ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 25, n° 1 (1989), p. 1-38
<http://www.numdam.org/item?id=AIHPB_1989__25_1_1_0>

Ann. Inst. Henri Poincaré,

Vol. 25, n° 1, 1989, p. 1-38. *Probabilités et Statistiques*

The weakly asymmetric simple exclusion process (*)

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ABSTRACT. — The one dimensional n.n. simple exclusion process with generator $\varepsilon^{-2} L_0 + \varepsilon^{-1} L_a$, $\varepsilon > 0$, is considered, L_0 and L_a being respectively the generators of the symmetric and totally asymmetric simple exclusion processes. Propagation of chaos and convergence to the Burgers equation with viscosity are proven in the limit when ε goes to zero. The density fluctuation field is shown to converge to a generalized Ornstein Uhlenbeck process with mean zero. The time asymptotic covariance kernel is explicitly computed for traveling wave profiles and the result indicates that the shock profile is stable while its space location fluctuates around its average position like a brownian motion. Its diffusion coefficient is explicitly computed.

Key words : Exclusion process, hydrodynamic limit, Burgers equation, shock waves.

RÉSUMÉ. — On considère le processus d'exclusion simple à de plus proches voisins de générateur $L_\varepsilon = \varepsilon^{-1} L_0 + L_a$, où L_0 et L_a sont respectivement les générateurs des processus d'exclusion simple symétrique et totalement asymétrique. La propagation du chaos et la convergence vers l'équation de Burgers avec viscosité sont démontrés pour $\varepsilon \to 0$. On montre que

(*) Partially supported by C.N.R. P.S. A.I.T.M., CNPq grant n. 401872-85 and the National Science Foundation under grant No: PHY82-17853 supplemented by funds for the National Areonautics and Space Admnistration.

Classification A.M.S. : 60 K 35.

le champ de fluctuation de la densité converge vers un processus généralisé de Ornstein Uhlenbeck, de moyenne 0. La covariance asymptotique est calculée explicitement pour des profils d'onde en mouvement et les résultats indiquent que le profil du choc est stable lorsque sa position fluctue comme un brownien. Autour de sa position moyenne, son coefficient de diffusion est calculé explicitement.

INTRODUCTION

The one dimensional Burgers equation

$$\partial_t \rho + 1/2\, \partial_r [\rho(1-\rho)] = \lambda\, \partial_r^2 \rho \qquad (1.1)$$

$0 \leqq \rho \leqq 1$, $r \in \mathbf{R}$, $t \geqq 0$, $\lambda \geqq 0$, is one of the simplest equations where the growth and the propagation of shock waves can be observed, *cf.* for instance Smoller, [26], and references quoted in. Non linear PDE's, like (1.1), describes to some extent and in particular situations the macroscopic behavior and the collective phenomena exhibited by several model systems, like cellular automata and stochastic interacting particle systems. In the last years such systems have been extensively studied, cellular automata for very fast computer simulations (*cf.* for instance [21], where eq. (1.1) was considered) and stochastic interacting particle systems for the possibility of a mathematically rigorous analysis, which seems beyond the present techniques for more realistic particles models of physical fluids.

The use of stochastic systems in the study of (1.1) goes back to McKean, [22], and carried out by Calderoni and Pulvirenti, [6], and Sznitman, [28], for systems of suitably interacting Brownian particles. In this paper we shall consider the exclusion process. The relation between such proces and (1.1) with $\lambda = 0$ is well known, *cf* [25], [20], [5], [11], [9], and [2], [29], [3] where a zero range process isomorphic to the exclusion process is considered.

The derivation of (1.1) with $\lambda > 0$ starting from the exclusion process is in a sense easier. In fact for drift and diffusion to have same strength, like in (1.1), one must suitably "weaken" the asymmetry in the exclusion process which can then be studied as a "perturbation" of the symmetric one. This can be done by applying general methods like the Fritz's technique, [14], *cf.* [15]. Alternatively one can use the Guo Papanicolaou and Varadhan approach, [17], which has the advantage of expliciting the connection with the large deviations theory. The derivation of (1.1) with

such techniques has been recently carried out in [19]. The simplest derivation of (1.1) is in our opinion due to Gartner [16] who considers a specific weakly asymmetric simple exclusion process and for this he is able to exploit at the particles' level the transformation which maps (1.1) into a linear differential equation. To conclude we mention that the steady state solutions to (1.1) can be derived directly, *cf.* [8], and that there is an interesting relation between (1.1) and some interface evolution problem, *cf.* Spohn [27].

In this paper we rederive (1.1) from the weakly asymmetric simple exclusion process, we refer to the next section for precise definitions and results.

We use the correlation functions technique, introducing some special correlation functions which solve a simplified hierarchy of equations and allow us to overcome the short time limitations usually present in the analysis of the hierarchy in the Grad-Boltzmann limit. Our approach is undoubtly lengthier but it has the advantage of providing very detailed information on the process, actually more than what needed for proving (1.1). This is not purely academic. Our aim is to go further in the analysis of the weakly asymmetric simple exclusion process beyond establishing (1.1) and to study the particle model in a much longer time scale than the one where (1.1) is proven. In particular we want to know at the particle level the stability properties of the traveling wave solutions to (1.1). How long does the shock move keeping its shape? according to (1.1) it is stationary, is that true for the microscopic model? Should we add to (1.1) correcting terms to catch the true limiting behavior? which ones in the case? the problem looks like the following. Consider X to be the space of all density profiles. Let $\rho \in X$ be a traveling wave solution to (1.1). Hence, as it is well known, the shape of ρ is determined by two parameters ρ_- and ρ_+ which denote the asymptotic densities to the left resp. right of the origin, $\rho_- < \rho_+$ (the drift in the process is directed toward the right). ρ is then completely determined by fixing its location in space. Call finally $M(\rho)$ the manifold in X obtained by rigidly shifting in all possible ways the profile ρ. Any such manifold is invariant under the evolution described by (1.1) and each point in the manifold moves with constant speed $c = 1 - \rho_- - \rho_+$. One might argue that the state of the particle model is only approximately described by the density profile of a traveling wave ρ, hence to a better approximation it should be represented in X by some "thin tube of profiles". In the long time regime such tube might be amplified and become macroscopic. What really happens is hard to decide by looking at (1.1) alone. From one side such equation seems to indicate that each traveling wave ρ is stable except for a neutral direction, that of $M(\rho)$. This would suggest that the only effect at long time should be a delocalization of the solution inside $M(\rho)$. However such

claim comes from the stability analysis w. r. t. localized initial perturbations, a change in the limiting densities does in fact produce a transition to some new manifold M and such disturbance would never vanish. It is not clear at all, by the way (1.1) is derived, if the particle system's evolution is in any sense related to what predicted by some stability analysis for the macroscopic equation and, if such is the case, what is the correct space where the stability problems should be considered.

Partial answers to such questions have been obtained for the asymmetric simple exclusion, as we report below, for more details we refer to [24]. If $\rho_- = 0$ it has been proven, [29] and [9], that $M(\rho)$ is stable and that the shock, moving with average speed c, fluctuates in space keeping the same shape, its motion being brownian with diffusion coefficient $D = c$. Even when $\rho_- > 0$ and $c = 0$ there are indications for the same behavior. Andjel Bramson and Liggett [1] have in fact proven that asymptotically in time the state of the system approaches a 1/2-1/2 mixture of the states with densities ρ_- and ρ_+, the left and right asymptotic densities at the shock, just what should be if the profile were rigidly fluctuating with Brownian motion. Finally we mention that a numerical analysis, [4], shows agreement with the above behavior also in other cases.

Let us now return to the weakly asymmetric simple exclusion process. The first step in analysing the stability of the shock is to look at the density fluctuations and that is what we have done here: we have found that the density fluctuation field (*cf.* the next section for precise definitions and results) converges to a generalized Ornstein-Uhlenbeck process with zero mean. Its covariance kernel diverges when $t \to \infty$ in the same way it should if the shock profile were stable but fluctuating around its average location like a Brownian with diffusion coefficient D. We have computed D and we have checked that its value agrees with that found in the asymmetric case for $\rho_- = 0$. We hope to complete the analysis of the long time behavior of the shocks for the weakly asymmetric exclusion process in a forthcoming paper.

We conclude this section by mentioning possible byproducts of our analysis which may be interesting in their own. We could easily consider the multidimensional case and quite arbitrary particles jumps, for notational simplicity we have restricted ourselves to the one dimensional case and to nearest neighbor jumps. We also have the possibility to derive equations of the form

$$\partial_t m = 1/2\, \partial_r^2 m - \partial_r f(m) - V'(m) \tag{1.2}$$

$m \in \mathbf{R}$, $r \in \mathbf{R}$, f and V smooth real functions. The case $V = 0$ in (1.2) is obtained by adding to the symmetric generator of the simple exclusion a small asymmetric perturbation described by the asymmetric generator of a "speed change exclusion process", which then determines the form of f.

To obtain the full equation (1.2) we interpret particles and empty sites as 1 and -1 spins, respectively. We then add to the previous generator a "small" generator describing local Glauber interactions among spins, see [7] where the equation with $f=0$ was derived. Our technique should apply quite straightforwardly also to these cases, we are indebted to Sznitman for very helpful discussions on these points. It is also worth mentioning that we can study cases where the "weak" asymmetric part of the generator of the simple exclusion is multiplied by a "slowly varying" factor which may depend both on space and time.

In Section 2 we state the main definitions and results and in Sections 3 and 4 their proofs.

2. DEFINITIONS AND RESULTS

Let $\Omega=\{0, 1\}^{Z}$ be the set of all particles configuration. We denote by η a generic element of Ω, so $\eta=\{\eta(x), x\in Z\}$. For any $\varepsilon>0$ and any cylinder function f on Ω let

$$L_\varepsilon f=L_0 f+\varepsilon L_a f \tag{2.1}$$

where

$$L_0 f(\eta)=1/2\,\Sigma_x[f(\eta^{x,x+1})-f(\eta)] \tag{2.2a}$$

$$L_a f(\eta)=\Sigma_x \eta(x)(1-\eta(x+1))[f(\eta^{x,x+1})-f(\eta)] \tag{2.2b}$$

$$\eta^{x,x+1}(z)=\begin{cases}\eta(z), & z\neq x,\ x+1\\ \eta(x), & z=x+1\\ \eta(x+1), & z=x\end{cases} \tag{2.2c}$$

We denote by $T_\varepsilon(t)$ the Markov semigroup with pregenerator L_ε, and we call such process the weakly asymmetric simple exclusion process (WASEP). We denote by $T^0(t)$ the Markov semigroup with generator L_0, such process is called the symmetric simple exclusion process (SEP).

We shall use the following notation. μ, ν denote measures on Ω, $\mu(\cdot)$ is the expectation w. r. t. μ while $E^\varepsilon_\mu(f(\eta_t))\equiv\mu(T_\varepsilon(t)f(\eta))$ denotes the expectation of f w. r. t. the WASEP with initial measure μ. Sometimes we shall simply write $E^\varepsilon(f)$.

2.1. THEOREM. — *Let ρ be a smooth function on R with values in $[0, 1]$. For $\varepsilon>0$ let μ^ε be the product measure on $\{0, 1\}^{Z}$ such that $\mu^\varepsilon(\eta(x))=\rho(\varepsilon x)$. Then for any $r\in R$ and $t>0$ and uniformly in the compacts of $R\times R_+$*

$$\lim_{\varepsilon\to 0}\left|E^\varepsilon_{\mu^\varepsilon}(\eta([\varepsilon^{-1}r], \varepsilon^{-2}t)-\rho(r, t)\right|=0 \tag{2.3}$$

where ρ solves

$$\partial_t \rho + \partial_r (\rho(1-\rho)) = 1/2\, \partial_r^2 \rho \qquad (2.4)$$
$$\rho(r, 0) = \rho(r)$$

For any $\eta \in \Omega$ define $\rho_\varepsilon(x, t \mid \eta)$ to be the solution of the following equation

$$\rho_\varepsilon(x, t \mid \eta) = \sum_z P_t(x \to z)\, \eta(z) + \varepsilon \int_0^t ds \sum_z P_{t-s}(x \to z) [\rho_\varepsilon(z-1, s \mid \eta) - \rho_\varepsilon(z, s \mid \eta) + \rho_\varepsilon(z, s \mid \eta) \{ \rho_\varepsilon(z+1, s \mid \eta) - \rho_\varepsilon(z-1, s \mid \eta) \}] \qquad (2.5)$$

where P_t is the probability kernel of the semigroup $T^0(t)$. Therefore $\rho_\varepsilon(x, t \mid \eta)$ is the discrete approximation to (2.4), x and t being "microscopic" variables in contrast to the macroscopic variables appearing in (2.4): namely microscopic space (resp. time) is ε^{-1} (resp. ε^{-2}) times the macroscopic space (resp. time). Define $\rho_\varepsilon(x, t \mid \mu^\varepsilon)$ as in (2.5) with $\eta(z)$ in the first term in the r. h. s. replaced by $\mu^\varepsilon(\eta(z)) \equiv \rho(\varepsilon z)$ (*cf.* Theorem 2.1). Define for any $t > 0$, $n \geqq 2$ for any $\underline{x} \equiv (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are distinct sites,

$$v_n^\varepsilon(\underline{x}, t \mid \eta) = E_\eta^\varepsilon \left(\prod_{i=1}^n (\eta(x_i, t) - \rho_\varepsilon(x_i, t \mid \eta)) \right) \qquad (2.6)$$

and

$$v_n^\varepsilon(\underline{x}, t \mid \mu^\varepsilon) = E_{\mu^\varepsilon}^\varepsilon \left(\prod_{i=1}^n (\eta(x_i, t) - \rho_\varepsilon(x_i, t \mid \mu^\varepsilon)) \right) \qquad (2.7)$$

2.2. PROPOSITION. — *With the above notation and definitions, for any* $n \geqq 1$ *and* $T > 0$ *there is* c *such that for any* η

$$\sup_{\underline{x}} |v_n^\varepsilon(\underline{x}, t \mid \eta)| \leqq c t^{-n/8}, \qquad \forall t \leqq \varepsilon^{-2} T \qquad (2.8\,a)$$

$$\sup_{\underline{x}} |v_n^\varepsilon(\underline{x}, t \mid \mu^\varepsilon)| \leqq c\, \varepsilon^{n/2}, \qquad \forall t \leqq \varepsilon^{-2} T \qquad (2.8\,b)$$

The above Proposition gives the key estimates for our analysis, all the results we obtain are more or less straight corollaries of Proposition 2.2. In fact using such result it is possible to derive a stronger version of Theorem 2.1 weakening the assumptions on the initial state. For instance it is possible to consider a family $\eta^\varepsilon \in \Omega$ such that for any $\varphi \in S(R)$

$$\lim_{\varepsilon \to 0} \varepsilon \sum_x \varphi(\varepsilon x)\, \eta(x) = \int dr\, \rho(r)\, \varphi(r)$$

and the same result as in Theorem 2.1 holds. We shall not discuss further this kind of generalizations and proceed in our analysis.

