(X(\sigma_1^0(X)) \in L_1)$.

Let $T=\sigma_1^0(X)$, and write $p_i(x)=P^x(X_T \in L_i)$ for $x \in [(0, 0), (1, 0)]$. If some of the S_j are filled, then some of the $p_i(x)$ will be zero.

It is necessary to treat each of the 7 cases given in (2.1) separately. We begin with (*d*), where we can use the same techniques as for the corner move. Reflecting in the line [(0, 0), (1, 1)] we have $p_4(x) \leqq p_1(x)$, $p_3(x) \leqq p_2(x)$. Reflecting on the last exit from [(0, 1/2), (1, 1/2)] we have $p_2(x) \leqq p_1(x)$, and thus $p_1(x) \geqq p_i(x)$ for $2 \leqq i \leqq 4$. Hence, in case (*d*)

$$p_1(x) \geqq 1/4 \quad \text{for all } x \in [(0, 0), (1/2, 0)]. \tag{2.3}$$

Some of the remaining cases [(c_1) and (*a*), for example] can be handled easily by this kind of argument. But (b_2), especially, is rather tricky, and so we adopt a more general approach.

Set

$$\Gamma = \bigcup_{i=0}^{3} \partial S_i,$$

$$A_i = \left[(0, 0), \left(\cos\frac{i\pi}{2}, \sin\frac{i\pi}{2}\right)\right] \quad \text{for} \quad i=0, \ldots, 3,$$

$$\eta_0 = 0, Y_0 = 0,$$

$$\eta_{r+1} = \inf\{t \geqq \eta_r : X_t \in \Gamma - A_{Y_r}\}, \qquad r \geqq 0,$$

$$N = \min\{r \geqq 1 : X_{\eta_r} \in \bigcup_{i=0}^{16} L_i\},$$

and if $r < N$, let Y_r be such that $X_{\eta_r} \in A_{Y_r}$. Define Y_N to be the $i \in \{0, 1, 2, 3\} - \{Y_{N-1}\}$ such that X_{η_N} is closest to A_i.

Since X does not hit 0, if $x \neq 0$ then the η_r do not accumulate P^x-a. s., and Y_r is uniquely defined for $1 \leqq r \leqq N$. Also, from standard properties of Brownian motion, it is clear $P^x(N < \infty) = 1$, for $x \neq 0$. Between times η_r and η_{r+1} the process X moves in one or two of the squares S_i: we pull this back using symmetry to give a process in the square S_0.

Set

$$\tilde{Z}_t^i = X_{\eta_i + t}, \qquad 0 \leqq t \leqq \eta_{i+1} - \eta_i, \quad 0 \leqq i \leqq N-1.$$

Let Z_t^i be $\tilde{Z}_t^i$ rotated and reflected to give a process in S_0: that is, $Z_t^i = f(\tilde{Z}_t^i, Y_i)$, where

$$f((x_1, x_2), i) = \left(x_1 \cos\frac{i\pi}{2} - x_2 \sin\frac{i\pi}{2}, \left|x_1 \sin\frac{i\pi}{2} + x_2 \cos\frac{i\pi}{2}\right|\right).$$

We set $Z_t^i = \partial$ for $t > \eta_{i+1} - \eta_i$. So Z^i is a process on $S_0 \cap \varphi(F_n)$, with reflection on the x-axis, and killed on hitting the rest of ∂S_0. Conditional on Z_0^i, Z^i is independent of $\sigma(Z_s^j, s \geqq 0, j \leqq i-1) \vee \sigma(Y_j, 0 \leqq j \leqq i)$. Also, by the symmetry of B, Y_r, $0 \leqq r \leqq N$, is a random walk on $\{0, 1, 2, 3\}$ and, conditional on N, is independent of the Z_t^i. By using information independent of the process X we can extend Y_r, $0 \leqq r \leqq N$, to a time homogeneous random Y_r, $r \geqq 0$. The exact law of Y_r depends on the arrangement of filled and unfilled squares: for example, in case (b_2) we have (using addition mod 4)

$$P(Y_{r+1} = i+1 \mid Y_r = i) = P(Y_{r+1} = i-1 \mid Y_r = i) = 1/2 \quad \text{if} \quad i = 0, 1,$$
$$P(Y_{r+1} = 1 \mid Y_r = 2) = P(Y_{r+1} = 0 \mid Y_r = 3) = 1.$$

However, we always have $Y_{2r} \in \{0, 2\}$, $Y_{2r+1} \in \{1, 3\}$.

Let $x \in A_0$, with $x \neq 0$, be fixed. Let

$$q_r(i) = P^x(Z^r_{\eta_{r+1} - \eta_r} \in L_i, r < N).$$

Since $N < \infty$ P^x-a. s., $\sum_{r=0}^{\infty} \sum_{i=0}^{3} q_r(i) = 1$. For $r \geqq 1$

$$\begin{aligned} P^x(N=r, X_{\eta_N} \in L_1) &= P^x(Y_{r-1}=0, Y_r=1, N > r-1, Z^i_{\eta_r - \eta_{r-1}} \in L_1) \\ &\quad + P^x(Y_{r-1}=1, Y_r=0, N > r-1, Z^i_{\eta_r - \eta_{r-1}} \in L_4) \\ &= q_{r-1}(1)\, P^x(Y_{r-1}=0, Y_r=1) + q_{r-1}(4)\, P^x(Y_{r-1}=1, Y_r=0). \end{aligned}$$

So,

$$\begin{aligned} p_1(x) &= \sum_{r=1}^{\infty} P^x(N=r, X_{\eta_N} \in L_1) \\ &= q_0(1)\, P^x(Y_0=0, Y_1=1) + \sum_{s=1}^{\infty} q_s(1)\, P^x(Y_s=0, Y_{s+1}=1) \\ &\quad + \sum_{s=1}^{\infty} q_s(4)\, P^x(Y_s=1, Y_{s+1}=0). \\ &= q_0(1)\, P^x(Y_0=0, Y_1=1) + \sum_{s=1}^{\infty} q_{2s}(1)\, P^x(Y_{2s}=0, Y_{2s+1}=1) \\ &\quad + \sum_{s=0}^{\infty} q_{2s+1}(4)\, P^x(Y_{2s+1}=1, Y_{2s+2}=0). \end{aligned}$$

Write

$$\alpha = q_0(1), \qquad \beta = \sum_{s=1}^{\infty} q_{2s}(1), \qquad \gamma = \sum_{s=0}^{\infty} q_{2s+1}(4).$$

Then,

$$\begin{aligned} p_1(x) \geqq \alpha\, P^x(Y_0=0, Y_1=1) &+ \beta \inf_{s \geqq 1} P(Y_{2s}=0, Y_{2s+1}=1) \\ &+ \gamma \inf_{s \geqq 0} P(Y_{2s+1}=1, Y_{2s+2}=0), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} p_1(x) \leqq \alpha\, P^x(Y_0=0, Y_1=1) &+ \beta \sup_{s \geqq 1} P(Y_{2s}=0, Y_{2s+1}=1) \\ &+ \gamma \sup_{s \geqq 0} P(Y_{2s+1}=1, Y_{2s+2}=0). \end{aligned} \tag{2.5}$$

While α, β, γ do not depend on the configuration of filled and unfilled squares, the probabilities relating to Y do. In case (*d*), $Y_{2r}=0$, $Y_{2r+1}=1$ a. s., and thus

$$p_1(x) = \alpha + \beta + \gamma.$$

So, by (2.5) and (2.3),

(2.6) $$\alpha+\beta+\gamma \geqq 1/4.$$

Write $I=\inf_{s\geqq 1} P(Y_{2s}=0,\ Y_{2s+1}=1)$, $J=\inf_{s\geqq 0} P(Y_{2s+1}=1,\ Y_{2s+2}=0)$. Routine calculations with the random walks Y give the following Table:

Case	I	J	Lower bound on $p_1(x)$ from (2.4)
a	1/4	1/4	$\frac{1}{2}\alpha+\frac{1}{4}\beta+\frac{1}{4}\gamma$
b_1	1/3	1/3	$\frac{1}{2}\alpha+\frac{1}{3}\beta+\frac{1}{3}\gamma$
b_2	1/3	1/4	$\frac{1}{2}\alpha+\frac{1}{3}\beta+\frac{1}{4}\gamma$
b_3	1/3	1/3	$\alpha+\frac{1}{3}\beta+\frac{1}{2}\gamma$
c_1	1/2	1/2	$\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\gamma$
c_2	1/2	1/2	$\alpha+\frac{1}{2}\beta+\frac{1}{2}\gamma$
d	1	1	$\alpha+\beta+\gamma$

THEOREM 2.2. — *For* $x \in [(0, 0), (1, 0)]$
$p_1(x) \geqq 1/16$ *in cases* (a) *and* (b_2),
$p_1(x) \geqq 1/12$ *in cases* (b_1) *and* (b_3),
$p_1(x) \geqq 1/8$ *in cases* (c_1) *and* (c_2),
$p_1(x) \geqq 1/4$ *in case* (d).

