ANNALES DE L'I. H. P., SECTION B

DANIEL OCONE
ETIENNE PARDOUX

A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations

Annales de l'I. H. P., section B, tome 25, nº 1 (1989), p. 39-71
<http://www.numdam.org/item?id=AIHPB_1989__25_1_39_0>

Ann. Inst. Henri Poincaré,

Vol. 25, n° 1, 1989, p. 39-71. *Probabilités et Statistiques*

A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations

by

Daniel OCONE [1]

Mathematics Department, Rutgers University,
New Brunswick, NJ 08903, U.S.A.

and

Etienne PARDOUX [2]

U.E.R. de Mathématiques, Université de Provence,
13331 Marseille Cedex 3, France

ABSTRACT. — We generalize the Itô-Ventzell formula to the case of anticipating integrands. We then apply that result to the study of a Stratonovich-type stochastic differential equation, where the initial condition and the "drift" term are allowed to anticipate the future of the driving Wiener process.

Key words : Stochastic differential equations, Non adapted solutions, Itô-Ventzell formula.

RÉSUMÉ. — Nous généralisons la formule d'Itô-Ventzell au cas où les intégrands ne sont pas adaptés. Ce résultat est ensuite utilisé pour étudier une équation différentielle stochastique de type Stratonovich, où la condition initiale et le terme de «dérive» anticipent le futur du processus de Wiener qui dirige l'équation.

Classification A.M.S. : 60 H 05, 60 H 10.

(1) Supported in part by N.S.F. grant # DMS-8703355.

(2) The research was carried out while this author was visiting the Institute for Advanced Study, Princeton N.J., and was supported by a grant from the R.C.A. corporation.

INTRODUCTION

Suppose $\{X_t \geqq 0\}$ is a d-dimensional Itô process of the form:

$$X_t = X_0 + \int_0^t A_s \, ds + \int_0^t B_s^i \, dW_s^i$$

where we use here and throughout the paper the convention of summation upon repeated indices, X_0, $\{A_t\}$, $\{B_t^1\}$, ..., $\{B_t^k\}$ are adapted to a filtration $\{\mathscr{F}_t, t \geqq 0\}$ with respect to which $W_t = (W_t^1, \ldots, W_t^k)'$ is a standard Wiener process. If $F \in C^{1, 2}(\mathbb{R}_+ \times \mathbb{R}^d)$, the Itô formula computes the differential of the process $F(t, X_t)$. Ventzell [17] has given the form of that differential when $\{F(t, x), t \geqq 0\}$ is an Itô process indexed by $x \in \mathbb{R}^d$, with certain regularity hypotheses. Rovovskii [10], Bismut [2], Kunita [5], Sznitman [14] and Ustunel [15] have proved various versions of the Itô-Ventzell formula, both in the Itô and in the Stratonovich form.

Recently, several authors have defined generalized stochastic integrals with anticipating integrands, and established generalized stochastic calculus rules. For an account and comparison of the various approaches, we refer the reader to the notes by Nualart [6].

In the first part of this paper, we use the results of Nualart-Pardoux [7] to establish a generalized Itô-Ventzell formula, and its analog in Stratonovich form.

In the second part, we apply that result to the study of a Stratonovich stochastic differential equation of the type:

$$X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma_i(s, X_s) \circ dW_s^i$$

where X_0 and $\{b(t, x), t \geqq 0, x \in \mathbb{R}^d\}$ are random and may depend on the whole path of $\{W_t, t \geqq 0\}$, while $\sigma(t, x)$ is a deterministic function of t and x, and the stochastic integrals are interpreted as generalized Stratonovich integrals, as defined in Nuarlart-Pardoux [7]. The main idea consists in using the result in Part I, in order to show that, if $\varphi_t(x)$ denotes the flow associated to the same equation with $b = 0$, then X_t solves the above equation if and only if $Y_t := \varphi_t^{-1}(X_t)$ solves a certain ordinary differential equation with random coefficients.

A similar equation has been considered by Ogawa [8] in dimension one, where only the initial condition is allowed to anticipate the future of the driving Wiener process. The same problem, with linear coefficients b and σ, but interpreted in the Skorohod-Itô sense, has been considered by Shiota [11] (*see* also Ustunel [16]), also with an anticipating initial condition. We will explain below (*see* Remarks I.1.9) why we think that solving the Itô-Skorohod version of our equation is a much harder problem than ours.

Another possible approach to our problem would be to use the enlargement of filtration technique, *see* e. g. Jeulin-Yor [4]. However, our approach allows us to treat cases where the enlargement of filtration technique does not work. Indeed, in the case $k=1$, the initial condition $X_0=\int_0^1 \exp\left(\frac{-1}{1-t}\right) dW_t$ satisfies our hypotheses. But $\{W_t\}$ is not a semimartingale in the corresponding enlarged filtration (*see* the criterion in Chaleyat-Maurel and Jeulin's paper in [4], p. 65).

PART I

THE GENERALIZED ITÔ-VENTZELL FORMULA

I.1. Generalized stochastic calculus

1.*a*. *Review of some results on generalized stochastic calculus*

In this section, we review those results from Nualart-Pardoux [7] which will be needed below.

We first define the underlying probability space, which will be fixed throughout this paper. $\Omega = C(\mathbb{R}_+; \mathbb{R}^k)$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$, $\mathscr{F}$ denotes the Borel σ-field over Ω, P is standard Wiener measure,

$$W_t(\omega) = (W_t^1(\omega), \ldots, W_t^k(\omega))' = \omega(t)$$

If $h \in L^2(\mathbb{R}_+)$, we denote by $\delta_i(h)$ the Wiener integral:

$$\delta_i(h) = \int_0^\infty h(t)\, dW_t^i$$

Let S denote the dense subset of $L^2(\Omega, \mathscr{F}, P)$ consisting of those (classes of) random variables F of the form:

$$F = f(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n)) \tag{1.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$, $i_1, \ldots, i_n \in \{1, \ldots, k\}$.

If F has the form (1.1), we define its derivative in the direction i as the process $\{D_t^i F, t \geq 0\}$ defined by:

$$D_t^i F = \sum_{\{l;\, i_l = i\}} \frac{\partial f}{\partial x_l}(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n))\, h_l(t)$$

More generally, we define the p-th order derivative of F:

$$D_{t_1 \ldots t_p}^{i_1 \ldots i_p} F = D_{t_p}^{i_p} \ldots D_{t_1}^{i_1} F$$

DF will stand for the k-dimensional process

$$\{D_t F = (D_t^1 F, \ldots, D_t^k F)', t \geqq 0\}.$$

PROPOSITION 1.1. – *For $i=1, \ldots, k$, D^i is an unbounded closable operator from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{R}_+)$. We identify D^i with its closed extension, and denote by $\mathbb{D}_i^{1,2}$ its domain. D^i is a local operator, in the sense that if $F \in \mathbb{D}_i^{1,2}$, then $D_t^i F = 0$ $dP \times dt$ a. e. on $\{F=0\} \times \mathbb{R}_+$.*

$\mathbb{D}^{1,2} = \bigcap_{i=1}^{k} \mathbb{D}_i^{1,2}$ is the domain of the closed unbounded operator D from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^k)$.

We shall use more generally the spaces $\mathbb{D}_i^{1,p}$ and $\mathbb{D}^{1,p} = \bigcap_{i=1}^{k} \mathbb{D}_i^{1,p}$ for $p \geqq 2$. $\mathbb{D}_i^{1,p}$ is the closure of S with respect to the norm:

$$\| F \|_{i,1,p} = \| F \|_p + \| \, \| D^i F \|_{L^2(\mathbb{R}_+)} \|_p$$

where $\| \, . \, \|_p$ denotes the norm in $L^p(\Omega)$.

We shall also use the spaces $\mathbb{D}_i^{2,p}$ and $\mathbb{D}^{2,p}$, again for $p \geqq 2$, which are respectively the completion of S with respect to:

$$\| F \|_{i,2,p} = \| F \|_p + \| \, \| D^i D^i F \|_{L^2(\mathbb{R}_+^2)} \|_p$$

and with respect to:

$$\| F \|_{2,p} = \| F \|_p + \| \sum_{i,j=1}^{k} \| D^i D^j F \|_{L^2(\mathbb{R}_+^2)} \|_p$$

We now introduce some classes of processes. For $i=1, \ldots, k$, $l=1$ or 2, $p \geqq 2$,

$$\mathbb{L}_i^{l,p} = L_{\text{loc}}^p(\mathbb{R}_+, dt; \mathbb{D}_i^{l,p})$$
$$\mathbb{L}^{l,p} = L_{\text{loc}}^p(\mathbb{R}_+, dt; \mathbb{D}^{l,p})$$

$\mathbb{L}_{i,C}^{1,p}$ will denote the set of those elements u of $\mathbb{L}_i^{1,p}$ which satisfy:

(i) For any $T>0$, the set of functions $\{s \to D_t^i u_s;\ s \in [0, T] - \{t\}\}_{t \in [0,T]}$ is equicontinuous with values in $L^p(\Omega)$.

(ii) $\text{ess sup}_{(s,t) \in [0,T]^2} E(|D_s^i u_t|^p) < \infty$, $\forall T > 0$, $(s, t) \in [0, T]^2$.

Moreover, $\mathbb{L}_C^{1,p} = \bigcap_{i=1}^{k} \mathbb{L}_{i,C}^{1,p}$ and $\mathbb{L}_C^{2,p} = \mathbb{L}_C^{1,p} \cap \mathbb{L}^{2,p}$. If $u \in \mathbb{L}_{i,C}^{1,p}$, we define:

$$(D_+^i u)_t = L^p(\Omega) - \lim_{s \to t, s>t} D_t u_s$$
$$(D_-^i u)_t = L^p(\Omega) - \lim_{s \to t, s<t} D_t u_s$$
$$(\nabla^i u)_t = (D_+^i u)_t + (D_-^i u)_t$$

$(\nabla u)_t$ will denote the k-dimensional vector $((\nabla^1 u)_t, \ldots, (\nabla^k u)_t)'$. We can now state:

PROPOSITION 1.2. — *For $1 \leqq i \leqq k$, $t>0$, we can define a linear continuous mapping from $\mathbb{L}_i^{1,2}$ into $L^2(\Omega)$ which to $u \in \mathbb{L}_i^{1,2}$ associates the Skorohod integral:*

$$\int_0^t u_s \, dW_s^i$$

This linear mapping is characterized by the two following properties:

$$E \int_0^t u_s \, dW_s^i = 0$$

$$E\left[\left(\int_0^t u_s \, dW_s^i\right)^2\right] = E \int_0^t u_s^2 \, ds + E \int_0^t \int_0^t D_s^i u_r \, D_r^i u_s \, ds \, dr$$

Moreover, this mapping is a local operator in the sense that if u, $v \in \mathbb{L}_i^{1,2}$, $\int_0^t u_s \, dW_s^i = \int_0^t v_s \, dW_s^i$ a. s. on $\{\omega, u_s(\omega) = v_s(\omega)$ for almost all $s \leqq t\}$.

DEFINITION 1.3. — *A measurable process $\{u_t, t \in [0, 1]\}$ is said to be Stratonovich integrable with respect to $\{W_t^i\}$ if the sequence*

$$\xi_{n,t}^i = \sum_{l=0}^{2^n - 1} (t_n^{l+1} - t_n^l)^{-1} (W_{t_n^{l+1}}^i - W_{t_n^l}^i) \int_{t_n^l \wedge t}^{t_n^{l+1} \wedge t} u_s \, ds$$

(with $t_n^l = l 2^{-n}$) converges in probability to a random variable ξ_t^i, for any $t>0$. We then write:

$$\xi_t^i = \int_0^t u_s \circ dW_s^i.$$

PROPOSITION 1.4. — *Let $u \in \mathbb{L}_{i,C}^{1,2}$. Then u is Stratonovich integrable with respect to $\{W_t^i\}$, and the Stratonovich integral is given by:*

$$\int_0^t u_s \circ dW_s^i = \int_0^t u_s \, dW_s^i + \frac{1}{2} \int_0^t (\nabla^i u)_s \, ds$$

PROPOSITION 1.5. — *Each of the following conditions implies that*

$\left\{\int_0^t u_s\, dW_s^i,\ t \geqq 0\right\}$ *has an a. s. continuous modification:*

(i) $\left\{ u \in \mathbb{L}_i^{1,2} \text{ and } \sup_{t\in[0,T]} E\left[\left(\int_0^T |D_s^i u_t|^2\, ds\right)^p\right] < \infty. \right.$ *for some* $p > 1$ *and all* $T > 0$

(ii) $u \in \mathbb{L}_i^{1,2}$ *and* $E\int_0^T \left(\int_0^t |D_s^i u_t|^2\, ds\right)^p dt < \infty$, *for some* $p > 2$ *and all* $T > 0$

(iii) $u \in \mathbb{L}_i^{2,2}$ *and* $\forall T > 0$, $\sup_{(s,t)\in[0,T]^2}\left(E|D_s^i u_t| + E\int_0^T |D_s^i D_r^i u_t|^2\, dr\right) < \infty$ *and moreover either* $\sup_{t\in[0,T]} E(|u_t|^p)$ *for some* $p > 2$ *or else* $E\int_0^T |u_t|^p\, dt < \infty$ *for some* $p > 4$.