Next we define the fluctuation fields. For φ as above and for μ^ε as in Theorem 2.1 we set

$$X_t^\varepsilon(\varphi)=\sqrt{\varepsilon}\sum_x \varphi(\varepsilon x)[\eta(x,\varepsilon^{-2}t)-E_{\mu^\varepsilon}^\varepsilon(\eta(x,\varepsilon^{-2}t))] \tag{2.9}$$

and let P^ε be the corresponding law on $D(R_+\to S'(R))$ induced by the process with initial measure μ^ε. We then have

2.3. THEOREM. — The law P^ε defined above converges weakly to the law of a mean zero generalized Ornstein-Uhlenbeck process with covariance kernel $C^*_{t,s}(r,r')$ determined by the equation

$$\frac{\partial}{\partial s}C^*_{t,t+s}(r',r)$$

$$=\frac{1}{2}\frac{\partial^2}{\partial r^2}C^*_{t,t+s}(r',r)-\frac{\partial}{\partial r}[1-2\rho(r,t+s)]C^*_{t,t+s}(r',r) \tag{2.10a}$$

$$C^*_{t,t}(r,r')=C_t(r,r')+\delta(r-r')\rho(r,t)[1-\rho(r.t)] \tag{2.10b}$$

$$C^*_{0,0}(r,r')=\delta(r-r')\rho(r)[1-\rho(r)] \tag{2.10c}$$

where C_t satisfies

$$\frac{\partial}{\partial t}C_t(r,r')=\left\{\frac{1}{2}\frac{\partial^2}{\partial r^2}C_t(r,r')-\frac{\partial}{\partial r}([1-2\rho(r,t)]C_t(r,r'))\right\}$$

$$+\left\{\frac{1}{2}\frac{\partial^2}{\partial r'^2}C_t(r,r')-\frac{\partial}{\partial r'}([1-2\rho(r',t)]C_t(r,r'))\right\}$$

$$+\delta(r-r')\left\{2\rho(r,t)[1-\rho(r,t)]\frac{\partial}{\partial r}\rho(r,t)-\left[\frac{\partial}{\partial r}\rho(r,t)\right]^2\right\} \tag{2.10d}$$

$$C_0(r,r')=0$$

Furthermore let ρ in Theorem 2.1 be a stationary shock wave solution to (2.4) such that

$$\lim_{r\to\pm\infty}\rho(r)=\rho_\pm,\qquad 0\leqq\rho_-<\rho_+\leqq 1 \tag{2.11}$$

namely $\rho(r,t)=\rho(r-ct)$ where the speed $c=1-\rho_--\rho_+$. Then for any r and r'

$$\lim_{t\to\infty}t^{-1}C_t(r+ct,r'+ct)=D\rho'(r)\rho'(r') \tag{2.12}$$

where ρ' is tthe first derivative of ρ and

$$D=\frac{1}{\rho_+-\rho_-}[\{1-\rho_+-\rho_-\}\{\rho_++\rho_-\}+2\rho_+\rho_-] \tag{2.13}$$

Notice that the different time covariance solves the linearized equation [*cf.* eq. (2.10)]. This is a general feature related to the validity of the Fluctuation-Dissipation theorem and proven rigorously for several stochastich interacting particle systems. The splitting of C* as in (2.10*b*) appears naturally when taking the expectation of the square of the fluctuation field. The diagonal terms when expanding the square give rise to the δ-term in (2.10*b*). The other term $C_t(r, r')$ is a smooth function whose evolution is again determined by the linearized Burgers equation to which a term is added, the last term in (2.10*d*) which is responsabile for the linear growth of C_t as $t \to \infty$. To relate such a behaviour to what discussed in the Introduction we argue as follows.

Let $\mu_{\varepsilon, r, t}$ be the product measure such that

$$\mu_{\varepsilon, r, t}(\eta(x)) = \rho(\varepsilon x - ct - r) \quad \text{for all } x \text{ in } Z \tag{2.14}$$

Therefore $\mu_{\varepsilon, r, t}$ is just the initial measure shifted by $\varepsilon^{-1}[ct+r]$. Let

$$\Gamma_{\varepsilon, t} = \int \gamma_{\varepsilon, t}(dr)\, \mu_{\varepsilon, r, t} \tag{2.15}$$

where $\gamma_{\varepsilon, t}(dr)$ is the law at time t of a Brownian motion starting from 0 and moving with diffusion coefficient εD, D being the same as in Theorem 2.3. Given $\varphi \in S(R)$ let $\varphi_t(r) = \varphi(r+ct)$, then, obviously,

$$\lim_{t \to \infty} t^{-1} \lim_{\varepsilon \to 0} E_{\Gamma_{\varepsilon, t}}(X_0^\varepsilon(\varphi_t) X_0^\varepsilon(\psi_t)) = D \int dr\, dr'\, \rho'(r)\, \rho'(r')\, \varphi(r)\, \psi(r') \tag{2.16}$$

Therefore the true covariance of the WASEP is the same as that produced by $\Gamma_{\varepsilon, t}$, hence as far as this computation is concerned it is like the initial measure were only shifted in space by $\varepsilon^{-1}[ct+r]$, the law of r being $\gamma_{\varepsilon, t}$. The above motivates the conjecture that for any $t > 0$ and r

$$\lim_{\varepsilon \to 0} \mu_\varepsilon^{\varepsilon^{-3} t, \varepsilon^{-1} r} = \int \gamma_t(dr')\, \nu_{\rho(r+r')}$$

where the l. h. s. denotes the law of the weakly asymmetric simple exclusion at time $\varepsilon^{-3} t$ shifted in space by the integer part of $\varepsilon^{-1} ct + \varepsilon^{-1} r$, if the initial measure is the same μ^ε as at the end of Theorem 2.3 and γ_t is the law of a Brownian motion with diffusion coefficient D.

3. SHORT TIME ESTIMATES

In this section we prove Proposition 2.2 for times $t \leqq \varepsilon^{-2+\beta}$, $\beta > 0$. We start with Lemma 3.1 where we derive an integral equation for the v-functions defined in eqs. (2.6), (2.7). To simplify notation we just write $v_n(\underline{x}, t)$ for either $v_n^\varepsilon(\underline{x}, t \mid \eta)$ or $v_n^\varepsilon(x, t \mid \mu^\varepsilon)$; the statements below hold for

both. We also define N_n, $n \geq 1$, as the set of all n-tuplets of ordered distinct sites of Z, $\underline{x}=(x_1, \ldots x_n)$ denoting the generic element in N_n and $\mathbf{E}_{\underline{x}}$ the expectation w. r. t. the SEP starting from $\underline{x}$.

3.1. LEMMA. — *For $n \geq 1$, $\underline{x} \in N_n$ and $t>0$ we have*

$$v_n(\underline{x}, t)=\int_0^t ds\, \mathbf{E}_{\underline{x}}[\psi(\underline{x}(t-s), s)] \tag{3.1a}$$

where

$$\begin{aligned}
\psi(\underline{x}, s)=&\varepsilon[\mathrm{R}\, v_n](\underline{x}, s)\\
&+\varepsilon \sum_{i=1}^{n} [\mathbf{1}(x_j \neq x_i+1, \forall j \neq i)\, v_{n+1}(\underline{x}, x_i+1, s)\\
&-\mathbf{1}((x_j \neq x_i-1, \forall j \neq i)\, v_{n+1}(\underline{x}, x_i-1, s)]\\
&+\sum_{i, j} \mathbf{1}(x_j=x_i+1)\, v_{n-2}(\underline{x}(i, j), s)\, a^{\varepsilon}(x_i, x_j, s)\\
&+\sum_{i, j} \mathbf{1}(x_j=x_i+1)[v_{n-1}(\underline{x}(j), s)-v_{n-1}(\underline{x}(i), s)]\, b^{\varepsilon}(x_i, x_j, s)\\
&+\sum_{i, j} \mathbf{1}(x_j=x_i+1)\, v_{n-1}(\underline{x}(j), s)\, c^{\varepsilon}(x_i, x_j, s)\\
&-\varepsilon \sum_{i, j} \mathbf{1}(x_j=x_i+1)\, v_n(\underline{x}, s)[\rho_{\varepsilon}(x_j, s)-\rho_{\varepsilon}(x_i, s)]
\end{aligned} \tag{3.1b}$$

and

$$\begin{aligned}
(\mathrm{R}\, v_n)(\underline{x}, s)=&\sum_{i=1}^{n} [\mathbf{1}\{x_j \neq x_i-1\}(1-\rho_{\varepsilon}(x_i, s))(v_n(\underline{x}-\underline{e}_i, s)-v_n(\underline{x}, s))\\
&+\mathbf{1}\{x_j \neq x_i+1\}\, \rho_{\varepsilon}(x_i, s)(v_n(\underline{x}+\underline{e}_i, s)-v_n(\underline{x}, s))\\
&+(\rho_{\varepsilon}(x_i+1, s)-\rho_{\varepsilon}(x_i-1, s))\, v_n(\underline{x}, s)]
\end{aligned} \tag{3.2a}$$

$$a^{\varepsilon}(x, x+1, s) \equiv \varepsilon[\rho_{\varepsilon}(x+1, s)-\rho_{\varepsilon}(x, s)]\, \rho_{\varepsilon}(x, s)[1-\rho_{\varepsilon}(x+1, s)]$$

$$-1/2[\rho_{\varepsilon}(x+1, s)-\rho_{\varepsilon}(x, s)]^2 \tag{3.2b}$$

$$b^{\varepsilon}(x, x+1, s) \equiv \varepsilon\rho_{\varepsilon}(x, s)[1-\rho_{\varepsilon}(x+1, s)]+[\rho_{\varepsilon}(x+1, s)-\rho_{\varepsilon}(x, s)]$$

$$+\varepsilon[\rho_{\varepsilon}(x+1, s)-\rho_{\varepsilon}(x, s)]\, \rho_{\varepsilon}(x, s) \tag{3.2c}$$

$$c^{\varepsilon}(x, x+1, s) \equiv \varepsilon\rho_{\varepsilon}(x, s)[\rho_{\varepsilon}(x+1, s)-\rho_{\varepsilon}(x, s)] \tag{3.2d}$$

We are using the following notation:

$$\text{for } \underline{x} \in Z^n, \quad \underline{x}(i) \equiv \underline{x} \setminus \{x_i\} \quad \text{and} \quad \underline{x}(i, j) \equiv \underline{x} \setminus \{x_i, x_j\} \tag{3.3a}$$

$\underline{e}_i \in Z^n$ is the unit vector in the positive i-direction, $i=1, \ldots, n$.

$\sum_{i, j}$ is the sum over all disjoint pairs i and j in $\{1, \ldots, n\}$ (3.3b)

Proof. — By definition for any $n \geqq 1$, $t>0$, $\underline{x} \in Z^n$

$$\frac{d}{dt} v_n(\underline{x}, t) = \mathbf{E}^\varepsilon [(L_0 + \varepsilon L_a) \prod_{i=1}^{n} (\eta(x_i, t) - \rho_\varepsilon(x_i, t))] - \sum_{i=1}^{n} v_{n-1}(\underline{x}(i), t) \frac{d}{dt} \rho_\varepsilon(x_i, t) \tag{3.4}$$

$\mathbf{E}^\varepsilon = \mathbf{E}^\varepsilon_{\mu^\varepsilon}$ or $\mathbf{E}^\varepsilon_\eta$ according to cases.

We use (2.2) and (2.5) distinguishing the cases when particles are n. n. from the others. We get terms containing products of the occupation number functions η (correlation functions) with n, $n+1$, $n-1$ and $n-2$-body. To recover the v functions we add and substract the missing ρ_ε's and we get

$$\begin{aligned}
\frac{d}{dt} v_n(\underline{x}, t) = & \sum_{i=1}^{n} \Big\{ \mathbf{1}\{x_j \neq x_i + 1, \forall j \neq i\} \frac{1}{2} [v_n(\underline{x} + \underline{e}_i, t) - v_n(\underline{x}, t)] \\
& + \mathbf{1}\{x_j \neq x_i - 1, \forall j \neq i\} \frac{1}{2} [v_n(\underline{x} - \underline{e}_i, t) - v_n(\underline{x}, t)] \Big\} \\
& + \varepsilon \sum_{i=1}^{n} \{ \mathbf{1}\{x_j \neq x_i - 1, \forall j \neq i\} [(1 - \rho_\varepsilon(x_i, t)) v_n(\underline{x} - \underline{e}_i, t) - \rho_\varepsilon(x_i - 1, t) v_n(\underline{x}, t)] \\
& - \mathbf{1}\{x_j \neq x_i + 1, \forall j \neq i\} [(1 - \rho_\varepsilon(x_i + 1, t)) v_n(\underline{x}, t) - \rho_\varepsilon(x_i, t) v_n(\underline{x} + \underline{e}_i, t)] \} \\
& + \varepsilon \sum_i [\mathbf{1}\{x_j \neq x_i + 1, \forall j \neq i\} v_{n+1}(\underline{x}, x_i + 1, t) \\
& - \mathbf{1}\{x_j \neq x_i - 1, \forall j \neq i\} v_{n+1}(\underline{x}, x_i - 1, t)] \\
& + \varepsilon \sum_{j,\, i} \mathbf{1}\{x_j = x_i + 1\} [\rho_\varepsilon(x_j, t) - \rho_\varepsilon(x_i, t)] \\
& \qquad \times [(1 - \rho_\varepsilon(x_j, t)) v_{n-1}(\underline{x}(j), t) - \rho_\varepsilon(x_i, t) v_{n-1}(\underline{x}(i), t)] \\
& + \sum_{j,\, i} \mathbf{1}\{x_j = x_i + 1\} [v_{n-1}(\underline{x}(j), t) - v_{n-1}(\underline{x}(i), t)] \\
& \qquad \times \{ \varepsilon \rho_\varepsilon(x_i, t)(1 - \rho_\varepsilon(x_j, t)) + [\rho_\varepsilon(x_j, t) - \rho_\varepsilon(x_i, t)] \} \\
& + \sum_{j,\, i} \mathbf{1}\{x_j = x_i + 1\} v_{n-2}(\underline{x}(i, j), t) a^\varepsilon(x_i, x_j, s) \\
& - \varepsilon \sum_{j,\, i} \mathbf{1}\{x_j = x_i + 1\} [\rho(x_i + 1, t) - \rho(x_i, t)] v_n(\underline{x}, t)
\end{aligned} \tag{3.5}$$

For $t \geqq 0$ let $\underline{v}_t$ be the following function defined on the subset $\aleph_n \in \{0, 1\}^Z$ of configurations with n particles. If $\eta \in \aleph_n$ then $\underline{v}_t(\eta) = v_n(\underline{x}, t)$ where n is the number of particles in the configuration η and $\underline{x}$ are their positions. Since v_n is symmetric under permutations of the x_i in $\underline{x}$ the above is well defined. For simplicity we identify the elements of $\aleph_n$ with points $\underline{x} \in N_n$ and write the first term in (3.5) as $(L_0 v_n)(\underline{x}, t)$. It is easy to see that the second one is $(R v_n)(\underline{x}, t)$. We consider next the terms with v_{n-1}. By adding and subtracting $\rho_\varepsilon(x_i, t) v_{n-1}(\underline{x}(j), t)$ we rewrite

the fourth sum in (3.5) as a sum of a "gradient" of v_{n-1} plus $v_{n-1}(\underline{x}(j), t)$ times c^ε, as defined in (3.2*d*). The gradients of v_{n-1} are multiplied by b^ε, as defined in (3.2*c*). From the above observations it follows that the time derivative of $v_n(\underline{x}, t)$ is equal to $(L_0 v_n)(\underline{x}, t)$ plus the other terms. Therefore we can use the process with generator L_0 to write an integral equation for v_n. We therefore get (3.1) since

$$v_n(\underline{x}, 0) = 0 \quad \text{for any } \underline{x}.$$

3.2. NOTATION. — We can interpret the right hand side of (3.1*a*) as the expectation with respect to a process describing n labeled stirring particles which starts from $\underline{x}$ move for a time $t-s$ and then undergo a branching process. [We are thinking of the degree of the v-function as a number of particles, their positions being specified by the arguments of the v-functions]. We recall that the stirring process is realized by independently exchanging the occupation numbers at all pairs of nearest neighbor sites after an exponential time of mean 2. The induced motion on the particles is the SEP, if the labels of the particles are ignored.