Proof. — If $x \neq 0$ this is immediate from (2.6), and the table above. The case $x=0$ follows by the continuity of hitting distributions for Brownian motion. □

3. A HARNACK INEQUALITY AND CONTINUITY OF HITTING DISTRIBUTIONS

Let $\tau=\tau(W^n)=\inf\{t \geqq 0: W^n_t \in \partial_u F_n\}$, and for $A \subseteqq \partial_u F_n$ set

$$h_n(x, A)=P^x(W^n_\tau \in A).$$

THEOREM 3.1 (Harnack's inequality). — *For each* $0 < \varepsilon \leqq 1/2$ *there exists a constant* θ_ε, *independent of n, such that*

$$\theta_\varepsilon^{-1} \leqq \frac{h_n(x, A)}{h_n(y, A)} \leqq \theta_\varepsilon \quad \textit{for all } x, y \in G_n(\varepsilon), \quad n \geqq 1.$$

The proof uses the following

LEMMA 3.2. — *Let $0 < \varepsilon \leqq 1/2$. There exists $\delta_\varepsilon > 0$ depending only on ε such that, if $x, y \in G_n(\varepsilon)$, and $\gamma(t)$, $0 \leqq t \leqq 1$, is a continuous curve from y to $\partial_u F_n$ contained in F_n, then*

$$P^x(W^n \textit{ hits } \gamma \textit{ before time } \tau) > \delta_\varepsilon.$$

Proof. — We consider first the case $\varepsilon = 1/3$ for the standard carpet ($k=3$, center cut out as in Figure 1). Let $B = [(1/3, 0), (1/3, 1/3)] \cup [(0, 1/3), (1/3, 1/3)]$: as the curve γ must cross B, it is sufficient to deal with the case $y \in B$, $\{\gamma(s), 0 < s < 1\} \cap B = \varnothing$. Also, by symmetry we can suppose $y \in [(1/3, 0), (1/3, 1/3)]$. It is also clearly sufficient to take $x \in B$.

Consider the case $x \in [(1/3, 0), (1/3, 1/6)]$. A "corner move" takes W^n to the line segment $[(1/3, 0), (1/2, 0)]$, and 4 additional moves (two corners and two knight's) takes W^n to, successively $[(2/3, 0), (2/3, 1/6)]$, $[(1/2, 0), (2/3, 0)]$, $[(2/3, 0), (2/3, 1/6)]$ (again) and finally $[(1/2, 1/3), (2/3, 1/3)]$. At this point W^n has moved from $[(1/2, 0), (2/3, 0)]$ to $[(1/2, 1/3), (2/3, 1/3)]$ inside the square $[1/3, 1] \times [0, 2/3]$, and so W^n must have crossed γ. As each move has a probability of at least 1/16, we have, in the case,

$$P^x(W^n \text{ hits } \gamma \text{ before } \tau) \geqq (16)^{-5}.$$

If x lies elsewhere in B then an appropriate sequence of moves takes W^n to the line segment $[(1/3, 0), (1/3, 1/6)]$: at most 3 moves are required. So, for any $x \in B$

$$P^x(W^n \text{ hits } \gamma \text{ before } \tau) \geqq (16)^{-8}.$$

The case of general ε, k is more complicated, but the idea is exactly the same. Choose r so that $4k^{-4} < \varepsilon$, then, given $x, y \in G_n(\varepsilon)$ and using the corners and knight's moves to move across the squares in $\mathcal{S}_r$, at most $\nu(x, y)$ moves will be required for W^n, starting from x, to circle y and intersect itself. Evidently $\nu_r = \max_{x, y} \nu(x, y)$ is finite: taking $\delta = 16^{-\nu_r}$ the result follows in the general case. □

Proof of Theorem 3.1. — Let $M_t = h_n(W^n_{t \wedge \tau}, A)$. Thus M is a P^y-martingale, and $0 \leqq M \leqq 1$. M is continuous, as W^n is a reflecting Brownian motion, and M is adapted to the filtration of W^n.

Let $\eta < 1$, and let

$$T = \inf\{t \geqq 0 : M_t < \eta\, h_n(y, A)\} \wedge \tau.$$

Then

$$\begin{aligned} h_n(y, A) &= E^y h_n(W^n_T, A) \\ &= E^y(h_n(W^n_\tau, A); T = \tau) + E^y(h_n(W^n_T, A); T < \tau) \\ &\leqq P^y(T = \tau) + \eta\, h_n(y, A)\, P^y(T < \tau). \end{aligned}$$

Rearranging, we have

$$P^y(T < \tau) \leqq \frac{1-h_n(y, A)}{1-\eta h_n(y, A)} < 1,$$

so that $P^y(T=\tau) > 0$. Thus there exists at least one curve γ with $\gamma(0)=y$, $\gamma(1) \in \partial_u F_n$ and satisfying $h_n(\gamma(t), A) \geqq \eta h_n(y, A)$ for $0 \leqq t \leqq 1$. Let $S=\inf\{t \geqq 0: W^n_t \in \{\gamma(s), 0 \leqq s \leqq 1\}\}$. By Lemma 2.2, $P^x(S < \tau) > \delta_\varepsilon$, and so, as M is also a P^x-martingale,

$$\begin{aligned} h_n(x, A) &= E^x h_n(W^n_{S \wedge \tau}, A) \\ &\geqq E^x(h_n(W^n_S, A); S < \tau) \\ &\geqq \delta_\varepsilon \eta h_n(y, A). \end{aligned}$$

Since η is arbitrary, the inequality above holds with $\eta=1$. Exchanging the roles of x and y, we have proved the theorem, with $\theta_\varepsilon=\delta_\varepsilon^{-1}$. □

These two results also apply if we run W^n in the large square $[-1, 1]^2$: the constants θ_ε, δ_ε are however different, but the proof is essentially identical. Using scaling we deduce.

COROLLARY 3.3. — *Let* $x \in F_n$, $r \geqq 1$, *and* $y \in D_r(x)$ *with* $d(y, \partial D_r(x)) > \varepsilon k^{-r}$. *Then*

$$1/\theta'_\varepsilon \leqq P^x(W^n(\sigma^r_1) \in A)/P^y(W^n(\sigma^r_1) \in A) \leqq \theta'_\varepsilon \quad \textit{for all } A \subseteqq \partial D_r(x).$$

For simplicity we now write θ_ε for the maximum of the constants introduced in Theorem 3.2, Corollary 3.3.

COROLLARY 3.4. — *There exists a measure* $\nu(dx)$ *on* $\partial_u F_n$ *such that for each* $y \in F_n - \partial_u F_n$, $h(y, .) \ll \nu$.

Proof. — It is enough to take $\nu(\cdot)=h(0, \cdot)$. □

In what follows, by harmonic function we mean harmonic with respect to the generator of W^n.

COROLLARY 3.5. — *Let g be harmonic on* $F_n - \partial_u F_n$, *and continuous on* F_n. *Then*

$$\theta_\varepsilon^{-1} \leqq g(x)/g(y) \leqq \theta_\varepsilon \quad \textit{for all } x, y \in G_n(\varepsilon).$$

Proof. — As $g(x)=E^x g(W^n_\tau)$, this is immediate from Theorem 2.1. □

We now apply the Harnack's inequality to deduce that bounded harmonic functions in F_n are uniformly Holder continuous away from the boundary. The proof uses the oscillation argument of Moser [12]. See also Krylov and Safonov [9].

For $f: F_n \to \mathbb{R}$, $x \in F_n$, set

$$O_m(x, f) = \sup_{y \in D_m(x) \cap F_n} f(y) - \inf_{y \in D_m(x) \cap F_n} f(y)$$

to be the oscillation of f in $D_m(x)$.

Let $\varepsilon_0 = \frac{1}{2} - 1/k$, and $p = 1 - (4\,\theta_{\varepsilon_0})^{-1}$.

LEMMA 3.8. — *Let f be bounded and harmonic on F_n. Let $x \in F_n$ with $D_m(x) \subseteq [-1, 1]^2$. Then*

$$O_{m+1}(x, f) \leqq p\, O_m(x, f).$$

Proof. — Note that $d(y, \partial D_m(x)) \geqq k^{-m}\left(\frac{1}{2} - 1/k\right)$ for all $y \in D_{m+1}(x)$. By rescaling f if necessary, it is enough to prove the result when $-1 \leqq f \leqq 1$, and $O_m(x, f) = 2$. Write $T = \inf\{t \geqq 0 : W_t^n \in \partial D_m(x)\}$, and let $A = \{y \in \partial D_m(x) : f(y) \leqq 0\}$. Now either $P^x(W_T^n \in A) \geqq 1/2$ or $P^x(W_T^n \in A^c) \geqq 1/2$; by multiplying f by -1 if necessary we may assume the first.