We finally state the change of variable formula under two different sets of hypotheses. The first statement is a minor variant of Corollary 6.5 in Nualart-Pardoux [7]. Both results can be proved by the technique used in [7]. Note that from now on we use the convention of summation over repeated indices.

PROPOSITION 1.6. – *Let* $\Phi \in C_b^2(\mathbb{R}^d)$ *and* X_0 *be a d-dimensional random vector,* $\{A_t, B_t^1, \ldots, B_t^k;\ t \geqq 0\}$ *be d-dimensional random processes such that:*

(i) $X_0 \in (\mathbb{D}^{1,4})^d$

(ii) $A \in (\mathbb{L}^{1,4})^d$

(iii) $B^i \in (\mathbb{L}^{2,p})^d$; $i = 1, \ldots, k$; *for some* $p > 4$.

Let

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s^i\, dW_s^i,\ t \geqq 0.$$

We then have:

$$\Phi(X_t) = \Phi(X_0) + \int_0^t (\Phi'(X_s), A_s)\, ds + \int_0^t (\Phi'(X_s), B_s^i)\, dW_s^i + \frac{1}{2}\int_0^t (\Phi''(X_s)(\nabla^i X)_s, B_s^i)\, ds$$

where

$$(\nabla^i X)_t = 2\, D_t^i X_0 + B_t^i + 2\int_0^t D_t^i A_s\, ds + 2\int_0^t D_t^i B_s^j\, dW_s^j$$

Note that X^j does not necessarily belong to $\mathbb{L}_C^{1,2}$, but we can define $\nabla^i X^j$ by:

$$(\nabla^i X^j)_t = \mathrm{P}-\lim_{\varepsilon \to 0, \varepsilon > 0} (D_t^i X_{t+\varepsilon}^j + D_t^i X_{t-\varepsilon}^j)$$

Part of the hypothese made on X_0, A, $B^1, \ldots, B^k$ are used in order to insure some properties of X_t. As we will see below, it is sometimes easier to check directly the required properties on X. This motivates the following version of the extended Itô formula:

PROPOSITION 1.7. — *Let* $\Phi \in C_b^2(\mathbb{R}^d)$, *and*

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s^i\, dW_s^i, \qquad t \geqq 0$$

The conclusion of Proposition 1.6 *is still valid under the following assumptions:*

(i) $X \in (\mathbb{L}_C^{1,4})^d$ *and is a. s. continuous*

(ii) $A \in (L_{loc}^2(\mathbb{R}^+))^d$ *a. s.*

(iii) $B^i \in (\mathbb{L}_i^{1,4})^d$; $i = 1, \ldots, k$

As in the adapted case, the Stratonovich integral obeys the ordinary rules of calculus.

PROPOSITION 1.8. — *Let* $\Phi \in C_b^2(\mathbb{R}^d)$, *and* $\{X_t, A_t, B_t^k;\ t \geqq 0\}$ *be d-dimensional random processes such that:*

(i) $X \in (\mathbb{L}_C^{1,4})^d$ *and is a. s. continuous.*

(ii) $A \in (L_{loc}^2(\mathbb{R}_+))^d$ *a. s.*

(iii) $B^i \in (\mathbb{L}_{i,C}^{1,4})^d$; $i = 1, \ldots, k$

and

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s^i \circ dW_s^i$$

or in other words

$$X_t = X_0 + \int_0^t \left(A_s + \frac{1}{2}(\nabla^i B^i)_s\right) ds + \int_0^t B^i\, dW_s^i$$

Then:

$$\Phi(X_t = \Phi(X_0) + \int_0^t (\Phi'(X_s), A_s)\, ds + \int_0^t (\Phi'(X_s), B_s^i) \circ dW_s^i.$$

1.*b*. *The localization procedure*

All the processes which have been integrated so far satisfied moment conditions which one would like to remove, as well as the boundedness condition imposed on Φ and its derivatives in Propositions 1.6, 1.7 and 1.8. This will be useful in the next sections and essential in Part II. In other words, we want to localize processes, which don't satisfy any moment requirement, within the above classes, and integrate them.

This is made possible by the local properties of D and of the Skorohod integral.

For $l=1, 2;\ p\geqq 2$, let us define $\mathbb{D}^{l,p}_{\text{loc}}$ as the set of random variables F which are such that there exists a sequence $\{(\Omega_n\ F_n),\ n\in\mathbb{N}\}\subset\mathcal{F}\times\mathbb{D}^{l,p}$ with the following two properties:

$$\text{(i)}\qquad \Omega_n\uparrow\Omega\ a.\,s.,\quad \text{as } n\to\infty$$
$$\text{(ii)}\qquad F=F_n\ a.\,s.\quad \text{on } \Omega_n,\qquad n\in\mathbb{N}$$

We then say that the sequence $\{F_n\}$ localizes F in $\mathbb{D}^{l,p}$, and $D_t F$ is defined without ambiguity (thanks to the last part of Proposition 1.1) by:

$$D_t F=D_t F_n\quad \text{on } \Omega_n\times\mathbb{R}_+,\qquad n\in\mathbb{N}$$

$\mathbb{D}^{l,p}_{i,\text{loc}}$ is defined analogously. We define $\mathbb{L}^{l,p}_{\text{loc}}$ as the set of measurable processes u which are such that for any $T>0$, there exists a sequence $\{(\Omega^T_n,\ u^T_n);\ n\in\mathbb{N}\}\subset\mathcal{F}\times\mathbb{L}^{l,p}$ such that:

$$\text{(i)}\qquad \Omega^T_n\uparrow\Omega\ a.\,s.$$
$$\text{(ii)}\qquad u=u^T_n\,dP\times dt\ a.\,e.\quad \text{on } \Omega^T_n\times[0,\,T],\qquad n\in\mathbb{N}$$

In that case, $\{u^T_n,\ n\in\mathbb{N}\}$ will be said to localize u in $\mathbb{L}^{l,p}$ on the time interval [0, T]. $\mathbb{L}^{l,p}_{i,\text{loc}}$, $\mathbb{L}^{l,p}_{C,\text{loc}}$ and $\mathbb{L}^{l,p}_{i,C,\text{loc}}$ are defined similarly.

If $u\in\mathbb{L}^{l,p}_{i,\text{loc}}$ then we define its Skorohod integral with respect to $\{W^i_t\}$ by:

$$\int_0^t u_s\,dW^i_s=\int_0^t u^T_{n,s}\,dW^i_s\quad \text{on } \Omega^T_n\times[0,\,T]$$

This definition is non ambiguous thanks to the last statement of Proposition 1.2.

Clearly the above results could be rephrased by localizing the hypotheses on the data. In particular, Propositions 1.6, 1.7 and 1.8 are true with $\Phi\in C^2(\mathbb{R}^d)$. We shall use in Part II a more restrictive localization procedure. Let us define $\mathbb{L}^{1,\text{loc}}$ as the set of measurable processes u such that for any $T>0$ there exists a sequence $\{\beta^T_n,\ n\in\mathbb{N}\}\subset\bigcap_{p\geqq 2}\mathbb{D}^{1,p}$ satisfying:

$$\text{(i)}\qquad \{\beta^T_n=1\}\uparrow\Omega\ a.\,s.$$
$$\text{(ii)}\qquad \gamma_T\,\beta^T_n\,u\in\bigcap_{p\geqq 2}\mathbb{L}^{1,p}\ \text{for every } n$$
$$\text{(iii)}\qquad \beta^T_n\,D_\cdot u_\cdot\in\bigcap_{p\geqq 2}L^p(\Omega;\ L^2([0,\,T]^2))\ \text{for every } n,$$

where $\gamma_T(t)=1_{[0,\,T]}(t)$.

$\mathbb{L}_C^{1,\text{loc}}$ is defined similarly with $\bigcap_{p\geq 2} \mathbb{L}^{1,p}$ in (ii) replaced by $\bigcap_{p\geq 2} \mathbb{L}_C^{1,p}$. The set of sequences $\{\beta_n^T\}_T$ will be called a localizer.

Note that $\mathbb{L}^{1,\text{loc}} \subset \mathbb{L}_{\text{loc}}^{1,p}$ and $\mathbb{L}_C^{1,\text{loc}} \subset \mathbb{L}_{C,\text{loc}}^{1,p}$, for all $p \geq 2$.

I.2. Generalized stochastic calculus for Hilbert-space valued processes

We will now construct the Skohorod integral of a process taking values in a Hilbert space, and prove an Itô formula. Our aim is not to develop a general theory, but only to present the material which will be needed in the next section in order to interpret and manipulate stochastic integrals of the form $\int_0^t u_s(x)\, dW_s$ depending on a parameter x as Hilbert space valued stochastic integrals.

Let $\mathbb{K}$ be a separable real Hilbert space. $(\Omega, \mathscr{F}, P)$ being defined as above, let $S(\mathbb{K})$ denote the dense subset of $L^2(\Omega, \mathscr{F}, P; \mathbb{K})$ consisting of those (classes of) random variables F of the form:

$$F = f(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n)) \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{K})$, $h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$, $i_1, \ldots, i_n \in \{1, \ldots, k\}$. If F has the form (2.1), we define $\{D_t^i F, t \geq 0\}$, its derivative in the direction i, exactly as in the scalar case; note that it is now à $\mathbb{K}$-valued process. Higher order derivatives are defined similarly. D^i is now a closed unbounded operator from $L^2(\Omega; \mathbb{K})$ into $L^2(\Omega \times \mathbb{R}_+; \mathbb{K})$, with domain denoted $\mathbb{D}_i^{1,2}(\mathbb{K})$. $\mathbb{D}_i^{1,p}(\mathbb{K})$, $\mathbb{D}^{1,p}(\mathbb{K})$, $\mathbb{D}_i^{2,p}(\mathbb{K})$ and $\mathbb{D}^{2,p}(\mathbb{K})$ are defined in a way similar to the scalar case.

For $p \geq 2$, $l = 1, 2$, we denote by $\mathbb{L}_i^{l,p}(\mathbb{K})$ the space $L_{\text{loc}}^p(\mathbb{R}_+; \mathbb{D}_i^{l,p}(\mathbb{K}))$ and by $\mathbb{L}^{l,p}(\mathbb{K})$ the space $L_{\text{loc}}^p(\mathbb{R}_+; \mathbb{D}^{l,p}(\mathbb{K}))$, and define $\mathbb{L}_{i,C}^{l,p}(\mathbb{K})$, $\mathbb{L}_C^{l,p}(\mathbb{K})$ as in the scalar case.

For $u \in \mathbb{L}_i^{1,2}(\mathbb{K})$ and $t \in \mathbb{R}_+$, as in the scalar case, we can define $\int_0^t u_s\, dW_s^i$ as the element of $L^2(\Omega; \mathbb{K})$ such that for any $F \in \mathbb{D}_i^{1,2}(\mathbb{K})$,

$$E\left\langle F, \int_0^t u_s\, dW_s^i \right\rangle = E \int_0^t \langle D_s^i F, u_s \rangle\, ds$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{K}$. Note that below $\|\cdot\|$ will denote the norm in $\mathbb{K}$. It follows easily from the definition that for any $v \in \mathbb{K}$, $\left\langle v, \int_0^t u_s\, dW_s^i \right\rangle = \int_0^t \langle v, u_s \rangle\, dW_s^i$. The same is true with $v \in \mathbb{K}'$ and $\langle \cdot, \cdot \rangle$ replaced by the duality product between $\mathbb{K}$ and $\mathbb{K}'$. Using the fact that $\mathbb{K}$ is separable, it is then easy to ptove many results by finite dimensional approximation. In particular, $E\int_0^t u_s\, dW_s^i = 0$ and

$$E\left(\left\|\int_0^t u_s\, dW_s^i\right\|^2\right) = E\int_0^t \|u_s\|^2\, ds + E\int_0^t\int_0^t \langle D_s^i u_r, D_r^i u_s\rangle\, ds\, dr$$

The definition of the operator ∇, Definition 1.3 and Proposition 1.4 can be reproduced word for word in the case of a $\mathbb{K}$-valued integrand. Moreover it is easy to adapt to this situation the proof of Theorem 5.3 in Nualart-Pardoux [7], so that Part (iii) of Proposition 1.5 is still valid.