Since ψ can be expressed as a linear combination of v-functions the different terms are interpreted as the outcome of the branching. To describe this structure we specify first of all the initial and the final number of particles. So the terms appearing in $R v_n$ are all of "type (n, n)", those in the first sum of (3.1*b*) are $(n, n+1)$, the successive ones are $(n, n-2)$, $(n, n-1)$ and (n, n) respectively. A finer description is needed to classify the single terms in each type. We start with the $(n, n-2)$ terms. We need two more labels which denote the particles appearing in the characteristic function in (3.1*b*). Hence each of these terms is determined by the multi-index $(n, n-2, j, i)$. The $(n, n-1)$ terms are singled out by adding the multi-index (j, i, σ) where j and i, as before, denote the particles involved in the characteristic function appearing in (3.1*b*) while σ equals ± 1 or 0. If $\sigma = 1$ we are considering the first term with $v_{n-1}(\underline{x}(j), s)$, if $\sigma = -1$ the term with $v_{n-1}(\underline{x}(i), s)$, if $\sigma = 0$ the last term with $v_{n-1}(\underline{x}(j), s)$.

The terms of type (n, n) are divided into two classes. Those in the first one are singled out by specifying a pair (j, i), *cf.* (3.1*b*). The others come from $R v_n$. To classify them one needs the pair (i, σ). The first index, i, labels a particle, *cf.* (3.2*a*), while $\sigma = 1$ refers to the first term in (3.2*a*), $\sigma = 2$ to the second, ..., $\sigma = 5$ to the fifth one.

Finally the $(n, n+1)$ terms can be written as

$$\varepsilon \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k} 1(x_{i_j} = x_{i_{j-1}} + 1, j = 2, \ldots, k;\ x_{i_1} - 1 \neq x_j, \forall j = 1, \ldots, n;\ x_{i_k} + 1 \neq x_j, \forall j = 1, \ldots, n) \\ [v_{n+1}(\underline{x}, x_{i_k} + 1, s) - v_{n+1}(\underline{x}, x_{i_1} - 1, s)] \tag{3.6}$$

where the sum over $i_1, \ldots, i_k$ is over all the subset with k elements of the set $(1, \ldots, n)$. Therefore these terms are classified by the specification $(i_1, \ldots, i_k, \sigma)$, where $i_1, \ldots, i_k$ are as above and $\sigma = 1$ selects the first term in the above sum while $\sigma = 2$ the second one.

Therefore the branching process occurring at time $(t-s)$ describes births, the $(n, n+1)$ terms, deaths, the $(n, n-2)$ and $(n, n-1)$ terms, as well as terms with same number of particles, the (n, n) terms. In some of these cases particles after the branching are displaced, and, as we shall see below, it will be convenient to relabel some of the particles after the branching. Of course new labels are needed for the newly born particles.

One more notation: to classify the various terms above we introduce a label λ which can take any of the multi-indexed values we have introduced above to single out the terms in (3.1). Therefore a value of λ specifies one of the terms in (3.1), in particular a v-function and a function of ε, $\underline{x}$ and σ which multiplies the v-function. This function will be denoted as $d_\varepsilon(\underline{x}, s, \lambda)$. For instance if $\lambda = (n, n-1, j, i, 0)$ then the corresponding d-function is $1(x_j = x_i + 1)\, c^\varepsilon(x_j, x_i, s)$.

Because of the presence of the terms $(n, n+1)$ the equation relating the v-functions is an infinite hierarchy of equations: the expression for v_n involves v_{n+1} which in turns involves v_{n+2} and so on. By looking at times smaller than $\varepsilon^{-2+\beta}$, $\beta > 0$, it is possible to control the above hierarchy. This is a consequence of the integral in (3.1 *a*): the time interval $\varepsilon^{-2+\beta}$ is "so short" that the successive iterates of (3.1 *a*) eventually become negligible. So we fix $\beta > 0$ and the degree m of the v-function for which we want to prove (2.8). We then iterate (3.1 *a*) N times, where N depends on m and β in a way which will be specified later on. At most, therefore, there will be $m + N$ particles, but we may need to introduce other particles, at most N, so that the set of necessary labels for all these particles is $(1, \ldots, m, m+1, \ldots, 2N)$. The first m labels are used for the initial m particles, hereafter referred to as the "old particles"; then each new particle appearing at a branching is named by using the next available label in the above list. According to the particular branch that we consider we may or may not use all the labels in the list. In particular no new label is needed for all the $(n, n-2)$, $(n, n-1)$, (n, n, j, i) and $(n, n, i, \sigma = 5)$ terms. For all the other (n, n) terms we add an extra particle. For the term $(n, n, i, 1)$ the particle i is at $x_i - 1$ and the new particle is placed at x_i. If $\sigma = 2$ the new particle is placed at $x_i - 1$; if $\sigma = 3$ the i particle is at $x_i + 1$ while the new particle is placed at x_i. Finally if $\sigma = 4$ the new particle is placed at $x_i + 1$.

For the $(n, n+1)$ terms we always add 2 particles one at the beginning of the cluster the other at its end, *cf.* Notation 3.2 above. One of them is "real", *i.e.* it appears in the argument of the v-function, the other one is fictitious.

One final notation which refers to the $(n, n-1, j, i, \sigma)$ when $j \leqq m$ and $i > m$ or $j > m$ and $i \leqq m$. In the first case if $\sigma = 1,0$ we label the particle in x_i as j particle and that in x_j (which is missing from the argument of the v-function) as particle i, so that the "old" particle survives. If $j > m$, $i \leqq m$ and $\sigma = -1$, we again switch labels between i and j.

Using the above notation we have

$$v_m(\underline{x}, t) = \sum_{k \leqq N} \sum_{\lambda_1, \ldots, \lambda_k} \int_0^t ds_1 \ldots \int_0^{s_{k-1}} ds_k \, E_{\underline{x}} \left(\prod_{i=1}^{k} d_\varepsilon(\underline{x}(t-s_i), s_i, \lambda_i) \right) v(\underline{x}(t-s_k)^+, s_k)) \quad (3.7)$$

where the functions d_ε are defined at the end of Notation 3.2; notice also that for notational simplicitly we have not explicited the degree of the v-functions which can be desumed from their argument. If $k < N$ then the sum is meant to be restricted to λ_k such that the final number of particles (after the k-th branching) is 0 hence at the end there is no v-function. If $k = N$ then the final number of particles, as specified by λ_N, might be different from 0, namely the configuration $\underline{x}(t-s_N)^+$, appearing after the last branching may be non empty. The sum in (3.7) is over all such possible $\lambda_1, \ldots, \lambda_k$ and the expectation refers to the stirring process in the time intervals $(t-s_i)$, $(t-s_{i+1})$ plus branchings at the times $t-s_i$ as specified by the values of λ_i. The labels of the particles are given according to the notational convention described before.

A separate bound on each of the terms in (3.7) would give the wrong result since there are important cancellations among the different terms. These origin from the differences among v-functions appearing in (3.1) and (3.2). We need to estimate differences like

$$v_n(\underline{x}(i), t) - v_n(\underline{x}(j), t), \qquad x_j - x_i = k \quad (3.8)$$

The case $k = 1$ appears in some of the $(n, n-1)$, (n, n) and $(n, n+1)$ terms, while $k > 1$ occurs in the terms $(n, n+1, i_1, \ldots, i_k, s)$, *cf.* (3.6).

From (3.1 *a*) we then have

$$v_n(\underline{x}(i), t) - v_n(\underline{x}(j), t) = \int_0^t ds \, E_{\underline{x}}(\psi(\underline{x}(t-s)(i), s) - \psi(\underline{x}(t-s)(j), s)) \quad (3.9)$$

where $\underline{x}(t-s)(i)$ is a configuration of $n-1$ stirring particles obtained from the configuration $\underline{x}(t-s)$ by subtracting particle i; $\underline{x}(t-s)(j)$ is defined analogously. Here we have used the fact that the marginal over the motion of a subset of stirring particles is still a process of stirring particles.

To estimate (3.9) we use a coupling between stirring and independent particles, *cf.* [12].

Coupling n stirring with n independent particles

Let $\partial_{+,i}$, $\partial_{-,i}$, $i=1,\ldots,n$ be the following operators on $N_n \equiv$ the subset of Z^n of n ordered distinct sites.

$$\partial_{\pm,i}\underline{x} = \begin{cases} \underline{x}+\underline{e}_i & \text{if } x_i \pm 1 \neq x_j \text{ for all } j \\ \underline{x}\pm\underline{e}_i-(\pm\underline{e}_j) & \text{if there is } j \text{ such that } x_i \pm 1 = x_j \text{ and } j>i \\ \underline{x} & \text{if there is } j \text{ such that } x_i \pm 1 = x_j \text{ and } j<i \end{cases} \quad (3.10)$$

If f is a real valued function on N_n, define $G_n f$ as

$$(G_n f)(\underline{x}) = \frac{1}{2}\sum_{i=1}^{n} \{[f(\partial_{+,i}\underline{x}) - f(\underline{x})] + [f(\partial_{-,i}\underline{x}) - f(\underline{x})]\} \quad (3.11)$$

then G_n is the generator of a Markov jump process whose law is the same as that of the stirring and the exclusion processes, for the last one the labels of the particles are neglected. In fact the marginal of such process over functions invariant under permutation of the labels is the SEP, since $G_n f(\underline{x}) = L_0 f(\underline{x})$ if f is symmetric.

The independent process has a generator G_n^0 which acts on the functions f on $Z^n (= N_n^0)$ as

$$(G_n^0 f)(\underline{x}^0) = \frac{1}{2}\sum_{i=1}^{n} \{[f(\underline{x}^0+\underline{e}_i) - f(\underline{x}^0)] + [f(\underline{x}^0-\underline{e}_i) - f(\underline{x}^0)]\} \quad (3.12)$$

Define a generator G_n^* on $N_n \times N_n^0$ as follows:

$$(G_n^* f)(\underline{x}, \underline{x}^0) = \frac{1}{2}\sum_{i=1}^{n} \{[f(\partial_{i,+}\underline{x}, \underline{x}^0+\underline{e}_i) - f(\underline{x}, \underline{x}^0)] + [f(\partial_{i,-}\underline{x}, \underline{x}^0-\underline{e}_i) - f(\underline{x}, \underline{x}^0)]\} \quad (3.13;)$$

It is easily seen that the marginals of the Markov process with generator G_n^* on N_n (resp. N_n^0) have same law as the stirring (resp. independent) process, hence the joint process is a coupling of the two. In particular from (3.13) it follows that with probability 1 for any i and t $x_i(t)$ is completely determined by the specification of $\{x_j^0(s), j=1,\ldots,i \text{ and } 0 \leqq s \leqq t\}$. In fact from (3.13) it follows that any jump of the stirring particle i necessarily induces a jump at the same time of some independent particle x_j^0 with $j \leqq i$. The rule is that whenever an independent particle, say particle i, jumps by ± 1 then the stirring particle i also jumps by ± 1, unless it would go to a site occupied by some particle j with $j<i$. In such case the jump is suppressed. On the other hand if $j>i$ the jump takes place and the stirring particle j makes the opposite jump, *i.e.* it moves by $-(\pm 1)$. We shall therefore say that particle i is of first class [or that it has priority] w. r. t. particle j if $j>i$. In this case we also say that j is second class w. r. t. i. Particle 1 is first class w. r. t. all the others and it moves just the same as the independent particle 1.

Other couplings can be introduced by simply changing the priority list: let $\pi=\{\pi(1), \ldots, \pi(n)\}$ be a permutation of $\{1, \ldots, n\}$. Define

$$\pi \underline{x}=(x_{\pi(1)}, \ldots, x_{\pi(n)})$$
$$\partial_{i, \pm, \pi} \underline{x}=\pi^{-1} \partial_{i, \pm} \pi \underline{x}$$

$$(G^*_{n, \pi} f)(\underline{x}, \underline{x}^0)=\frac{1}{2} \sum_{i=1}^{n} \{[f(\partial_{i, +, \pi} \underline{x}, \underline{x}^0+\underline{e}_{\pi(i)})-f(\underline{x}, \underline{x}^0)]$$
$$+[f(\partial_{i, -, \pi} \underline{x}, \underline{x}^0-\underline{e}_{\pi(i)})-f(\underline{x}, \underline{x}^0)]\}$$

$G^*_{n, \pi}$ is easily seen to define again a coupling of the stirring and independent processes, which will be called the π-coupled process. In such process particle $\pi(1)$ has the highest priority, it moves the same as the independent particle $\pi(1)$. Then comes particle $\pi(2)$ with second highest priority and so on. Therefore $x_{\pi(i)}(t)$ is specified (modulo zero) by $(x^0_{\pi(1)}(s), \ldots, x^0_{\pi(i)}(s)$ for all $0 \leqq s \leqq t)$.

We now go back to (3.8) and introduce for i, j, k as in (3.8) and any $s>0$ the following stopping time on the space $(N_n \times N_n^0)^{R+}$

$$\tau^{(k)}_{i, j}(s)=\inf\{t>s: x^0_j(t)-x^0_j(s)-x^0_i(t)+x^0_i(s)=-k\} \qquad (3.14)$$

and for $T>s$ we set

$$g^{(k)}_{i, j}(s, T)=1(\tau^{(k)}_{i, j}(s)>s+(T-s)/2) \qquad (3.15)$$

Then from (3.6), setting $k=x_j-x_i$,

$$v_n(\underline{x}(i), t)-v_n(\underline{x}(j), t)=\int_0^t ds\, E_{\underline{x}, \underline{x}^0}$$
$$(g^{(k)}_{i, j}(0, t-s)[\psi(\underline{x}(t-s)(i), s)-\psi(\underline{x}(t-s)(j), s)]) \qquad (3.16)$$

where the expectation refers to the coupled process with i and j having top priority; $\underline{x}^0$ is an arbitrary configuration of n independent particles.