For $y \in D_{m+1}(x)$

$$\begin{aligned} f(y) &= E^y f(W_T^n) \\ &= E^y(f(W_T^n);\ W_T^n \in A^c) + E^y(f(W_T^n);\ W_T^n \in A) \\ &\leqq P^y(W_T^n \in A^c) + 0 . P^y(W_T^n \in A^c) \\ &= 1 - P^y(W_T^n \in A). \end{aligned}$$

By Corollary 3.3. $P^y(W_T^n \in A) \geqq \theta_{\varepsilon_0}^{-1} P^x(W_T^n \in A)$, and so, for any $y \in D_{m+1}(x)$,

$$\begin{aligned} f(y) &\leqq 1 - \theta_{\varepsilon_0}^{-1} P^x(W_T^n \in A) \\ &\leqq 1 - \frac{1}{2}\theta_{\varepsilon_0}^{-1}. \end{aligned}$$

Hence $O_{m+1}(x, f) \leqq 2 - \frac{1}{2}\theta_{\varepsilon_0}^{-1} = p\, O_m(x, f)$. □

THEOREM 3.9. — *Let $\varepsilon > 0$. There exist constants β_1, c_ε such that, if f is bounded and harmonic on F_n, then*

$$(3.1)\qquad |f(x) - f(y)| \leqq c_\varepsilon |x - y|^{\beta_1} \|f\|_\infty \quad \textit{for all } x, y \in G_n(\varepsilon).$$

Proof. — Let $x, y \in G_n(\varepsilon)$. Choose m such that

$$\frac{1}{2}k^{-(m+1)} < |x - y| \leqq \frac{1}{2}k^{-m},$$

and r such that $2k^{-r} \leqq \varepsilon \leqq 2k^{-(r-1)}$. Thus $y \in D_m(x)$, and $D_r(x) \cap \partial_u F_n = \varnothing$. So if $m \geqq r$

$$\begin{aligned} |f(x) - f(y)| &\leqq O_m(x, f) \\ &\leqq p^{m-r} O_r(x, f) \end{aligned}$$

$$\leqq 2p^{-r}p^m \|f\|_\infty$$
$$\leqq c_\varepsilon |x-y|^{\beta_1} \|f\|_\infty$$

where c_ε depends only on r, and $\beta_1 = \log p^{-1}/\log k$. If $m \leqq r$ then $|x-y| \geqq \varepsilon/4k^2$, and $|f(x)-f(y)| \leqq 2\|f\|_\infty$, and so, adjusting c_ε we have (3.1). □

Remark. — Note that the proof shows us we can take $c_\varepsilon = K\varepsilon^{-\beta}$ for some constants K and β.

4. INEQUALITIES

Let $n \geqq 1$, $\tau = \tau(W^n)$ be the first exit of W^n from F_n, and let

$$g_n(x) = E^x \tau, \qquad x \in F_n,$$
$$\alpha_n = \sup_{x \in F_n} g_n(x), \qquad \beta_n = \inf_{x \in G_n(1/2)} g_n(x), \qquad \gamma_n = \sup_{x \in G_n(1/2)} g_n(x).$$

As W^n is a reflecting Brownian motion on a Lipschitz domain, g_n is continuous, and so the sups and infs above are attained. In this section we obtain inequalities relating α_n, β_n, γ_n: unless indicated otherwise all constants will be independent of n.

By elementary inclusion

$$(4.1) \qquad \beta_n \leqq \gamma_n \leqq \alpha_n.$$

Let $x \in F_n$, and let y be the center of the square $D_m(x)$. Define $\psi: \mathbb{R}^2 \to \mathbb{R}^2_+$ by $\psi((z^{(1)}, z^{(2)})) = (k^m |z^{(1)} - y^{(1)}|, k^m |z^{(2)} - y^{(2)}|)$, so that ψ maps $D_m(x)$ onto $[0, 1]^2$. Set $Y_t = \psi(W^n(tk^{2m}))$: under P^x, Y is a reflecting Brownian motion on F_{n-m}, and thus equal in law to W^{n-m} under $P^{\psi(x)}$. We deduce that

$$(4.2) \qquad k^{-2m}\beta_{n-m} \leqq E^x \sigma_1^m(W^n) \leqq k^{-2m}\gamma_{n-m}.$$

Remark. — By the same kind of reflection argument as in Section 2 we can show that g_n attains its maximum in $G_n(1/2)$, so that $\alpha_n = \gamma_n$. But we do not need this.

Let $k_0 = [k/2]$. Grom $G_n(1/2)$ it is necessary to cross at least k_0 squares in $\mathscr{S}_1$ to leave F_n. So,

$$(4.3) \qquad \beta_n \geqq k_0 k^{-2} \beta_{n-1},$$

and more generally, as $\frac{1}{2}k^r - 1 \geqq \frac{1}{2}k^{r-1}$

$$(4.4) \qquad \beta_n \geqq (2k)^{-1} k^{-r} \beta_{n-r}.$$

By the knight's and corner moves of Section 2 and scaling there exist m, $\eta > 0$ (independent of n) such that, for all $x \in F_n$, $P^x(\sigma_m^1 > \tau) > \eta$.

Hence writing $\sigma=\sigma_m^1(W^n)$,

$$\begin{aligned} g_n(x) &= E^x \tau \\ &= E^x(\tau;\ \tau > \sigma) + E^x(\tau;\ \tau \leqq \sigma) \\ &\leqq E^x(1_{(\tau > \sigma)} E^{W_\sigma^n} \tau) + E^x \sigma \\ &\leqq (1-\eta)\,\alpha_n + mk^{-2}\gamma_{n-1}. \end{aligned}$$

Take the supremum in x of the left hand side to obtain

$$\alpha_n \leqq c_1 \gamma_{n-1}, \tag{4.5}$$

where $c_1 = m/k^2 \eta$.

Our final inequality is more delicate. Set $r_n=\gamma_n/\beta_n$, and let $x_1, x_2 \in G_n(1/2)$ be points with $\beta_n=g_n(x_1)$, $\gamma_n=g_n(x_2)$. (Note that x_1, x_2 will in general depend on n). As $k \geqq 3$, $\sigma_1^1 < \tau$, P^{x_i}-a. s., and so, writing $S=\sigma_1^1$,

$$g_n(x_i) = E^{x_i} S + E^{x_i} g_n(W_S^n). \tag{4.6}$$

LEMMA 4.1. — *There exists a constant $c_2 > 0$ such that*

$$c_2^{-1} E^{x_1} g_n(W_S^n) \leqq E^{x_2} g_n(W_S^n) \leqq c_2 E^{x_1} g_n(W_S^n). \tag{4.7}$$

Proof. — This is similar to the proof of Theorem 3.1. Note, however, that the measures $P^{x_i}(W_S^n \in \cdot)$ may have disjoint supports.

Let $A=\{y \in \bigcup_{Q \in \mathscr{S}_1} \partial Q:\ D_1(y)=D_1(x_1)\}$. Using the knight and corner moves of Section 2 there exist $m \geqq 1$, $\eta > 0$ such that $P^{x_2}(W^n(\sigma_m^1) \in A,\ \sigma_m^1 < \tau) > \eta$. Since $y \to E^y g_n(W_S^n)$ is harmonic in $D_1(x_1)$, by Theorem 3.1, writing $\theta=\theta_{1/2}$,

$$\theta^{-1} E^{x_1} g_n(W_S^n) \leqq E^y g_n(W_S^n) \leqq \theta E^{x_1} g_n(W_S^n) \quad \text{for all } y \in A.$$

So,

$$\begin{aligned} E^{x_2} g_n(X_S) &= E^{x_2}(\tau - S) \\ &\geqq E^{x_2}(\tau-\sigma_m^1;\ W^n(\sigma_m^1) \in A,\ \sigma_m^1 < \tau) \\ &= E^{x_2}(1_{(W^n(\sigma_m^1) \in A,\ \sigma_m^1 < \tau)} E^{W^n(\sigma_m^1)} \tau) \\ &\geqq \eta\theta^{-1} E^{x_1} g_n(W_S^n). \end{aligned}$$

Reversing the roles of x_1 and x_2 gives the other side of (4.7). □

From (4.2), (4.6) and (4.7) we have

$$\begin{aligned} \gamma_n &\leqq k^{-2}\gamma_{n-1} + E^{x_2} g_n(W_S^n) \\ &\leqq k^{-2} r_{n-1}\beta_{n-1} + c_2 E^{x_1} g_n(W_S^n) \\ &\leqq (r_{n-1} \vee c_2)(k^{-2}\beta_{n-1} + E^{x_1} g_n(W_S^n)). \end{aligned}$$

However,

$$\beta_n \geqq k^{-2} \beta_{n-1} + E^{x_1} g_n(W^n_S),$$

and so $\gamma_n \leqq (r_{n-1} \vee c_2)\, \beta_n$. Hence $r_n \leqq r_{n-1} \vee c_2$, so that, writing $c_3 = r_0 \vee c_2$, we have

(4.8) $$\gamma_n \leqq c_3 \beta_n \quad \text{for } n \geqq 1.$$

Collecting these inequalities together, we obtain

PROPOSITION 4.2. — *These exist constants c_1, c_4, $c_5 > 0$ independent of n such that*

(4.9) $$\beta_n \leqq \gamma_n \leqq \alpha_n \leqq c_4 \beta_n \qquad \text{for} \quad n \geqq 0$$

(4.10) $$c_5 k^{-r} \alpha_{n-r} \leqq \alpha_n \leqq c_1^r \alpha_{n-r} \qquad \text{for} \quad n \geqq 0,\ 0 \leqq r \leqq n.$$

Proof. — (4.9) is evident from (4.1), (4.5), (4.8) and (4.3). The right hand side of (4.10) is immediate from (4.5) and (4.1). From (4.9) and (4.4), $\alpha_n \geqq \beta_n \geqq (2k)^{-1} k^{-r} \beta_{n-r} \geqq (2kc_4)^{-1} k^{-r} \alpha_{n-r}$. □

Remarks. — 1. The value of the constant c_4 is not very significant. What is important is the value of the constants relating $(\alpha_n, \beta_n, \gamma_n)$ to $(\alpha_{n-r}, \beta_{n-r}, \gamma_{n-r})$. The estimates we have given in (4.10) are very poor, and while they could be improved somewhat fairly easily, the problem of obtaining satisfactory estimates seems difficult.