Let us now prove the Itô formula for the norm squared.

THEOREM 2.1. – *Let* $X \in \mathbb{L}_C^{1,4}(\mathbb{K})$ *be a. s. continuous, and suppose there exist* $A \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{K})$ *a. s.,* $B^i \in \mathbb{L}_i^{1,4}(\mathbb{K})$, $i = 1, \ldots, k$, *such that:*

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s^i\, dW_s^i, \qquad t \geqq 0,$$

We then have:

$$\|X_t\|^2 = \|X_0\|^2 + 2\int_0^t \langle X_s, A_s\rangle\, ds + 2\int_0^t \langle X_s, B_s^i\rangle\, dW_s^i + \int_0^t \langle (\nabla^i X)_s, B_s^i\rangle\, ds$$

Proof. – Let $\{e_l, l \in \mathbb{N}\}$ be an orthonormal basis of $\mathbb{K}$. We may apply Proposition 1.7 to $\langle X_t, e_l\rangle^2$ and obtain:

$$\langle X_t, e_l\rangle^2 = \langle X_0, e_l\rangle^2 + 2\int_0^t \langle X_s, e_l\rangle\langle A_s, e_l\rangle\, ds + 2\int_0^t \langle X_s, e_l\rangle\langle B_s^i, e_l\rangle\, dW_s^i + \int_0^t \langle (\nabla^i X)_s, e_l\rangle\langle B_s^i, e_l\rangle\, ds.$$

It remains to sum from $l = 0$ to N, and let N tend to ∞. The convergence of the ds integrals follows easily from the fact that $\int_0^t \|X_s\|\,\|A_s\|\, ds < \infty$ and $\int_0^t \|(\nabla^i X)_s\|\,\|B_s^i\|\, ds < \infty$ *a. s.* The convergence of the Skorohod integral follows from the fact that $\langle X, B^i\rangle \in \mathbb{L}_i^{1,2}$, $i = 1, \ldots, k$. ■

Note that the Itô formula for $\langle X_t, Y_t\rangle$, with $\{X\}$ and $\{Y\}$ both satisfying the asumptions of Theorem 2.1, follows easily from this theorem applied to $X + Y$, X and Y.

I.3. The generalized Itô-Ventzell formula

The aim of this section is to give an Itô-Ventzell-type formula fo $F_t(X_t)$, when:

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s^i\, dW_s^i \tag{3.1}$$

$$F_t(x) = F_0(x) + \int_0^t G_s(x)\, ds + \int_0^t H_s^i(x)\, dW_s^i \tag{3.2}$$

where $\{X_t, A_t, B_t^1, \ldots, B_t^k;\ t \geqq 0\}$ satisfy the assumptions of Proposition 1.7 with the exponent 4 replaced by 8, and $\{F_t(.), G_t(.), H_t^1(.), \ldots, H_t^k(.)\, t \geqq 0\}$ are $L^2(\mathbb{R}^d;\ \mu)$-valued random processes, where μ is a measure which is absolutely continuous with respect to Lebesgue measure, with a smooth and everywhere strictly positive density q. We suppose that $\{F, G, H^1, \ldots, H^k\}$ satisfy the hypotheses of Theorem 2.1, with $\mathbb{K}$ replaced by $L^2(\mathbb{R}^d;\ \mu)$. Note that (3.2) is interpreted as an equality in $L^2(\mathbb{R}^d;\ \mu)$.

Let us now formulate a set of hypotheses which will be supposed to hold below.

(3.3) $\begin{cases} F \in \mathbb{L}_C^{1,4}(L^2(\mathbb{R}^d;\ \mu)); \quad G \in L^2_{loc}(\mathbb{R}_+;\ L^2(\mathbb{R}^d;\ \mu)); \\ H^i \in \mathbb{L}_i^{1,4}(L^2(\mathbb{R}^d;\ \mu)), \quad i = 1, \ldots, k \end{cases}$

(3.4) For any $t \geqq 0$ and a. s., $F_t \in C^2(\mathbb{R}^d)$

(3.5) $F' \in \mathbb{L}^{1,2}((L^2(\mathbb{R}^d;\ \mu))^d)$

(3.6) $\begin{cases} (t, \omega) \text{ a. e.}, \quad (\nabla^i F)_t \in C^1(\mathbb{R}^d), \\ G_t \in C^0(\mathbb{R}^d), \quad H_t^i \in C^1(\mathbb{R}^d), \quad i = 1, \ldots, k \end{cases}$

(3.7) $\begin{cases} (t, s, \omega) \text{ a. e.}, \quad D_s(F_t') \in (C^0(\mathbb{R}^d))^d, \\ D_s^i H_t^i \in C^0(\mathbb{R}^d), \quad i = 1, \ldots, k \end{cases}$

(3.8) $\forall n \in \mathbb{N}$, for any compact subset $\mathbf{K}$ of $\mathbb{R}^d$, the following holds:

(3.8.a) $E \int_0^t \sup_{x \in \mathbf{K}} |F_s'(x)|^4\, ds < \infty, \qquad E \int_0^t \sup_{x \in \mathbf{K}} |F_s''(x)|^4\, ds < \infty$

(3.8.b) $E \int_0^t \left(\int_0^t \sup_{x \in \mathbf{K}} |D_u^i F_s'(x)|^2\, du \right)^2 ds < \infty$

(3.8.c) $\int_0^t \sup_{x \in \mathbf{K}} |G_s(x)|\, ds < \infty, \qquad \int_0^t \sup_{x \in \mathbf{K}} |(\nabla^i F_s)'(x)|^{4/3}\, ds < \infty$ a. s.

For $i=1, \ldots, k$,

$$(3.8.d)\quad \left\{\begin{array}{l} E\int_0^t \sup_{x\in\mathbf{K}} |H_s^i(x)|^2\, ds<\infty, \qquad E\int_0^t \sup_{x\in\mathbf{K}} |H_s^{i'}(x)|^4\, ds<\infty \\ E\int_0^t\int_0^t \sup_{x\in\mathbf{K}} |D_u^i H_s^i(x)|^2\, ds\, du<\infty \end{array}\right.$$

THEOREM 3.1. — *Let* $\{X_t\}$ *and* $\{F_t\}$ *be respectively an* $\mathbb{R}^d$ *and a* $L^2(\mathbb{R}^d; \mu)$*-valued process satisfying* (3.1) *and* (3.2). *We suppose that* $\{X_t, A_t, B_t^1, \ldots, B_t^k;\ t\geqq 0\}$ *satisfy the assumptions of Proposition* 1.7 *with the exponent* 4 *replaced by* 8, *and that the conditions* (3.3) *to* (3.8) *are in force. Then the processes* $\{F_t'(X_t)B_t^i,\ t\geqq 0\}$ *and* $\{H_t^i(X_t),\ t\geqq 0\}$, $i=1, \ldots, k$, *are elements of* $\mathbb{L}_{\text{loc}}^{1,2}$, *and the following holds:*

$$(3.9)\quad F_t(X_t)=F_0(X_0)+\int_0^t (F_s'(X_s), A_s)\, ds+\int_0^t (F_s'(X_s), B_s^i)\, dW_s^i$$
$$+\frac{1}{2}\int_0^t (F_s''(X_s)(\nabla^i X)_s, B_s^i)\, ds+\int_0^t G_s(X_s)\, ds+\int_0^t H_s^i(X_s)\, dW_s^i$$
$$+\frac{1}{2}\int_0^t ((\nabla^i F)_s'(X_s), B_s^i)\, ds+\frac{1}{2}\int_0^t (H_s^{i'}(X_s), (\nabla^i X)_s)\, ds.$$

Proof. — We are going to use the same technique as in Bismut [2] and Sznitman [13]. Let $\varphi\in C_C^\infty(\mathbb{R}^d, \mathbb{R}_+)$, such that $\int_{\mathbb{R}^d}\varphi(x)\, dx=1$. For $\varepsilon>0$, we define $\varphi_\varepsilon(x)=\varepsilon^{-d}\varphi\left(\frac{x}{\varepsilon}\right)$. It follows from Corollary 1.7 that:

$$\varphi_\varepsilon(X_t-x)=\varphi_\varepsilon(X_0-x)+\int_0^t (\varphi_\varepsilon'(X_s-x), A_s)\, ds$$
$$+\int_0^t (\varphi_\varepsilon'(X_s-x), B_s^i)\, dW_s^i+\frac{1}{2}\int_0^t (\varphi_\varepsilon''(X_s-x)(\nabla^i X)_s, B_s^i)\, ds.$$

We multiply each term of the above equality by $q^{-1}(x)$, yielding:

$$(3.10)\quad q^{-1}(x)\varphi_\varepsilon(X_t-x)=q^{-1}(x)\varphi_\varepsilon(X_0-x)$$
$$+\int_0^t q^{-1}(x)(\varphi_\varepsilon'(X_s-x), A_s)\, ds$$
$$+\int_0^t q^{-1}(x)(\varphi_\varepsilon'(X_s-x), B_s^i)\, dW_s^i$$
$$+\frac{1}{2}\int_0^t q^{-1}(x)(\varphi_\varepsilon''(X_s-x)(\nabla^i X)_s, B_s^i)\, ds.$$

and then regard (3.10) as an equality between processes with values in $L^2(\mathbb{R}^d; \mu)$. Indeed, if $\mathbf{Q}$ is a countable dense subset of $\mathbb{R}^d$, there exists a set $N \in \mathscr{F}$ s. t. $P(N)=0$ and (3.10) holds outside N for any $x \in \mathbf{Q}$. On the other hand, each term in (3.10) can be considered as a random variable taking values in the Sobolev space $H^n(\mathbb{R}^d)$, for any $n \in \mathbb{N}$, and therefore is almost surely continuous in x, from the Sobolev embedding theorem with $n > d/2$ (*see e.g.* Adams [1]). Therefore, the equality in $L^2(\mathbb{R}^d; \mu)$ will follow from the equality at each point of $\mathbf{Q}$. It just remains to check that the random element of $H^n(\mathbb{R}^d)$ (with $n > d/2$)

$$\int_0^t q^{-1}(.)(\varphi_\varepsilon'(X_s - .), B_s^i)\, dW_s^i$$

evaluated at x equals a. s. the $\mathbb{R}$-valued Skorohod integral

$$\int_0^t q^{-1}(x)(\varphi_\varepsilon'(X_s - x), B_s^i)\, dW_s^i.$$

This follows from one of the basic properties of Hilbert space valued Skohorod integrals, since evaluating an element of $H^n(\mathbb{R}^d)$ at x means taking its pairing with $\delta_x \in H^{-n}(\mathbb{R}^d)$.