To prove (3.16) we observe that the displacements of x_i and x_j are the same as those of x^0_i and x^0_j till time $\tau^{(k)}_{i, j}(0)$, since the particles i and j have top priority. At time $\tau^{(k)}_{i, j}(0)^+$ the distribution of the stirring particles is symmetric under the permutation of particles i and j. Therefore if $\tau^{(k)}_{i, j}(0)<t-s$ then the distribution of $\underline{x}(t-s)$ is symmetric under the permutation of the positions of i and j, hence such a subset of trajectories does not contribute to the difference $v_m(\underline{x}(i), t)-v_m(\underline{x}(j), t)$. *A fortiori* this occurs when $\tau^{(k)}_{i, j}(0)<(t-s)/2$, so that (3.16) is proven.

We can now write (3.7) as

$$v_m(\underline{x}, t)=\sum_{k \leqq N} \sum_{\lambda_1, \ldots, \lambda_k} \int_0^t ds_1 \ldots \int_0^{s_{k-1}} ds_k$$
$$\times \mathscr{E}_{\underline{x}}\left(\prod_{h=1}^{k} d_\varepsilon(\underline{x}(t-s_h), s_h, \lambda_h) v(\underline{x}(t-s_k)^+, s_k) \prod_{h \neq N} g(h)\right) \qquad (3.17)$$

where the following notation have been used: firstly $g(h)$ stands for $g^{(k)}_{i,j}(t-s_h, t-s_{h+1})$ where i, j, k are specified by λ_h if λ_h refers to a branching where a difference of v-functions appears, otherwise it equals 1. $\mathscr{E}_{\underline{x}}$ denotes the expectation with respect to a process whose law is specified once $\lambda_1, \ldots, \lambda_k$ and $s_1, \ldots, s_k$ are given. In the time intervals $(t-s_h, t-s_{h+1})$ this is a coupled stirring+independent process whose priority list is specified by λ_h. If at $t-s_h$ there was not a v-difference then the priority list is chosen in an arbitrary but fixed way. If on the contrary there was a difference of v-functions then two particles are involved in such a difference and the priority list has these 2 particles with highest priority. The independent and the stirring particles start both from $\underline{x}$. The branching for the stirring particles is specified by the values λ_h as described above. The whole process is then determined by saying that when a new stirring particle is added at some of the branchings then an independent particle with same label is added on the same site and when a stirring particle dies the corresponding independent particle also disappears. The fictitious particles, which are added to take into account the v-differences, play a role only in the time intervals $t-s_h$, $t-s_h+(s_h-s_{h+1})/2$, if at the h-th branchincing such a v-difference was present. So we say that they disappear/die at the end of this time interval.

We use the following key estimate proven in [10]:

3.3. PROPOSITION. — *Fix* n, $\underline{x}=(x_1, \ldots, x_n)\in \mathbf{N}_n$, $\alpha>1/4$ *and let* $\mathrm{P}_{\underline{x}}$ *be the law of any of the couplings previously defined between the stirring and the independent processes, both starting from* $\underline{x}$. *Then for any* m *there is* c *independent of* x *such that for all* $\mathrm{T}>0$

$$\mathrm{P}_{\underline{x}}(\text{there exists } t>\mathrm{T} \text{ and } i\in\{1, \ldots, n\} : |x_i(t)-x_i^0(t)|>t^{\alpha}) \leqq c\,\mathrm{T}^{-m} \quad (3.17)$$

As a corollary of the above Proposition we have

3.4. COROLLARY. — *Let* $k\leqq \mathbf{N}$, $s_1, \ldots, s_k$, $\lambda_1, \ldots, \lambda_k$, $\mathscr{E}_{\underline{x}}$ *be as in* (3.17). *Let* a *be any positive number. Then for any* n *there is* c *so that*

$$\mathscr{E}_{\underline{x}}(1-h)<c\,\varepsilon^n \quad (3.18\,a)$$

where h *is the characteristic function of the event*

$$\{\,|[x_i(t-s)-x_i(t-s')]-[x_i^0(t-s)-x_i^0(t-s')]| \leqq |s-s'|^{1/4+a}\,\forall\, i,\ s,\ s' : s-s'\geqq\varepsilon^{-a}\} \quad (3.18\,b)$$

We fix a sequence $s_1, \ldots, s_k, \lambda_1, \ldots, \lambda_k$ and denote by J the corresponding term in (3.17). Calling G the set of values of λ_i for which there is a difference of v-functions, then using the Corollary 3.4

$$|\mathrm{J}|\leqq \mathscr{E}_{\underline{x}}\Big(\prod_{\lambda_i\in \mathrm{G}} g(\lambda_i)\,\mathrm{J}_{\mathscr{G}}\Big)+c\,\varepsilon^n \quad (3.19)$$

where

$$J_{\mathcal{G}} = \mathcal{E}_{\underline{x}}(h \Pi_{i=1,\, k} \left| d'_\varepsilon(\underline{x}(t-s_i),\, s_i,\, \lambda_i) \right\| \mathcal{G}) \tag{3.20}$$
$$d_\varepsilon(\underline{x}(t-s_i),\, s_i,\, \lambda_i) = d'_\varepsilon(\underline{x}(t-s_i),\, s_i,\, \lambda_i)\, g(\lambda_i)$$

and

$$\mathcal{G} = \sigma\text{-algebra generated by} \bigcup_{i\,:\,\lambda_i \in A} \left\{ \underline{x}^0(t') - \underline{x}^0(t-s_i) \right.$$
$$\left. t' \in \left[t-s_i,\, t-s+\frac{1}{2}\Delta S_i \right] \right\} \tag{3.21}$$

The contribution to (3.17) of the second term in (3.19) is bounded by $c\,\varepsilon^{n-2N}$, since there are at most N integrals each ranging in the interval $[0, \varepsilon^{-2+\beta}]$. Hence choosing n large enough this can be made smaller than the left side of (2.8), so we can ignore it in this proof. Our aim now is to estimate $J_{\mathcal{G}}$ uniformly on the conditioning so that, after inserting this bound, the expectation in (3.19) only involves independent random walks. The main difficulty consists in taking proper advantage of those characteristic functions which may appear in d'_ε and which require that certain pairs of particles should be at neighbor sites. We shall use Corollary 3.4 to reduce these to conditions involving independent particles. Even after this, however, the analysis will not be very simple because of the branching structure of the process which creates correlations in the motion of the particles. To have a simpler structure we shall get rid of some of these conditions. We can do this in some of the terms without affecting the bound we aim to prove. With this in mind we give the following

DEFINITION. — (i) A particle is called auxiliary if it is not "old" and it dies together with an old particle. Particles which are neither old nor auxiliary are called "normal".

(ii) Let $e_\varepsilon(\underline{x}, s, \lambda)$ be the function obtained from $d'_\varepsilon(\underline{x}, s, \lambda)$ by dropping those characteristic functions possibly present in the latter which only involve normal particles.

(iii) Let i be the label of an auxiliary particle, then $o(i)$ denotes the label of the old particle which dies with particle i. Furthermore $t(i)$ denotes the time when i was born, $t'(i)$ when it dies and x_i, $x_{o(i)}$ the position at time $t(i)$ of particles i and $o(i)$.

We can now relax the conditions on the death of the old and the auxiliary particles, remind that we have dropped the conditions on the normal particles. Let us consider for instance a condition referring to the death of an old and an auxiliary particle, say i, $o(i)$ and let $t(i)$, $t'(i)$ be as in the definition above. Then if $h=1$ and $t'(i)-t(i) \geqq \varepsilon^{-a}$

$$\left| [\Delta x_i^0 - \Delta x_{o(i)}^0] - [x_i - x_{o(i)}] \right| \leqq c \left| t'(i) - t(i) \right|^{1/4+a} \tag{3.22}$$

where Δx_i^0 and $\Delta x_{o(i)}^0$ are respectively the increments of the independent particles i and $o(i)$ in the time interval $t(i)$, $t'(i)$. Analogous expression holds for the condition involving two old particles, the initial positions being however fixed by $\underline{x}$. If $t'(i)-t(i) \leqq \varepsilon^{-a}$ we just drop the characteristic function which is therefore bounded by 1.

What we have now is not yet an expression involving only the evolution of the independent particles since the stirring particles still appear in (3.22). We consider then the largest among the times $t(i)$, let it be $t(j)$. We condition on the whole process up to time $t(j)$; we also condition on all the increments of the independent particles after time $t(j)$, except for the increments of the independent particle j. After such conditioning, using the fact that $t(j)$ is the largest time of birth (for auxiliary particles), then all the characteristic functions are fixed except for the following one:

$$1(|\Delta x_j^0 - C| \leqq c\,|t'(j)-t(j)|^{1/4+a})$$

where C is a suitable constant specified by the conditioning, and c is a constant related to that appearing in Corollary 3.4. Since we are also conditioning on G the variable Δx_j^0 is the sum of two quantities: one is the sum of the increments in the time intervals $t-s_i$, $t-s_i+(s_i+s_{i+1})/2$ with $t(j)<t-s_i<t'(j)$ and i is such that $\lambda_i \in G$. These increments are fixed by G, while the others have the distribution of a symmetric nearest neighbor random walk with intensity 1 which moves for a time not smaller than $[t'(j)-t(j)]/2$. Therefore the contribution of this event to the conditional expectation is bounded by

$$c\,|t'(j)-t(j)|^{-1/4+a}$$

uniformly on x_j, $x_{o(j)}$ and the conditioning in G. Note that if $t'(j)-t(j) \leqq \varepsilon^{-a}$ then the characteristic function has been dropped, according to what said before. However we can always write the following bound valid in both cases

$$c\,|t'(j)-t(j)|^{-1/4+a}[c^{-1}\varepsilon^{-a[1/4-a]}].$$

We further increase the above bound by replacing $t'(j)-t(j)$ by the length of the interval which starts at $t(j)$. Since the dependence on x_j and $x_{o(j)}$ has now disappeared we can iterate the above procedure to the next auxiliary particle. We keep doing this till all the characteristic functions concerning auxiliary particles have been estimated. We may still have characteristic functions involving deaths of old particles alone. But now the branching structure of the process has been lost and we can proceed like in [10], we omit the details and simply refer to [10]. We need the following estimates on the coefficients appearing in (3.1). These are obtained by studying the equation satisfied by ρ_ε, we refer to Section 4 for a detailed analysis from which it is easy to derive the bounds below which hold uniformly on the initial configuration η (c here and in the sequel

denotes a constant whose value changes from line to line)

$$\sup_{x\in Z}\left|a^{\varepsilon}(x,\,x+1,\,t)\right|<c\left[\frac{1}{t}+\frac{\varepsilon}{\sqrt{t}}\right] \tag{3.23a}$$

$$\sup_{x\in Z}\left|b^{\varepsilon}(x,\,x+1,\,t)\right|<c\frac{1}{\sqrt{t}} \tag{3.23b}$$

$$\sup_{x\in Z}\left|c^{\varepsilon}(x,\,x+1,\,t)\right|<c\frac{\varepsilon}{\sqrt{t}} \tag{3.23c}$$

If the initial measure is μ^{ε} then

$$\sup_{x\in Z}\left|a^{\varepsilon}(x,\,x+1,\,t)\right|<c\,\varepsilon^2 \tag{3.24a}$$

$$\sup_{x\in Z}\left|b^{\varepsilon}(x,\,x+1,\,t)\right|<c\,\varepsilon \tag{3.24b}$$

$$\sup_{x\in Z}\left|c^{\varepsilon}(x,\,x+1,\,t)\right|<c\,\varepsilon \tag{3.24c}$$

(*i.e.* the same ε-dependence holds also at small times!). Recalling that $t\leqq\varepsilon^{-2+\beta}$ we then have from (3.19)

$$\begin{aligned}\left|J\right|\leqq c\{&[\prod_{\lambda_i\in A_1}(t-s_i)^{-1/4+a}(s_i-s_{i+1})^{-1/2}s_i^{-1/2}][\prod_{\lambda_i\in A_2}(s_i-s_{i+1})^{-1/2}s_i^{-1/2}]\\&[\prod_{\lambda_i\in B_1}(t-s_i)^{-1/4+a}s_i^{-1+a}][\prod_{\lambda_i\in B_2}s_i^{-1+a}]\\&\varepsilon^{|C|}\prod_{\lambda_i\in C_1}(s_i-s_{i+1})^{-1/2-1/4+a}\;\mathrm{P}_{\lambda_i\in C_2}(s_i-s_{i+1})^{-1/2}\}\,\varepsilon^{-aN/4}+c\,\varepsilon^{n}\end{aligned} \tag{3.25}$$

where A_1 denotes the terms of type $(n,\,n-1)$ where an old particle dies, (and by our conventions this was close to another old particle, so that the characteristic function of the corresponding event was not dropped) A_2 the others. For the $(n,\,n-2)$ terms B_1 refers to cases where 2 old particle dies while B_2 covers the remaining cases. The $(n,\,n)$ and $(n,\,n+1)$ terms are grouped in $C=C_1+C_2$ and C_1 refers to the case when an auxiliary particle is born. The factor $\varepsilon^{-aN/4}$ arises from the fact that some of the characteristic functions might be missing if the time intervals between branching is too short, (less than ε^{-a}) see above. We have also used that for $\lambda_i\in G$

$$\left|\mathscr{E}_{\underline{x}}(g(\lambda_i)\right|\leqq c\,(s_i-s_{i+1})^{-1/2})$$

since this estimates the probability that two random walks starting at fixed distance do not meet before time $(s_i-s_{i+1})/2$.

The analysis follows now very closely that in [10]. We distinguish the case (i) when $k<N$ or $k=N$ and λ_N determines a branching where no particle survives from the other case (ii) when after λ_N there are particles which survive. In the first case all the old particles die and we get after some simple algebra the following bound for the corresponding integral

over $s_1, \ldots, s_k$ (the bound can be easily understood by using a dimensional analysis)

$$t^{(-1/4+a)(|A_1|+|B_1|+|C_1|)} t^{a|B|} \varepsilon^{-aN/4} (\varepsilon t^{1/2})^{|C|} \tag{3.26}$$

This gives the desired estimate if $|C|>0$, after choosing a small enough since m, the total number of old particles equals $|A_1|+|B_1|/2+|C_1|$; in fact in the case we are considering all the old particles die and they can die only in the A_1, B_1 and C_1 cases, two particles die in the B_1 terms (the "worst case"). On the other hand if $|C|=0$ the branching is the same as in the symmetric simple exclusion so we can refer to [10] where (2.8) was proven for the SEP.

It remains therefore only the case when after λ_N there are still particles. This can be treated just as above the only difference is that we might miss a factor ε arising from the last difference of v-functions (if λ_N refers to this case). Here we do not have anymore the small factor arising from the death of all the old particles what is now small is the factor $(\varepsilon t^{1/2})^{|C|}$ which gives the desired estimate after choosing N sufficiently large. In fact by definition

$$|A|+|B|+|C|=N$$

The total number of particles is bounded by $m+|C|$ hence

$$\begin{gathered} |A|+2|B| \leqq m+|C| \\ |A|+|B| \leqq m+|C| \\ N=|A|+|B|+|C| \leqq m+2|C| \end{gathered}$$

From (3.26) and the last inequality we get

$$\varepsilon^{-1} \varepsilon^{-2aN} (\varepsilon^{1-\beta/2}) (N-m)/2$$

[the first factor comes from the already remarked fact that we might be missing a factor ε because of a possible v-difference after the last N branching the second one is an obvious bound for the second factor in (3.26)].