2. As we remarked in the introduction, both intuition and computer simulation suggest that $\alpha_n \to \infty$ as $n \to \infty$. The estimates here, however, are not sharp enough to establish this. *See* Section 8, problem 2.

We now turn to examining the lower tail of τ.

LEMMA 4.3. — *There exists a constant $c_6 \in (0, 1)$ with*

$$P^x(\tau \leqq s) \leqq c_6 + s\,\alpha_n^{-1} \qquad \textit{for} \quad s \geqq 0, \quad x \in G_n(1/2).$$

Proof. — Let $t > 0$. Then as $\tau \leqq t + (\tau - t)\, 1_{(\tau > t)}$,

$$\begin{aligned} E^x \tau &\leqq t + E^x 1_{(\tau > t)} E^{W^n_t} \tau \\ &\leqq t + \alpha_n P^x(\tau > t). \end{aligned}$$

Thus $\alpha_n P^x(\tau \leqq t) \leqq \alpha_n + t - E^x \tau$, and since $E^x \tau \geqq \beta_n \geqq c_4^{-1} \alpha_n$, this gives $P^x(\tau \leqq t) \leqq \alpha_n^{-1} (\alpha_n + t - c_4^{-1} \alpha_n)$, as desired. □

PROPOSITION 4.4. — *There exist constants γ, c_{11}, $c_{12} > 0$ such that*

$$P^x(\tau(W^n) \leqq \alpha_n s) \leqq c_{12} e^{-c_{11} s^{-\gamma}} \qquad \textit{for} \quad s \geqq 0, \quad n \geqq 3, \quad x \in G_n(1/2).$$

Proof. — Let $x \in G_n(1/2)$, let $3 \leqq r \leqq n$, and let $N = \min\{i : \sigma_i^r \geqq \tau\}$. As in the proof of (4.4) we must have $N \geqq \frac{1}{2} k^r$. Let $m_r = \frac{1}{2} k^r - 2 \geqq \frac{1}{3} k^r$, and let $Y_i = \sigma^r_{i+1} - \sigma^r_i$ for $i = 1, \ldots, m_r$. Then

$$\tau \geqq \sum_{i=1}^{m_r} Y_i. \tag{4.11}$$

By scaling, the P^x law of σ^r_1 is the P^y law of $k^{-2r}\tau(W^{n-r})$ for some $y \in G_{n-r}(1/2)$. So, by Lemma 4.3

$$\begin{aligned} P^x(Y_i \leqq t \mid \sigma(Y_j, j \leqq i-1)) &\leqq \sup_{y \in G_n(1/2)} P^y(\tau(W^{n-r}) < k^{2r} t) \\ &\leqq c_6 + t k^{2r} \alpha_{n-r}^{-1}. \end{aligned}$$

Therefore by Lemma 1.1,

$$\begin{aligned} P^x(\tau \leqq \alpha_n s) &\leqq \exp\left(2\left(\frac{k^{2r} m_r \alpha_n s}{\alpha_{n-r} c_6}\right)^{1/2} - m_r \log c_6^{-1}\right) \\ &\leqq \exp(c_7 (c_1 k^3)^{r/2} s^{1/2} - c_8 k^r). \end{aligned}$$

Choose $c_9 \geqq (c_1 k)^{1/2}$ so that $c_9 > 1$, and let $f_1(r) = c_7 s^{1/2} (c_9 k)^r$, and $f_2(r) = c_8 k^r$. Then if $r_0 = \log(c_8 c_7^{-1} s^{-1/2})/\log c_9$, $f_1(r_0) = f_2(r_0) = c_{10} s^{-\gamma}$, where $\gamma = \log k / 2 \log c_9$. There exists a constant $s_0 > 0$ such that, if $0 < s \leqq s_0$, then $r_0 \geqq 5$. If $s < s_0$, then let $r = [r_0 - 1]$, so that $r_0 - 2 \leqq r \leqq r_0 - 1$. Then $f_1(r) - f_2(r) = f_2(r_0)((c_9 k)^{r_0 - r} - k^{r_0 - r}) \leqq -c_{11} s^{-\gamma}$, where $c_{11} > 0$.

Choosing $c_{12} = \exp(c_{11} s_0^\gamma)$ completes the proof. □

By scaling we deduce

COROLLARY 4.5. — *For all* $x \in F_n$, $3 \leqq r \leqq n$, $s \geqq 0$, $k \geqq 1$,

$$P^x(\sigma^r_{k+1}(W^n) - \sigma^r_k(W^n) \leqq \alpha_{n-r} s) \leqq c_{12} \exp(-c_{11} (9^r s)^{-\gamma}).$$

5. TIGHTNESS AND RESOLVENTS

In this section, and the next, we fit the uniform estimates on the W^n together to construct a process on $F = \bigcap_n F_n$. We begin with some definitions:

DEFINITION. — *Let* $X^n_t = W^n_{\alpha_n t}$, $t \geqq 0$, *and let* P^x_n *be the probability distribution on* $\mathscr{C}(\mathbb{R}_+, F)$ *corresponding to* X^n *with* $X^n_0 = x$. *We write* X *for the coordinate process on* $\mathscr{C}(\mathbb{R}_+, F)$.

THEOREM 5.1. — *Let* x_n *be a sequence, with* $x_n \in F_n$. *Then* $\{P^{x_n}_n, n \geqq 1\}$ *is tight in* $\mathscr{D}(\mathbb{R}_+, F_0)$.

Proof. — From Corollary 4.5, we have

$$\sup_{i \geqq 1} P_n^{x_n}(\sigma_{i+1}^r(X) - \sigma_i^r(X) \leqq s) \leqq c_{12} \exp(-c_{11}(9^r s)^{-\gamma}),$$

for $3 \leqq r \leqq n$. Hence, applying Ethier and Kurtz [5], Proposition 3.8.3, Lemma 3.8.1 and Theorem 3.7.2, we deduce that $(P_n^{x_n})$ is tight in $\mathscr{D}(\mathbb{R}_+, F_0)$ (the space of cadlag functions from $\mathbb{R}_+$ to F_0).

Since X is $P_n^{x_n}$-a. s. continuous, it follows from [5], Theorem 10.2, that if Q is any limit point of $\{P_n^{x_n}, n \geqq 1\}$ then X is Q-a. s. continuous. Thus $\{P_n^{x_n}, n \geqq 1\}$ is tight in $\mathscr{C}(\mathbb{R}_+, F_0)$. □

DEFINITION. — *For* $x \in F_n$, $n \geqq 0$, f *bounded, set*

$$U_n f(x) = E_n^x \int_0^\tau f(X_s)\, ds.$$

From the definition of X^n and α_n we have the immediate bound

$$U_n f(x) \leqq \| f \|_\infty E_n^x \tau \leqq \| f \|_\infty. \tag{5.1}$$

Note that, by scaling and (4.10) we have

$$E_n^x(\sigma_1^r(X)) \leqq k^{-2r} \frac{\alpha_{n-r}}{\alpha_n} \leqq c_5^{-1} k^{-r}. \tag{5.2}$$

We begin by examining $E_n^x \tau = U_n 1(x)$ as $x \to \partial_u F_n$.

THEOREM 5.2. — *There exists constants* β_2 *and* $c_{13} > 0$ *such that*

$$E_n^x \tau \leqq c_{13}(d(x, \partial_u F))^{\beta_2} \quad \text{for} \quad n \geqq 0, \quad x \in F_n.$$

Proof. — Let $n \geqq 0$ be fixed, and let $H_r = F_n - G_n(k^{-r})$, so that H_r is the union of the $2k^r - 1$ squares in $\mathscr{S}_r$ with upper edges on $\partial_u F_n$. Set

$$h_r = \sup_{x \in H_r} E_n^x \tau.$$

Let $S = \sigma_1^r(X) \wedge \tau$, and note that $X_S \in H_{r-1}$ P^x-a. s. if $x \in H_r$. By the estimates of Section 2 there exists $\delta > 0$ (independent of n and r) such that

$$P_n^x(S = \tau) > \delta \quad \text{for all } x \in H_r.$$

So, as $S \leqq \tau$,

$$\begin{aligned} E_n^x \tau &= E_n^x S + E_n^x 1_{(S < \tau)} E^{X_S} \tau \\ &\leqq E_n^x S + (1 - \delta) h_{r-1}. \end{aligned}$$

So, taking the supremum over x and using (5.2) we deduce

$$h_r \leqq c_5^{-1} k^{-r} + (1 - \delta) h_{r-1}. \tag{5.3}$$

Now $h_0 \leqq 1$, and so, setting $c_{13} = 1 + c_5^{-1}(k(1-\delta) - 1)^{-1}$ we have

$$h_r \leqq c_{13}(1-\delta)^r - (c_{13} - 1) k^{-r} \leqq c_{13}(1-\delta)^r;$$

the desired result then follows easily. □

THEOREM 5.3. — *There exist constants* β_2, c_{14} *such that*

$$|U_n f(x) - U_n f(y)| \leqq c_{14} \| f \|_\infty |x-y|^{\beta_2} \tag{5.4}$$

for all bounded f *on* F_n, $x, y \in F_n$.