It now follows from Theorem 2.1 that:

$$\begin{aligned}
\int_{\mathbb{R}^d} F_t(x)\,\varphi_\varepsilon(X_t - x)\,dx = & \int_{\mathbb{R}^d} F_0(x)\,\varphi_\varepsilon(X_0 - x)\,dx \\
& + \int_0^t \int_{\mathbb{R}^d} F_s(x)(\varphi_\varepsilon'(X_s - x), A_s)\,dx\,ds \\
& + \int_0^t \int_{\mathbb{R}^d} F_s(x)(\varphi_\varepsilon'(X_s - x), B_s^i)\,dx\,dW_s^i \\
& + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} F_s(x)(\varphi_\varepsilon''(X_s - x)(\nabla^i X)_s, B_s^i)\,dx\,ds \\
& + \int_0^t \int_{\mathbb{R}^d} G_s(x)\,\varphi_\varepsilon(X_s - x)\,dx\,ds \\
& + \int_0^t \int_{\mathbb{R}^d} H_s^i(x)\,\varphi_\varepsilon(X_s - x)\,dx\,dW_s^i \\
& + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} (\nabla^i F)_s(x)(\varphi_\varepsilon'(X_s - x), B_s^i)\,dx\,ds \\
& + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} H_s^i(x)(\varphi_\varepsilon'(X_s - x), (\nabla^i X)_s)\,dx\,ds.
\end{aligned}$$

We now integrate by parts all integrals where derivatives of φ_ε appear, yielding:

$$\begin{aligned}\int_{\mathbb{R}^d} F_t(x)\,\varphi_\varepsilon(X_t-x)\,dx = \int_{\mathbb{R}^d} F_0(x)\,\varphi_\varepsilon(X_0-x)\,dx \\ + \int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(F'_s(x),\ A_s)\,dx\,ds \\ + \int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(F'_s(x),\ B^i_s)\,dx\,dW^i_s \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(F''_s(x)(\nabla^i X)_s,\ B^i_s)\,dx\,ds \\ + \int_0^t \int_{\mathbb{R}^d} G_s(x)\,\varphi_\varepsilon(X_s-x)\,dx\,ds \\ + \int_0^t \int_{\mathbb{R}^d} H^i_s(x)\,\varphi_\varepsilon(X_s-x)\,dx\,dW^i_s \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)((\nabla^i F)'_s(x),\ B^i_s)\,dx\,ds \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(H^{i'}_s(x),\ (\nabla^i X)_s)\,dx\,ds.\end{aligned}$$

We want finally to let ε tend to zero. Since we could have replaced F, G, $H^1, \ldots, H^k$ by the same quantities multiplied by any element of $C_c^\infty(\mathbb{R}^d)$, we can and will suppose below that conditions (3.8.a), ..., (3.8.d) hold with $\sup_{x \in K}$ replaced by $\sup_{x \in \mathbb{R}^d}$. Thus the arguments below will establish (3.10) with F, G, $H^1, \ldots, H^k$ multiplied say by an element of $C_C^\infty(\mathbb{R}^d)$ which is one on a ball of arbitrary radius since $\{X_t\}$ is continuous, this will establish the result. We can now let ε tend to zero. The convergence in the two first terms follows from the a. s. continuity of $F_t(x)$ in x for each t fixed. Again from the continuity of F' in x for fixed (ω, s), we have that $\int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(F'_s(x), A_s)\,dx$ tends to $(F'(X_s), A_s)$ when ε tends to zero, for fixed (ω, s). Moreover,

$$\left|\int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)(F'_s(x),\ A_s)\,dx\right| \leq \sup_{x \in \mathbb{R}^d} |(F'_s(x),\ A_s)|$$

The convergence then follows from Lebesgue's dominated convergence theorem and (3.8.a). The convergence in the other ds integrals follows similarly from (3.8.a), (3.8.c) and (3.8.d). The convergence in the stochastic integrals will follow from the following convergences:

$$E\int_0^t\left|\int_{\mathbb{R}^d} F'_s(x)\,\varphi_\varepsilon(X_s-x)\,dx - F'_s(X_s)\right|^2 (B^i_s)^2\,ds \to 0$$

$$E\int_0^t\int_0^t\left|\int_{\mathbb{R}^d} D^i_u F'_s(x)\,\varphi_\varepsilon(X_s-x)\,dx - D^i_u F'_s(X_s)\right|^2 (B^i_s)^2\,ds\,du \to 0$$

$$E\int_0^t\int_0^t\left|\int_{\mathbb{R}^d} F''_s(x)\,\varphi_\varepsilon(X_s-x)\,dx - F''_s(X_s)\right|^2 (D^i_u X_s B^i_s)^2\,ds\,du \to 0$$

$$E\int_0^t\int_0^t\left|\int_{\mathbb{R}^d} F'_s(x)\,\varphi_\varepsilon(X_s-x)\,dx - F'_s(X_s)\right|^2 (D^i_u B^i_s)^2\,ds\,du \to 0$$

$$E\int_0^t\left|\int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)\,H^i_s(x)\,dx - H^i_s(X_s)\right|^2 ds \to 0$$

$$E\int_0^t\int_0^t\left|\int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)\,H^{i'}_s(x)\,dx - H^{i'}_s(X_s)\right|^2 (D_u X_s)^2\,ds\,du \to 0$$

$$E\int_0^t\int_0^t\left|\int_{\mathbb{R}^d} \varphi_\varepsilon(X_s-x)\,D^i_u H^i_s(x)\,dx - D^i_u H^i_s(X_s)\right|^2 ds\,du \to 0$$

These convergences follow from the same argument as above, using the hypotheses (3.8.*a*) to (3.8.*d*). They establish both the fact that $F'(X)\,B^i$, $H^i(X)\in\mathbb{L}^{1,2}$ and the convergence of the stochastic integrals. ■

Remark 3.2. — *The usual appoach to the Itô-Ventzell formula is to impose conditions insuring that each process appearing in* (3.2) *has a version which is continuous in x, then use an Itô formula for the product* $\varphi_\varepsilon(X_t-x)\,F_t(x)$, *integrate with respect to dx, interchange the dx and ds, the dx and* dW_s *integrals, etc. Our approach, using an Itô formula for Hilbert space valued processes, avoids having to do explicity the interchange of integrations, and does not require the existence of continuous (in x) versions of the stochastic integrals. We refer the reader to Ustunel* [15] *for still another proof.* ■

I.4. The generalized Itô-Ventzell formula in Stratonovich language

We now suppose that $A\in L^2_{loc}(\mathbb{R}_+;\ \mathbb{R}^d)\,B^i\in\mathbb{L}^{1,8}_{i,C}(\mathbb{R}^d)$, $i=1,\ \ldots,\ k$, $G\in L^2_{loc}(\mathbb{R}_+;\ L^2(\mathbb{R}^d;\ \mu))$, and $H^i\in\mathbb{L}^{1,4}_{i,C}(L^2(\mathbb{R}^d;\ \mu))$, $i=1,\ \ldots,\ k$, and moreover that all the hypotheses of Theorem 3.1 are satisfied, with A replaced by $A+\frac{1}{2}\nabla^i B^i$, and G replaced by $G+\frac{1}{2}\nabla^i H^i$. We have in particular:

$$X_t = X_0 + \int_0^t A_s\,ds + \int_0^t B^i_s\circ dW^i_s$$

$$F_t(x) = F_0(x) + \int_0^t G_s(x)\,ds + \int_0^t H^i_s(x)\circ dW^i_s$$

THEOREM 4.1. — *We assume that the above hypotheses hold and also:*

$$F' \in \mathbb{L}_C^{1,2}((L^2(\mathbb{R}^d;\mu))^d) \tag{4.1}$$

$$(\nabla^i(F'))_t \in (C^0(\mathbb{R}^d))^d, \ t \text{ a. e.} \tag{4.2}$$

Suppose moreover that for any compact subset $\mathbf{K} \subset \mathbb{R}^d$, *for* $i=1, \ldots, k$,

$$E\int_0^t \sup_{x\in\mathbf{K}} |\nabla^i(F'_s)(x)|^{8/3}\,ds < \infty, \qquad E\int_0^t \sup_{x\in\mathbf{K}} |\nabla^i H_s^i(x)|^2\,ds < \infty \tag{4.3}$$

Then $(F'_t(X_t), B_t^i)$ *and* $H_t^i(X_t)$ *are elements of* $\mathbb{L}^{1,2}_{i,C,\mathrm{loc}}$, $1 \leqq i \leqq k$, *and*

$$F_t(X_t) = F_0(X_0) + \int_0^t (F'_s(X_s), A_s)\,ds + \int_0^t (F'_s(X_s), B_s^i)\circ dW_s^i + \int_0^t G_s(X_s)\,ds + \int_0^t H_s^i(X_s)\circ dW_s^i \tag{4.4}$$

Proof. — The first statement follows from (4.5) below, (4.3) and the hypotheses of Theorem 2.1. Let us prove (4.4). We have:

$$X_t = X_0 + \int_0^t \left[A_s + \frac{1}{2}(\nabla^i B^i)_s\right] ds + \int_0^t B_s^i\,dW_s^i$$

$$F_t(x) = F_0(x) + \int_0^t [G_s(x) + \frac{1}{2}(\nabla^i H^i)_s(x)]\,ds + \int_0^t H_s^i(x)\,dW_s^i$$

It then follows from Theorem 3.2 that:

$$\begin{aligned} F_t(X_t) = F_0(X_0) &+ \int_0^t (F'_s(X_s), A_s)\,ds + \int_0^t (F'_s(X_s), B_s^i)\,dW_s^i \\ &+ \frac{1}{2}\int_0^t [(F'_s(X_s), (\nabla^i B^i)_s) + (F''_s(X_s)(\nabla^i X)_s, B_s^i) + ((\nabla^i F)'_s, B_s^i)]\,ds \\ &+ \int_0^t G_s(X_s)\,ds + \int_0^t H_s^i(X_s)\,dW_s^i \\ &+ \frac{1}{2}\int_0^t [(\nabla^i H^i)(X_s) + ((H_s^i)'(X_s), (\nabla^i X)_s)]\,ds \end{aligned}$$

The result now follows from Proposition 1.4, provided:

$$(\nabla^i(F'))_t(X_t) = (\nabla^i F)'_t(X_t), \ t \text{ a. e.} \tag{4.5}$$

Let us first verify that:

$$D_s^i(F'_t) = (D_s^i F_t)', \ (s, t) \text{ a. e.} \tag{4.6}$$

Note that from (3.3) and (4.1), $F \in \mathbb{L}_C^{1,2}(H^1(\mathbb{R}^d;\mu))$. Here $H^1(\mathbb{R}^d;\mu)$ denotes the Sobolev space of functions which, together with their first order distributional derivatives, belong to $L^2(\mathbb{R}^d;\mu)$. The set of u's of the form:

$$u_t(x) = \sum_{j=1}^{n} g_j(x)\, v_j(t)$$

with $n \in \mathbb{N}$, $g_j \in \mathrm{H}^1(\mathbb{R}^d; \mu)$; $v_j \in \mathbb{L}^{1,2}$; $j=1, \ldots, n$, is dense in $\mathbb{L}^{1,2}(\mathrm{H}^1(\mathbb{R}^d; \mu))$. Clearly (4.6) holds for such u's and consequently also for F. Since $\mathrm{F} \in \mathbb{L}_{\mathrm{C}}^{1,2}(\mathrm{H}^1(\mathbb{R}^d; \mu))$ and $\mathrm{F}' \in \mathbb{L}_{\mathrm{C}}^{1,2}(\mathrm{L}^2(\mathbb{R}^d; \mu))$, it follows that

$$(\nabla^i(\mathrm{F}'))_t = (\nabla^i \mathrm{F})'_t, \; t \text{ a. e.}$$

where the above equality is an equality in $\mathrm{L}^2(\mathbb{R}^d; \mu)$. But from (4.2) and (4.6) both terms are continuous in x for almost all (t, ω). (4.5) follows. ■

Note that in the particular case where $\mathrm{B}^1 = \ldots = \mathrm{B}^k = 0$, the index 4 of Proposition 1.7 need not be replaced by 8, and one could also weaken the hypotheses on F in Theorem 3.1 and Theorem 4.1.

PART II

A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH ANTICIPATING COEFFICIENTS

II.1. Statement of the problem and main result

$(\Omega, \mathscr{F}, \mathrm{P}, \{\mathrm{W}_t\})$ being defined as in section I, we consider the stochastic differential equation in $\mathbb{R}^d$:

$$\mathrm{X}_t = \mathrm{X}_0 + \int_0^t b(s, \mathrm{X}_s)\, ds + \int_0^t \sigma^i(s, \mathrm{X}_s) \circ d\mathrm{W}_s^i, \tag{1.1}$$

where b: $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and σ^i: $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$.

Define $m(t, x) = \frac{1}{2} \sum_{i=1}^{k} \frac{\partial \sigma^i(t, x)}{\partial x} \sigma^i(t, x)$ and let $\varphi_t(x)$ denote the flow defined by the adapted equation:

$$\begin{aligned} \varphi_t(x) &= x + \int_0^t \sigma^i(s, \varphi_s(x)) \circ d\mathrm{W}_s^i \\ &= x + \int_0^t m(s, \varphi_s(x))\, ds + \int_0^t \sigma^i(s, \varphi_s(x))\, d\mathrm{W}_s^i. \end{aligned} \tag{1.2}$$

Under conditions to be stated below $\left[\frac{\partial \varphi_t}{\partial x}\right]^{-1}(x)$ exists and we define

$$(\varphi_t^{*-1} b)(t, \omega, x) = \left[\frac{\partial \varphi_t}{\partial x}\right]^{-1}(x)\, b(t, \omega, \varphi_t(x)).$$

A formal calculation based on Theorem I.4.1. shows that $\{\varphi_t(Y_t), t \geqq 0\}$ is a solution to (1.1) if

$$(1.3) \qquad \frac{dY_t}{dt} = (\varphi_t^{*-1} b)(t, Y_t), \qquad Y_0 = X_0.$$

Our main theorem shows under which conditions this is correct. We need the following hypotheses:

$$(1.4) \qquad X_0 \in \mathbb{D}_{\text{loc}}^{1,p} \text{ and } 1_{\{|X_0^i| \leqq n\}} \sup_{s \leqq T} |D_s X_0^i| \in L^p(\Omega)$$

for any $p \geqq 2$, $T > 0$, $n < \infty$, $1 \leqq i \leqq d$.