By choosing N large enough and a sufficiently small we derive the desired estimate so (2.8 *a*) is proven at times smaller than $\varepsilon^{-2+\beta}$.

4. PROOFS

NOTATION. — Throughout this section τ will denote an arbitrary but fixed positive number.

We shall extend the estimates of the previous section to the whole time interval $[0, \varepsilon^{-2}\tau]$, firstly by proving that for some $\delta>0$ and for any n

there is c (depending on n and δ) such that for all initial configuration η

$$\sup_{\underline{x} \in \aleph n} |v_n^\varepsilon(\underline{x}, t | \eta)| \leqq ct^{-\delta n} \quad \text{for all } t \leqq \varepsilon^{-2}\tau \tag{4.1}$$

From this Theorem 2.1 and then Proposition 2.2 will follow; Theorem 2.3 will be easy consequence of (2.8).

Let β be a positive number less than 2. We divide $\mathbf{R}_+$ into intervals $(t(k), t(k+1)]$ where

$$t(k) = k\varepsilon^{-2+\beta} \tag{4.2}$$

(we are dropping the dependence on ε to have lighter notation). For any integer $k \geqq 0$ we denote by η_k the configuration at time $t(k)$, *i.e.*

$$\eta_k = (\eta(x, t(k)), x \in \mathbf{Z}). \tag{4.3}$$

Sketch of the Proof of (4.1)

First of all we observe that in Section 3 it has been proven that

$$\sup_{\eta_k} \sup_{t(k) < t \leqq t(k+1)} \sup_{\underline{x} \in \aleph n} |v_n^\varepsilon(\underline{x}, t | \eta_k)| \leqq c\varepsilon^{n(2-\beta)/8}. \tag{4.4}$$

Let $\mathscr{F}(k)$ be the σ-algebra that specifies the hystory of the process up to time $t(k)$ and let $E_{\eta, k}(.)$ be the $\mathscr{F}(k)$-conditional expectation with respect to the process starting at time 0 with the configuration η. Given $t > 0$ let k be such that $t(k) < t \leqq t(k+1)$. Let $\rho_\varepsilon(.,. | \eta_k)$ [respectively $\rho_\varepsilon(.,. | \eta_0)$] be the solution to (2.5) with initial value η_k at time $t(k)$ (resp. η_0 at $t=0$). Then, by definition and by (4.4) we have that

$$\begin{aligned}
|v_n^\varepsilon(\underline{x}, t | \eta_0)| &= \Bigg| E_{\eta_0}\Bigg(\prod_{i=1}^{n} [(\eta(x_i, t) - \rho_\varepsilon(x_i, t | \eta_k) \\
&\qquad + (\rho_\varepsilon(x_i, t | \eta_k) - \rho_\varepsilon(x_i, t | \eta_0)] \Bigg| \\
&\leqq \sum_{\Delta \subset \{1, \ldots, n\}} E_{\eta_0}\Big(\Big| E_{\eta_0, k}\Big(\prod_{i \in \Delta} (\eta(x_i, t) - \rho_\varepsilon(x_i, t | \eta_k)) \Big| \\
&\qquad \times \Big| \Big(\prod_{j \notin \Delta} \rho_\varepsilon(x_j, t | \eta_k) - \rho_\varepsilon(x_j, t | \eta_0) \Big) \Big| \Big) \\
&\leqq \sum_{\Delta \subset \{1, \ldots, n\}} c\varepsilon^{|\Delta|(2-\beta)/8} E_{\eta_0}\Big(\Big| \prod_{i \notin \Delta} (\rho_\varepsilon(x_i, t | \eta_k) - \rho_\varepsilon(x_i, t | \eta_0) \Big| \Big)
\end{aligned} \tag{4.5}$$

Therefore, in order to prove (4.1) we need to estimate the expectation in the r. h. s. of (4.5). It would be sufficient to prove that for some $\zeta > 0$

$$\begin{aligned}
\text{Prob}\Big\{ \sup_{i \in \{1, \ldots, n\}} &|\rho_\varepsilon(x_i, t | \eta_k) \\
&- \rho_\varepsilon(x_i, t | \eta_0)| \leqq \varepsilon^\zeta \Big\} > 1 - c\varepsilon^m \text{ for any } m
\end{aligned} \tag{4.6}$$

We have imposed that the probability in (4.6) goes to one faster that any power of ε because we want an estimate of the expectation in the r. h. s. of (4.5) of the form $\varepsilon^{\lambda(n-|\Delta|)}$ with some $\lambda>0$. For the same reason we need the sup over $i\in\{1, \ldots, n\}$.

To understand why (4.6) is true, let us consider the difference in the modulus in (4.6) and let us rewrite it using (2.5). [We add and subctract $\rho_\varepsilon(., t(k)\,|\,\eta_{k-1})$ in (4.7) below where $\rho_\varepsilon(., t\,|\,\eta_{k-1})$ is defined analogously to $\rho_\varepsilon(., t\,|\,\eta_k)$].

$$\rho_\varepsilon(x, t\,|\,\eta_k)-\rho_\varepsilon(x, t\,|\,\eta_0)=\sum_z P_{t-t(k)}(x\to z)\,[\eta(z, t(k))-\rho_\varepsilon(z, t(k)\,|\,\eta_{k-1})]$$
$$+\sum_z P_{t-t(k)}(x\to z)\,[\rho_\varepsilon(z, t(k)\,|\,\eta_{k-1})-\rho_\varepsilon(z, t(k)\,|\,\eta_0)]$$
$$+\varepsilon\int_{t(k)}^{t} ds\sum_z P_{t-s}(x\to z)\,g(z, s, k) \quad (4.7a)$$

where

$$g(z, s, k)\equiv[\rho_\varepsilon(z-1, s\,|\,\eta_k)-\rho_\varepsilon(z-1, s\,|\,\eta_0)]-[\rho_\varepsilon(z, s\,|\,\eta_k)-\rho_\varepsilon(z, s\,|\,\eta_0)]$$
$$+[\rho_\varepsilon(z, s\,|\,\eta_k)\,(\rho_\varepsilon(z+1, s\,|\,\eta_k)-\rho_\varepsilon(z-1, s\,|\,\eta_k))]$$
$$-[\rho_\varepsilon(z, s\,|\,\eta_0)\,(\rho_\varepsilon(z+1, s\,|\,\eta_0)-\rho_\varepsilon(z-1, s\,|\,\eta_0))] \quad (4.7b)$$

Of course the difference $\eta(z, t(k))-\rho_\varepsilon(z, t(k)\,|\,\eta_{k-1})$ is not small, but if we rewrite the first term in the r. h. s. of (4.7*a*) as follows

$$\sum_w P_{t-t(k)-\varepsilon^{-1/2}}(x\to w)\sum_z P_{\varepsilon^{-1/2}}(w\to z)\,(\eta(z, t(k))-\rho_\varepsilon(z, t(k)\,|\,\eta_{k-1}) \quad (4.8)$$

then the sum over z in (4.8) is indeed small with probability that goes to one faster than any power of ε, [this follows from Chebishev's inequality and from (4.4)]. In order to write (4.8) we need $t>t(k)+\varepsilon^{-1/2}$ but this is not a problem as we shall see. Notice that the value $\varepsilon^{-1/2}$ is not the only possible, any negative power of ε would be sufficient. If for all $h\leqq k$ η_h is such that the sum over z in (4.8) is less then, say, ε^ζ with $\zeta>0$, then the fact that the difference in the l.h.s. of (4.7*a*) is small is a property of the viscous Burgers equation which can be proven iterating (4.7*a*).

Sumarizing we proceed as follows. For any $T\leqq\varepsilon^{-2}\tau$ and any $x\in Z$, we define below (see Definition 4.1) the set $i(\zeta, T, x)$ of "good configurations", *i.e.* of trajectories $\underline{\eta}\equiv\{\eta_k, k\leqq T\}$ for which the sum over z in (4.8) is less than ε^ζ uniformly in $k\leqq T$ and in all w with $|x-w|\leqq\varepsilon^{-2}$, [it is obvious that the typical w in (4.8) are at distance of order ε^{-1} from x, therefore $|x-w|\leqq\varepsilon^{-2}$ is more than what needed]. We therefore prove, in Lemma 4.2 below that the "good configurations" have a probability that goes to one faster than any power of ε. In Definition 4.1 we also introduce the function $\rho^*_\varepsilon(., t\,|\,\underline{\eta})$ which is the solution to (2.5) with initial value η_k at $t(k)$ if $t(k)<t<t(k+1)$ and then in Lemma 4.3 below we prove that if $\underline{\eta}\in i(\zeta, T, x)$ then $\rho^*_\varepsilon(., t\,|\,\underline{\eta})$ is sufficiently close to $\rho_\varepsilon(., t\,|\,\eta_0)$ for all

$t \leqq T$. Finally we notice that the proofs would be simpler if we were considering the process in the "finite volume" $[0, \varepsilon^{-1}]$ with periodic boundary conditions.

4.1. DEFINITION. — We set β (*cf.* Section 3) so small that there is $\zeta > 0$ for which

$$0 < \beta < \zeta < \frac{1}{8}(2-\beta) \tag{4.9}$$

and ζ will be chosen smaller than $\frac{1}{8}$ (the reason for such choices will become clear in the sequel).

For $2\,\varepsilon^{-2+\beta} \leqq T \leqq \varepsilon^{-2}\tau$ we define an integer number $N(T) \geqq 1$ such that

$$N(T): \quad (N(T)+1)\,\varepsilon^{-2+\beta} \leqq T \leqq (N(T)+2)\,\varepsilon^{-2+\beta}. \tag{4.10}$$

We let

$$\underline{\eta} \equiv (\eta_k)_{k \leqq N(T)}, \qquad \eta_k \text{ is defined in (4.3)} \tag{4.11}$$

For such $\underline{\eta}$ we define $\rho_\varepsilon^*(x, t \,|\, \underline{\eta})$, $x \in Z$, $t \leqq T$, as follows. When $t \in (t(k), t(k+1)]$ and $k < N(T)$, then $\rho_\varepsilon^*(x, t \,|\, \underline{\eta})$ is the solution to (2.5) with initial value η_k at time $t(k)$ and for $t \in (t(N(T)), T]$ it is again solution to (2.5) with value $\eta_{N(T)}$ at time $t(N(T))$ [it is important to remember that according to our notation $\rho_\varepsilon^*(., t(k) \,|\, \underline{\eta}) \neq \eta_k(.)$]. We denote by $\rho_\varepsilon(x, t \,|\, \eta)$, $\eta \in \{0,1\}^Z$, $t \leqq T$, the solution to (2.5) with initial condition η at time $t=0$. In Lemma 4.3 below we shall compare $\rho_\varepsilon(x, t \,|\, \eta_0)$ and $\rho_\varepsilon^*(x, t \,|\, \underline{\eta})$ for special choices of $\underline{\eta}$, namely $\underline{\eta}$ belonging to the set

$$i(\zeta, T, x)$$

$$= \{\underline{\eta} : \sup_{1 \leqq n \leqq N(T)} \sup_{|y-x| \leqq \varepsilon^{-2}} \| \eta(., t(n)) - \rho_\varepsilon^*(., t(n) \,|\, \underline{\eta}) \|_y \leqq \varepsilon^\zeta \} \tag{4.12a}$$

where for any two functions g and g' on Z

$$\| g(.) - g'(.) \|_y = \left| \sum_z P_{\varepsilon^{-1/2}}(y \to z)\,[g(z) - g'(z)] \right| \tag{4.12b}$$

(again, to have lighter notation we do not explicit the dependence on ε).

4.2. LEMMA. — *Let β and ζ be as in* 4.9. *Then for any n there is c such that*

$$P_\eta^\varepsilon(i(\zeta, \varepsilon^{-2}\tau, x)) \geqq 1 - c\,\varepsilon^n \tag{4.13}$$

uniformly in $x \in Z$ and $\eta \in \{0,1\}^Z$, P_η^ε being the law of the WASEP starting at time zero from η, obviously same inequality holds for $T \leqq \varepsilon^{-2}\tau$.

Proof. — For notational simplicity we set $x=0$ and simply write $i(\zeta)$ for $i(\zeta, \varepsilon^{-2}\tau, 0)$. We have, ($A^c$ denoting below the complement of the

set A),

$$P^{\varepsilon}_{\eta}(i(\zeta)^c) \leqq \varepsilon^{-2} N(\varepsilon^{-2}\tau) \sup_{\eta'} P^{\varepsilon}_{\eta'}\Big(\Big| \sum_z P_{\varepsilon^{-1/2}}(0 \to z)[\eta(z, \varepsilon^{-2+\beta}) - \rho_\varepsilon(z, \varepsilon^{-2+\beta} | \eta')] \Big| > \varepsilon^{\zeta} \Big)$$

where $\rho_\varepsilon(x, t | \eta')$ as usual denotes the solution to (2.5) with initial datum η'. To bound the above probability we use Chebishev inequality with power $2m$; we get

$$\varepsilon^{-\zeta 2m} \sum_{z_1 \dots z_{2m}} P_{\varepsilon^{-1/2}}(0 \to z_1) \dots P_{\varepsilon^{-1/2}}(0 \to z_{2m}) \times \left| E^{\varepsilon}_{\eta'}\left(\prod_{i=1}^{2m} \{\eta(z_i, \varepsilon^{-2+\beta}) - \rho_\varepsilon(z_i \varepsilon^{-2+\beta} | \eta')\} \right) \right| \leqq c \max(\{\varepsilon^{-2\zeta} \sum_z P_{\varepsilon^{-1/2}}(0 \to z)^2\}^m,$$

$$\varepsilon^{-\zeta 2m} \sup_{\underline{z} \in \aleph_{2m}} \left| E^{\varepsilon}_{\eta'}\left(\prod_{i=1}^{2m} \{\eta(z_i, \varepsilon^{-2+\beta}) - \rho_\varepsilon(z_i, \varepsilon^{-2+\beta} | \eta')\} \right) \right| \leqq c \max\{[\varepsilon^{1/4-2\zeta}]^m, \varepsilon^{2m(((2-\beta)/8)-\zeta)}\}$$

By our choices of β and ζ if m is large enough we get (4.13), so the Lemma is proven.

