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Limit laws for a coagulation model of interacting random particles

by

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ABSTRACT. — We consider in $\mathbb{R}^d$, n brownian particles which interact with each other and eventually die. Each particle is represented by the process $(x_i^{(n)}, \xi_i^{(n)})$ $(i=1, \ldots, n)$ where the $x_i^{(n)}$ is brownian motion and $\xi_i^{(n)}$ is a process in $D([0, T]; \{0, 1\})$ which defines the state of the particle: death or life.

We prove propagation of chaos and a fluctuation theorem for the empirical distribution using a martingale method.

Key words : Martingale, empirical distributions, propagation of chaos.

RÉSUMÉ. — On considère n particules browniennes en $\mathbb{R}^d$ qui réagissent réciproquement entre elles et finalement meurent. Chaque particule est représenté par le processus $(x_i^{(n)}, \xi_i^{(n)})$ $(i=1, \ldots, n)$ où les $x_i^{(n)}$ sont des mouvements browniens et les $\xi_i^{(n)}$ sont des processus dans $D([0, T]; \{0, 1\})$ qui définissent l'état des particules : mort ou vie.

On prouve la propagation du chaos et un théorème sur les fluctuations des distributions empiriques en utilisant une méthode de martingales.

Classification A.M.S. : 60 K 35.

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0. INTRODUCTION

Problems concerning interacting processes have been extensively treated since Kac ([10], [11]). McKean formulated them in terms of non linear Markov processes ([14], [15]).

A large number of authors has worked on such problems in several contexts with different methods. M. Metivier [17], Oelschläger [19], Leonard [13], proved propagation of chaos using a martingale method. Sznitman [21] proved with a similar approach propagation of chaos for a system of interacting particles in a bounded domain with reflecting normal conditions. He dealt with fluctuations using an argument based on Girsanov formula.

Dawson in [5] examines the dynamics and the fluctuations of a collection of anharmonic oscillators in a two-well potential with an attractive mean field interaction. Some critical phenomena appear in the limit behaviour depending on diffusion constant.

This list is not exaustive. In all the papers quoted the interaction is of mean field type: weak interaction. Sometimes these limit problems are called Vlasov-limits [6].

The model we consider consists of a system of n independent brownian particles in $\mathbb{R}^d$. Every particle can die according to a rate which depends on the configuration of the particles still alive; the dependence is of mean field type. We describe the change by a spin process which assumes values 0 (dead) or 1 (alive). We represent the model by a process $(x_i^{(n)}, \xi_i^{(n)})_{i=1}^n$ satisfying:

$$x_i^{(n)}(t) = x_i^{(n)}(0) + w_i(t)$$

$$\xi_i^{(n)}(t) = \xi_i^{(n)}(0) - \beta_i\left(\int_0^t \xi_i^{(n)}(s)\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^n q(x_i^{(n)}(s) - x_j^{(n)}(s))\,\xi_j^{(n)}(s)\,ds\right) \quad (0.1)$$

$$i = 1, \ldots, n, \qquad t \geqq 0$$

where

w_i are independent brownian motions,

β_i are independent Poisson processes of parameter 1,

q is a continuous, non negative function with compact support.

$(x_i^{(n)}(0), \xi_i^{(n)}(0))$, $i = 1, \ldots, n$ are independent identically distributed random variables with values in $\mathbb{R}^d \times \{0, 1\}$.

We are interested in the asymptotic behaviour of the system as the number of particles goes to infinity.

We show the propagation of chaos, that is a law of large numbers for the empirical distribution of the processes. Moreover we get fluctuations results for the empirical distributions of the particles which are still alive.

Our model differs from those previously described because the processes of the system have both diffusion and jumps components.

We chose the Wiener process as the diffusion component to semplify the calculations.

One can easily extend our results to the case of diffusions weakly interacting, making suitable hypotheses on the coefficients.

An interesting related problem is to increase the strength of the interaction q and to shrink its range, in the sense that q depends on n:

$$q_n(x) = n^\beta q(n^\beta/d), \qquad \beta > 0.$$

In the moderately interacting case $0 < \beta < \frac{d}{d-2}$, the mascroscopic equation for the limit probability density v of the alive particles solves:

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - cv^2$$

where $c = \int q(x)\,dx$ (in this direction, for different models, *see* [3], [20]).

In the strong interaction case $\beta = \frac{d}{d-2}$ the constant c in the macroscopic equation should take a different value. This case, for the Smoluchowsky model of colloids, has been studied by Lang-Xanh [12] by means of the B.B.G.K.Y. hierarchy and by Sznitman [23] with a Wiener sausage approach.

We state the main results in paragraph 2. In paragraph 3 we analyze some properties of the model (0.1). In paragraph 4 there is the proof of the propagation of chaos and paragraph 5 deals with fluctuations. The problem of moderate interaction is considered in a forthcoming paper.

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1. NOTATIONS

If Z is a complete, separable metric space:

$\mathscr{B}(Z)$ is the Borel σ-field.

$C(Z)$ is the Banach space of bounded continuous functions from Z to $\mathbb{R}$ with sup norm $\| \cdot \|_\infty$.

$M(Z)$ is the space of measures μ of bounded variation $\|\mu\|_{BV}$ with the topology induced by $C(Z)$.

$M^1(Z) \subset M(Z)$ is the subset of probability measures.

$C([0, T]; Z)$ is the Polish space of continuous trajectories from $[0, T]$ to Z with the uniform topology.

$D([0, T]; Z)$ is the Polish space of trajectories continuous on the right with limits on the left, from $[0, T]$ to Z with the Skorohod topology.

If η is a random variable with values in Z.

$\mathscr{L}(\eta) \in M^1(Z)$ is the law of η.

$S = C([0, T]; \mathscr{R}^d) \times D([0, T]; \{0, 1\})$; $(S, \mathscr{F})$ is a measurable space filtered with the canonical filtration:

$$\mathscr{F}_t = \bigcap_{s>t} \sigma\{(x(r), \xi(r)), r \leq s\}, \qquad \mathscr{F} = \bigvee_{0 \leq t \leq T} \mathscr{F}_t$$

where $\omega = (x, \xi)$ is the canonical process in S.

If $\nu \in M(S)$ is a measure on $(S, \mathscr{F})$.

$\nu(s) = \nu_s$ is the measure on $\mathbb{R}^d \times \{0, 1\}$

$$\nu_s(A, B) = \nu(x(s) \in A, \xi(s) \in B), \qquad A \in \mathscr{B}(\mathbb{R}^d), \quad B \subseteq \{0, 1\}$$

$\nu^1(s) = \nu_s^1$ is the measure on $\mathbb{R}^d$

$$\nu_s^1(A) = \nu_s(A, 1), \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

If $Z = \mathbb{R}^k$.

$C_p = C_p(\mathbb{R}^k)$ is the space of continuous functions f such that

$$|f|_p = \sup_{x \in \mathbb{R}^k} \frac{|f(x)|}{(1+|x|^2)^{p/2}} < \infty, \quad p > 0; \qquad \text{for} \quad p=0, \quad C_0 = C(\mathbb{R}^k)$$

$M_p = M_p(\mathbb{R}^k)$ is the space of measures μ on borelians of $\mathbb{R}^k$ such that $|\mu|_p^* = \int (1+|x|^2)^{p/2} \|\mu\|_{BV}(dx) < \infty$, M_p is the topological dual of C_p.

$C_0^\infty = C_0^\infty(\mathbb{R}^k)$ is the space of infinitely differentiable functions with compact support.

$\mathcal{S} = \mathcal{S}(\mathbb{R}^k)$ is the Schwartz space of rapidly decreasing C^∞ functions.

$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^k)$ is the dual of $\mathcal{S}$.

$W_p^j = W_p^j(\mathbb{R}^k)$ is the space of measurable functions f such that

$$\|f\|_{0,p}^2 = \sum_{|\alpha| \leqq j} \int_{\mathbb{R}^d} \frac{|D_\alpha f(x)|^2}{(1+|x|^2)^{p/2}} dx < \infty, \qquad p > 0, \quad j \in \mathbb{N}$$

where D_α denotes the distribution derivative and $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \geqq 0$ is a multiindex and $|\alpha| = \sum_{i=1}^{k} \alpha_i$.

$W_p^{-j} = W_p^{-j}(\mathbb{R}^k)$ is the dual of W_p^j.