(1.5 i) b: $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable mapping such that (t, ω) *a. e.*,

$b(t, \omega, .) \in C^2(\mathbb{R}^d)$; for some measure μ defined as in section I.3, b, $b'_{x_1}, \ldots, b'_{x_d} \in \mathbb{L}_{\text{loc}}^{1,2}((L^2(\mathbb{R}^d; \mu))^d)$, and moreover (s, t, ω) a. e.,

$$D_s b(t, \omega, .), D_s b'_{x_1}(t, \omega, .), \ldots, D_s b'_{x_d}(t, \omega, .) \in C(\mathbb{R}^d).$$

(1.5 ii) $\forall T > 0$, $\forall \varepsilon > 0$, $\exists C_{T,\varepsilon}$ s. t.

$$|b(t, \omega, x)| \leqq C_{T,\varepsilon}(1 + |x|^{1-\varepsilon}), \quad \forall (t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^d$$

and $\exists p$ and $C_{T,p}$ s. t.:

$$|D_s b(t, \omega, x)| + |b'_x(t, \omega, x)| + |b''_{xx}(t, \omega, x)| + |D_s b'_x(t, \omega, x)| \leqq C_{T,p}(1 + |x|^p), \quad \forall (s, t, \omega, x) \in [0, T]^2 \times \Omega \times \mathbb{R}^d$$

(1.6 i) For $1 \leqq i \leqq k$, σ^i: $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable mapping s. t. $\sigma^i(t, .) \in C^7(\mathbb{R}^d; \mathbb{R}^d)$ for $t > 0$; and $\sigma(t, 0)$ is bounded on compact subsets of $\mathbb{R}_+$.

(1.6 ii) The partial derivatives of m and $\sigma^1, \ldots, \sigma^k$ with respect to x of order j are bounded on $[0, T] \times \mathbb{R}^d$, for $1 \leqq j \leqq 6$ and any $T > 0$.

Note that the assumption (1.6) will allow us to verify that $\varphi_t(x)$ satisfies the hypotheses imposed on $F_t(x)$ in the Itô-Ventsell formula of theorem I.4.1. Assumption (1.5) is needed to insure that Y_t satisfies the hypotheses imposed on X_t in Theorem I.4.1. The sublinear growth of b in x will be used in order to show that $\varphi^{*-1} b$ grows at most linearly in x, and hence the solution to (1.3) does not explode. For the existence proof below, $C_{T,\varepsilon}$ could be random, and the uniqueness would still be true with a random $C_{T,\varepsilon}$ satisfying an assumption similar to that imposed on X_0.

THEOREM 1.1. — *Under conditions* (1.4), ..., (1.6), *the equation* (1.3) *possesses a unique non exploding solution* $\{Y_t\}$. *If* $X_t = \varphi_t(Y_t)$, $t \geqq 0$, *then* X *is the unique a. s. continuous process in* $\mathbb{L}_C^{1,\text{loc}}$ *which solves* (1.1).

Remark 1.2. — *The technique of transforming an equation like* (1.1) *into* (1.2), (1.3) *has been used by other authors,* see e. g. *Bismut, Michel* [3], *from whom we borrow the notation* φ^{*-1}. *However, the estimates in Lemma* 2.1

below which we obtain via Sobolev's embedding theorem, following Kunita [5], *seem to be new.*

Remark 1.3. — *In the case where* $\sigma^i(t, x)$, $1 \leqq i \leqq k$ *are affine functions of* x, $\varphi_t(x)$ *is also affine and it is clear that we may take* $\varepsilon = 0$ *in* (1.5 ii): *that is we may allow linear growth of* b *in* x.

The next three sections are devoted to the proof of Theorem 1.1.

II.2. Preliminary Lemmas

The most important tool in the proof of Theorem 1.1, besides the Itô-Ventzell formula, is a series of estimates of $\varphi_t(x)$ and its derivatives, which we state in a general context.

LEMMA 2.1. — *Assume that* $\sigma^i(t, .) \in C^{r+2}(\mathbb{R}^d; \mathbb{R}^d)$ *for* $t \geqq 0$, $1 \leqq i \leqq d$ *and that the partial derivatives with respect to* x *order* j *of* $\sigma^i(t, .)$ *are bounded on* $[0, T] \times \mathbb{R}^d$, *for* $1 \leqq j \leqq r$. *Then there exists a version* $\varphi_t(x)$ *of the solution to* (1.2) *such that* φ *is a. s. jointly continuous* (t, x), $\varphi_t(.)$ *is a* C^r*-diffeomorphism for every* t, *and for every* $\delta > 0$, *there exists* $\zeta(\delta) \in \bigcap_{p \geqq 1} L^p(\Omega)$ *s. t. a. s.*

$$\sup_{t \leqq T} |\varphi_t(x)| \leqq \zeta(\delta)(1 + |x|^2)^{(1/2)+\delta}, \qquad x \in \mathbb{R}^d \tag{2.1}$$

$$\sup_{t \leqq T} |\varphi_t^{-1}(x)| \leqq \zeta(\delta)(1 + |x|^2)^{(1/2)+\delta}, \qquad x \in \mathbb{R}^d \tag{2.2}$$

$$\sup_{t \leqq T} \left| \left(\frac{\partial \varphi_t}{\partial x} \right)^{-1} (x) \right| \leqq \zeta(\delta)(1 + |x|^2)^{\delta}, \qquad x \in \mathbb{R}^d \tag{2.3}$$

$$\sup_{t \leqq T} \left| \frac{\partial^j \varphi_t}{\partial x^j}(x) \right| \leqq \zeta(\delta)(1 + |x|^2)^{\delta}, \qquad x \in \mathbb{R}^d, \quad 1 \leqq j \leqq r-1 \tag{2.4}$$

$$\sup_{t \leqq T} \left| \frac{\partial^j}{\partial x^j} \left[\left(\frac{\partial \varphi_t}{\partial x} \right)^{-1} \right](x) \right| \leqq \zeta(\delta)(1 + |x|^2)^{\delta}, \qquad x \in \mathbb{R}^d, \quad 1 \leqq j \leqq r-2. \tag{2.5}$$

Proof: First note that the notation $\frac{\partial^j \varphi}{\partial x^j}$ is a shorthand for the tensor of the j-th order derivatives of the components of φ.

The statement about the smoothness and diffeomorphism properties of φ_t may be found in Kunita [5]. In particular, we may obtain differential equations for the higher order derivatives by differentiating equation (1.2). In this way, we obtain:

$$\frac{\partial^j \varphi_t}{\partial x^j}(x) = \eta_j + \int_0^t \left[\frac{\partial m}{\partial x}(\varphi_s(x)) \frac{\partial^j \varphi_s}{\partial x^j}(x) + q_s^j \right] ds + \int_0^t \left[\frac{\partial \sigma^i}{\partial x}(\varphi_s(x)) \frac{\partial^j \varphi_s}{\partial x^j}(x) + p_s^{j,i} \right] dW_s^i, \tag{2.6}$$

where $\eta_1=1$, $q^1=p^{1,i}=0$ and for $j>1$, $\eta_j=0$ and

$$q_s^j=\rho^j\left(\frac{\partial m}{\partial x}(\varphi_s(x)),\ \ldots,\ \frac{\partial^j m}{\partial x^j}(\varphi_s(x)),\ \frac{\partial \varphi_s}{\partial x}(x),\ \ldots,\ \frac{\partial^{j-1}\varphi_s}{\partial x^{j-1}}(x)\right)$$
$$p_s^{j,i}=\pi^{j,i}\left(\frac{\partial \sigma^i}{\partial x}(\varphi_s(x)),\ \ldots,\ \frac{\partial^j \sigma^i}{\partial x^j}(\varphi_s(x)),\ \frac{\partial \varphi_s}{\partial x}(x),\ \ldots,\ \frac{\partial^{j-1}\varphi_s}{\partial x^{j-1}}(x)\right),$$

where ρ^j and $\pi^{j,i}$ are polynomial functions.

Let $U_t(x)=\left[\frac{\partial \varphi_t}{\partial x}\right]^{-1}(x)$. Using Itô's formula and (2.6) for $j=1$, we obtain:

$$U_t(x)=I+\int_0^t U_s(x)\left[\sum_{i=1}^d\left(\frac{\partial \sigma^i}{\partial x}\right)^2(\varphi_s(x))-\frac{\partial m}{\partial x}(\varphi_s(x))\right]ds$$
$$-\int_0^t U_s(x)\frac{\partial \sigma^i}{\partial x}(\varphi_s(x))\,dW_s^i$$

$\frac{\partial^j U_t}{\partial x^j}$ exists for $j\leqq r-2$, and satisfies an equation whose coefficients depend on $\frac{\partial^l m}{\partial x^l}$ and $\frac{\partial^l \sigma^i}{\partial x^l}$ for $l\leqq j+1$, $\frac{\partial^l \varphi}{\partial x^l}$ for $l\leqq j$, and $\frac{\partial^l U}{\partial x^l}$ for $l\leqq j-1$.

A standard estimate (*see e.g.* Stroock [12]) gives that for any $p\geqq 2$, $\exists\, c_p$ s. t.:

$$E\sup_{t\leqq T}|\varphi_t(x)|^p\leqq c_p(1+|x|^2)^{p/2},\qquad \forall\, x\in\mathbb{R}^d.$$

A similar estimate applied recursively to equation (2.6) for $j=1, 2, \ldots, r-1$, and to the equation satisfied by $\frac{\partial^l U_t}{\partial x^l}$ for $l=0, 1, 2, \ldots, r-2$, bearing in mind that initial conditions are constant, and that the derivatives of m and $\sigma^1, \ldots, \sigma^k$ are bounded, yields:

$$E\left(\sup_{t\leqq T}\left|\frac{\partial^j \varphi_t}{\partial x^j}(x)\right|^p+\sup_{t\leqq T}\left|\frac{\partial^l U_t}{\partial x^l}(x)\right|^p\right)\leqq c_p,\qquad \forall\, x\in\mathbb{R}^d,$$

for $p \geqq 2$, $1 \leqq j \leqq r-1$, $1 \leqq l \leqq r-2$. We thus obtain that for any $q > \frac{d}{2}$ and $p \geqq 2$,

$$(2.7\,a) \qquad \mathrm{E}\left(\sup_{t \leqq \mathrm{T}} \int_{\mathbb{R}^d} \frac{|\varphi_t(x)|^p \rho(x) + |\mathrm{U}_t(x)|^p}{(1+|x|^2)^q} dx \right) < \infty$$

$$(2.7\,b) \quad \mathrm{E}\left(\sup_{t \leqq \mathrm{T}} \int_{\mathbb{R}^d} \frac{\sum_{j=1}^{r-1} |(\partial^j \varphi_t/\partial x^j)(x)|^p + \sum_{l=0}^{r-2} |(\partial^l \mathrm{U}_t/\partial x^l)(x)|^p}{(1+|x|^2)^q} dx \right) < \infty$$

where $\rho(x) = (1+|x|^2)^{-p/2}$. We now use Sobolev's inequality (*see e.g.* Adams [1, Theorem 5.4.1.*c*]) which implies that for any $p > d$ there exists a constant c_p s. t.

$$(2.8) \qquad \sup_{x \in \mathbb{R}^d} |v(x)| \leqq c_p \| v \|_{1,p}, \qquad \forall v \in \mathrm{C}^1(\mathbb{R}^d),$$

where $\| v \|_{1,p}^p = \int_{\mathbb{R}^d} \left(|v(x)|^p + \left| \frac{\partial v}{\partial x}(x) \right|^p \right) dx$. Let

$\tilde{\varphi}_{\alpha,t}(x) = (1+|x|^2)^{-\alpha-1/2} \varphi_t(x)$.