4.3. LEMMA. — *Let* β, ζ *and* T *be as in Definition* 4.1. *Referring below to Definition* 4.1, *eqs.* (4.2) *and* (4.10) *for notation, we set*

$$t(T) = \bigcup_{0 \leqq k < N(T)} (t(k) + \varepsilon^{-1/2}, t(k+1)] \cup (t(N(T)), T]$$

then there is c such that for all x

$$\sup_{\underline{\eta} \in i(\zeta, T, x)} \sup_{t \in t(T)} | \rho^*_\varepsilon(x, t | \underline{\eta}) - \rho_\varepsilon(x, t | \eta_0) | \leqq c \varepsilon^{\delta}, \qquad \delta = \zeta - \beta \quad (4.14)$$

Proof. — Let $t(k) + \varepsilon^{-1/2} \leqq t \leqq t(k+1)$, $k < N(T)$ or, when $k = N(T)$, $t(k) + \varepsilon^{-1/2} \leqq t \leqq T$. Then for any $y \in Z$ we have

$$| \rho^*_\varepsilon(y, t | \underline{\eta}) - \sum_z P_{t-t(k)}(y \to z) \eta(z, t(k)) + \varepsilon \int_{t(k)+\varepsilon^{-1/2}}^{t} ds \sum_z P_{t-s}(y \to z) \nabla_- f(z) | \leqq 2\sqrt{\varepsilon}$$

where

$$\nabla_- f(z) = f(z) - f(z-1), \qquad \nabla_+ f(z) = f(z+1) - f(z)$$

and

$$f(z) = \rho^*_\varepsilon(z, s | \underline{\eta})[1 - \rho^*_\varepsilon(z+1, s | \underline{\eta})]$$

On the other hand if $|y-x| \leqq \frac{1}{2}\varepsilon^{-2}$

$$|\sum_z P_{t-t(k)}(y \to z)[\eta(z, t(k)) - \rho_\varepsilon^*(z, t(k)|\underline{\eta}|$$

$$\leqq \sum_z P_{t-(\varepsilon^{-1/2}+t(k))}(y \to z) \| \eta(., t(k)) - \rho_\varepsilon^*(., t(k)|\underline{\eta}) \|_z$$

$$\leqq \sum_{z:|z-y|>\varepsilon^{-2}/2} P_{t-(\varepsilon^{-1/2}+t(k))}(y \to z) - \varepsilon^\zeta \leqq c\varepsilon^\zeta$$

because $\underline{\eta} \in i(\zeta, T, x)$ and the probability that a random walk moves by $\frac{1}{2}\varepsilon^{-2}$ in a time $\leqq \varepsilon^{-2}\tau$ is smaller than any power of ε, as $\varepsilon \to 0$. Therefore posing

$$h_\varepsilon(y, t) = \rho_\varepsilon^*(y, t|\underline{\eta}) - \rho_\varepsilon(y, t|\eta_0)$$

we get

$$|h_\varepsilon(y, t) - \sum_z P_{t-t(k)}(y \to z) h_\varepsilon(z, t(k)) + \varepsilon \int_{t(k)+\varepsilon^{-1/2}}^t ds \sum_z P_{t-s}(y \to z) \nabla_- g$$

$$(g = [h_\varepsilon(z, s) - h_\varepsilon(z, s)\rho_\varepsilon(z+1, s|\underline{\eta}) - h_\varepsilon(z+1, s)\rho_\varepsilon(z, s|\eta_0)]|)$$

$$\leqq c\varepsilon^\zeta + 4\varepsilon^{1/2}$$

(according to our notation c is a constant whose value changes from line to line, so:)

$$\leqq c\varepsilon^\zeta \tag{4.15}$$

We denote below by E_y the expectation w. r. t. the law P_y of a single random walk $x(t)$ which at time 0 starts from y and we set

$$\{\tau = k\} = \left\{ |x(t-t(j)) - x| \leqq \frac{1}{2}\varepsilon^{-2}, j = N(t), \ldots, k+1 \right.$$

$$\left. \text{and} \quad |x(t-t(k)) - x| > \frac{1}{2}\varepsilon^{-2} \right\} \tag{4.16a}$$

Then by iterating (4.15) we get for $|y-x| < \frac{1}{2}\varepsilon^{-2}$

$$\left| h_\varepsilon(y, t) - \sum_{k=1}^{N(t)} E_y(\mathbf{1}(\tau = k) h_\varepsilon(x(t-t(k)), t(k)) + \varepsilon \sum_k \int_{\substack{t(k) \\ s \in t}}^t ds\, E_y(\mathbf{1}(\tau = k) \nabla_- g) \right.$$

$$\left. + \varepsilon \int_{\substack{0 \\ s \in t}}^t ds\, E_y(\mathbf{1}(\tau \neq k, \forall k) \nabla_- g) \right| \leqq c\varepsilon^{-\beta}\varepsilon^\zeta = c\varepsilon^\delta \tag{4.16b}$$

where t above and below shorthands $t(T)$.

If $|y-x| \leqq \frac{1}{4}\varepsilon^{-2}$ $\left[\frac{1}{4}$ and not $\frac{1}{2}$ here!$\right]$ then it is convenient to bound h_ε as follows, using for this purpose (4.16) and recalling that $N(t) \leqq N(\varepsilon^{-2}\tau)$,

$$|h_\varepsilon(y, t)| \leqq c\varepsilon^{-1} N(\varepsilon^{-2}\tau)\,[\sum_k P_y(\tau = k)] + c\varepsilon^\delta + \varepsilon \int_{\substack{0 \\ s \in t}}^t ds \sum_z |\nabla_+ P_{t-s}(y \to z)| \{2|h_\varepsilon(z, s)| + |h_\varepsilon(z+1, s)|\}$$

where

$$\nabla_+ P_t(x \to z) = P_t(x \to z+1) - P_t(x \to z)$$

The factor ε^{-1} appearing in the first term comes from bounding each integral in the sum over k in (4.9 *b*) by $c\varepsilon^{-2}$, which, being multiplied by ε, behaves like ε^{-1}. Such term bounds also the second term in the l. h. s. of (4.16 *b*). Since $P_y(\tau = k)$, for any k, vanishes faster than any power of ε $\left($because the times involved are $\leqq \varepsilon^{-2}\tau$ and $|x-y| \leqq \frac{1}{4}\varepsilon^{-2}\right)$ we then have

$$|h_\varepsilon(y, t)| \leqq c\varepsilon^\delta + \varepsilon \int_{s \in t} ds \sum_z |\nabla_+ P_{t-s}(y \to z)| \times (2|h_\varepsilon(z, s)| + |h_\varepsilon(z+1, s)|) \quad (4.17)$$

We can use again (4.17) if z and $z+1$ are closer to x than $\frac{1}{4}\varepsilon^{-2}$ and for notational simplicity we only give the argument as if $h_\varepsilon(z+1, s)$ were not present at all. Then we get (all the integrals below are extended to domains contained in t)

$$|h_\varepsilon(x, t)| \leqq c \sum_k \sum_{z_1 \ldots z_k} \varepsilon \int_0^t ds_1 |\nabla_+ P_{t-s_1}(x \to z_1)| \ldots$$

$$\ldots \int_0^{s_{k-1}} ds_k |\nabla_+ P_{s_{k-1}-s_k}(z_{k-1} \to z_k)| \left[\varepsilon^\delta + \mathbf{1}\left(|z_k - x| > \frac{1}{4}\varepsilon^{-2}\right)\right] \quad (4.18)$$

The first term vanishes like ε^δ, since

$$\sum_z |\nabla_+ P_t(x \to z)| \leqq \frac{c}{\sqrt{t}}$$

For the second term we notice that if $|z_k - x| > \frac{1}{4}\varepsilon^{-2}$ then there is $j < k$ such that

$$|z_j - z_{j+1}| > \frac{1}{4k}\varepsilon^{-2} \qquad (z_0 = x)$$

This term in (4.18) is therefore bounded by

$$\sum_k \sum_{j=0}^{k-1} \varepsilon \int_0^t ds_1 \frac{c}{\sqrt{t-s_1}} \cdots \varepsilon \int_0^{s_{j-2}} ds_{j-1} \frac{c}{\sqrt{s_{j-2}-s_{j-1}}} \times \varepsilon \int_0^{s_{j-1}} ds_j \, P_0\left(|x(s_{j+1})-x(s_j)| > \frac{\varepsilon^{-2}}{4k} \right) \varepsilon \int_0^{s_j} ds_{j+1} \frac{c}{\sqrt{s_j-s_{j+1}}} \cdots \varepsilon \int_0^{s_{k-1}} ds_k \frac{c}{\sqrt{s_{k-1}-s_k}} \leqq \sum_k k \frac{c^k}{[(k-1)/4]!} \varepsilon^{-1} \exp\left\{ -\frac{1}{2\varepsilon^{-2}\tau} \left[\frac{\varepsilon^{-2}}{4k} \right]^2 \right\} \quad (4.19)$$

where k counts the possible values of j, $\varepsilon^{-1} \exp\{.\}$ bounds the contribution coming from the integral over s_j. The other factor in (4.19) arises from the following estimate, uniform in $t \leqq \varepsilon^{-2}\tau$ and $\varepsilon > 0$,

$$\sup_{t \leqq \varepsilon^{-2}\tau} \varepsilon \int_0^t ds_1 \frac{c}{\sqrt{t-s_1}} \cdots \varepsilon \int_0^{s_{h-1}} ds_h \frac{c}{\sqrt{s_{h-1}-s_h}} \leqq \frac{c^h}{[h/2]!}$$

It is easy to see that the r. h. s. in (4.19) vanishes faster than any power of ε and this completes the proof that $|h_\varepsilon(x, t)| \leqq c\varepsilon^\delta$.

4.4. LEMMA. — *Let* β *and* ζ *be as in Definition* 4.1 *and* $\delta = \beta - \zeta$ (cf. *Lemma* 4.3). *Then for any n there is c such that for all* $\underline{x} \in \aleph_n$ *and for all t:* $2\varepsilon^{-2+\beta} \leqq t \leqq \varepsilon^{-2}\tau$ (*the case* $t < 2\varepsilon^{-2+\beta}$ *is included in the estimates of Section* 3)

$$|v_n^\varepsilon(\underline{x}, t)| \leqq c\varepsilon^{\delta n}$$

where v_n^ε *is defined for a process which starts from a single configuration* $\eta \in \{0,1\}^{\mathbf{Z}}$ *and c does not depend on the choice of* η.

Proof. — Given t as in the Lemma and $N(t)$ as in Definition 4.1 we define $\Sigma(N(t))$ as the σ-algebra generated by $\{\eta(x, s),\ x \in \mathbf{Z},\ s \leqq N(t)\varepsilon^{-2+\beta}\}$. We then have

$$v_n^\varepsilon(\underline{x}, t) = E_\eta^\varepsilon\left(E_\eta^\varepsilon\left(\prod_{i=1}^n \{[\eta(x_i, t) - \rho_\varepsilon(x_i, t|\underline{\eta})] + [\rho_\varepsilon(x_i, t|\underline{\eta}) - \rho_\varepsilon(x_i, t|\eta)]\} \,|\, \Sigma(N(t)) \right) \right) \quad (4.20)$$

We expand the product in (4.20), the factors coming from the second square bracket are constant w. r. t. the conditional expectation, those from the first one can be bounded using the estimates of Section 3, hence

$$|v_n^\varepsilon(\underline{x}, t)| \, c \max\left\{ \varepsilon^{(2-\beta)n/8}, \left| E_\eta^\varepsilon\left(\prod_{i=1}^n [\rho_\varepsilon(x_i, t|\underline{\eta}) - \rho_\varepsilon(x_i, t|\eta)] \right) \right| \right\}$$

We set

$$i(\zeta, t, \underline{x}) = \cap i(\zeta, t, x_i)$$

and the Lemma follows from Lemmas 4.3 and 4.1.

Proof of Theorem 2.1. — Let $\rho(r)$ and μ^ε be as in Theorem 2.1. Let $\rho_\varepsilon(x, t) \equiv \rho_\varepsilon(x, t \mid \mu^\varepsilon)$, *cf.* (2.5) and below. Then for all m there is c so that

$$\mu^\varepsilon(\cdot \sup_{|y| \leqq \varepsilon^{-2}} \| \eta(.) - \rho_\varepsilon(., 0) \|_y \leqq \varepsilon^\zeta) \geqq 1 - c\varepsilon^m$$

With arguments used for proving Lemma 4.3 we easily get that for any positive r

$$\mu^\varepsilon(\sup_{|x| \leqq r\varepsilon^{-1}} \sup_{t \in t(\varepsilon^{-2}\tau)} |\rho_\varepsilon(x, t) - \rho_\varepsilon(x, t \mid \eta_0)| > c\varepsilon^\delta) \leqq c\varepsilon^m \quad (4.21)$$

Hence by Lemma 4.4, choosing $m > \delta$ and, as usual, letting c be a constant whose value changes from inequality to the other, we get

$$\begin{aligned} &\sup_{|x| \leqq r\varepsilon^{-1}} \sup_{t \in t(\varepsilon^{-2}\tau)} |E^\varepsilon_{\mu^\varepsilon}(\eta(x, t)) - \rho_\varepsilon(x, t)| \\ &\quad \leqq \sup_{|x| \leqq r\varepsilon^{-1}} \sup_{t \in t(\varepsilon^{-2}\tau)} |E^\varepsilon_{\mu^\varepsilon}(|E^\varepsilon_\eta(\eta(x, t) - \rho_\varepsilon(x, t \mid \eta))|) \\ &\qquad + c\varepsilon^\delta + c\varepsilon^m \leqq c\varepsilon^\delta \end{aligned} \quad (4.22)$$

To prove Theorem 2.1 therefore we need only to show that $\rho_\varepsilon([\varepsilon^{-1}r], \varepsilon^{-2}\tau)$ converges uniformly on the compacts to $\rho(r, t)$, solution to (2.4) with initial datum $\rho(r)$. We define

$$a_\varepsilon(r, t) = \rho_\varepsilon([\varepsilon^{-1}r], \varepsilon^{-2}t)$$

when $\varepsilon^{-1}r \in Z$ and $t \leqq \tau$ and linearly interpolate for the other values of r. Obviously $0 \leqq a_\varepsilon(r, t) \leqq 1$ and proceeding like before it is easy to see that $a_\varepsilon(r, t)$ is also equicontinuous as $\varepsilon \to 0$. Hence it only remains to prove that $a_\varepsilon(r, t) \to \rho(r, t)$ for any fixed (r, t). Let Φ be a smooth function on R with compact support and let $0 < T < t \leqq \tau$. Then

$$\begin{aligned} &\Big| \varepsilon \sum_x \Phi(\varepsilon x) a_\varepsilon(\varepsilon x, t) - \varepsilon \sum_x \Phi(\varepsilon x) \sum_z P_{\varepsilon^{-2}t}(x \to z) a_\varepsilon(\varepsilon z, 0) \\ &\qquad - \int_0^T ds\, \varepsilon \sum_x \Phi(\varepsilon x) \varepsilon^{-1} \sum_z [P_{\varepsilon^{-2}(t-s)}(x \to z+1) \\ &\qquad - P_{\varepsilon^{-2}(t-s)}(x \to z)] a_\varepsilon(\varepsilon z, s)[1 - a_\varepsilon(\varepsilon(z+1), s)] \Big| \\ &\qquad \leqq c \int_T^t ds \frac{c}{\sqrt{t-s}} = c\sqrt{t-T} \end{aligned}$$

Using the fact that, (i) $P_t(x \to z+1) = P_t(x-1 \to z)$ for any $t > 0$, (ii) the first derivative of Φ is continuous, (iii) the local central limit theorem, we

have that for any limiting point $a(.,.)$ of $a_\varepsilon(.,.)$ [denoting below by $G_t(r \to r')$ the heat kernel]

$$\left| \int dr\, \Phi(r)\, a(r, t) - \int dr\, \Phi(r) \int dr'\, G_t(r \to r')\, \rho(r') \right.$$
$$\left. - \int_0^T ds \int dr\, \Phi(r) \int dr' \left[\frac{\partial}{\partial r'} G_{t-s}(r \to r') \right] a(r', s)[1 - a(r', s)] \right| \leqq c\sqrt{t-T}.$$

Since $a(r, t)$ and $\partial/\partial r\, G_{t-s}(r \to r')$ are both continuous functions, and the above inequality is true for any smooth Φ, we have

$$\left| a(r, t) - \int dr'\, G_t(r \to r')\, \rho(r') \right.$$
$$\left. - \int_0^T ds \int dr' \left[\frac{\partial}{\partial r'} G_{t-s}(r \to r') \right] a(r', s)[1 - a(r', s)] \right| \leqq c\sqrt{t-T}.$$

Letting $T \to t$ we then see that $a(r, t)$ solves (2.4), hence Theorem 2.1 is proven

Proof of Proposition 2.2. — So far we have proven that v_n^ε is bounded by $c\varepsilon^{\delta n}$ with $\delta > 0$. To prove Proposition 2.2 we need to show that we can improve the above result and have a bound like in (2.8 *a*) and (2.8 *b*). We shall only prove (2.8 *b*), the proof of (2.8 *a*) being analogous. We shall need later the estimate in (2.8 *b*) to prove Theorem 2.3 on the convergence of the fluctuation field.