$W^s = W^s(\mathbb{R}^k)$, $s \in \mathbb{R}$, is the space of functions $f \in \mathcal{S}'$ such that

$$\|f\|_s^2 = \int_{\mathbb{R}^k} [\hat{f}(\lambda)(1+|\lambda|^2)^{s/2}]^2 \, d\lambda < \infty$$

where $\hat{f}$ is the Fourier transform.

For $s=j$, $j \in \mathbb{N}$, W^j coincide with W_0^j and $\|\cdot\|_j = \|\cdot\|_{j,0}$.

$*$ denotes convolution: $q * \mu(x) = \int q(x-y)\,\mu(dy)$.

$(\ ,\ )$ denotes duality: we drop the dependence on the spaces, but it will be clear from the context. The same symbol is used for scalar product.

If M, N are martingales with real values.

$\langle M \rangle_t$ is the Meyer process associated to M: the unique predictable process such that $M_t^2 - \langle M \rangle_t$ is a martingale.

$\langle M, N \rangle_t$ is the unique predictable process such that $M_t N_t - \langle M, N \rangle_t$ is a martingale.

2. RESULTS

We denote by

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i^{(n)}, \xi_i^{(n)})} \tag{2.1}$$

the empirical distribution of the process in the system (0.1); ν_n is a random variable with values in $M^1(S)$.

We consider the operator

$$(L_\mu f)(x, \xi) = \frac{1}{2} \Delta_x f(x, \xi) + \xi (q * \mu)(x) [f(x, 0) - f(x, \xi)] \quad (2.2)$$

with $f(., \xi) \in C^2(\mathbb{R}^d)$, $\xi = 0, 1$ and $\mu \in M(\mathbb{R}^d)$.

We say that the probability ν on $(S, \mathscr{F})$ is solution of the non linear martingale problem (2.3) if and only if:

$$\begin{aligned} &(a) \qquad \nu(x(0) \in A, \xi(0) \in B) = \nu_0(A \times B), \\ &\qquad\qquad A \in \mathscr{B}(\mathbb{R}^d), \qquad B \subseteq \{0, 1\} \\ &(b) \quad f(x(t), \xi(t)) - f(x(0), \xi(0)) - \int_0^t (L_{\nu^1(s)} f)(x(s), \xi(s))\, ds \end{aligned} \quad (2.3)$$

is a ν-martingale for every f such that $f(., \xi) \in C^2(\mathbb{R}^d)$, $\xi = 0, 1$.

[*See* paragraph 1 for the definition of $\nu^1(s)$].

We are able to state a law of large numbers for the empirical distributions.

THEOREM 1. — *The sequence* $\{\mathscr{L}(\nu_n)\}$ *converges to* δ_ν *in* $M^1(M^1(S))$ *where* ν *is a measure in* $M^1(S)$ *such that*

(i) ν *is the unique solution of the martingale problem* (2.3).

(ii) $\nu^1(t)$ *has density* $u(t)$ *which is the solution of*

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= \frac{1}{2} \Delta u(t) - (q * u(t))\, u(t) \\ u(0) &= \frac{d}{dx} \nu^1(0) \end{aligned} \quad (2.4)$$

provided that

(A) *the initial conditions of* (0.1) *are identical independent random variables, with* $\xi_i^{(n)}(0) = 1$, $n \in \mathbb{N}$, $i = 1, \ldots, n$ *such that* $\{\nu_n(0)\}$ *is convergent in law to* ν_0, *where* $\nu_0 = \nu^1(0) \times \delta_{\{1\}}$.

(B) $\nu^1(0)$ *is absolutely continuous w. r. t. Lebesgue measure.* ■

We want to point out that we consider ν_n as random variables in $M^1(S)$, instead of considering them as processes in $D([0, T]; M^1(\mathbb{R}^d \times \{0, 1\})$.

This is due to the fact that we want to exploit the simmetry properties of system (1.1) in the propagation of chaos (*see* for details paragraph 4.1).

The next stage is the central limit theorem for

$$\nu_n^1(t) = \frac{1}{n} \sum_{i=1}^{n} \xi_i^{(n)}(t)\, \delta_{x_i^{(n)}(t)} \quad (2.5)$$

For each $t \leq T$, $\nu_n^1(t)$ is a positive measure on $\mathbb{R}^d$ which counts only the particles still alive at time t, and ν_n^1 is a process in $D([0, T]; M(\mathbb{R}^d))$.

We investigate the limit of:

$$Y_n(t)=\sqrt{n}\,(v_n^1(t)-v^1(t)) \tag{2.6}$$

as n goes to infinity, i. e. the fluctuations around the "living" particles measure $v^1(t)$ of point (ii) of Theorem 1.

THEOREM 2. — *Suppose that the hypotheses of Theorem* 1 *are satisfied,* $q\in C_0^\infty$, *the sequences* $\{E[|v_n^1(0)|_{2\,p}^*]\}$ *and* $\{E[\|Y_n(0)\|_{-j}^2]\}$ *are uniformly bounded for some* $p>d$, *and* $j>\max\left(d+1, \frac{d}{2}+2\right)$ *and the sequence* $\{\mathscr{L}(Y_n(0)\}$ *converges to* $\mathscr{L}(Y(0))$ *in* $M^1(\mathscr{S}')$, *then the sequence* $\{Y_n\}$, *considered as a sequence of processes in* $D([0, T]; \mathscr{S}')$, *converges in law to the generalized gaussian random process* Y, *solution of the linear stochastic equation:*

$$\begin{aligned}(f, Y(t))=(f, Y(0))&+\int_0^t\left(\frac{1}{2}\Delta f, Y(s)\right)ds\\ &-\int_0^t (f, q * v^1(s)\,Y(s))\,ds\\ &-\int_0^t (f, q * Y(s))\,v^1(s)\,ds+(f, N(t));\quad f\in\mathscr{S}.\end{aligned}$$

The process N *is a continuous gaussian martingale with independent increments, with zero mean and covariance:*

$$\begin{aligned}&E[(h, N(t))\,(g, N(t'))]\\ &\quad=\int_0^{t\wedge t'}(\nabla h.\nabla g, \mu^1(s))\,ds+\int_0^{t\wedge t'}((q * v^1(s))\,hg, u(s))\,ds.\end{aligned}$$

Moreover if the hypotheses on the initial conditions are satisfied with $j>\max\left(d+1, \frac{d}{2}+2\right)+2$ *and* $\{\mathbb{E}[\|Y_n(0)\|_{-l,p'}^2\}$ *is uniformly bounded for some* $p'>d$ *and* $l>0$, *then the sequence* $\{Y_n\}$ *considered in* $D([0, T], W^{-j})$ *converges in law to the* Y *process realized in* $D([0, T], W^{-j})$. ■

3. PROPERTIES OF THE n-PARTICLE SYSTEM

In this paragraph we will examine some properties of system (0.1) that we will use in the sequel.

System (0.1) has a unique solution $(x_i^{(n)}, \xi_i^{(n)})_{i=1}^n$ in $\mathbb{R}^+$ for every initial condition $(x_i^{(n)}(0), \xi_i^{(n)}(0))_{i=1}^n$. However we consider the process in a bounded interval of time $[0, T]$ and we will always refer to the canonical solution on the space S^n endowed with the filtration:

$$\mathcal{F}_t^n = \bigcap_{s>t} \sigma\{(x_i(r), \xi_i(r))_{i=1}^n, r \leqq s\}$$

and with the probability measure P^n equal to the law of the solution of (0.1). For semplicity of notations when writing expectations sometimes we will drop out the dependence on n and on the initial conditions.

For the same reason we will always assume (without mentioning) that $q(0)=0$. In particular this assumption allows to write:

$$\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n} q(x_j^{(n)}(s)-x_i^{(n)}(s))\,\xi_j^{(n)}(s)=(q * \nu_n^1(s))(x_i^{(n)}(s)). \tag{3.1}$$

Our system defines a Markov process $(\underline{x}^{(n)}, \underline{\xi}^{(n)})$ in $(\mathbb{R}^d)^n \times \{0, 1\}^n$ whose generator L_n acts as follows:

$$L_n \Phi(\underline{x}, \underline{\xi}) = \frac{1}{2}\Delta_{\underline{x}}\Phi(\underline{x}, \underline{\xi}) + \sum_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n} q(x_j-x_i)\,\xi_i\,\xi_j[\Phi(\underline{x}, \underline{\xi}^i)-\Phi(\underline{x}, \underline{\xi})] \tag{3.2}$$

[here $(\underline{x}, \underline{\xi}) \in (\mathbb{R}^d)^n \times \{0, 1\}^n$ and the components of $\underline{\xi}^i$ are defined by $\xi_j^i=\xi_j$ if $j \neq i$ and $\xi_i^i=0$] on the family $\mathcal{D}=\{\Phi: (\mathbb{R}^d)^n \times \{0, 1\}^n \to \mathbb{R}$ such that the function $\Phi(., \underline{\xi}) \in C^2(C(\mathbb{R}^d)^n)$ for every $\underline{\xi}$ fixed in $\{0, 1\}^n\}$.