Proof. — Let f be bounded on F_n, $x, y \in F_n$ with $|x-y| = \delta$, and let r be chosen so that

$$\frac{1}{4} k^{-(r+1)} \leqq |x-y| < \frac{1}{4} k^{-r}.$$

If $d(x, \partial_u F) \leqq 2k\delta$ then, by Theorem 5.2,

$$\begin{aligned} |U_n f(x) - U_n f(y)| &\leqq c_{13} \| f \|_\infty ((2k\delta)^{\beta_2} + ((2k+1)\delta)^{\beta_2}) \\ &\leqq c \| f \|_\infty \delta^{\beta_2}. \end{aligned} \tag{5.5}$$

So now suppose that $d(x, \partial_u F) > 2k\delta$; then $D_r(x) \subseteqq F_0$. Let $S = \inf\{t \geqq 0 : X_t \in D_r(x)^c\}$. Since $S \leqq \tau$ for $z \in D_r(x)$

$$\begin{aligned} U_n f(z) &= E_n^z \int_0^S f(X_t)\, dt + E_n^z \int_S^\tau f(X_t)\, dt \\ &= E_n^z \int_0^S f(X_t)\, dt + E_n^z U_n f(X_S). \end{aligned}$$

Now $E_n U_n f(X_s)$ is harmonic in $D_r(x)$, $d(y, \partial D_r(x)) \geqq \frac{1}{4} k^{-r}$, and so, by Theorem 3.9

$$|E_n^x U_n f(X_S) - E_n^y U_n f(X_S)| \leqq c_{13} |x-y|^{\beta_1} \| f \|_\infty.$$

By (5.1) and (5.2), if $z = x$ or y,

$$E_n^z \int_0^S f(X_t)\, dt \leqq \| f \|_\infty E_n^z S \leqq c_5^{-1} k^{-r} \| f \|_\infty.$$

So,

$$\begin{aligned} |U_n f(x) - U_n f(y)| &\leqq (2 c_5^{-1} k^{-r} + c_{13} |x-y|^{\beta_1}) \| f \|_\infty \\ &\leqq c |x-y|^{\beta'} \| f \|_\infty \end{aligned} \tag{5.6}$$

for a suitable β'. Combining (5.5) and (5.6) gives (5.4). □

For each n, X under P_n^x is a time change of a reflecting Brownian motion on a Lipschitz domain. So, U_n has a symmetric potential kernel density $u_n(x, y)$ with respect to μ_n, and u_n is jointly continuous in x, y away from the diagonal.

THEOREM 5.4. — *For each* $\varepsilon > 0$ *there exists* $M_\varepsilon < \infty$ *such that* $u_n(x, y) \leqq M_\varepsilon$ *for all* $n \geqq 0$, $x, y \in G_n(\varepsilon)$ *with* $|x-y| > \varepsilon$.

Proof. — Let $\varepsilon > 0$, $n \geqq 1$ be fixed, and let $x \in F_n$, $y \in G_n(\varepsilon)$ with $|x-y| > \varepsilon$. Choose r so that $6k^{-r} < \varepsilon \leqq 6k^{-r+1}$; then $D_r(x) \subseteqq B(x, 3k^{-r}) \subseteqq B(x, \varepsilon)$, and $D_r(y) \subseteqq F_n$, $D_r(x) \cap D_r(y) = \varnothing$.

Now,

$$\int_{F_n} u_n(x, y)\,\mu_n(dy) \leqq E_n^x \tau,$$

and so, as $\mu_n(D_{r+1}(y)) \geqq k^{-2r-2}$ there exists $y_0 \in D_{r+1}(y)$ with $u_n(x, y_0) \leqq k^{2r-2} E_n^x \tau$. However, $u_n(x, \cdot)$ is harmonic in $D_r(y)$, and so, by Corollary 3.3

$$u_n(x, z) \leqq \theta\, u_n(x, y_0) \quad \text{for all } z \in D_{r+1}(y).$$

In particular, this holds for $z = y$, and as $E_n^x \tau \leqq 1$, and r depends only on ε, this completes the proof. □

6. CONSTRUCTION OF THE PROCESS

We have constructed processes $\{X_t, P_n^x\}$ with state space F_n, and we want to take a weak limit as $n \to \infty$. To do that it is more convenient to have processes and resolvents defined on all of F_0. First, in this section, we identify all the points $[0, 1]^2 - [0, 1)^2$ as Δ and make the convention that $f(\Delta) = 0$ for all functions f. Thus, saying f is continuous on F_n automatically implies that $f(x) \to 0$ uniformly as $x \to \Delta$.

For $x \in F_0 - F_n$, define P_n^x to be the probability measure for the process that behaves like standard Brownian motion up until the time of hitting F_n, and then behaves like X_t^n thereafter. More precisely, if

$$\tau_n = \inf\{t > 0 : X_t \in F_n\},$$

Q^x is standard Wiener measure on paths in $\mathbb{R}^2$, $A \in \mathscr{F}_{\tau_n}$, then define

$$P_n^x(A \cap (B \circ \theta_{\tau_n})) = E_Q^x(P_n^{X_{\tau_n}}(B); A).$$

This suffices to define P_n^x on all of $\mathscr{F}_\infty$. We now extend U_n, etc., to the whole of F_0.

We have uniform bounds on $|U_n f(x) - U_n f(y)|$ when $x, y \in F_n$, but we need bounds for $x, y \in F_0$.

PROPOSITION 6.1. — *Suppose $f : F_0 \to \mathbb{R}$ is bounded. Then there exists a function $\omega(\delta)$ that tends to 0 as $\delta \to 0$ such that*

$$\sup_n \sup_{\substack{x, y \in F_0 \\ |x-y| < \delta}} |U_n f(x) - U_n f(y)| \leqq \omega(\delta)\, \|f\|_\infty.$$

Proof. — We first make the following observation. Let ε, $\eta > 0$. If B is a square of side r and $\tau_B = \inf\{t : X_t \notin B\}$, then there exists $\delta > 0$ (independent of r) such that

(6.1)
$E_Q^x \tau_B < \varepsilon$ and $Q^x(|X_{\tau_B} - x| > \eta) < \varepsilon$ whenever $x \in B$ and $d(x, B^c) < \delta$.

This follows easily from the results of [13], Ch. 2, Sect. 3.

Let $\varepsilon > 0$. Suppose we show that for each x_0 there exists a δ (depending on x_0) such that

$$|U_n f(x) - U_n f(x_0)| \leqq 4\varepsilon \|f\|_\infty \tag{6.2}$$

whenever $|x - x_0| < \delta$. Then by the compactness of F_0, we will have the desired proposition.

Suppose first that $x_0 = \Delta$. Choose η sufficiently small so that $E_n^x \tau < \varepsilon$ if $x \in F_n$ and $d(x, \Delta) < 2\eta$. We can do this independently of n by Theorem 5.2. Then choose $\delta \in (0, \eta)$ sufficiently small so that (6.1) holds. If $d(x, \Delta) < \delta$ and $x \in F_n$, of course $E_n^x \tau < \varepsilon$, while if $x \notin F_n$,

$$\begin{aligned} E_n^x \tau &\leqq E_Q^x \tau_n + E_Q^x E_n^{X_{\tau_n}} \tau \\ &\leqq \varepsilon + \varepsilon \sup_{x \in F_n} E_n^x \tau + \sup_{\substack{x \in F_n \\ d(x, \Delta) < \eta + \delta}} E_n^x \tau \leqq 3\varepsilon. \end{aligned}$$

Hence

$$|U_n f(x) - U_n f(x_0)| = |U_n f(x)| \leqq \|f\|_\infty E_n^x \tau \leqq 3\varepsilon \|f\|_\infty.$$

Next suppose $x_0 \notin F$. Then there exists m and η such that $B_\eta(x_0) \cap F_n = \varnothing$ whenever $n \geqq m$. Let $\rho = \inf\{t : X_t \notin B_\eta(x_0)\}$. Choose η sufficiently small so that $\sup_{x \in B_\eta(x_0)} E_Q^x \rho < \varepsilon$. If g is bounded, $E_Q^x g(X_\rho)$ is harmonic in x in the interior of $B_\eta(x_0)$, and hence there exists $\delta < \eta/2$ such that

$$|E_Q^x g(X_\rho) - E_Q^{x_0} g(X_\rho)| \leqq \varepsilon \|g\|_\infty$$

if $|x - x_0| < \delta$. For $x \in B_\eta(x_0)$ we have

$$U_n f(x) = E_Q^x \int_0^\rho f(X_s)\, ds + E_Q^x U_n f(X_\rho),$$

and hence

$$\begin{aligned} |U_n f(x) - U_n f(x_0)| &\leqq 2\|f\|_\infty \sup_{x \in B_\eta(x_0)} E_Q^x \rho \\ &\quad + |E_Q^x U_n f(X_\rho) - E_Q^{x_0} U_n f(X_\rho)| \\ &\leqq 2\varepsilon \|f\|_\infty + \varepsilon \|U_n f\|_\infty \\ &\leqq 3\varepsilon \|f\|_\infty. \end{aligned}$$

Finally, suppose $x_0 \in F$. Pick η so that if $|x-x_0| < 2\eta$, $x \in F_n$, then

$$|U_n f(x) - U_n f(x_0)| < \varepsilon \| f \|_\infty .$$

This can be done independently of n by Theorem 5.3. Pick $\delta < \eta$ so that (6.1) holds. The estimate (6.2) holds if $x \in F_n$, while if $x \notin F_n$ and $|x-x_0| < \delta$, then

$$\begin{aligned}
|U_n f(x) - U_n f(x_0)| &\leqq \left| E_Q^x \int_0^{\tau_n} f(X_s)\, ds \right| + |E_Q^x U_n f(X_{\tau_n}) - U_n f(x_0)| \\
&\leqq \| f \|_\infty E_Q^x \tau_n + 2 \| U_n f \|_\infty P_Q^x (|X_{\tau_n} - x| > \eta) \\
&\quad + \sup_{\substack{y \in F_n \\ |y-x_0| < 2\eta}} |U_n f(y) - U_n f(x)| \\
&\leqq 4\varepsilon \| f \|_\infty . \quad \square
\end{aligned}$$

We also need a tightness estimate for P_n^x, $x \notin F_n$, but this is routine.