The proof of (2.8 *b*) is obtained bu iterating eq. (3.1), we do not have problems of convergence [of the iteration] because we can use Lemma 4.4 when the number of bodies is large. In this way we can avoid the "short times assumption" of Section 3. We prove below only (2.8 *b*) the proof of (2.8 *a*) being analogous. We fix a $T > 0$ and for any $t \leqq \varepsilon^{-2} T$ we define for all $n \geqq 0$

$$a(\varepsilon, n, t) = \sup_{\underline{x}} |v_n^\varepsilon(\underline{x}, t)|, \qquad a(\varepsilon, 0, t) = 1 \tag{4.23 a}$$

$$d(\varepsilon, n, t) = \sup_{\underline{x}} |v_n^\varepsilon(\underline{x}, t) - v_n^\varepsilon(\underline{x} + e_1, t)|, \qquad d(\varepsilon, 0, t) = 0 \tag{4.23 b}$$

where the sup is over all $\underline{x} \in \aleph_n$. We use (3.1) to write an integral equation for the a's. It is convenient to rewrite the first two terms in (3.1) after "integrating by parts", as it is going to be explained below. From (3.2 *a*)

we get

$$\begin{aligned}\mathbf{E}_{\underline{x}}[(\mathrm{R}\, v_n^\varepsilon)(\underline{x}(t-s), s)] = \sum_i \sum_{\underline{z}} v_n^\varepsilon(\underline{z}, s) \\ \times [\mathrm{P}_{t-s}(\underline{x} \to \underline{z}+\underline{e}_i)\mathbf{1}(z_j \neq z_i, \forall j \neq i)\{1-\rho_\varepsilon(z_i+1, s)\} \\ -\mathrm{P}_{t-s}(\underline{x} \to \underline{z})\mathbf{1}(z_j \neq z_i - 1, \forall j \neq i)\{1-\rho_\varepsilon(z_i, s)\}] \\ -\sum_i \sum_{\underline{z}} v_n^\varepsilon(\underline{z}, s)[\mathrm{P}_{t-s}(\underline{x} \to \underline{z})\mathbf{1}(z_j \neq z_i + 1, \forall j \neq i)\rho_\varepsilon(z_i, s) \\ -\mathrm{P}_{t-s}(\underline{x} \to \underline{z}-\underline{e}_i)\mathbf{1}(z_j \neq z_i, \forall j \neq i)\rho_\varepsilon(z_i - 1, s)] \\ +\sum_i \sum_{\underline{z}} v_n^\varepsilon(\underline{z}, s)\mathrm{P}_{t-s}(\underline{x} \to \underline{z})[\rho_\varepsilon(z_i+1, s) - \rho_\varepsilon(z_i-1, s)] \qquad (4.24)\end{aligned}$$

while for the second term in (3.1 *b*) we get

$$\begin{aligned}\varepsilon \sum_i \sum_{\underline{z}} v_{n+1}^\varepsilon(\underline{z}, z_i - 1, s)[\mathrm{P}_{t-s}(\underline{x} \to \underline{z}-\underline{e}_i)\mathbf{1}(z_j \neq z_i, \forall j \neq i) \\ -\mathrm{P}_{t-s}(\underline{x} \to \underline{z})\mathbf{1}(z_j \neq z_i - 1, \forall j \neq i)]. \qquad (4.25)\end{aligned}$$

We shall prove below that $\forall t \geqq 0$, $\forall i = 1, \ldots, n$

$$\sup_{\underline{x}} \sum_{\underline{z}} |(\mathrm{P}_t(\underline{x} \to \underline{z}+\underline{e}_i) - \mathrm{P}_t(\underline{x} \to \underline{z})| \leqq \frac{c}{\sqrt{t+1}} \qquad (4.26)$$

Recalling that $|\rho_\varepsilon(z, s) - \rho_\varepsilon(z+1, s)| \leqq c\varepsilon$ and that

$$\sup_{\underline{x}} \sum_{\underline{z}} \mathrm{P}_t(\underline{x} \to \underline{z})\mathbf{1}(\text{there are } j \text{ and } i: |z_j - z_i| = 1) \leqq \frac{c}{\sqrt{t+1}} \qquad (4.27)$$

we get a bound for (4.17) and (4.18) which goes like

$$cn\left[\frac{1}{\sqrt{t-s+1}} + \varepsilon\right] a(\varepsilon, n, s) + \frac{cn}{\sqrt{t-s+1}} a(\varepsilon, n+1, s) \qquad (4.28)$$

Notice that $\varepsilon < c[t-s+1]^{-1/2}$ so that we can and will drop the term with ε in (4.28) from now on. From (3.1), (3.7), (4.28) and (4.26) it follows that for any $0 \leqq t^* \leqq t$

$$\begin{aligned}|v_n^\varepsilon(\underline{x}, t)| \leqq a(\varepsilon, n, t^*) + c\int_{t^*}^t ds \frac{\varepsilon}{\sqrt{t-s+1}}[a(\varepsilon, n, s) + a(\varepsilon, n+1, s)] \\ + c\int_{t^*}^t ds \frac{1}{\sqrt{t-s+1}}[\varepsilon a(\varepsilon, n-2, s) + \varepsilon d(\varepsilon, n-1, s) \\ + \varepsilon^2 a(\varepsilon, n-1, s) + \varepsilon^2 a(\varepsilon, n, s)] \qquad \text{for} \quad n \geqq 2 \quad (4.29\,a)\end{aligned}$$

while

$$|v_1^\varepsilon(\underline{x}, t)| \leqq a(\varepsilon, 1, t^*) + c\int_{t^*}^t ds \frac{\varepsilon}{\sqrt{t-s+1}}[a(\varepsilon, 1, s) + a(\varepsilon, 2, s)] \quad (4.29\,b)$$

To write an integral equation for $d(\varepsilon, n, t)$ we use eqs. (3.13) and (3.1) (we do not need to "integrate by parts" as we did before) so we get for

any $0 \leqq t^* \leqq t$

$$\begin{aligned}
|v_n^\varepsilon(\underline{x}(1), t) - v_n^\varepsilon(\underline{x}(2), t)| &\leqq \mathrm{P}_{\underline{x}}\left(\tau_{1,2}(0, t-t^*) > \frac{t-t^*}{2}\right) a(\varepsilon, n, t^*) \\
&+ \int_{t^*}^t ds\, \mathrm{E}_{\underline{x}}\left(\mathbf{1}\left\{\tau_{1,2}(0, t-s) > \frac{t-s}{2}\right\}\right) \varepsilon [d(\varepsilon, n, s) + d(\varepsilon, n+1, s)] \\
&+ \int_{t^*}^t ds\, \mathrm{E}_{\underline{x}}^\varepsilon \Bigg[\mathbf{1}\left\{\tau(0, t-s) > \frac{t-s}{2}\right\} \Big[\sum_{i, j \neq 1} \mathbf{1}\{|x_j(t-s) - x_j(t-s)| = 1\} \\
&\qquad + \sum_{i, j \neq 2} \mathbf{1}\{|x_j(t-s) - x_i(t-s)| = 1\} \Big] \\
&\qquad [\varepsilon^2 a(\varepsilon, n-2, s) + \varepsilon d(\varepsilon, n-1, s) \\
&\qquad + \varepsilon^2 a(\varepsilon, n-1, s) + \varepsilon^2 a(\varepsilon, n, s)] \qquad \text{for} \quad n \geqq 2 \qquad (4.30a)
\end{aligned}$$

$$\begin{aligned}
|v_1^\varepsilon(x_1, t) - v_1^\varepsilon(x_2, t)| &\leqq \frac{c}{\sqrt{t-t^*}} a(\varepsilon, 1, t^*) \\
&+ c \int_{t^*}^t ds \frac{\varepsilon}{\sqrt{t-s}} [d(\varepsilon, 1, s) + d(\varepsilon, 2, s)] \qquad (4.30b)
\end{aligned}$$

We condition on the σ-algebra which specifies $\underline{x}(s')$ for $s' \leqq (t-s)/2$. We set $\underline{y} = \underline{x}((t-s)/2)$ and we proceed like before, with $\underline{y}$ in place of $\underline{x}$. We obtain a bound independent of $\underline{y}$ and then we estimate the probability that $\mathbf{1}\{\tau_{1,2}(0, t-s) > (t-s)/2\}$ bounding it by $c(t-s+1)^{-1/2}$, assuming that $|x_1 - x_2| = 1$. Therefore we get for any $0 \leqq t^* \leqq t$

$$\begin{aligned}
d(\varepsilon, n, t) &\leqq \frac{c}{\sqrt{t-t^*}} a(\varepsilon, n, t^*) \\
&+ c \int_{t^*}^t ds \frac{\varepsilon}{\sqrt{t-s+1}} [d(\varepsilon, n, s) + d(\varepsilon, n+1, s)] \\
&+ c \int_{t^*}^t ds \frac{1}{t-s+1} [\varepsilon^2 a(\varepsilon, n-2, s) + \varepsilon d(\varepsilon, n-1, s) \\
&\qquad + \varepsilon^2 a(\varepsilon, n-1, s) + \varepsilon^2 a(\varepsilon, n, s)] \qquad \text{for} \quad n \geqq 2 \qquad (4.31a)
\end{aligned}$$

$$\begin{aligned}
d(\varepsilon, 1, t) &\leqq \frac{c}{\sqrt{t-t^*}} a(\varepsilon, 1, t^*) \\
&+ c \int_{t^*}^t ds \frac{\varepsilon}{\sqrt{t-s+1}} [d(\varepsilon, 1, s) + d(\varepsilon, 2, s)] \qquad (4.31\,b)
\end{aligned}$$

The eq. (2.8 *b*) easily follows from eqs. (4.29) and (4.31). We give below one possible way to show the estimate (2.8 *b*).

We fix $\lambda > 0$ small enough (how small will become clear later) and, to simplify notation we choose it so that $\lambda^{-1}\tau$ is an integer. Let n be the

integer for which we wish to prove Proposition 2.2. Set

$$I(m)=m\wedge n \tag{4.32}$$

and

$$a^*(\varepsilon, m, k)=\varepsilon^{-I(m)/2}\sup_{0\leqq\sigma\leqq k\lambda} a(\varepsilon, m, \varepsilon^{-2}\sigma), \qquad a^*(\varepsilon, 0, k)=1 \tag{4.33a}$$

$$d^*(\varepsilon, m, k)=\varepsilon^{-(I(m)/2)-1}(\log\varepsilon^{-1})^{-2}\sup_{0\leqq\sigma\leqq k\lambda} d(\varepsilon, m, \varepsilon^{-2}\sigma), \qquad d^*(\varepsilon, 0, k)=0 \tag{4.33b}$$

We shall only consider $m\leqq N+1$ where

$$\varepsilon^{\delta(N+1)}\leqq\varepsilon^{(1/2)n+2} \tag{4.34}$$

and δ is given in Lemma 4.4. Hence

$$a^*(\varepsilon, N+1, k)\leqq c\,\varepsilon^2, \qquad d^*(\varepsilon, N+1, k)\leqq c(\log\varepsilon^{-1})^2\varepsilon \tag{4.35}$$

for $k=\lambda^{-1}\tau$ (hence for all $k\leqq\lambda^{-1}\tau$) and uniformly in ε. We shall now find bounds for $m\leqq N$. From (4.22) and (4.24) with $t^*=\varepsilon^{-2}k\lambda$ and $t=\varepsilon^{-2}(k+1)\lambda$ we get for $m\leqq N$

$$\begin{aligned} a^*(\varepsilon, m, k)\leqq{}& a^*(\varepsilon, m, k-1)\\ &+c\lambda^{1/2}[a^*(\varepsilon, m, k)+a^*(\varepsilon, m+1, k)+a^*(\varepsilon, m-2, k)\\ &\quad+\varepsilon^{1/2}(\log\varepsilon^{-1})^2 d^*(\varepsilon, m-1, k)\\ &+\varepsilon^{1/2}a^*(\varepsilon, m-1, k)+\varepsilon a^*(\varepsilon, m, k)] \qquad \text{for} \quad m\geqq 2 \end{aligned} \tag{4.36a}$$

$$a^*(\varepsilon, 1, k)\leqq a^*(\varepsilon, 1, k-1)+c\lambda^{1/2}[a^*(\varepsilon, 1, k)+a^*(\varepsilon, 2, k) \tag{4.36b}$$

$$\begin{aligned} d^*(\varepsilon, m, k)\leqq{}& c\lambda^{-1/2}(\log\varepsilon^{-1})^{-2}a^*(\varepsilon, m, k-1)\\ &+c\lambda^{1/2}[d^*(\varepsilon, m, k)+d^*(\varepsilon, m+1, k)]\\ &+c(\log\varepsilon^{-1})^{-1}[a^*(\varepsilon, m-2, k)+\varepsilon^{1/2}(\log\varepsilon^{-1})^2 d^*(\varepsilon, m-1, k)\\ &+\varepsilon^{1/2}a^*(\varepsilon, m-1, k)+\varepsilon^1 a^*(\varepsilon, m, k)] \qquad \text{for} \quad m\geqq 2 \end{aligned} \tag{4.36c}$$

$$\begin{aligned} d^*(\varepsilon, 1, k)\leqq{}& c\lambda^{-1/2}(\log\varepsilon^{-1})^{-2}a^*(\varepsilon, 1, k-1)\\ &+c\lambda^{1/2}[d^*(\varepsilon, 1, k)+d^*(\varepsilon, 2, k)] \end{aligned} \tag{4.36d}$$

where $a^*(\varepsilon, -1, k)=0$, $a^*(\varepsilon, N+1, k)$ and $d^*(\varepsilon, N+1, k)$ are bounded as in (4.35). We can rewrite (4.36) in vector form. Let $\underline{x}_\varepsilon(k)$ be the vector whose first N component are $a^*(\varepsilon, 1, k), \ldots, a^*(\varepsilon, N, k)$ and the second N components are $d^*(\varepsilon, 1, k), \ldots, d^*(\varepsilon, N, k)$.

$$\underline{x}_\varepsilon(k)\leqq A_\varepsilon\,\underline{x}_\varepsilon(k-1)+\Lambda(\varepsilon, \lambda)\,\underline{x}_\varepsilon(k)+\underline{u}_\varepsilon(k) \quad \text{(componentwise)}$$

where $\underline{u}_\varepsilon(k)$ is the vector arising from the terms with $a^*(\varepsilon, 0, k)$, $a^*(\varepsilon, N+1, k)$ and $d^*(\varepsilon, N+1, k)$, the other terms are defined so to make this inequality valid. From (4.36*a*) the first component of $\underline{u}_\varepsilon(k)$ equals 0, the second one $c\lambda^{1/2}$, the N-th $c\lambda^{1/2}c\varepsilon^2$ [*cf.* (4.35)]. From (4.29*b*) the (N+1)-th component equals 0 the (N+2)-th $c(\log\varepsilon^{-1})^{-1}$ the 2 N-th equals

$c\lambda^{1/2}c(\log\varepsilon^{-1})^2\varepsilon$. By choosing λ small enough $\|\Lambda(\varepsilon,\lambda)\|\leqq\frac{1}{2}$ uniformly in ε as $\varepsilon\to 0$. Furthermore $\|A_\varepsilon\|\leqq 1$ and each component of $\underline{u}_\varepsilon(k)$ is less than some constant c, uniformly in ε. From this it easily follows that all the components of $\underline{x}_\varepsilon(k)$ are uniformly bounded in ε and for all $k\leqq\lambda^{-1}\tau$.