Moreover for every Φ in this family we can define the process

$$\mathcal{M}_\Phi^n(t)=\Phi(\underline{x}^{(n)}(t), \underline{\xi}^{(n)}(t))-\Phi(\underline{x}^{(n)}(0), \underline{\xi}^{(n)}(0)) - \int_0^t (L_n\Phi)(\underline{x}^{(n)}(s), \underline{\xi}^{(n)}(s))\,ds \tag{3.3}$$

The process $\mathcal{M}_\Phi^n$ is a real valued martingale with trajectories in $D([0, T]; \mathbb{R})$ in the filtered probability space $(S^n, (\mathcal{F}_t^n)_{t\in[0,T]}, P^n)$ and its associated Meyer process is [9]:

$$\langle \mathcal{M}_\Phi^n \rangle_t = \int_0^t [L_n \varphi^2 - 2\Phi L_n \Phi](\underline{x}^{(n)}(s), \underline{\xi}^{(n)}(s))\,ds \tag{3.4}$$

With the particular choice of $\Phi(\underline{x}, \underline{\xi}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i, \xi_i)$, $f(., \xi) \in C^2(\mathbb{R}^d)$, we set $\mathcal{M}_\Phi^n = M_f^n$ and recalling (3.1) from (3.3):

$$M_f^n(t)=(f, \nu_n(t))-(f, \nu_n(0))-\int_0^t (L_{\nu_n^1(s)} f, \nu_n(s))\,ds \tag{3.5}$$

where $L_{\nu_n^1(s)}$ is defined in (2.2).

Then from (3.4)

$$\langle M_f^n \rangle_t = \frac{1}{n} \int_0^t \{((\nabla_{\underline{x}} f)^2, \nu_n(s)) + (q * \nu_n^1(s)) [f^0 - f]^2, \nu_n(s))\} \, ds \quad (3.6)$$

where $f^{(0)}(x, \xi) = f(x, 0)$ for every $x \in \mathbb{R}^d$, $\xi \in \{0, 1\}$.

4. PROPAGATION OF CHAOS

4.1. Methodology

Propagation of chaos for system (0.1) means that there exists a $\nu \in M^1(S)$ such that

$$\lim_{n \to \infty} \int_{S^n} \prod_{i=1}^{k} \varphi_i(x_i, \xi_i) \, P^n(dx_1, d\xi_1, \ldots, dx_n, d\xi_n) = \prod_{i=1}^{k} \int_S \varphi_i(x, \xi) \, \nu(dx, d\xi)$$

for every choice of $\varphi_1, \ldots, \varphi_k$ in $C(S)$.

This is the same as asking that there exists the limit measure P^∞ on $S^{\mathbb{N}}$ of the sequence $\{P^n\}$ and that

$$P^\infty(d\tilde{\omega}) = \prod_{i=1}^{\infty} \nu(d\omega_i), \qquad \tilde{\omega} = (\omega_i)_{i=1}^{\infty}. \quad (4.1)$$

The main tool is the n-exchangeability of the system: if $(x_i^{(n)}(0), \xi_i^{(n)}(0))$ $(i = 1, \ldots, n)$ are independent identically distributed random variables, then for every permutation σ of the indexes $\{1, 2, \ldots, n\}$ the law of $(x_{\sigma(i)}^{(n)}, \xi_{\sigma(i)}^{(n)})$ $(i = 1, \ldots, n)$ is the same as the law P^n of $(x_i^{(n)}, \xi_i^{(n)})$ $(i = 1, \ldots, n)$.

Then any limit point P^∞ of the sequence $\{P^n\}$ is excheangeable, or symmetric, namely it is n-exchangeable for every $n \in \mathbb{N}$. One can therefore apply De Finetti's theorem ([2], p. 51) and a characterization of the limit probability ([2], p. 55) to state that propagation of chaos is equivalent to the Law of Large Numbers for the empirical distributions $\{\nu_n\}$, namely $\mathscr{L}(\nu_n) \to \delta_{\{\nu\}}$.

Therefore to achieve the thesis of Theorem 1 we need to show the tightness of the empirical distributions (*see* Proposition 4.1) and the uniqueness of the limit points for $\{\mathscr{L}(\nu_n)\}$, which is divided in two steps:

(i) every limit point of $\{\mathscr{L}(\nu_n)\}$ in $M^1(M^1(S))$ has support on the set of solutions of the non linear martingale problem (2.3) (*see* Proposition 4.2).

(ii) the martingale problem (2.3) has a unique solution ν (*see* Proposition 4.3); this assures both the convergence of $\{\mathscr{L}(\nu_n)\}$ and the propagation of chaos, since the support of the limit law is then the singleton $\{\nu\}$.

4.2. Tightness

For sake of semplicity in this paragraph we will suppose that $\xi_i^{(n)}(0)=1$, $n\in\mathbb{N}$, $i=1, \ldots, n$.

PROPOSITION 4.1. — *Under the condition* A *of Theorem* 1 *the sequence* $\{\nu_n\}$ *is tight in* $M^1(S)$. ■

Proof. — It is sufficient to show tightness of the processes (x_1^n, ξ_1^n) since by Sznitman result ([22], lemma 3.2), the sequence $\{\mathscr{L}(\nu_n)\}$ is tight in $M^1(M^1(S))$ if and only if the sequence $\{\mathscr{L}(x_1^{(n)}, \xi_1^{(n)})\}$ is tight in $M^1(S)$.

As far as the first component $x_1^{(n)}(t)=x_1^{(n)}(0)+w_1(t)$ is concerned, tightness follows immediately from the convergence of the initial conditions. To show the tightness of $\xi_1^{(n)}=I_{[0,\tau_1^n)}$ in $D([0, T]; \{0, 1\})$, where τ_1^n is the instant of death for the first particle, it is sufficient to show

$$P(\tau_1^n \wedge T < \delta \text{ or } 0 < T-\tau_1^n \wedge T < \delta) \to 0 \quad \text{as } \delta \to 0 \tag{4.2}$$

Relation (4.2) follows from

$$P(\tau_1^n \wedge T < \delta) \leqq 1-e^{-\delta \|q\|_\infty}$$

and Markov property.

4.3. Identification

We characterize the limit points of $\{\mathscr{L}(\nu_n)\}$:

PROPOSITION 4.2. — *Under the condition* A *of Theorem* 1, *the support of any limit point* Q *of* $\{\mathscr{L}(\nu_n)\}$ *in* $M^1(M^1(S))$ *is contained in the set of measures* $\nu\in M^1(S)$ *such that* ν *is a solution of the nonlinear martingale problem* (2.3). ■

Proof. — The proof is based on the following idea.

Let us set

$$F(\nu) \doteq \int_S \nu(dx, d\xi)\, G(x, \xi) \times\left[f(x(t), \xi(t))-f(x(s), \xi(s))-\int_s^t L_{\nu^1(r)} f(x(r), \xi(r))\, dr\right] \tag{4.3}$$

where $G: S \to \mathbb{R}$ is continuous, $\mathscr{F}_s$-measurable and bounded by 1 and $f(.,\xi)\in C^2(\mathbb{R}^d)$ for $\xi=0, 1$.

Then ν is a solution of the martingale problem (2.3) if and only if $F(\nu)=0$ for every choice of f, G and $s, t\in[0, T]$.

The thesis is therefore equivalent to require that $F(\nu)=0$ Q. a. s. that is:

$$\int_{M^1(S)} Q(d\nu)\, F^2(\nu)=0. \tag{4.4}$$

The functionals F are continous Q. a. s., since for all $t>0$ $\nu((x, \xi)/\xi(t) \neq \xi(t^-)\}=0$ Q. a. s. and bounded.

$$|F(\nu)| \leqq 2\|f\|_\infty (1+T\|q\|_\infty)+\frac{T}{2}\|\Delta_x f\|_\infty$$

Moreover, since Q is the limit of some subsequence of $\{\mathscr{L}(\nu_n)\}$, we have:

$$\lim_{k \to \infty} E[F^2(\nu_{n_k})]=\int_{M^1(S)} Q(d\nu) F^2(\nu).$$

Then (4.4), and *a fortiori* the thesis, is proven once we show that:

$$\limsup_{n \to \infty} E[F^2(\nu_n)]=0. \tag{4.5}$$

We refer to the following lemma for its proof.