We define

$$U_n^\lambda f(x) = E_n^x \int_0^\tau e^{-\lambda s} f(X_s)\, ds \tag{6.3}$$

for f bounded. Of course, $U_n^0 \equiv U_n$, and for any λ,

$$|U_n^\lambda f(x)| \leqq \| f \|_\infty E_n^x \tau \leqq \| f \|_\infty . \tag{6.4}$$

By [2], V. 5.10,

$$U_n^\lambda f = \sum_{i=0}^{\infty} (\beta - \lambda)^i (U_n^\beta)^{i+1} f, \qquad \beta \geqq 0, \quad |\beta - \lambda| < 1. \tag{6.5}$$

PROPOSITION 6.2. — *Suppose* $f: F_0 \to \mathbb{R}$, $\| f \|_\infty < \infty$, *and* $\lambda > 0$. *Then*

$$\lim_{\delta \downarrow 0} \sup_n \sup_{\substack{x,\, y \in F_n \\ |x-y| < \delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| = 0.$$

Proof. — Suppose first that $\lambda \leqq 1/2$. By Theorem 5.3,

$$\sup_{\substack{x,\, y \in F_n \\ |x-y| < \delta}} |U_n g(x) - U_n g(y)| \leqq \| g \|_\infty \omega(\delta), \tag{6.6}$$

where $\omega(\delta) \to 0$ as $\delta \to 0$ independently of n. By applying (6.6) to $g = (U_n)^i f$ and using (6.4), we see that

$$\sup_{\substack{x,\, y \in F_n \\ |x-y| < \delta}} |(U_n)^{i+1} f(x) - (U_n)^{i+1} f(y)| \leqq \| f \|_\infty \omega(\delta). \tag{6.7}$$

Let $\varepsilon > 0$. Choose i_0 so that $\sum_{i=i_0}^{\infty} \lambda^i < \varepsilon/4$, and then δ so that $\omega(\delta) < \varepsilon/2 i_0$. Using (6.4) (with $\lambda = 0$) shows

$$\sum_{i=i_0}^{\infty} \lambda^i \,|\, (U_n)^{i+1} f(x) - (U_n)^{i+1} f(y) \,| \leqq \varepsilon \| f \|_\infty / 2.$$

By (6.5) and (6.7) then, with $\beta = 0$, we have

$$\sup_{\substack{x,\, y \in F_n \\ |x-y| < \delta}} |\, U_n^\lambda f(x) - U_n^\lambda f(y) \,| \leqq (\varepsilon/2 + i_0 \,\omega(\delta)) \| f \|_\infty < \varepsilon \| f \|_\infty.$$

This proves the proposition when $\lambda \leqq 1/2$. If $\lambda \in [1/2, 1]$, repeat the proof, replacing U_n by U_n^β, where $\beta = 1/2$. Repeat again with $\beta = 1$ to obtain the proposition for $\lambda \in [1, 3/2]$, etc. □

PROPOSITION 6.3. — *There is a subsequence n_j such that $U_{n_j}^\lambda f$ converges uniformly for each $\lambda \in [0, \infty)$, f continuous on F_0. The limit, U^λ, satisfies the resolvent identity and $\| U^\lambda \|_\infty \leqq \lambda^{-1}$.*

Proof. — Let f_m be a sequence of continuous functions on F_0 such that $\| f_m \|_\infty \leqq 1$ and such that the closure of the linear span of $\{f_m\}$ consists of all continuous functions on F_0. (Recall $f_m = 0$ on Δ). Let λ_i be a countable dense subset of $[0, \infty)$. Fix i and m. By the Ascoli-Arzela theorem, a subsequence of $U_n^{\lambda_i} f_m$ converges uniformly. By a diagonalization procedure, we can choose a subsequence n_j so that $U_{n_j}^{\lambda_i} f_m$ converges uniformly for each i and m. Call the limit $U^{\lambda_i} f_m$.

Each $U_n^{\lambda_i}$ satisfies $\| U_n^{\lambda_i} \|_\infty \leqq \lambda_i^{-1}$ and the same is true for U^{λ_i}. It follows that $U_{n_j}^{\lambda_i} f$ converges uniformly, say to $U^{\lambda_i} f$, for each f continuous on F_0. Each U_n^λ satisfies the resolvent equation:

$$U_n^\lambda - U_n^\beta = (\beta - \lambda)\, U_n^\lambda U_n^\beta,$$

and by a limiting argument U^λ does also, provided λ is one of the λ_i's. Since $\| U_n^\lambda - U_n^\beta \|_\infty \leqq \dfrac{\beta - \lambda}{\lambda \beta}$, the same is true for $U^\lambda - U^\beta$ provided λ and $\beta \in \{\lambda_i\}$, and it then follows easily that for all $\lambda \in [0, \infty)$, all f continuous on F_0, $U_{n_j}^\lambda f$ converges uniformly, say to $U^\lambda f$; also U^λ satisfies the resolvent equation and $\| U^\lambda \|_\infty \leqq \lambda^{-1}$. □

We could use the Hille-Yosida theorem at this point, but it is easier to construct the desired P^x's directly. Before doing so, we state the following well-known elementary lemma.

LEMMA 6.4. — *Suppose g and g_m, $m = 1, 2, \ldots$ are functions on F_0 with the property that $g_m(y_m) \to g(y)$ whenever $y_m \to y$. Then g is continuous and $g_m \to g$ uniformly.*

PROPOSITION 6.5. — *Suppose* $x_j \to x$ *with* $\{x_j\}$, $\{x\} \subseteq F_0$. *Then* $P_{nj}^{x_j} \xrightarrow{w}$.

Remark. — We will call the limit P^x.

Proof. — The sequence $\{P_{nj}^{x_j}\}$ is tight by Theorem 5.1 and the remark following the proof of Proposition 6.1, and it suffices to show any two limit points agree. Let f be continuous on F_0. If a subsequence of $P_{nj}^{x_j}$ converges weakly, say to P', then (recall $f=0$ on Δ)

$$U_{n_j}^{\lambda} f(x_j) = E_{nj}^{x_j} \int_0^{\infty} e^{-\lambda s} f(X_s^{n_j})\, ds \to E' \int_0^{\infty} e^{-\lambda s} f(X_s)\, ds.$$

On the other hand, using the equicontinuity of $U_{n_j}^{\lambda} f$, we have $U_{n_j}^{\lambda} f(x_j) \to U^{\lambda} f(x)$. If P'' is another subsequential limit point of $P_{nj}^{x_j}$, we get

$$E' \int_0^{\infty} e^{-\lambda s} f(X_s)\, ds = U^{\lambda} f(x) = E'' \int_0^{\infty} e^{-\lambda s} f(X_s)\, ds. \tag{6.8}$$

By the uniqueness of the Laplace transforms,

$$E' f(X_s) = E'' f(X_s).$$

for almost all s. Since f is continuous and X_s is continuous a. s. under both P' and P'', we get equality for all s. The usual limiting argument gives (6.6) for bounded f, and hence the one-dimensional distributions of X_t are equal under P' and P''.

Denote $E' f(X_s)$ by $P_s f(x)$. We have $P_s^{n_j} f(x_j) \to P_s f(x)$ whenever $x_j \to x$. Using Lemma 6.4, $P_s f$ is continuous on F_0, and the convergence is uniform.

If we have times $s < t$ and continuous functions f and g, by the Markov property for P_{nj}^x,

$$\begin{aligned} E_{nj}^{x_j} g(X_s) f(X_t) &= E_{nj}^{x_j} ((P_{t-s}^{n_j} f) g)(X_s) \\ &= E_{nj}^{x_j} ((P_{t-s} f) g)(X_s) + E_{nj}^{x_j} ((P_{t-s} f - P_{t-s}^{n_j} f) g)(X_s). \end{aligned} \tag{6.9}$$

The first term on the right of (6.9) converges to $E'((P_{t-s} f)\, g)(X_s)$ by the first part of the proof, while the second term tends to 0 since $P_{t-s}^{n_j} f \to P_{t-s} f$ uniformly. Repeating this argument, we see that the finite dimensional distributions under $P_{nj}^{x_j}$ converge. This with tightness proves the proposition. □

In fact, the proof shows the following.

COROLLARY 6.6. — *If f is continuous, $P_t f$ is continuous.*

We next show

PROPOSITION 6.7. — *$\{P^x\}$ is a strong Markov family of probability measures.*

Proof. — For each n, we have $P_t^n P_s^n = P_{t+s}^n$, since $\{P_n^x\}$ form Markov processes. If f is continuous, we have from Corollary 6.6 that $P_s^{n_j} f \to P_s f$, uniformly and $P_0 f$ is continuous, so $P_{t+s}^{n_j} f \to P_{t+s} f$, and

$$\| P_t^{n_j} P_s^{n_j} f - P_t P_s f \|_\infty \leqq \| P_s^{n_j} f - P_s f \|_\infty + \| P_t^{n_j} P_s f - P_t P_s f \|_\infty \to 0$$

as $j \to \infty$. Hence $P_{t+s} f = P_t P_s f$ for f continuous. The usual limiting arguments give this for all bounded and measurable f.