To conclude the proof of Proposition 2.2 we need to prove (4.26). We go back to the definition of the Stirring Process, *cf.* Section 3. It is convenient to realize such process as follows.. We consider products of independent Poisson processes of mean $\frac{1}{2}$, indexed by $(x, x+1)$, $x\in Z$. An event at $(x, x+1)$ is the time when a "mark" between x and $x+1$ appears. With probability $\frac{1}{2}$ the mark is "active" and with same probability "passive". All such events are mutually independent. The Stirring Process is then obtained by stating that a particle at (x, t) moves if at time t there is an active mark involving x, namely an active mark at $(x-1, x)$ resp. $(x, x+1)$. In such case the particle goes to $x-1$, resp. $x+1$. Notice that in this way we may realize in the same space all stirring processes with different number of particles. We denote by $Y(x, s)$ the position at time s of the stirring particle which at time 0 was in x, *i.e.* $Y(x, 0)=x$. For any given $t>0$, we set $Z(z, s)=Y(x, t-s)$, where z is such that $Y(z, t)=x$, the law of $\{Z(x, s), x\in Z, s\leqq t\}$ is then the same as that of $\{Y(x, s), x\in Z, s\leqq t\}$. We set $i=1$ in (4.19) for notational simplicity. We then use the translational invariance property of the process to write

$$P_t(\underline{x}\to\underline{z})=\int dP\,\mathbf{1}(Z(0, t)=x_1-z_1)\,\mathbf{1}(Z(z_i-z_1, t)=x_i-z_1, \forall i>1)$$

$$P_t(\underline{x}\to\underline{z}+e_1)=\int dP\,\mathbf{1}(Z(1, t)=x_1-z_1)\,\mathbf{1}(Z(z_i-z_1, t)=x_i-z_1, \forall i>1)$$

Call $g=\{$there is a first time $t'<t$ such that $|Z(0, t')-Z(1, t')|=1$ and at t' there is a mark, either active or passive, at the sites $\{Z(0, t'), Z(1, t')\}$. Then

$$\begin{aligned}&\Sigma_{\underline{z}}\left|P_t(\underline{x}\to\underline{z})-P_t(\underline{x}\to\underline{z}+e_1)\right|\\&\leqq\Sigma_{\underline{z}}\int dP(1-\mathbf{1}(g))\,\mathbf{1}(Z(0, t)=x_1-z_1)\,\mathbf{1}(Z(z_i-z_1, t)=x_i-z_1, \forall i>1)\\&\quad+\Sigma_{\underline{z}}\int dP(1-\mathbf{1}(g))\,\mathbf{1}(Z(1, t)=x_1-z_1)\,\mathbf{1}(Z(z_i-z_1, t)=x_i-z_1, \forall i>1)\end{aligned}$$

We sum first over $z_2, \ldots, z_n$ keeping z_1 fixed. Since by definition the event

$$\{ Z(z_i - z_1, t) = x_i - z_1 \} = \{ Y(x_i - z_1, t) = z_i - z_1 \}$$

and the events $\{ Y(x_i - z_1, t) = z_i - z_1, \forall i > 1 \}$ are disjoint for different $z_2, \ldots, z_n$ we get

$$\Sigma_{\underline{z}} | P_t(\underline{x} \to \underline{z}) - P_t(\underline{x} \to \underline{z} + e_1) |$$

$$\leqq \Sigma_z \int dP(1 - \mathbf{1}(g)) \mathbf{1}(Z(0, t) = x_1 - z)$$

$$+ \Sigma_z \int dP(1 - \mathbf{1}(g)) \mathbf{1}(Z(1, t) = x_1 - z) \leqq c(t+1)^{-1} \quad ^2$$

Proof of Theorem 2.3. — We shall prove that $X^\varepsilon_\tau(\varphi)$ defined in (2.9) converges weakly in the space $D(R_+ \to \mathscr{S}'(R))$ to a generalized Ornstein-Unhlebeck process with the covariance given in (2.10). With the estimates on v^ε_n obtained in Lemma 4.4 it is easy to prove that the moments of X^ε_t, at each fixed t, converge to those of the limiting process as described in Theorem 2.3. The proof however for the moments at different times is more involved, so we prefer to use the more "traditional" theory of Holley and Stroock, [18]. We need to prove the following 3 statements.

S 1. The sequence $\underline{X}^\varepsilon \equiv \{ X^\varepsilon_\tau(\varphi), \ \tau \geqq 0, \ \varphi \in \mathscr{S}(R) \}$ is tight in $D(R_+ \to \mathscr{S}'(R))$ and any limiting point $\underline{X}$ is supported by $C^0(R_+ \to \mathscr{S}'(R))$.

S 2. Any limiting point $\underline{X}$ of $\underline{X}^\varepsilon$ satisfies the following "martingale problem". For any $\varphi \in \mathscr{S}(R)$, and any $F \in C^\infty(R)$

$$F(X_t(\varphi)) - \int_0^t ds\, F'(X_s(\varphi)) X_s(A_s \varphi) - \int_0^t ds \frac{1}{2} \| B_s \varphi \|^2 F''(X_s(\varphi)) \quad (4.37)$$

is a martingale with respect to the canonical filtration in $C^0(R_+ \to \mathscr{S}'(R))$. In (4.37) F′ (resp. F″) denotes the first (resp. the second) derivative of the function F, while the operators A_s and B_s are the following.

$$(A_s \varphi)(r) = \frac{1}{2} \frac{d^2 \varphi(r)}{dr^2} - [1 - 2\rho(r, s)] \frac{d}{dr} \varphi(r) \quad (4.38a)$$

$\rho(r, s)$ is the solution of the Burgers equation (4.38*b*)

$$\| B_s \varphi \|^2 \equiv \int_{-\infty}^{+\infty} dr\, \psi'(r) \rho(r, s)(1 - \rho(r, s) \varphi'(r) \quad (4.38c)$$

S 3. The law of $\underline{X}_0$ (*i.e.* the law of the limiting process at time 0) is gaussian with covariance kernel $C^*_0(r, r')$, *cf.* Theorem 2.3, eq. (2.10).

Once S 1 to S 3 are proven we know that $\underline{X}$ is a gaussian process and all its moments are finite. So we can rewrite (4.37) with $F(r)=r^2$ and r to derive an explicit expression for the covariance which is readily seen to agree with the desired expression. For the reader's convenience we report some details of the computations. From the martingale equation we have

$$\int_{R^2} C_t^*(r, r')\varphi(r)\varphi(r')\,dr\,dr' = \int_0^t ds \int_{R^2} C_s^*(r, r')\varphi(r) 2A_s\varphi(r')\,dr\,dr' + \int_0^t ds \int_R \{\varphi'(r)\}^2 \rho_s(1-\rho_s)$$

where A_s is the adjoint of the linearized Burgers operator, *i. e.*,

$$A_s\varphi(r)=\frac{1}{2}\frac{d^2\varphi(r)}{dr^2}+(1-2\rho_s)\frac{d\varphi}{dr}$$

Taking the time derivative in the above equation and using eq. (2.10*b*) we get

$$\int_{R^2} \varphi(r)\varphi(r')\frac{\partial}{\partial t}C_t\,dr\,dr' = \int_{R^2} \varphi(r)\varphi(r') 2L_s C_s - \int_R \varphi^2(1-2\rho_t)\left[\frac{1}{2}\rho'' - \frac{\partial}{\partial r}(\rho_t(1-\rho_t))\right] + \int_R 2\rho_t(1-\rho_t)\varphi\left[\frac{1}{2}\varphi''+(1-2\rho_t)\varphi'\right] + \int_R (\varphi')^2\rho_t(1-\rho_t)$$

where L_s is the linearized Burgers operator acting on C_t as a function of r'. Now using the following identities

$$\int_R \rho(1-\rho)\varphi\varphi'' = -\int_R \rho(1-\rho)(\varphi')^2 + \int_R \frac{1}{2}\varphi^2\frac{\partial}{\partial r^2}\{\rho(1-\rho)\}$$

$$\frac{1}{2}\frac{\partial}{\partial r^2}\{\rho(1-\rho)\} = \frac{1}{2}\rho''(1-2\rho)-(\rho')^2$$

it is easy to reconstruct the δ-term in the equation (2.10*d*).

We are left with the proof of S 1 to S 3. S 3 is obviously satisfied. The support properties of the limiting measure follow from standard arguments after noticing that the jumps of $X_t^\varepsilon(\varphi)$ are bounded by $c\varepsilon^{1/2}$. To prove the remaining statements in S 1 and S 2 we notice that

$$F(X_t^\varepsilon(\varphi)) - \int_0^{\varepsilon^{-2}\tau} ds\, L^\varepsilon F(X_s^\varepsilon(\varphi)) = F(X_t^\varepsilon(\varphi)) - \int_0^\tau ds\,\varepsilon^{-2} L^\varepsilon F(X_{s\varepsilon^2}^\varepsilon(\varphi)) \quad (4.39)$$

is a martingale, L_ε being the generator of the process as defined in (2.1). Hence

$$L_\varepsilon F(X_t^\varepsilon(\varphi)) = F'(X_t^\varepsilon(\varphi))\gamma_1^\varepsilon(t,\varphi) + F''(X_t^\varepsilon(\varphi))\gamma_2^\varepsilon(t,\varphi) + R^\varepsilon(t,\varphi) \quad (4.40)$$

where

$$\varepsilon^{-2} R^\varepsilon(t,\varphi) \text{ is uniformly bounded in } \varepsilon \quad (4.41)$$

$$\gamma_1^\varepsilon(t,\varphi) = X_\tau^\varepsilon\left(\frac{1}{2}\varphi''\right) + Y_t^\varepsilon(\varphi') \quad (4.42)$$

and

$$Y_\tau^\varepsilon(\varphi') = -\sqrt{\varepsilon}\sum_{x\in Z}\varphi'(\varepsilon x)\{(\eta(x,\varepsilon^{-2}\tau)(1-\eta(x+1,\varepsilon^{-2}\tau) - E^\varepsilon[\eta(x,\varepsilon^{-2}\tau)(1-\eta(x+1,\varepsilon^{-2}\tau)]\} \quad (4.43)$$

$$\gamma_2^\varepsilon(t,\varphi) = \frac{1}{2}\varepsilon\sum_{x\in Z}\varphi'(\varepsilon x)^2[\eta(x,\varepsilon^{-2}t) - \eta(x+1,\varepsilon^{-2}t)]^2 \quad (4.44)$$

Tightness follows from the fact that by (2.8*b*) the L_2 norms of γ_i^ε, $i=1, 2$, are bounded uniformly in ε and $t\leqq\tau$, *cf.* [23]. Using (2.8) and the same argument introduced in [13] it is possible to show that

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0} E_{\mu_\varepsilon}^\varepsilon\left[\left(\frac{1}{T}\int_t^{t+T} ds\{Y_s^\varepsilon(\varphi) - X_s^\varepsilon((1-2\rho(.,s))\varphi)\}\right)^2\right] = 0 \quad (4.45)$$

From (4.45) eq. (4.37) follows, we omit the details and refer to [13].

To conclude the proof of Theorem 2.3 we prove (2.12). To this purpose notice that the solution of (2.10) can be written as follows.

$$C_t(r+ct, r'+ct) = \int_0^t ds\int_{-\infty}^{+\infty} dz\, K_{t-s}(z\to r)K_{t-s}(z\to r') \times[2\rho'(z)\{\rho(z)(1-\rho(z)\} - \rho'(z)^2] \quad (4.46)$$

where ρ' is the first derivative of the function ρ and $K_t(r\to z)$ is the probability kernel for a Brownian motion with diffusion coefficient 1 and drift $1-2\rho-c$, namely as a function of r solves the forward Kolmogorov equation

$$\frac{\partial}{\partial t}K_t(z\to r) = \frac{1}{2}\frac{\partial^2}{\partial r^2}K_t(r\to z) - \frac{\partial}{\partial r}\{[1-2\rho(r)-c]K_t(z\to r)\} \quad (4.47a)$$

$$K_0(z\to r) = \delta(r-z) \quad (4.47b)$$

The drift $1-2\rho(r)-c$ is strictly decreasing (resp. increasing) when $r<0$ (resp. $r>0$) so there is a unique invariant measure μ for this evolution and convergence to μ is exponentially fast. To find μ we notice that since ρ is a stationary traveling wave solution to the Burgers equation then its

space derivative ρ' solves the linearized equation hence

$$\mu(dr) = \frac{1}{\rho_+ - \rho_-} \rho'(r)\, dr \tag{4.48}$$

Therefore

$$\lim_{t\to\infty} \frac{1}{t} C_t(r, r') = \rho'(r)\rho'(r') \int_{-\infty}^{\infty} dz \frac{1}{(\rho_+ - \rho_-)^2} \times \{2\rho'(z)[1-\rho(z)]\rho(z) - \rho'(z)^2\} \tag{4.49}$$

An explicit computation of the integral in dz shows that eq. (2.12) holds with D given by (2.13).

Remark. — The good mixing properties of the operator K have a microscopic counterpart and can be exploited to investigate the long time behavior of the v-functions. This is the main ingredient that we use in trying to prove the conjecture formulated in (2.17).

ACKNOWLEDGEMENTS

We are indebted to A. Sznitman and E. Orlandi for many helpful discussions. E.P. acknowledges very kind hospitality at the Laboratoire des Probabilités of Paris-VI where a preliminary version of the present paper was prepared and A.D.M. acknowledges very kind hospitality at the École Polytechnique of Palaiseau. We finally acknowledge Dave Wick for very useful comments.

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(*Manuscrit reçu le* 7 *avril* 1987)
(*révisé le* 4 *juillet* 1988.)