LEMMA 4.1. — *Under the condition* A *of theorem* 1, *for every* F *defined as in* (4.3)

$$\limsup_{n \to \infty} E[F^2(\nu_n)]=0. \quad \blacksquare$$

Proof. — If we denote by $\Phi_i(\underline{x}, \underline{\xi})$ the function $f(x_i, \xi_i)$, taking into account (3.2) and (3.3), we can write down explicitly $F(\nu_n)$

$$F(\nu_n)=\frac{1}{n}\sum_{i=1}^n G(x_i^{(n)}, \xi_i^{(n)})[\mathscr{M}_{\Phi_i}^n(t)-\mathscr{M}_{\Phi_i}^n(s)].$$

Then, setting $G_i=G(x_i^{(n)}, \xi_i^{(n)})$ we have

$$E[F^2(\nu_n)]=\frac{1}{n^2}E[\Sigma_{i,j} G_i G_j E[(\mathscr{M}_{\Phi_i}^n(t)-\mathscr{M}_{\Phi_i}^n(s))(\mathscr{M}_{\Phi_j}^n(t)-\mathscr{M}_{\Phi_j}^n(s)) \mid \mathscr{F}_s^n]]$$

$$=\frac{1}{n^2}E[\Sigma_{i,j} G_i G_j E[\langle \mathscr{M}_{\Phi_i}^n, \mathscr{M}_{\Phi_j}^n\rangle_t-\langle \mathscr{M}_{\Phi_i}^n, \mathscr{M}_{\Phi_j}^n\rangle_s \mid \mathscr{F}_s^n]]$$

$$=\frac{1}{n^2}E[\Sigma_{ij} G_i G_j(\langle \mathscr{M}_{\varphi_i}^n, \mathscr{M}_{\Phi_j}^n\rangle_t-\langle \mathscr{M}_{\Phi_i}^n, \mathscr{M}_{\Phi_j}^n\rangle_s)].$$

We can compute $\langle \mathscr{M}_{\Phi_i}^n, \mathscr{M}_{\Phi_j}^n\rangle$ through (3.4) by the polarization identity and we get that for $i \neq j$ the martingales are orthogonal while for $i=j$ we get, through (3.4):

$$\langle \mathscr{M}_{\Phi_i}^n\rangle_t=\int_0^t (\nabla_x f(x_i^{(n)}(r), \xi_i^{(n)}(r)))^2\, dr$$

$$+\int_0^t \xi_i^{(n)}(r)(q * \nu_n^1(r))(x_i^{(n)}(r))[f(x_i^{(n)}(r), 0)-f(x_i^{(n)}(r), \xi_i^{(n)}(r))]^2\, dr.$$

Therefore

$$\mathbb{E}[F^2(\nu_n)] \leqq \frac{1}{n} T(\|(\nabla_x f)^2\|_\infty + 4\|q\|_\infty \|f^2\|_\infty).$$

4.4. Uniqueness

PROPOSITION 4.3. — *The martingale problem* (2.3) *has a unique solution for every* $\nu_0 \in \mathscr{C}$ *where*

$\mathscr{C} = \{\nu_0 \in M^1(\mathbb{R}^d \times \{0, 1\})$ *such that for* $\xi = 0, 1$

$\nu_0(., \xi)$ *is absolutely continuous*

w.r.t. Lebesgue measure$\}$. ■

Proof. — The basic idea of the proof is based on the following facts:

(i) the class $\mathscr{C}$ is invariant for the martingale problem (2.3), that is if $\nu_0 \in \mathscr{C}$ then, for any solution ν of the problem (2.3), the distribution ν_t of ν at time t is still in the class $\mathscr{C}$.

Here the idea is to note that if $\nu \in M^1(S)$ is a solution of (2.3), then its marginal λ on $C([0, T]; \mathbb{R}^d)$, defined by $\lambda(B) = \nu(B \times D([0, T]; \{0, 1\}))$ for B belonging to $\mathscr{B}(C([0, T]; \mathbb{R}^d))$, is the Wiener measure with initial distribution

$$\lambda_0(A) = \nu_0(A, \{0, 1\}) = \nu_0(A, 0) + \nu_0(A, 1); \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

Then

$$\lambda_t(A) = \nu_t(A, 0) + \nu_t(A, 1), \qquad t \in [0, T]. \tag{4.6}$$

(ii) If ν and ν^1 are solutions of (2.3), with ν_0 in the class $\mathscr{C}$, then the distributions at time t, ν_t and ν_t^1, coincide (*see* Lemma 4.2).

By a slight modification of the usual argument (*see* [7]) one can then prove that properties (i) and (ii) imply uniqueness of the solution for the martingale problem (2.3) for any $\nu_0 \in \mathscr{C}$.

We observe that, since (4.6), uniqueness of ν_t is equivalent to uniqueness of ν_t^1 or equivalently of its density $u(t)$. Moreover, for any $g \in C^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} g(x) u(t, x)\, dx = \int_{\mathbb{R}^d} g(x) u(0, x)\, dx + \int_0^t ds \int_{\mathbb{R}^d} \left[\frac{1}{2}\Delta g(x) - (q * u(s))(x) g(x)\right] u(s, x)\, dx$$

since ν is a solution of the martingale problem (2.3).

This is the mild form of the Cauchy problem (2.4).

LEMMA 4.2. — *The Cauchy problem* (2.4) *has a unique solution* $u(t)$ *such that* $u(.) \in C([0, T]; L^1(\mathbb{R}^d))$, $u(t) \geqq 0$. ■

Proof. — Suppose that u' and u'' are both solutions of the Cauchy problem (2.4) and set

$$v=u'-u''$$

From the mild form of (2.4) we get that

$$\|v(t)\|_{L^1} \leqq \int_0^t \|q\|_\infty \|v(s)\|_{L^1} (\|u'(s)\|_{L^1} + \|u''(s)\|_{L^1})\, ds$$

$$\leqq \|q\|_\infty \sup_{0 \leqq s \leqq T} (\|u'(s)\|_{L^1} + \|u''(s)\|_{L^1}) \int_0^t \|v(s)\|_{L^1}\, ds$$

and by Gronwall inequality $\|v(t)\|_{L^1}=0$, $t\in[0, T]$.

5. FLUCTUATIONS

5.1. Preliminary results

In order to prove the results on the fluctuations (Th. 2) we consider the Y_n as processes with values in W_p^{-j} for suitable j and p.

More precisely, we prove that Y_n can be represented (Prop. 5.1) as a semimartingale. In Proposition 5.2 we derive the Meyer process associated to the martingale part of Y_n.

The definition of the Meyer process is recalled in the Appendix toghether with the Sobolev inequalities which we largely use in the following.

PROPOSITION 5.1. — *The process* Y_n *admits the following semimartingale representation in* W^{-j} *with* $j > \frac{d}{2}+2$.

$$Y_n(t)=Y_n(0)+\int_0^t \left[\frac{1}{2}\Delta Y_n(s)-(q * \mu_n(s))\, Y_n(s) - (q * Y_n(s))\, \mu(s)\right] ds + N^n(t) \quad (5.1)$$

where $N^{(n)}(t)$ *is a* P^n*-martingale with values in* W^{-j}. ■

Proof. — From the martingale formulation for $\nu_n(t)$ stated in (3.5), for functions $f(x, \xi)=\xi h(x)$ with $h\in W^j$ (remark that $W^j \subsetneq C^2$ since Sobolev embedding theorem [1]), we have that:

$$(h, \nu_n^1(t))=(h, \nu_n^1(0))+\int_0^t \left(\frac{1}{2}\Delta h-(q * \nu_n^1(s))\, h,\ \nu_n^1(s)\right) ds + M_h^n(t). \quad (5.2)$$

With little abuse of notation we wrote M_h^n instead of M_f^n.

For each $h \in W^j$, M_h^n is a real P^n-martingale.