Clearly,

$$\| \tilde{\varphi}_{\alpha,t} \|_{1,p} \leqq \mathrm{K}(\alpha) \left[\left(\int_{\mathbb{R}^d} |(1+|x|^2)^{-\alpha-1/2} \varphi_t(x)|^p dx \right)^{1/p} + \left(\int_{\mathbb{R}^d} |(1+|x|^2)^{-\alpha-1/2} \frac{\partial \varphi_t}{\partial x}(x)|^p dx \right)^{1/p} \right]$$

It then follows from (2.7) and Hölder's inequality that if

$$\zeta(p, q) = \sup_{t \leqq \mathrm{T}} \| \varphi_{q/p,t} \|_{1,p}$$

$$\mathrm{E}[\zeta^n(p, q)] < \infty, \qquad \forall n \in \mathbb{N},$$

provided $p \geqq 2$, $q > \frac{d}{2}$; and from (2.8):

$$|\varphi_t(x)| \leqq \zeta(p, q)(1+|x|^2)^{(q/p)+(1/2)}$$

(2.1) follows. (2.3), (2.4) and (2.5) are proved in the same way.

It remains to show (2.2). Remark that $\frac{\partial}{\partial x}[\varphi_t^{-1}](x) = \mathrm{U}_t(\varphi_t^{-1}(x))$. Consequently, if we define $g(r) = \sup_{|x| \leqq r} |\varphi_t^{-1}(x)|$, we deduce from (2.3):

$$g(r) \leqq |\varphi_t^{-1}(0)| + [\sup_{|x| \leqq r} \mathrm{U}_t(\varphi_t^{-1}(x))] r$$

$$\leqq |\varphi_t^{-1}(0)| + \zeta(\delta) r + \zeta(\delta) g(r)^{2\delta} r$$

But from Young's inequality, $\exists c(\delta)$ s. t.:

$$\xi^{2\delta} \leqq \frac{1}{2}\frac{\xi}{\alpha} + c(\delta)\alpha^{\varepsilon}$$

where $\varepsilon = \frac{2\delta}{1-2\delta}$, $\xi \geqq 0$, $\alpha > 0$. We apply this inequality with $\xi = g(r)$, $\alpha = \zeta(\delta) r$. It follows that:

$$g(r) \leqq 2|\varphi_t^{-1}(0)| + 2\zeta(\delta) r + 2c(\delta)(\zeta(\delta) r)^{1+\varepsilon}$$

(2.2) will follow if we show that $|\varphi_t^{-1}(0)|$ belongs to all $L^p(\Omega)$.

$$\varphi_t^{-1}(0) = -\int_0^t U_s(\varphi_s^{-1}(0))\,\sigma^i(0) \circ dW_s^i$$

The joint quadratic variation between $U_t(\varphi_t^{-1}(0))$ and W_t^i is deduced from the (adapted) Itô-Ventzell formula applied to $U_t(x)$ and $\varphi_t^{-1}(0)$, yielding:

$$\varphi_t^{-1}(0) = \int_0^t U_s(\varphi_s^{-1}(0))\frac{\partial\sigma^i}{\partial x}(0)\,\sigma^i(0)\,ds$$
$$+\int_0^t \frac{\partial U_s}{\partial x}(\varphi_s^{-1}(0)).\,[U_s(\varphi_s^{-1}(0))\,\sigma^i(0)]\,\sigma^i(0)\,ds - \int_0^t U_s(\varphi_s^{-1}(0))\,\sigma^i(0)\,dW_s^i$$

The required estimate now follows from (2.3), (2.5) and Gronwall's Lemma. ■

We shall need similar bounds on $D_s\varphi_t(x)$, $D_s\varphi_t^{-1}(x)$, ... For fixed s, $D_s^i\varphi_t(x) = 0$ if $t < s$, and for $t > s$:

$$D_s^i\varphi_t(x) = \sigma^i(\varphi_t(x)) + \int_s^t \frac{\partial m}{\partial x}(\varphi_r(x))\,D_s^i\varphi_r(x)\,dr$$
$$+\int_s^t \frac{\partial\sigma^j}{\partial x}(\varphi_r(x))\,D_s^i\varphi_r(x)\,dW_r^j.$$

A rigorous derivation of this formula can be found e. g. in Stroock [13]. Clearly,

$$D_s^i\varphi_t(x) = \frac{\partial\varphi_t}{\partial x}(x)\left[\frac{\partial\varphi_s}{\partial x}(x)\right]^{-1}\sigma^i(\varphi_s(x)) \tag{2.9}$$

Similarly, one obtains:

$$(2.10)\qquad D_s^i \frac{\partial^j \varphi_t}{\partial x^j}(x) = \frac{\partial^j}{\partial x^j}\left\{\frac{\partial \varphi_t}{\partial x}(.)\left[\frac{\partial \varphi_s}{\partial x}(.)\right]^{-1}\sigma^i(\varphi_s(.))\right\}(x)$$

Since $D_s\left\{\frac{\partial \varphi_t}{\partial x}\left[\frac{\partial \varphi_t}{\partial x}\right]^{-1}\right\}=0,$

$$(2.11)\qquad D_s^i U_t(x) = -U_t(x)\, D_s^i\left(\frac{\partial \varphi_t}{\partial x}(x)\right) U_t(x)$$

and $D_s^i \frac{\partial^j U_t}{\partial x^j}(x)$ may be obtained by differentiating (2.11).

We now have:

LEMMA 2.2. — *Under the hypotheses of Lemma* 2.1, *there exist* $\zeta \in \bigcap_{p \geqq 1} L^p(\Omega)$; $q_0, q_1, \ldots, q_{r-2} \in \mathbb{R}_+$ *and versions of*

$$D_s^i \frac{\partial^j \varphi_t}{\partial x^j}(x),\ D_s^i \frac{\partial^j U_t}{\partial x^j}(x) \qquad \text{and} \qquad D_s^i \varphi_t^{-1}(x)$$

satisfying:

$$(2.12)\qquad \sup_{s \leqq t \leqq T}\left|D_s^i \frac{\partial^i \varphi_t}{\partial x^j}(x)\right| \leqq \zeta(1+|x|^2)^{q_j}$$
$$x \in \mathbb{R}^d, \qquad 1 \leqq i \leqq k, \qquad 0 \leqq j \leqq r-2$$

$$(2.13)\qquad \sup_{s \leqq t \leqq T}\left|D_s^i \frac{\partial^j U_t}{\partial x^j}(x)\right| \leqq \zeta(1+|x|^2)^{q_j}$$
$$x \in \mathbb{R}^d, \qquad 1 \leqq i \leqq k, \qquad 0 \leqq j \leqq r-3$$

$$(2.14)\qquad \sup_{s \leqq t \leqq T}|D_s^i \varphi_t^{-1}(x)| \leqq \zeta(1+|x|^2)^{q_0}$$
$$x \in \mathbb{R}^d, \qquad 1 \leqq i \leqq k$$

Proof. — (2.12) and (2.13) follow immediately from (2.9), (2.10), (2.11) and Lemma 2.1. Moreover,

$$\varphi_t^{-1}(x) = x - \int_0^t U_s(\varphi_s^{-1}(x))\,\sigma^i(x)\circ dW_s^i$$

$$\begin{aligned} D_s^i \varphi_t^{-1}(x) = {} & -U_s(\varphi_s^{-1}(x))\,\sigma^i(x) \\ & - \int_s^t (D_s^i U_r)(\varphi_r^{-1}(x))\,\sigma^j(x)\circ dW_r^j \\ & - \int_s^t \frac{\partial U_r}{\partial x}(\varphi_r^{-1}(x))\, D_s^i \varphi_r^{-1}(x)\,\sigma^j(x)\circ dW_r^j, \qquad t \geqq s \end{aligned}$$

The (adapted) Itô-Ventzell formula allows to check that

$$-U_t(\varphi_t^{-1}(x))(D_s^i \varphi_t)(\varphi_t^{-1}(x))$$

is a version of $D_s^i \varphi_t^{-1}(x)$. (2.14) then follows from (2.13) and Lemma 2.1. ■

Finally, we want to give a formula for $D_s[F(\omega, X(\omega))]$.

LEMMA 2.3. — *Let* $\{F(\omega, x), x \in \mathbb{R}^d\}$ *be a random field and* $q>2$ *such that* $\psi F \in \mathbb{D}^{1, q}(L^2(\mathbb{R}^d))$ *for any* $\psi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. *Assume that:*

$$\text{(i)} \qquad \begin{cases} F(\omega, .) \in C^1(\mathbb{R}^d), \ \omega \ a.s. \\ D_t F(\omega, .) \in C^0(\mathbb{R}^d), \ (t, \omega) \ a.e. \end{cases}$$

(ii) *For every compact set* $K \subset \mathbb{R}^d$ *and* $T>0$, *there exists* $M_{K, T} \in L^q(\Omega)$ *s. t.:*

$$\sup_{t \leq T, \ x \in K} (|F(\omega, x)| + |F'_x(\omega, x)| + |D_t F(\omega, x)|) \leq M_{K, T}(\omega)$$

Then, if $X \in \mathbb{D}_{loc}^{1, p}(\mathbb{R}^d)$ *for all* $p \geq 2$, $F(., X) \in \mathbb{D}_{loc}^{1, r}$ *for* $2 \leq r < q$ *and:*

$$D_t[F(., X)] = (D_t F)(., X) + F'_x(., X) D_t X$$

Proof. — Choose $r \in [2, q)$, and fix $p=(r-q)^{-1} rq$. Let $\{X_n\}$ be a localizing sequence of X in $\mathbb{D}^{1, p}(\mathbb{R}^d)$, and let $\{\rho_n, n \in \mathbb{N}\} \subset C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ satisfy $\rho_n(x)=x$ for $|x| \leq n$. It then suffices to show that $F(., \rho_n(X_n))$ localizes $F(., X)$ in $\mathbb{D}^{1, r}$ and that:

$$D_t[F(., \rho_n(X_n))] = (D_t F)(., \rho_n(X_n)) + F'_x(., \rho_n(X_n)) \rho'_n(X_n) D_t X_n$$

These facts are shown by approximating $F(., \rho_n(X_n))$ by

$$\int \varphi_\varepsilon(X_n - y) F(., \rho_n(y)) \, dy$$

where φ_ε is defined as in the proof of Theorem I.3.1, and using the fact that D is a closed operator. ■

The same method shows:

LEMMA 2.4. — *Let* $\{F(t, \omega, x), t \geq 0, x \in \mathbb{R}^d\}$ *be a random field and* $q>2$ *such* $\psi F \in \mathbb{L}^{1, q}(L^2(\mathbb{R}^d))$ *for any* $\psi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. *Assume that:*

$$\text{(i)} \qquad \begin{cases} F(t, \omega, .) \in C^1(\mathbb{R}^d), \ (t, \omega) \ a.e. \\ D_s F(t, \omega, .) \in C^0(\mathbb{R}^d), \ (s, t, \omega) \ a.e. \end{cases}$$

(ii) *For every compact set* $K \subset \mathbb{R}^d$ *and* $T>0$, *there exists* $M_{K, T} \in L^q(\Omega)$ *s. t.:*

$$\sup_{s, \ t \leq T, \ x \in K} (|F(t, \omega, x)| + |F'_x(t, \omega, x)| + |D_s F(t, \omega, x)|) \leq M_{K, T}(\omega)$$

Then if $X \in \mathbb{L}^{1,p}_{\text{loc}}(\mathbb{R}^d)$ *for any* $p \geqq 2$, $\{F(t, ., X_t), t \geqq 0\} \in \mathbb{L}^{1,r}_{\text{loc}}$ *and* $\int_0^T F(t, ., X_t)\,dt \in \mathbb{D}^{1,r}_{\text{loc}}$ *for* $2 \leqq r < q$, $T > 0$, *and:*

$$D_s \int_0^T F(t, ., X_t)\,dt = \int_0^T [(D_s F)(t, ., X_t) + F'_x(t, ., X_t)\, D_s X_t]\,dt. \quad \blacksquare$$

II.3. Proof of uniqueness

Let X be an a.s. continuous process in $\mathbb{L}_C^{1,\text{loc}}$ which solves equation (1.1). Let $\{\beta_n^T\}$ be its localizer. Then:

$$\beta_n^T X_t = \beta_n^T X_0 + \int_0^t \beta_n^T b(s, X_s)\,ds + \int_0^t \beta_n^T \sigma^i(s, X_s) \circ dW_s^i$$

On the other hand,

$$\varphi_t^{-1}(x) = x - \int_0^t U_s(\varphi_s^{-1}(x))\,\sigma^i(s, x) \circ dW_s^i$$

Lemma 2.1 and 2.2 allow us to use the Itô-Ventsell formula of Theorem I.4.1, yielding:

$$\begin{aligned}\varphi_t^{-1}(\beta_n^T X_t) = \beta_n^T X_0 &+ \int_0^t \frac{\partial \varphi_s^{-1}}{\partial x}(\beta_n^T X_s)\,\beta_n^T b(s, X_s)\,ds \\ &+ \int_0^t \frac{\partial \varphi_s^{-1}}{\partial x}(\beta_n^T X_s)\,\beta_n^T \sigma^i(s, X_s) \circ dW_s^i \\ &- \int_0^t U_s(\varphi_s^{-1}(\beta_n^T X_s))\,\sigma^i(s, \beta_n^T X_s) \circ dW_s^i\end{aligned}$$

But $\dfrac{\partial \varphi_t^{-1}}{\partial x} = U_t(\varphi_t^{-1}(.))$ and so from the local property of the Stratonovich integral, on the set $\{\beta_n = 1\}$ we have:

$$\varphi_t^{-1}(X_t) = X_0 + \int_0^t (\varphi^{*-1} b)(s, \varphi_s^{-1}(X_s))\,ds, \qquad t \leqq T$$

Consequently, $Y_t = \varphi_t^{-1}(X_t)$ solves equation (1.3), and uniqueness follows from:

PROPOSITION 3.1. — *Equation* (1.3) *has a unique, non-exploding solution* $\{Y_t, t \geqq 0\}$.