Since the $P_{n_j}^x$ are tight with weak limit P^x, we know P^x (paths of X_t are continuous) $= 1$ and $P^x(X_0 = x) = 1$. Hence $P_t f(x) = E^x f(X_t) \to f(x)$ as $t \to 0$ if f is continuous. Furthermore, $P_t f$ is continuous when f is. So by [2], Th. I. 9.4 and proof, $\{X_t, P^x\}$ is in fact strong Markov. . □

After all this work, it would be upsetting if our process X_t turned out to be degenerate, *i. e.*, if $P^x(X_t = x$ for all $t) = 1$. But let

$$\tau_\varepsilon = \inf \{t : X_t \notin [0, 1-\varepsilon]^2\}.$$

Then for any x and all t and ε,

$$\begin{aligned} P^x(\tau_\varepsilon > t) &= P^x(\sup_{s \leqq t} (X_s^{(1)} \vee X_s^{(2)}) \leqq 1-\varepsilon) \\ &\leqq \lim_{j \to \infty} \sup P_{n_j}^x (\sup_{s \leqq t} (X_s^{(1)} \vee X_s^{(2)} \leqq 1-\varepsilon/2) \\ &= \lim_{j \to \infty} \sup P_{n_j}^x (\tau_{\varepsilon/2} > t), \end{aligned}$$

where $X_s^{(1)}, X_s^{(2)}$ are the $x^{(1)}$ and $x^{(2)}$ coordinates of X_s. Hence,

$$\begin{aligned} E^x \tau_\varepsilon &= \int_0^\infty P^x(\tau_\varepsilon > s)\, ds \leqq \limsup \int_0^\infty P_{n_j}^x (\tau_{\varepsilon/2} > s)\, ds \\ &= \limsup E_{n_j}^x \tau_{\varepsilon/2} \leqq \limsup E_{n_j}^x \tau \leqq 1. \end{aligned}$$

By monotone convergence, $E^x \tau \leqq 1$, hence $\tau < \infty$, a. s. So our process is nowhere degenerate.

Applying a similar argument to $\sigma_1^r(X)$ and using (5.2) shows that $E^x \sigma_1^r(X) \to 0$ as $r \to \infty$. Hence, by the Blumenthal 0-1 law,

(6.10) $$P^x (\text{for some } \varepsilon > 0,\ X_t = x \text{ for } t \in [0, \varepsilon]) = 0.$$

The following is also obviously desirable:

PROPOSITION 6.8. — *If* $x \in F$, $P^x(X_t \notin F$ *for some* $t < \tau) = 0$.

Proof. — If $x \in F \subseteq F_n$ and $n > m$, P_n^x (X_t ever hits $F_0 - F_m) = 0$. A routine argument using limits and the regularity of the sets $F_0 - F_m$ shows P^x (X_t ever hits $F_0 - F_m) = 0$ for all m, which yields the proposition. □

We observe that (X_t, P^x) has a strong Feller resolvent.

PROPOSITION 6.9. — *If* f *is bounded, then* $U^\lambda f$ *is continuous on* F.

Proof. — Suppose f is continuous. Then

$$(6.11) \qquad \sup_{\substack{x,\, y \in F_n \\ |x-y| < \delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| \leqq \omega(\delta) \| f \|_\infty,$$

where $\omega(\delta) \to 0$ as $\delta \to 0$, independent of n. Taking a limit along the subsequence n_j shows that (6.11) still holds if U_n^λ is replaced by U^λ. The bound $\omega(\delta) \| f \|_\infty$ does not depend on the modulus of continuity of f, and so a limiting argument shows (6.11) holds for f bounded, which is what was required. □

Finally, note that X_t is invariant under certain transformations of the state space. For example, if $S \in \mathcal{S}_r$, with $S \cap F \neq \emptyset$, and we consider $X_{t \wedge T}$ where $T = \inf\{t: X_t \notin S\}$, then the P^x law of $X_{t \wedge T}$ equals the $P^{\varphi(x)}$ law of $\varphi(X_{t \wedge T})$, where φ is (a) rotation of S by 90° or (b) reflection of S through a diagonal or midline or (c) φ is the translation of S to S', another square in $\mathcal{S}_r$ such that $S' \cap F \neq \emptyset$. These follow immediately from the corresponding facts for X_t^n. The one transformation for which we have not established invariance but believe invariance exists is scaling (*see* Section 8): is $\varphi(X_{t \wedge T})$ a time change of $X_{t \wedge \tau}$, where $S \in \mathcal{S}_r$ with lower left corner at (0, 0), and φ is the natural dilation of S to F?

7. ESTIMATES ON GREEN'S FUNCTION

Now that we have constructed a process $\{X_t, P^x\}$ on F, we would like to say something about its properties. Recall that P_n^x is the law on Ω induced by X_t^n, where $X_t^n = W_{\alpha_n t}^n$. Hence X_t^n has a symmetric Green's function $u_n(x, y)$ with respect to μ_n: for all $f \geqq 0$, for all $x \in F_n$,

$$E_n^x \int_0^\tau f(X_s)\, ds = \int f(y)\, u_n(x, y)\, \mu_n(dy).$$

Moreover, for all $\varepsilon > 0$, $u_n(x, y)$ is Hölder continuous and bounded on $\{(x, y): |x-y| > \varepsilon\}$. In this section we occasionally use the notation σ_r for $\sigma_1^r(X)$.

Our first goal is to give a bound that is uniform in n.

THEOREM 7.1. — *Suppose* $\varepsilon > 0$. *Then there exists* M such that

$$|u_n(x, y)| \leqq M \textit{ whenever } n > 0,\ |x-y| > \varepsilon,\ x, y \in F_n.$$

Proof. — Let $\varepsilon \in (0, 1/8k^2)$ and write G_n for $G_n(\varepsilon/2)$. If $x, y \in G_n$, our result is just Theorem 5.4.

Suppose $x \in G_n$, $y \in F_n - G_n$, and $|x-y| > \varepsilon$. If $z \in G_n \cap cl(F_n - G_n)$, then $|z-x| > \varepsilon/2$. The function $u_n(y, x)$ is harmonic in y and 0 when $y \in \Delta$. So by the maximum principle and Theorem 5.4 we get

$$u_n(y, x) \leqq \sup_{z \in G_n \cap cl(F_n - G_n)} u_n(z, x) \leqq M.$$

Now suppose $x, y \in F_n - G_n$, $|x-y| \geqq \varepsilon$. Let $A \subseteq B_{\varepsilon/2}(x)$ be a neighborhood of x such that $A \cap G_n = \varnothing$. Since $U_n(y, A)$ is harmonic, by the maximum principle

$$U_n(y, A) \leqq \sup_{|y-x|=\varepsilon} U_n(y, A),$$

and so there is no loss of generality in assuming $|x-y|=\varepsilon$.

There is also no loss of generality in assuming $x^{(2)}, y^{(2)} \geqq 1/2$. If both $x^{(1)}, y^{(1)} \leqq 1/2$, let $\hat{x}, \hat{y}, \hat{A}$ be the reflection of x, y, and A across the line $[(0, 1/2), (1, 1/2)]$. Otherwise, let $\hat{x}, \hat{y}, \hat{A}$ be the reflection of x, y, and A across the line $[(0, 1), (1, 0)]$. In either case, $\hat{x}, \hat{y}$ are in G_n, and

$$\begin{aligned} E_n^y \int_0^{\sigma_1} 1_A(X_s)\,ds &\leqq E_n^{\hat{y}} \int_0^{\sigma_1} 1_{\hat{A}}(X_s)\,ds \leqq E_n^{\hat{y}} \int_0^{\tau} 1_{\hat{A}}(X_s)\,ds \\ &= U_n(\hat{y}, \hat{A}) \leqq M\,\mu_n(\hat{A}) = M\,\mu_n(A). \end{aligned}$$

There exists a δ (independent of n) such that $P_n^y(\sigma_1=\tau) > \delta$. Let $N = \sup_{\substack{y \in F_n - G_n \\ |x-y|=\varepsilon}} U_n(y, A)$. We then have that if $|y-x|=\varepsilon$, $y \in F_n - G_n$,

$$\begin{aligned} U_n(y, A) = E_n^y \int_0^{\tau} 1_A(X_s)\,ds &= E_n^y \int_0^{\sigma_1} 1_A(X_s)\,ds + E_n^y U_n(X_{\sigma_1}, A) \\ &\leqq M\,\mu_n(A) + N P_n^y(\sigma_1 < \tau) \\ &\leqq M\,\mu_n(A) + (1-\delta)\,N, \end{aligned}$$

where the first inequality follows from the maximum principle.

Taking sups over y in $F_n - G_n$,

$$N \leqq M\,\mu_n(A) + (1-\delta)\,N.$$

But then

$$U_n(y, A) \leqq N \leqq (M/\delta)\,\mu_n(A);$$

letting A shrink down to x and using the continuity of $u_n(y, x)$ gives $u_n(y, x) \leqq M/\delta$. □

Next we want to get the uniform Hölder continuity of $u_n(x, y)$ away from the diagonal. Again, the points near the boundary Δ are a nuisance.