We want to prove that the application $h \mapsto M_h^n$ is a linear continuous functional on W^j, so there exists for each t an element $M^n(t) \in W^{-j}$ such that

$$(h, M^n(t)) = M_h^n(t).$$

The linearity is straight forward and for the boundedness we have that:

$$\begin{aligned} |M_h^n(t)| \leqq |(h, v_n^1(t))| + |(h, v_n^1(0))| \\ + \int_0^t \left[\frac{1}{2} |(\Delta h, v_n^1(s))| + |(h, (q * v_n^1(s)) v_n^1(s))| \right] ds \\ \leqq \| h \|_j (\| v_n^1(t) \|_{-j} + \| v_n^1(0) \|_{-j}) \\ + \frac{1}{2} \| \Delta h \|_{j-2} \int_0^t \| v_n^1(s) \|_{-(j-2)} ds \\ + \| h \|_j \int_0^t \| (q * v_n^1(s)) v_n^1(s) \|_{-j} ds. \end{aligned}$$

Taking into account that from Sobolev inequalities:

$$\| \mu \|_{-j} \leqq b_1 \| \mu \|_{-(j-2)} \leqq b_2 \| \mu \|_{BV} \quad \text{for } j > \frac{d}{2} + 2 \tag{5.3}$$

we have that for some constants C_1 and C_2

$$|M_h^n(t)| \leqq (C_1 + C_2 T) \| h \|_j.$$

Therefore for $v_n^1(t)$ the following representation holds in W^{-j}

$$v_n^1(t) = v_n^1(0) + \int_0^t \left[\frac{1}{2} \Delta v_n^1(s) - (q * v_n^1(s)) v_n^1(s) \right] ds + M^n(t) \tag{5.4}$$

where the martingale $M^n(t) \in W^{-j}$.

Moreover we have that:

$$v^1(t) = v^1(0) + \int_0^t \left[\frac{1}{2} \Delta v^1(s) - (q * v^1(s)) v^1(s) \right] ds. \tag{5.5}$$

Then the representation (5.1) for $Y_n = \sqrt{n}(v_n^1 - v^1)$ follows defining $N^n = \sqrt{n}\, M^n$.

PROPOSITION 5.2. — *If* $j > \max\left(\frac{d}{2} + 2, d+1\right)$ *and* $E[| v_n^1(0) |_{2p}^*]$ *is finite,* $p > d$, *then* $\ll N^n \gg_t$, *defined as*

$$(h, \ll N^n \gg_t g) = \int_0^t (\nabla h \cdot \nabla g + q * v_n^1(s) hg, v_n^1(s)) ds \tag{5.6}$$

for h, g in W^j, *is the Meyer process associated to* $N^n(t)$ *in* W^{-j}. ■

Proof. – From Proposition 5.1 and from (3.6) by using polarization identity we have that:

$$\langle (h, N^n), (g, N^n) \rangle_t = \int_0^t (\nabla h . \nabla g + q * \nu_n^1(s)\, hg,\ \nu_n^1(s))\, ds.$$

To get the thesis we have to show that if j is suitably chosen the operator $\ll N^n \gg_t$ defined as

$$(h, \ll N^n \gg_t g) = \langle (h, N^n), (g, N^n) \rangle_t \qquad h, g \in W^j \tag{5.7}$$

is nuclear.

To this end we show that $\ll N^n \gg_t : W_p^l \to W_p^{-l}$ is a linear, bounded operator, for $p > d$ and $l > \frac{d}{2} + 1$; indeed the Sobolev inequalities assure that W^{l+q} is Hilbert Schmidt embedded in W_p^l if $q > \frac{d}{2}$ and $p > d$ (Th. A.3), therefore setting $j = l + q$ we have:

$$\begin{aligned} &|(h, \ll N^n \gg_t g)| \\ &\leq \int_0^t (|\nabla h \cdot \nabla g|_{2p} |\nu_n^1(s)|_{2p}^* + |hg|_{2p} |(q * \nu_n^1(s)\, \nu_n^1(s)|_{2p}^*)\, ds \\ &\leq (|(|\nabla h|)|_p |(|\nabla g|)_p + |h|_p |g|_p \|q\|_\infty) \int_0^t |\nu_n^1(s)|_{2p}^* ds \\ &\leq k_{l,p} \|h\|_{l,p} \|g\|_{l,p} (1 + \|q\|_\infty) \int_0^t |\nu_n^1(s)|_{2p}^* ds \end{aligned} \tag{5.8}$$

Therefore to estimate $|(h, \ll N^n \gg_t g)|$ is sufficient to give an estimate for $|\nu_n^1(s)|_{2p}^*$

$$E[\sup_{0 \leq s \leq t} |\nu_n^1(s)|_{2p}^*] \leq E[\sup_{0 \leq s \leq t} |\bar{\nu}_n^1(s)|_{2p}^*]. \tag{5.9}$$

Here $\bar{\nu}_n^1(s) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}(s)}$, i.e. the empirical distribution of n independent brownian motions.

$$\begin{aligned} E[\sup_{0 \leq s \leq t} |\bar{\nu}_n^1(s)|_{2p}^*] &= E\left[\sup_{0 \leq s \leq t} \frac{1}{n} \sum_{i=1}^n (1 + |x_i^{(n)}(s)|^2)^p\right] \\ &\leq E[\sup_{0 \leq s \leq t} (1 + |x_1^{(n)}(s)|^2)^p] \leq C_p E[(1 + |x_1^{(n)}(t)|^2)^p] \\ &\leq C_{p,t} E[(1 + |x_1^{(n)}(0)|^2)^p] = C_{p,t} E[|\nu_n^1(0)|_{2p}^*] \end{aligned} \tag{5.10}$$

where $C_p C_{p,t}$ are constants.

The last inequality holds because Doob's inequality for submartingale.

Therefore, since (5.8), (5.9), (5.10):

$$\mathrm{E}[|(h, \ll \mathrm{N}^n \gg_t g)|] \leqq \mathrm{K}_\mathrm{T} \| h \|_{l,p} \| g \|_{l,p} \mathrm{E}[| \nu_n^1(0) |_{2p}^*] \tag{5.11}$$

where K_T is a constant.

We stress only the dependence on T even though it depends also on $l, p, \| q \|_\infty$.

Therefore from the hypothesis on the initial conditions we get the thesis.

Remark. — The results stated in Proposition 5.1 and 5.2 hold in $\mathrm{W}_{p'}^{-j}$ for $p' > 0$ provided we make stronger assumptions on the initial conditions, respectively: for Proposition 5.1 $\mathrm{E}[| \nu_n^1(0) |_{p'}^*]$ is finite and for Proposition 5.2 $\mathrm{E}[| \nu_n^1(0) |_{2(p+p')}^*]$ is finite.

The proof goes in the same way, taking into account the Sobolev inequalities in Theorem A.3.

5.2. Convergence of martingales

In this section we prove the convergence of the martingales N^n as processes in $\mathscr{D}([0,\mathrm{T}], \mathrm{W}^{-j})$ and we characterize the limit points. The tightness of N^n, the martingale part of Y_n, is given in terms of the associated Meyer process $\ll \mathrm{N}^n \gg$. Sufficient conditions needed to prove the tightness of N^n are recalled in the Appendix (Th. A.2.).

PROPOSITION 5.3. — *If* $\mathrm{E}[| \nu_n^1(0) |_{2p}^*]$, $p > d$, *is bounded uniformly in* n, *then the sequence* $\{\mathrm{N}^n\}$ *is tight in* $\mathrm{D}([0, \mathrm{T}], \mathrm{W}^{-j})$ *with*

$$j > \max\left(d+1, \frac{d}{2}+2\right). \quad \blacksquare$$

PROPOSITION 5.4. — *Under the hypotheses of Theorem* 1 *and moreover if* $\mathrm{E}[| \nu_n^1(0) |_{2p}^*]$, $p > d$, *is uniformly bounded we have that the sequence of the martingale* $\{\mathrm{N}^n\}$ *converges weakly in* $\mathrm{D}([0, \mathrm{T}]; \mathrm{W}^{-j})$ *to the continuous, gaussian process with independent increments* N, *with covariance given by*

$$\int_0^t (\nabla h \cdot \nabla g + (q * \nu^1(s)) hg, \nu^1(s)) \, ds \tag{5.12}$$

where ν^1 *is the measure defined in* (ii) *of Theorem* 1. ■

Proof of Proposition 5.3. — We have only to verify that the conditions (A.3) and (A.4) of Theorem A.2 hold.

We note that we can always chose on orthonormal basis $\{h_k\}$ in W^{l+q}, l, q as in Proposition 5.2 such that $\sum_{k=1}^{\infty} \| h_k \|_{l,p}^2 < \infty$ because W^{l+q} is Hilbert-Schmidt embedded in W_p^l.