Proof. — Existence and uniqueness follow from the fact that $\varphi_t^{*-1} b(t, \omega, x)$ is a.e. C^1 in x and a.s. locally bounded together with its

x-derivative. Non explosion follows from

$$(3.1)\qquad \begin{gathered}\forall T>0,\ \exists \varepsilon>0 \text{ and } k_{T,\varepsilon}\in \bigcap_{p>1} L^p(\Omega) \text{ such that:}\\ |\varphi_t^{*-1} b(t,\omega,x)| \leqq k_{T,\varepsilon}(\omega)(1+|x|^{1-\varepsilon})\end{gathered}$$

Indeed, from (1.5 ii) and Lemma 2.1:

$$\begin{aligned}|(\varphi_t^*)^{-1} b(t,\omega,x)=&\left|\left[\frac{\partial\varphi_t}{\partial x}\right]^{-1}(x)\, b(t,\omega,x)\right|\\ &\leqq \zeta(\delta)(1+|x|^2)^\delta C_{T,\varepsilon}(1+|\varphi_t(x)|^{1-\varepsilon})\\ \leqq \zeta(\delta) C_{T,\varepsilon}(1+|x|^2)^\delta &(1+\zeta(\delta)^{1-\varepsilon}(1+|x|^2)^{((1/2)+\delta)(1-\varepsilon)}) \qquad \text{for } t\leqq T.\end{aligned}$$

By choosing δ small enough we can clearly achieve (3.1). Note that the ε in (3.1) is strictly smaller than the one for which (1.5 ii) holds. ■

II.4. Proof of existence

Let $\{Y_t\}$ denote again the unique solution of equation (1.3). We are going to use the following result, whose proof will be given at this end of the section.

LEMMA 4.1. – *$Y\in \bigcap_{p\geqq 2} \mathbb{L}^{1,p}_{C,\,loc}$ and $D_s Y_t$ satisfies*

$$(4.1)\qquad D_s Y_t = D_s X_0 + \int_0^t (D_s \varphi_r^{*-1} b)(r, Y_r)\, dr + \int_0^t (\varphi_r^{*-1} b)'_x (r, Y_r)\, D_s Y_r\, dr. \quad ■$$

Let us now define a localizer $\{\beta_n^T\}$ for Y:

$$(4.2)\qquad \beta_n^T = \alpha_n(|X_0|^2)\, \alpha_n\left(\int_0^T |(\varphi_t^{*-1} b)'_x(t, Y_t)|^2\, dt\right)$$

where $\alpha_n\in C_c^\infty(\mathbb{R}; [0, 1])$, $\alpha_n(x)=1$ for $|x|\leqq n$ and $\alpha_n \uparrow 1$. Clearly $\beta_n^T\uparrow 1$.

From (3.1), Holder's and Gronwall's inequalities,

$$(4.3)\qquad E(\sup_{0\leqq t\leqq T} \beta_n^T |Y_t|^p) \leqq \infty \quad \text{for any } p\geqq 2.$$

Consequently, from Lemmas 2.1 and 2.2 and hypothesis (1.5 ii),

$$(4.4)\qquad E[\sup_{0\leqq s,\, t\leqq T} (\beta_n^T |(D_s \varphi_t^{*-1} b)(t, Y_t)|^p)] < \infty \quad \text{for any } p\geqq 2.$$

These inequalities would still be true with β_n^T replaced by $\alpha_n(|X_0|^2)$. We now need the full definition of β_n^T to deduce from Gronwall's Lemma that:

$$\beta_n^T DY \in \bigcap_{p \geqq 2} L^p(\Omega; L^2((0, T)^2)) \tag{4.5}$$

It remains to bound $D\beta_n^T$ in $L^p(\Omega \times (0, T))$ and $D(\beta_n^T) Y$ in $L^p(\Omega \times (0, T)^2)$. These bounds follow exactly from the above estimates, since (4.3), (4.4) and (4.5) hold with α_n in the definition of β_n^T replaced by α'_n.

We can now apply theorem I.4.1 to:

$$\beta_n^T Y_t = \beta_n^T X_0 + \int_0^t \beta_n^T (\varphi_s^{*-1} b)(s, Y_s)\, ds$$

$$\varphi_t(x) = x + \int_0^t \sigma^i(s, \varphi_s(xy)) \circ dW_s^i$$

yielding:

$$\varphi_t(\beta_n^T Y_t) = \beta_n^T X_0 + \int_0^t \beta_n^T \varphi'_s(\beta_n^T Y_s)(\varphi_s^{*-1} b)(s, Y_s)\, ds + \int_0^t \sigma^i(s, \varphi_s(\beta_n^T Y_s)) \circ dW_s^i$$

which implies that X solves equation (1.1) on $[0, T] \times \{\beta_n^T = 1\}$ for any T and n. It remains to show that $X = \varphi(Y) \in \mathbb{L}_C^{1,\,\mathrm{loc}}$. All we need to show is that $\{\beta_n^T\}$ is a localizer for X, which follows from Lemmas 2.1 and 2.2, and again (4.3), (4.4) and (4.5). The proof is complete.

Proof of Lemma 4.1. — The first step is to control $(\varphi_t^{*-1} b)(t, \omega, x)$. For $n \in \mathbb{N}_+$, let $\rho_n \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\psi_n \in C_c^\infty(\mathbb{R}^{d \times d}; \mathbb{R}^{d \times d})$ be chosen such that

$$\rho_n(x) = x \quad \text{for } |x| \leqq n$$
$$\psi_n(z) = z \quad \text{for } z \in \mathbb{R}^{d \times d}, \quad |z| \leqq n.$$

Define

$$f^n(t, \omega, x) = \psi_n\left(\left[\frac{\partial \varphi_t}{\partial x}\right]^{-1}(x)\right) b(t, \omega, \rho_n(\varphi_t(x))).$$

Let $\bar{Y}^n$ be the solution to

$$\begin{aligned} \frac{d\bar{Y}_t^n}{dt} &= f^n(t, \omega, \bar{Y}_t^n), \quad 0 \leqq t \leqq T \\ \bar{Y}_0^n &= \rho_n(X_0) \\ \bar{Y}_t &= 0, \quad t \geqq T \end{aligned} \tag{4.6}$$

Note that $\rho_n(X_0)$ is a localizing sequence for X_0 in $\mathbb{D}^{1,p}$ because of hypothesis (1.4). Since

$$\rho_n(X_0) \leqq \| \rho_n \|_\infty = \sup_{x \in \mathbb{R}^d} | \rho_n(x) |$$

and

$$| f^n(t, x) | \leqq C_{T,\varepsilon}(1 + \| \rho_n \|_\infty^{1-\varepsilon}) \| \psi_n \|_\infty,$$

where $C_{T,\varepsilon}$ comes from hypothesis (1.5 ii),

$$\sup_{t \leqq T} | \bar{Y}_t^n | \leqq r_n := \| \rho_n \|_\infty (1 + 2TC_{T,\varepsilon}) \| \psi_n \|_\infty$$

We indicate formally how to control $D_s \bar{Y}_t^n$ before deriving the rigorous result. We shall show that:

$$\frac{d}{dt} D_s^i \bar{Y}_t^n = (D_s^i f^n)(t, \bar{Y}_t^n) + f_x^{n\prime}(t, \bar{Y}_t^n) D_s \bar{Y}_t^n$$

To apply the Gronwall Lemma, we need to bound

$$f_x^{n\prime}(t, x) = \psi_n'(U_t(x)) \frac{\partial U_t}{\partial x}(x)\, b(t, \rho_n(\varphi_t(x)))$$
$$+ \psi_n(U_t(x)) \frac{\partial b}{\partial x}(t, \rho_n(\varphi_t(x)))\, \rho_n'(\varphi_t(x)) \frac{\partial \varphi_t}{\partial x}(x)$$

for $|x| \leqq r_n$. To do this let $q \geqq 2$ and define

$$\eta_n = \int_0^T \int_{|x| \leqq r_n} \left(\left| \frac{\partial \varphi_t}{\partial x}(x) \right|^2 + \left| \frac{\partial^2 \varphi_t}{\partial x^2}(x) \right|^2 + \left| \frac{\partial U_t}{\partial x}(x) \right|^2 + \left| \frac{\partial^2 U_t}{\partial x^2}(x) \right|^2 \right) dx dt.$$

Let $k(t) = \sup_{|x| \leqq r_n} | f_x^{n\prime}(t, x) |$. By the Sobolev embedding theorem and the uniform boundedness of $b(t, \omega, x)$ and $b_x'(t, \omega, x)$ for

$$(t, \omega, x) \in [0, T] \times \omega \times \{ |x| \leqq r_n \}$$

(*see* hypothesis (1.5 ii) $\int_0^t k(t)\, dt \leqq C \eta_n^{1/2}$ for some constant C. Thus we want to cut off η_n. Accordingly, let

$$\nu_{l,n} = \alpha_l(\eta_n)$$

where α_n is as in (4.2). Note that Lemmas 2.1 and 2.2 imply that $\eta_n \in \bigcap_{p \geqq 2} \mathbb{D}^{1,p}$. We now consider a Picard iteration for equation (4.6), which we now write without the superscript n for simplicity of notation:

$$\text{(4.7)} \qquad \frac{d\bar{Y}_t}{dt}(\omega) = f(t, \omega, \bar{Y}_t(\omega)), \qquad \bar{Y}_0(\omega) = \rho(X_0(\omega))$$

Thus, we define the sequence

$$Z_t^0 \equiv \rho(X_0)$$
$$Z_t^{m+1} = \rho(X_0) + \int_0^t f(s, Z_s^m)\, ds.$$

We can show by recursion, and Lemmas 2.1, 2.2 and 2.4 that $Z^m \in \mathbb{L}_C^{1,p}$ for all $p \geq 2$ and all m and

$$D_s Z_t^{m+1} = D_s \rho(X_0) + \int_0^t (D_s f)(r, Z_r^m)\, dr + \int_0^t f'_x(r, Z_r^m)\, D_s Z_r^m\, dr.$$

Clearly, $(\bar{V}_{s,t}^n)$ being the solution of:

$$\bar{V}_{s,t}^n = D_s \rho(X_0) + \int_0^t (D_s f)(r, \bar{Y}_r^n)\, dr + \int_0^t f'_x(r, \bar{Y}_r^n)\, \bar{V}_{s,r}^n\, dr,$$

$$\lim_{m \to \infty} \Big(\sup_{t \leq T} |Z_t^m - \bar{Y}_t^n| + \sup_{s,\, t \leq T} |D_s Z_t^m - \bar{V}_{s,t}^n|\Big) = 0 \ a.s. \tag{4.8}$$

Also because $\sup_{t \leq T} |Z_t^m| \leq r_n$ a. s. for all m, the convergence of the first term in (4.8) holds in all $L^p(\Omega)$. Moreover, for any $p \geq 2$,

$$\lim_{m \to \infty} E \int_0^T \int_0^T |D_s(\nu_{l,n} Z_t^m) - \nu_{l,n} \bar{V}_{s,t}^n - D_s \nu_{l,n} \bar{Y}_t^n|^p\, ds dt = 0$$

since $\int_0^t |f'_x(r, \bar{Y}_r^n)|\, dr$ is bounded on $\{\nu_{l,n} \neq 0\}$. From the fact that D is closed, $\nu_{l,n} \bar{Y}^n \in \mathbb{L}^{1,p}$ and

$$D_s(\nu_{l,n} \bar{Y}_t^n) = (D_s \nu_{l,n}) \bar{Y}_t^n + \nu_{l,n} \bar{V}_{s,t}^n$$

and also $\nu_{l,n} \bar{Y}^n \in \mathbb{L}_C^{1,p}$. This implies that $\bar{Y}^n \in \bigcap_{p \geq 2} \mathbb{L}_{C,\,\mathrm{loc}}^{1,p}$ and $D\bar{Y}^n = \bar{V}^n$. Since $\{Y_t = \bar{Y}_t^n,\ 0 \leq t \leq T\} \uparrow \Omega$ a. s., the result follows.