THEOREM 7.2. — *Suppose* $\varepsilon > 0$. *There exists* K, α *and* $\beta > 0$ *independent of* ε *such that* $|u_n(x, y_1) - u_n(x, y_2)| \leqq K\varepsilon^{-\beta}|y_1 - y_2|^{\alpha}$ *whenever* $n > 0$, $|x-y_1|, |x-y_2| > \varepsilon$, $x, y_1, y_2 \in F_n$.

Proof. — Let $\varepsilon \in (0, 1/8k^2)$, $G_n = G_n(k^{-2})$. The function $u_n(x, y)$ is harmonic in y on $|y-x| > \varepsilon$, and so we have our result for $y_1, y_2 \in G_n$ by Theorem 3.9 and the remark following it. So we may suppose at least one of $y_1, y_2 \in F_n - G_n$. Suppose $|y_1 - y_2| < \varepsilon/4$.

The function $u_n(x, y_1) - u_n(x, y_2)$ is harmonic in x, and so by the maximum and minimum principles, it suffices to obtain a bound on $|u_n(x, y_1) - u_n(x, y_2)|$ for x such that $\varepsilon/2 \leqq |x - y_1|, |x - y_2| \leqq \varepsilon$.

For $i = 1, 2$, let A_i be neighborhoods of y_i contained in $B_{\varepsilon/2}(y_i)$. Assume $x^{(2)} \geqq 1/2$, and let $\hat{x}, \hat{y}_1, \hat{y}_2, \hat{A}_1, \hat{A}_2, \hat{D}_1(x)$ be the reflections of x, y_1, y_2, A_1, A_2, and $D_1(x)$ across either the line $[(0, 1/2), (1, 1/2)]$ or $[(0, 1), (1, 0)]$ as in the proof of Theorem 7.1. Let $S = \inf \{t : X_t \notin \hat{D}_1(x) \cap (0, 1]^2\}$. Provided A_1 and A_2 are sufficiently small, the first paragraph of the proof tells us that

$$|\mu_n(\hat{A}_1)^{-1} U_n(z, \hat{A}_1) - \mu_n(\hat{A}_2)^{-1} U_n(z, \hat{A}_2)| \leqq 2 |u_n(z, \hat{y}_1) - u_n(z, \hat{y}_2)| \leqq 2 K \varepsilon^{-\beta} |y_1 - y_2|^\alpha,$$

uniformly for $z \in G_n$. Since

$$E_n^{\hat{x}} \int_0^S 1_{\hat{A}_i}(X_s)\, ds = U_n(\hat{x}, \hat{A}_i) - E^{\hat{x}} U_n(X_S, \hat{A}_i),$$

we then get

$$(7.1) \quad \left| \mu_n(\hat{A}_1)^{-1} E_n^{\hat{x}} \int_0^S 1_{\hat{A}_i}(X_s)\, ds - \mu_n(\hat{A}_2)^{-1} E_n^{\hat{x}} \int_0^S 1_{\hat{A}_2}(X_s)\, ds \right| \leqq 4 K \varepsilon^{-\beta} |y_1 - y_2|^\alpha.$$

Let $f(z) = \mu_n(A_1)^{-1} U_n(z, A_1) - \mu_n(A_2)^{-1} U_n(z, A_2)$. f is harmonic in z for $|z - y_1|, |z - y_2| \geqq \varepsilon/2$, and so by the maximum and minimum principles,

$$\sup_{|z-y_1|,\, |z-y_2| \geqq \varepsilon/2} |f(z)| \leqq \theta = \sup_{\varepsilon \geqq |z-y_1|,\, |z-y_2| \geqq \varepsilon/2} |f(z)|.$$

As in the proof of Theorem 7.1, there exists $\delta > 0$ independent of n such that $P_n^x(\sigma_1 = \tau) > \delta$ for $x \in F_n - G_n$.

By the strong Markov property, for $i = 1, 2$

$$(7.2) \quad E_n^x \int_0^\tau 1_{A_i}(X_s)\, ds = E_n^x \int_0^{\sigma_1} 1_{A_i}(X_s)\, ds + E_n^x U_n(X_{\sigma_1}, A_i)$$

$$= E_0^{\hat{x}} \int_0^S 1_{\hat{A}_i}(X_s)\, ds + E_n^x [U_n(X_{\sigma_1}, A_i);\ \sigma_1 < \tau].$$

Let z be such that $\varepsilon \geqq |z-y_1|, |z-y_2| \geqq \varepsilon/2$. Then by (7.1) and (7.2),

$$|f(z)| \leqq 4 K \varepsilon^{-\beta} |y_1-y_2|^\alpha + E_n^x [f(X_{\sigma_1}); \sigma_1 < \tau]$$
$$\leqq 4 K \varepsilon^{-\beta} |y_1-y_2|^\alpha + \theta(1-\delta).$$

Taking the sup over such z's gives

$$\theta \leqq 4 K \delta^{-1} \varepsilon^{-\beta} |y_1-y_2|^\alpha.$$

Finally, since $f(x) \leqq \theta$, we can let A_1, A_2 shrink down to y_1, y_2 to get

$$|u_n(x, y_1)-u_n(x, y_2)| \leqq 4 K \delta^{-1} \varepsilon^{-\beta} |y_1-y_2|^\alpha. \quad \square$$

If f is a continuous function on F, it may be extended to a continuous function on F_0. Then, as $n_j \to \infty$,

$$\text{(7.3)} \qquad \int u_{n_j}(x, y) f(y) \mu_{n_j}(dy) = E_{n_j}^x \int_0^\tau f(X_s) ds \to E^x \int_0^\tau f(X_s) ds.$$

Recall, also, that if g is continuous,

$$\text{(7.4)} \qquad \int g(y) \mu_n(dy) \to \int g(y) \mu(dy)$$

as $n \to \infty$. And if $n \geqq r$,

$$\text{(7.5)} \quad E_n^x \int_0^\tau f(X_s) ds \leqq \|f\|_\infty \sup_n E_n^x \sigma_r + E_n^x E_n^{X_{\sigma_r}} \int_0^\tau f(X_s) ds$$
$$\leqq \|f\|_\infty \sup_n E_n^x \sigma_r + \sup_{|z-x|>k^{-r}/2} E_n^z \int_0^\tau f(X_s) ds.$$

We can now prove:

THEOREM 7.3. — *There exists a symmetric function $u(x, y)$ which is bounded and Hölder continuous on $\{(x, y): |x-y| > \varepsilon\}$ for each ε and which serves as a Green's function for $\{X_t, P^x\}$: if $f \geqq 0$, $x \in F$,*

$$E^x \int_0^\tau f(X_s) ds = \int u(x, y) f(y) \mu(dy).$$

Proof. — The proof is a routine application of limits, Theorem 7.1 and 7.2, (7.3) and (7.4). The uniform bounds on $E_n^x \sigma_r$ and (7.5) are used to show that $E^x \int_0^\tau 1_{\{x\}}(X_s) ds = 0$ for each x. $\square$

THEOREM 7.4. — *There exists a symmetric function $p_t(x, y)$ which is the transition density of X (killed at time τ) with respect to μ:*

$$p_t(x, y) = p_t(y, x) \quad \textit{for all } x, y \textit{ in } F$$

and

$$P^x(X_t \in A,\ t < \tau) = \int_A p_t(x, y)\,\mu(dy) \quad \textit{for all } A \subseteq F.$$

Proof. – By Fukushima [6], Theorem 4.3.4, the transition semigroup P_t of X is absolutely continuous with respect to μ. As P_t is self-adjoint with respect to μ, the symmetry of its density p_t follows by [16], Corollary 1.2. □

8. OPEN PROBLEMS

We mention four problems that we consider to be both nontrivial and interesting.

1. *Higher dimensional analogues.* A key feature of our construction was the proof of the Harnack inequality in Section 3, which in term relied upon the fact that a curve in $\mathbb{R}^2$ separates the plane into two parts. To construct Brownian motion on fractals imbedded in $\mathbb{R}^d$ with $d \geq 3$, some other approach to the Harnack inequality, perhaps a more analytic one, is needed.

2. *Local times.* We conjecture that for every fractal F constructed as in Section 1, the Brownian motion with state space F is point recurrent and has jointly continuous local times. We remark that if we could show that the constants α_n defined in Section 3 satisfied $\alpha_{n+1} \geq \alpha_n$ for all n, this would imply that the potential kernel density u of X was bounded on the diagonal. It could then be shown that X hit points and had jointly continuous local times.

3. *Uniqueness.* We did not prove that the Brownian motions that we constructed had scale invariance (*see* Section 6), If the P_n^x could be shown to converge (instead of just a subsequence), scale invariance would be a consequence. Even the convergence of hitting distributions, together with the use of Blumenthal-Getoor-McKean [3], would suffice.

We have also not proved that there is only one Brownian motion on F. This uniqueness is important: other approaches to the construction of a Brownian motion on F are possible, and without uniqueness one would not know that the various constructions gave the same process.

In fact there is a close link between these two problems. Uniqueness would imply scale invariance immediately, while most approaches to the problem of proving scale invariance involve some kind of uniqueness argument.

This problem seems guite hard.

4. *Random state space.* One can easily come up with ways to construct fractals according to some random mechanism $\mathscr{P}$. Can one construct Brownian motions on these fractals, $\mathscr{P}$-a.s.?

Note added in proof. In a work in preparation the authors have shown there exists $c > 0$ independent of n and r such that $\alpha_{n+r}/\alpha_n \geqq c$ and that this suffices to establish the conjecture in problem 2 above.

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