Therefore from the (5.11) we have that

$$\sum_{k=m}^{\infty} \mathrm{E}[(h_k, \ll \mathrm{N}^n \gg_t h_k)] \leqq \mathrm{K_T} \sum_{k=m}^{\infty} \| h_k \|_{l,p}^2 \, \mathrm{E}[| \nu_n^1(0) |_{2p}^*] \tag{5.13}$$

This implies both conditions in (A.3), since the hypothesis on the initial conditions.

For the condition (A.4), from Cheybeshev inequality, it is enough to show that:

$$\sup_{0 \leqq \theta \leqq \delta} \mathrm{E}[| \operatorname{tr} \ll \mathrm{N}^n \gg_{\tau_n+\theta} - \operatorname{tr} \ll \mathrm{N}^n \gg_{\tau_n} |] \to 0$$

as $\delta \to 0$, uniformly in n.

From the explicit form of $\ll \mathrm{N}^n \gg$ (5.6) we have that for the same basis $\{h_k\}$, using (5.8) and (5.10)

$$\begin{aligned}
&\mathrm{E}[| \operatorname{tr} \ll \mathrm{N}^n \gg_{\tau_n+\theta} - \operatorname{tr} \ll \mathrm{N}^n \gg_{\tau_n} |] \\
&\qquad \leqq \sum_{k=1}^{\infty} \mathrm{E}[| (h_k, \ll \mathrm{N}^n \gg_{\tau_n+\theta} h_k) - (h_k, \ll \mathrm{N}^n \gg_{\tau_n} h_k) |] \\
&\qquad \leqq k_{l,p}(1 + \| q \|_\infty) \sum_{k=1}^{\infty} \| h_k \|_{l,p}^2 \, \mathrm{E}\left[\int_{\tau_n}^{\tau_n+\theta} | \nu_n^1(s) |_{2p}^* \, ds \right] \\
&\qquad\qquad \leqq \mathrm{K} \delta \, \mathrm{E}[\sup_{0 \leqq s \leqq \mathrm{T}} | \nu_n^1(s) |_{2p}^*] \leqq \mathrm{K}' \delta \, \mathrm{E}[| \nu_n^1(0) |_{2p}^*].
\end{aligned} \tag{5.14}$$

From the uniform boundedness hypothesis on the initial condition, (A.4) is proven.

Proof of Proposition 5.4. – We show that each limit point Q^∞ of $\mathrm{Q}^n = \mathscr{L}(\mathrm{N}^n)$ is solution of the following martingale problem on $\mathrm{C}([0, \mathrm{T}]; \mathrm{W}^{-j})$: setting $\mathscr{N}$ the canonical process in $\mathrm{C}([0, \mathrm{T}]; \mathrm{W}^{-j})$, for each $g \in \mathrm{W}^j$:

(i) the real process $(g, \mathscr{N}(t))$ is a Q^∞-local martingale;

(ii) $(g, \mathscr{N}(t))^2 - \Phi_g^\nu(t)$ is a Q^∞-local martingale where

$$\Phi_g^\nu(t) \doteq \int_0^t ((\nabla g)^2 + (q * \nu^1(s)) g^2, \nu^1(s)) \, ds.$$

The operator $\ll \mathrm{N} \gg_t$ is deterministic and then from the characterization [7] of continuous Gaussian processes with independent increments as the unique solution of (i) and (ii) the thesis follows.

The fact that

$$\| \mathrm{N}^n(t) - \mathrm{N}^n(t^-) \|_{-j} = \sqrt{n} \| \nu_n^1(t) - \nu_n^1(t^-) \|_{\mathrm{BV}} \leqq \frac{1}{\sqrt{n}}$$

implies that the limit Q^∞ has support on $\mathrm{C}([0, \mathrm{T}]; \mathrm{W}^{-j})$.

In order to verify point (i) and (ii) we apply Ito's formula to the real martingale $(g, \mathrm{N}^n(t))$: for $\varphi \in \mathrm{C}^2$

$$\varphi((g, \mathrm{N}^n(t))) = \frac{1}{2}\int_0^t \varphi''((g, \mathrm{N}^n(s^-)))((\nabla g)^2 + q * \nu_n^1(s) g^2, \nu_n^s(s))\, ds$$
$$+ \sum_{s \leqq t} [\varphi((g, \mathrm{N}^n(s)) - \varphi((g, \mathrm{N}^n(s^-))) - \varphi'((g, \mathrm{N}^n(s^-)))(g, \mathrm{N}^n(s) - \mathrm{N}^n(s^-))]$$
$$- \frac{1}{2}\sum_{s \leqq t} \varphi''((g, \mathrm{N}^n(s^-)))(g, \mathrm{N}^n(s) - \mathrm{N}^n(s^-))^2 + \mathcal{M}_t^{n, \varphi, g} \quad (5.15)$$

with $\mathcal{M}_t^{n, \varphi, g}$ a martingale.

We need also to consider the following stopping times on S^n and $\mathrm{D}([0, \mathrm{T}]; \mathrm{W}^{-j})$:

$$\tau_\alpha^n = \inf\{t : \|\mathrm{N}^n(t)\|_{-j} > \alpha\}$$
$$\tau_\alpha = \inf\{t : \|\mathcal{N}(t)\|_{-j} > \alpha\} \qquad (\text{for } \alpha > 0).$$

For the proof of point (i) we consider $\varphi \in \mathrm{C}^2$ such that:

$$\varphi_\alpha(x) = x \qquad \text{if} \quad |x| \leqq \|g\|_j(\alpha + 1).$$

The Ito's formula (5.15) for such a choice of φ_α gives

$$(g, \mathrm{N}^n(t)) = \mathcal{M}_t^{n, \varphi_\alpha, g}, \qquad \forall t < \tau_\alpha^n. \quad (5.16)$$

Consider $\mathrm{G} \in \mathcal{G}_{\tau_\alpha \wedge s}$ ($\mathcal{G}_s = \sigma\{\mathcal{N}(r), r \leqq s\}$) and set $\mathrm{G}_n = \{\omega : \mathrm{N}^n(\omega) \in \mathrm{G}\}$

$$\mathrm{E}^{\mathrm{Q}^\infty}[\mathrm{I}_\mathrm{G}\{(g, \mathcal{N}(t \wedge \tau_\alpha)) - (g, \mathcal{N}(s \wedge \tau_\alpha))\}]$$
$$= \mathrm{E}^{\mathrm{Q}^\infty}[\mathrm{I}_\mathrm{G}\{\varphi_\alpha((g, \mathcal{N}(t \wedge \tau_\alpha))) - \varphi_\alpha((g, \mathcal{N}(s \wedge \tau_\alpha)))\}]$$
$$= \lim_{n \to \infty} \mathrm{E}^{\mathrm{Q}^n}[\mathrm{I}_\mathrm{G}\{\varphi_\alpha((g, \mathcal{N}(t \wedge \tau_\alpha))) - \varphi_\alpha((g, \mathcal{N}(s \wedge \tau_\alpha)))\}]$$
$$= \lim_{n \to \infty} \mathrm{E}[\mathrm{I}_{\mathrm{G}_n}\{(g, \mathrm{N}^n(t \wedge \tau_\alpha^n)) - (g, \mathrm{N}^n(s \wedge \tau_\alpha^n))\}]$$

since (5.16).

To show that $(g, \eta(t))$ is a local martingale it is enough to notice that:

$$\lim_{\alpha \to \infty} \mathrm{Q}^\infty(\tau_\alpha \neq +\infty) = \lim_{\alpha \to \infty} \mathrm{Q}^\infty(\tau_\alpha \leqq \mathrm{T})$$
$$\leqq \lim_{\alpha \to \infty} \sup_n \mathrm{Q}^n(\tau_\alpha \leqq \mathrm{T})$$
$$= \lim_{\alpha \to \infty} \sup_n \mathrm{Q}^n(\sup_{0 \leqq t \leqq \mathrm{T}} \|\mathcal{N}(t)\|_{-j} \geqq d)$$
$$\leqq \lim_{\alpha \to \infty} \sup_n \frac{1}{\alpha^2}\mathrm{E}[\sup_{0 \leqq t \leqq \mathrm{T}} \|\mathrm{N}^n(t)\|_{-j}^2]$$
$$\leqq \lim_{\alpha \to \infty} \sup_n \frac{4}{\alpha^2}\mathrm{E}[\|\mathrm{N}^n(\mathrm{T})\|_{-j}^2] = \lim_{\alpha \to \infty} \sup_n \frac{4}{\alpha^2}\mathrm{E}[\mathrm{tr} \ll \mathrm{N}^n \gg_\mathrm{T}] = 0.$$

The last equality holds since (5.13) for $m = 1$ and the hypothesis on the initial conditions.