II.5. Application to time-reversed stochastic differential equations

In this section, we want to indicate how our results apply to the equation satisfied by the time reversal (at fixed time) of the adapted solution of a classical stochastic differential equation. Let us first prove a general result about time reversal of Skorohod and Stratonovich integrals. In this section, the time interval is restricted to $[0, 1]: \Omega = C([0, 1]; \mathbb{R}^k)$, F is the Borel σ-field on Ω, P = Wiener measure, $W_t(\omega) = (W_t^1(\omega), \ldots, W_t^k(\omega))' = \omega(t)$; $\mathcal{S}$ denotes the subset of $L^2(\Omega)$ of "simple" random variables of the form:

$$F = f(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n)), \tag{5.1}$$

where $n \in \mathbb{N}$; $i_1, \ldots, i_n \in \{1, \ldots, k\}$, $h_1, \ldots, h_n \in L^2(0, 1)$; and

$$\delta_i(h) = \int_0^1 h(t)\, dW_t^i$$

We will use the same notation δ_i to denote the Skorohod integral, which can be defined as follows (see Nualart-Pardoux [7]). For $F \in \mathcal{S}$, F of the form (5.1), define:

$$D_t^i F = \sum_{\{l;\ i_l = i\}} \frac{\partial f}{\partial x_l}(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n))\, h_l(t)$$

and Dom δ_i as the subset of $L^2(\Omega \times (0, 1))$ consisting of those u's to which we can associate a constant c s. t.:

$$\left| E \int_0^1 u_t D_t^i F\, dt \right| \leqq c \sqrt{E(F^2)}, \qquad \forall F \in \mathcal{S}$$

$\delta_i(u)$ is then the unique class of r. v. which satisfies:

$$E[F\, \delta_i(u)] = E \int_0^1 u_t D_t^i F\, dt, \qquad \forall F \in \mathcal{S}$$

and whose existence follows from Riesz's theorem. Let us now consider the processes:

$$\tilde{W}_t^i = W_{1-t}^i - W_1^i,\ 0 \leqq t \leqq 1; \qquad i = 1, \ldots, k.$$

Clearly, any element $F \in \mathcal{S}$ of the form (4.1) can be rewritten as:

$$F = f\left(\int_0^1 k_1(t)\, d\tilde{W}_t^{i_1}, \ldots, \int_0^1 k_n(t)\, d\tilde{W}_t^{i_n} \right)$$

We then define:

$$\tilde{D}_t^i F = \sum_{\{l;\ i_l = i\}} \frac{\partial f}{\partial x_l} \left(\int_0^1 k_1(t)\, d\tilde{W}_t^{i_1}, \ldots, \int_0^1 k_n(t)\, d\tilde{W}_t^{i_n} \right) k_l(t)$$

We finally define Dom $\tilde{\delta}_i$ and $\tilde{\delta}_i$ exactly as Dom δ_i, δ_i, except that D_t^i is replaced by $\tilde{D}_t^i$. To any $u \in L^2(\Omega \times (0, 1))$, we associate $\bar{u} \in L^2(\Omega \times (0, 1))$ by:

$$\bar{u}_t = u_{1-t}.$$

We then have:

LEMMA 5.1. — *(i)* $u \in$ Dom δ_i *if and only if* $\bar{u} \in$ Dom $\tilde{\delta}_i$, *and in that case:*

$$\tilde{\delta}_i(\bar{u}) = -\delta_i(u)$$

(ii) *u is Stratonovich integrable with respect to* $\{W_t^i\}$ *over the interval* [0, 1] *if and only if $\bar{u}$ is Stratonovich integrable with respect to* $\{\tilde{W}_t^i\}$ *over* *the interval* [0, 1] *and:*

$$\int_0^1 \bar{u}_t \circ d\tilde{W}_t^i = -\int_0^1 u_t \circ dW_t^i.$$

Proof. — (ii) is an immediate consequence of Definition 1.3.

It is clear from the definitions of Dom δ_i, δ_i, Dom $\tilde{\delta}_i$, $\tilde{\delta}_i$ that the two statements in (i) will follow from the following equality:

$$(5.2) \qquad E \int_0^1 D_t^i F\, u_t\, dt = -E \int_0^1 \tilde{D}_t^i F\, \bar{u}_t\, dt$$

for any $F \in \mathcal{S}$. We then restrict ourself to F's of the form:

$$F = f(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n))$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in L^2(0, 1)$, $i_1, \ldots, i_n \in \{1, \ldots, k\}$. Since $\delta_{i_l}(h_l)$ is a Stratonovich integral, it follows from (ii) that:

$$F = f(-\tilde{\delta}_{i_1}(\bar{h}_1), \ldots, -\tilde{\delta}_{i_n}(\bar{h}_n))$$

$$\tilde{D}_t^i F = -\sum_{l;\ i_l = i} \frac{\partial f}{\partial x_l}(\delta_{i_1}(h_1), \ \ldots, \ \delta_{i_n}(h_n))\, h_l(1-t)$$

$$= -D_{1-t}^i F$$

and (5.2) is established. ■

Let now b: $[0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping satisfying:

(5.3 i) $$\sup_{0 \leq t \leq 1} |b(t, 0)| < \infty$$

(5.3 ii) $b(t, .)$ is of class $C^1(\mathbb{R}^d, \mathbb{R}^d)$, $t > 0$

(5.3 iii) $\dfrac{\partial b}{\partial x}$ is bounded on $[0, 1] \times \mathbb{R}^d$

and σ^i, $i = 1, \ldots, k$, be measurable mappings from $[0, 1] \times \mathbb{R}^d$ into $\mathbb{R}^d$ which satisfy (1.6).

Let $x_0 \in \mathbb{R}^d$. Each of the following equations has an unique $\mathcal{F}_t^W$-adapted solution, which belongs to $\mathbb{L}_C^{1,q}$, $\forall q \geq 2$:

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma^i(s, X_s)\, dW_s^i$$

$$Y_t = x_0 + \int_0^t b(s, Y_s)\, ds + \int_0^t \sigma^i(s, Y_s) \circ dW_s^i$$

Let $\bar{X}_t = X_{1-t}$, $\bar{Y}_t = Y_{1-t}$, $\tilde{W}_t^i = W_{1-t}^i - W_1^i$; $i = 1, \ldots, k$; $t \in [0, 1]$. It follows from Lemma 5.1 that we have:

$$\bar{X}_t = X_1 - \int_0^t b(1-s, \bar{X}_s)\,dx + \int_0^t \sigma^i(1-s, \bar{X}_s)\,d\tilde{W}_s^i \tag{5.4}$$

$$\bar{Y}_t = Y_1 - \int_0^t b(1-s, \bar{Y}_s)\,ds + \int_0^t \sigma^i(1-s, \bar{Y}_s) \circ dW_s^i \tag{5.5}$$

Note that (5.4) can be rewritten in Stratanovich form as:

$$\bar{X}_t = X_1 - \int_0^t \left(b - \frac{1}{2}\sigma^{i'}\sigma^i\right)(1-s, \bar{X}_s)\,ds + \int_0^t \sigma^i(1-s, \bar{X}_s) \circ dW_s^i \tag{5.4'}$$

We have:

PROPOSITION 5.2. — $\{\bar{X}_t, t \in [0, 1]\}$ *(resp.* $\{\bar{Y}_t, t \in [0, 1]\}$*) is the unique solution of* (5.4') [*resp.* (5.5)] *in* $\mathbb{L}_C^{1, \text{loc}}$.

Proof. — The result follows from the uniqueness part of Theorem 3.1. Note that b here satisfies (1.3 i) with $\varepsilon = 0$, $\varepsilon > 0$ was required only in the existence part of the proof of Theorem 3.1. ■

We have no uniqueness result concerning equation (5.4).

Remark 5.3. — *Note that the stochastic integral in* (5.4) *is a backward Itô integral. In the case where* $\tilde{W}_t$ *is a* $\mathcal{G}_t = \sigma\{X_1; \tilde{W}_s, 0 \leq s \leq t\}$ *semi-martingale* (see e. g. *Pardoux* [9] *for a sufficient condition), then one can rewrite* (5.4) *in terms of a* $\mathcal{G}_t$ *semi-martingale stochastic integral, which differs from the one in* (5.4), *since it is a forward integral.* ■

REFERENCES

[1] R. A. ADAMS, *Sobolev Spaces*, Acad. Press, 1975.

[2] J. M. BISMUT, A Generalized Formula of Itô and Some Other Properties of Stochastic Flows, *Z. Wahrschein. Werw. Geb.*, Vol. **55**, 1981, pp. 331-350.

[3] J. M. BISMUT and D. MICHEL, Diffusions conditionnelles, *J. Funct. Anal.*, Vol. **44**, 1981, pp. 174-211 and **45**, 1982, pp. 274-292.

[4] T. JEULIN and M. YOR Eds., Grossissements de filtrations: exemples et applications, *Lecture Notes in Mathematics*, Vol. **1118**, Springer Verlag, 1985.

[5] H. KUNITA Stochastic Differential Equations and Stochastic Flow of Diffeomorphisms, in *Ecole d'été de Probabilités de St-Flour*, P. L. Hennequin Ed., *Lecture Notes in Mathematics*, Vol. **1097**, Springer-Verlag, 1984, pp. 144-300.

[6] D. NUALART Noncausal Stochastic Integrals and Calculus, in *Stochastic Analysis and Related Topics*, H. KOREZLIOGLU and A. S. USTUNEL Eds., *Lecture Notes in Mathematics*, No. 1316, Springer-Verlag, 1988, pp. 80-129.

[7] D. NUALART and E. PARDOUX, Stochastic Calculus with Anticipating Integrands *Prob. Theory and Rel. Fields*, Vol. **78**, 1988, pp. 535-581.

[8] S. OGAWA, Sur la question d'existence de solutions d'une équation différentielle stochastique de type noncausal, *J. Math. Kyoto Univ.*, Vol. **24**, 1984, pp. 699-704.

[9] E. PARDOUX, Grossissement d'une filtration et retournement du temps d'une diffusion, in *Séminaire de Probabilités XX*, J. AZÉMA and M. YOR Eds., *Lecture Notes in Mathematics*, Vol. **1204**, Springer Verlag, 1986, pp. 48-55.
[10] B. L. ROZOVSKII, On the Itô-Ventzell Formula, *Vestnik Mosk. Univ. Mat.*, Vol. **28**, 1973, pp. 26-32.
[11] Y. SHIOTA, A Linear Stochastic Integral Equation Containing the Extended Itô Integral, *Math. Rep. Toyama Univ.*, Vol. **9**, 1986, pp. 43-65.
[12] D. W. STROOCK, *Topics in Stochastic Differential Equations*, Tata Institute, Springer-Verlag, 1982.
[13] D. W. STROOCK, Some Applications of Malliavin Calculus to Partial Differential Equations, in *Ecole d'été de Probabilités de St-Flour, Lecture Notes in Math.*, Vol. **976**, Springer-Verlag, 1983.
[14] A. S. SZNITMAN, Martingales dépendant d'un paramètre: une formule d'Itô, *Z. Wahrschein. Verw. Geb.*, Vol. **60**, 1982, pp. 41-70.
[15] A. S. USTUNEL Some Applications of Stochastic Integration in Infinite Dimensions, *Stochastics*, Vol. **7**, 1982, pp. 255-288.
[16] A. S. USTUNEL, Some Comments on the Filtering of Diffusions and the Malliavin Calculus, in *Stochastic Analysis and Related Topics*, H. KOREZLIOGLU and A. S. USTUNEL Eds, *Lecture Notes in Mathematics*, No. 1316, Springer-Verlag, 1988.
[17] A. D. VENTZELL, On the Equations of the Theory of Conditional Markov Processess, *Theory of Probability and Applic.*, Vol. **10**, 1965, pp. 357-361.

(*Manuscrit reçu le* 4 *juin* 1987)
(*révisé le* 26 *avril* 1988.)