Similarly, for proving (ii), we use the fact that

$$\lim_{n \to \infty} E[I_{G_n}\{(g, N^n(t \wedge \tau_\alpha^n))^2 - (g, N^n(s \wedge \tau_\alpha^n))^2 - [\Phi_g(t \wedge \tau_\alpha^n) - \Phi_g(s \wedge \tau_\alpha^n)]\}] = 0 \quad (5.17)$$

One can get (5.17) applying Ito's formula (5.15) to $\varphi = \psi_\alpha$, $\psi_\alpha \in C^2$

$$\psi_\alpha(x) = x^2 \qquad \text{if} \quad |x| \leqq \|g\|_j(1+\alpha)$$

and showing that:

$$\lim_{n \to \infty} E[|(g, \ll N^n \gg_t g) - \Phi_g(t)|] = 0. \quad (5.18)$$

The relation (5.18) follows immediately from the weak convergence of ν_n to ν and from the fact that $F: M^1(S) \to \mathbb{R}$,

$$\lambda \mapsto \int_0^t ((\nabla g)^2 + q * \lambda_s(., 1) g^2, \lambda_s(., 1))\, ds$$

is a continuous bounded functional and that

$$F(\nu_n) = (g, \ll N^n \gg_t g) \qquad \text{and} \qquad F(\nu) = \Phi_g(t).$$

5.3. Convergence of fluctuations

So far we have shown that the martingale part of the fluctuations converges in W^{-j} to a continuous gaussian martingale with independent increments.

This and the estimates given in the following Proposition 5.5, enable us to prove easily Theorem 2.

PROPOSITION 5.5. — *For the sequence of processes $\{Y_n\}$ the following estimate holds:*

$$\sup_n \sup_{0 \leqq t \leqq T} E[\|Y_n(t)\|_{-j}^2] < C; \qquad j > \max\left(d+1, \frac{d}{2}+2\right)$$

provided that the sequences $\{E|\nu_n^1(0)|_{2p}^*\}$, $p > d$ and $\{E[\|Y_n(0)\|_{-j}]\}$ *are uniformly bounded.* ■

Proof. — From the semimartingale representation of ψ_n of Proposition 5.1, and from Ito's formula applied to the function

$f(y)=(y, y)$ we have that:

$$
\begin{aligned}
(Y_n(t), Y_n(t)) &= (Y_n(0), Y_n(0)) \\
&+ \int_0^t \left\{ \left(Y_n(s), \frac{\Delta}{2} Y_n(s) \right) + \left(\frac{\Delta}{2} Y_n(s), Y_n(s) \right) \right\} ds \\
&- \int_0^t \{ (Y_n(s), q * v_n^1(s) Y_n(s)) + (q * v_n^1(s) Y_n(s), Y_n(s)) \} ds \\
&- \int_0^t \{ (Y_n(s), q * Y_n(s) v^1(s)) + (q * Y_n(s) v^1(s), Y_n(s)) \} ds \\
&+ [N^n]_t + \int_0^t \{ (Y_n(s), dN^n(s)) + (dN^n(s), Y_n(s)) \} \quad (5.19)
\end{aligned}
$$

Here $[N^n]$ is the quadratic variation of N^n.

Taking the expectation of (5.19) and using the monotonicity of the Laplacian in W^{-j} we have:

$$
\begin{aligned}
E[\| Y_n(t) \|_{-j}^2] &\leqq E[\| Y_n(0) \|_{-j}^2] \\
&+ 2 \int_0^t E[|(Y_n(s), q * v_n^1(s) Y_n(s))|] \, ds \\
&+ 2 \int_0^t E[|(Y_n(s), q * Y_n(s) v^1(s))|] \, ds + E[\operatorname{tr} \ll N^n \gg_t]. \quad (5.20)
\end{aligned}
$$

But

$$
E[|(Y_n(s), q * v_n^1(s) Y_n(s))|] \leqq C_1 E[\| Y_n(s) \|_{-j}^2] \quad (5.21)
$$

and

$$
E[|(Y_n(s), q * Y_n(s) v^1(s))|] \leqq C_2 E[\| Y_n(s) \|_{-j}^2] \quad (5.22)
$$

and

$$
E[\operatorname{tr} \ll N^n \gg_t] \leqq C_3 E[| v_n^1(0) |_{2p}] \leqq C_4
$$

because of (5.13) with $m=1$ and the hypothesis on the initial conditions.

Therefore (5.20) is estimated as:

$$
E[\| Y_n(t) \|_{-j}^2] \leqq E[\| Y_n(0) \|_{-j}^2] + (C_1 + C_2) \int_0^t E[\| Y_n(s) \|_{-j}^2] \, ds + C_4.
$$

The thesis follows from Gronwall inequality.

Remark. — The result of Proposition 5.5 holds in $W_{p'}^{-j}$, $p' > 0$ provided that $E[| v_n^1(0) |_{2(p+p')}^*]$ and $E[\| Y_n(0) \|_{-j, p'}^2]$ are uniformly bounded.

Proof of Theorem 2. — In order to prove the convergence in $D([0, T]; \mathcal{S}')$ we show that:

(A) the processes Y_n are tight in $D([0, T]; \mathcal{S}')$.

(B) any limit point Y is solution of the following stochastic equation

$$(f, Y(t)) = (f, Y(0)) + \int_0^t \left(\frac{1}{2}\Delta f + q * v^1(s) f, Y(s)\right) ds$$
$$- \int_0^t (f, q * Y(s) v^1(s)) ds + (f, N(t)), \qquad f \in \mathcal{S} \quad (5.23)$$

Here $N(t)$ is the continuous gaussian process with independent increments defined in Proposition 5.4.

(C) the uniqueness of the solution of (5.23).

To prove conditions (A) we use Mitoma result [18] that is:

if the sequence of processes $z_n(t) \doteq (f, Y_n(t))$ is tight in $D([0, T]; \mathbb{R})$ for each $f \in \mathcal{S}$ then $\{Y_n\}$ is tight in $D([0, T]; \mathcal{S}')$.

For the tightness of z_n in $D([0, T]; \mathbb{R})$ we use Theorem A.1 and Proposition 5.5:

$$E[|z_n(t)|^2] \leqq \|f\|_j^2 E[\|Y_n(t)\|_{-j}^2] \leqq C \|f\|_j^2$$

and

$$E[|z_n(\tau_n + \theta) - z_n(\tau_n)|]$$
$$\leqq E\left[\int_{\tau_n}^{\tau_n+\delta} \left|\left(\frac{1}{2}\Delta f - q * v_n^1(s) f, Y_n(s)\right) - (f, q * Y_n(s) v^1(s))\right| ds\right]$$
$$+ E[(f, N^n(\tau_n + \theta)) - (f, N^n(\tau_n))] \quad (5.24)$$

It is easy to see that:

$$\left|\left(\frac{1}{2}\Delta f - q * v_n^1(s) f, Y_n(s)\right) - (f, (q * Y_n(s)) v^1(s))\right|$$
$$\leqq C_1 (\|\Delta f\|_j + \|f\|_j) \|Y_n(s)\|_{-j} \leqq c_2 \|Y_n(s)\|_{-j}.$$

Therefore (5.24) is less or equal to:

$$C_2 \delta^{1/2} E\left[\left(\int_{\tau_n}^{\tau_n+\delta} \|Y_n(s)\|_{-j}^2 ds\right)^{1/2}\right]$$
$$+ (E[\{(f, N^n(\tau_n + \theta)) - (f, N^n(\tau_n))\}^2])^{1/2}$$
$$\leqq C_3 \delta^{1/2} + (E[(f_1 [\ll N^n \gg_{\tau_n+\theta} - \ll N^n \gg_{\tau_n}] f)])^{1/2}.$$

Using (5.8), (5.10) and the hypothesis on the initial conditions we have that

$$E[|z_n(\tau_n+\theta)-z_n(\tau_n)|] \leqq C_3\,\delta^{1/2}+C_4\left(E\left[\int_{\tau_n}^{\tau_n+\delta}|v_n^1(s)|_{2\,p}^{*}\,ds\right]\right)^{1/2} \leqq k\,\delta^{1/2}.$$

To identify the limit we add and subtract to (5.1)

$$\int_0^t (q * v^1(s))\, Y_n(s)\, ds.$$

Therefore we have that:

$$\begin{aligned}(f, Y_n(t)) = (f, Y_n(0)) &+ \int_0^t \left(\frac{1}{2}\Delta f - q * v^1(s) f, Y_n(s)\right) ds \\ &- \int_0^t (f, q * Y_n(s) v^1(s))\, ds - \frac{1}{\sqrt{n}} \int_0^t (f, (q * Y_n(s))\, Y_n(s))\, ds \\ &+ (f, N^n(t)).\end{aligned} \quad (5.25)$$

The point (B) follows from the continuity of (5.25) and the boundedness of the non linear term (Prop. 5.5)

$$|(f, q * Y_n(s)\, Y_n(s))| \leqq \|f\|_j \|q\|_\infty \|Y_n(s)\|_{-j}^2.$$

The condition (C) is implied by Proposition 5.4 and by the linearity of (5.23).

For the convergence of $\{Y_n\}$ in $D([0, T], W^{-j})$ we have only to show tightness. The identification of the limit point and the uniqueness follows as previously.

The (i) of (A.1) consists of

$$\forall t \in [0, T], \quad \sup_n E[\|Y_n(t)\|_{-j}^2] < \infty$$

and it is verified for $j > \max\left(d+1, \frac{d}{2}+2\right)$ (Prop. 5.5).

For the (ii) of (A.1) we note that if $\{h_k\}$ is an orthonormal basis in W^j

$$P\left(\sum_{k=m}^{\infty} (h_n, Y_n(t))^2 > \rho\right) \leqq \frac{1}{\rho} E\left[\sum_{k=m}^{\infty} |(h_k, Y_n(t))|^2\right]$$

$$\leqq \frac{1}{\rho} E[\|Y_n(t)\|_{-l,p'}^2]\left(\sum_{k=m}^{\infty} \|h_k\|_{l,p'}^2\right).$$

If we choose $p' > d$, $j = l+q$, $q > \frac{d}{2}$, $l \geqq 0$ then the embedding $W^j \subsetneq W_{p'}^l$ is Hilbert-Schmidt and therefore one can choose $\{h_k\}$ such that

$\sum_{k=1}^{\infty} \| h_k \|_{l,p'}^2 < \infty$. Then the uniform boundedness of $E[\| Y_n(t) \|_{-l,p'}^2]$ (*see* Remark of Proposition 5.5) proves (ii).

For the Aldous condition (A.2) we have that, for $\theta < \delta$

$$E[\| Y_n(\tau_n+\theta) - Y_n(\tau_n) \|_{-j}] = E[\sup_{\| f \|_j = 1} |(f, Y_n(\tau_n+\theta) - Y_n(\tau_n))|]$$

$$\leqq E\left[\sup_{\| f \|_j = 1} \int_{\tau_n}^{\tau_n+\theta} \left\{ \left| \left(\frac{1}{2}\Delta f, Y_n(s)\right) \right| + |(f, q * v_n^1(s) Y_n(s))| + |(f, q * Y_n(s) v^1(s))| \right\} ds \right] + E[\| N^n(\tau_n+\theta) - N^n(\tau_n) \|_{-j}]$$

$$\leqq E\left[\sup_{\| f \|_j = 1} \int_{\tau_n}^{\tau_n+\theta} \left\{ \frac{1}{2} \| \Delta f \|_{-(j-2)} \cdot \| Y_n(s) \|_{-(j-2)} + \| F \|_j \| q \|_\infty \| Y_n(s) \|_{-j} + \| f \|_j C_1 \| Y_n(s) \|_{-j} \right\} ds \right] + (E[\| N^n(\tau_n+\theta) - N^n(\tau_n) \|_{-j}^2])^{1/2}$$

$$\leqq E\left[\left(C\delta \int_0^T \| Y_n(s) \|_{-(j-2)}^2 ds \right)^{1/2}\right] + (E[\operatorname{tr} \ll N^n \gg_{\tau_n+\theta} - \operatorname{tr} \ll N^n \gg_{\tau_n}])^{1/2}$$

$$\leqq \left(C\delta E\left[\int_0^T \| Y_n(s) \|_{-(j-2)}^2 ds \right] \right)^{1/2} + (k\delta E[| v_n^1(0) |_{2p}^*])^{1/2}.$$

The last inequality follows from (5.14).

Then since $E\left[\int_0^T \| Y_n(s) \|_{-(j-2)}^2\right]$ is bounded for

$$j > \max\left(d+1, \frac{d}{2}+2\right)+2$$

(Prop. 5.5), condition (A.2) holds.

APPENDIX

In order to make the paper selfcontained we formulate the theorems we use in proving tightness for processes in $D([0, T]; \mathscr{H})$, where $\mathscr{H}$ is a separable Hilbert space with scalar product $(.,.)$ and norm $\| . \|$.

THEOREM A 1 ([9], [17]). — *A sequence of processes* $\{z^{(n)}\}$ *in* $D([0, T]; \mathscr{H})$ *is tight if the following conditions are satisfied:*

$$\text{(A.1)} \quad \begin{cases} \text{(i)} & \forall t, \quad \sup_n E[\| z^{(n)}(t) \|^2] < \infty \\ \text{(ii)} & \lim_{m \to \infty} \limsup_{n \to \infty} P\left(\sum_{k=m}^{\infty} (z^{(n)}(t), h_k)^2 > \rho \right) = 0, \qquad \forall \rho > 0 \end{cases}$$

where $\{h_k\}$ *is an orthonormal basis.*

(A.2) *For each* $\eta > 0$ *and for each family of stopping times* $\{\tau_n\}$, $\tau_n \leqq T$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{0 \leqq \theta \leqq \delta} P(\| z^{(n)}(\tau_n) - z^{(n)}(\tau_n + \theta) \| \geqq \eta) = 0. \quad \blacksquare$$

If $z^{(n)} = N^n$ is a martingale process one can use the Rebolledo idea that the tightness of $\{N^n\}$ is implied by the tightness of the associated Meyer process $\ll N^n \gg$.

In our case $\mathscr{H} = W^{-j}$, so the Meyer process $\ll N^n \gg$ is defined as the unique predictable process with values in the space of nuclear operators from W^j to W^{-j}, with the property that

$$(h, N^n(t))(g, N^n(t)) - (h, N^n(0))(g, N^n(0)) - (h, \ll N^n \gg_t g)$$

is a local martingale for any $h, g \in W^j$.

Since $\ll N^n \gg_t$ is a nuclear operator we can define the operator $\ll N^n \gg_t^{1/2}$ which is Hilbert-Schmidt; we remind that in this case $\mathrm{tr}(\ll N^n \gg_t) = \sum_{k=1}^{\infty} (h_k, \ll N^n \gg_t h_k)$.

THEOREM A.2 [17]. — *If the sequence of martingales* $\{N^n\}$ *with values in* $\mathscr{H}$ *verifies*

(A.3) *There exists on orthonormal basis* $\{h_k\}$ *in* $\mathscr{H}$ *such that* $\forall \eta > 0$, $\forall t \leqq T$

$$\text{(i)} \quad \limsup_{m \to \infty} \sup_n P\left(\sum_{k=m}^{\infty} (h_k, \ll N^n \gg_t h_k) > \eta \right) = 0$$

$$\text{(ii)} \quad \lim_{\rho \to \infty} \sup_n P\left(\sum_{k=1}^{\infty} (h_k, \ll N^n \gg_t h_k) > \rho \right) = 0$$

(A.4) *For* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each sequence of stopping times* $\{\tau_n\}$, $\tau_n \leqq T$

$$\limsup_{n \to \infty} \sup_{0 \leqq \theta \leqq \delta} P(| \mathrm{tr} \ll N^n \gg_{\tau_n + \theta} - \mathrm{tr} \ll N^n \gg_{\tau_n} | > \varepsilon) = 0$$

then $\{\mathscr{L}(N^n)\}$ *is tight in* $D([0, T]; \mathscr{H})$. $\blacksquare$

We state the results of Sobolev inequalities in the following theorem.

THEOREM A.3 [1]. — *If* $l > \frac{d}{2}$, $r \geqq 0$ *then*

$$W_r^l \hookrightarrow C_r \qquad \textit{and} \qquad |f|_r \leqq K_{l,r} \| f \|_{l,r}$$

therefore $M_r \hookrightarrow W_r^{-l}$.

If $l \geqq 0$, $q > \frac{d}{2}$, $r \geqq 0$, $s > d$ *then the embedding* $W_r^{l+q} \hookrightarrow W_{r+s}^l$ *is Hilbert-Schmidt, therefore also the embedding* $W_{r+s}^{-l} \hookrightarrow W_r^{-(l+q)}$ *is Hilbert-Schmidt.* ■

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