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Hydrodynamical limit for the asymmetric zero-range process

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ABSTRACT. — In [3] we proved a strong law of large numbers for the rescaled asymmetric nearest neighbor simple exclusion process. The method is based on the properties of the limiting first order nonlinear partial differential equation.

In this article we show that the same method applies to the asymmetric zero-range process, the limiting equation being of the same type. An improvement of [3] enables us to remove the nearest neighbor assumption.

We obtain the local equilibrium as an easy consequence.

Key words : Zero-range process, hydrodynamical limit, first order quasilinear P.D.E., entropy condition.

RÉSUMÉ. — Dans [3] nous avons démontré une loi des grands nombres pour le processus d'exclusion simple asymétrique, dans le cas où les particules ne sautent qu'aux sites voisins. La méthode est basée sur les propriétés de l'équation aux dérivées partielles limite qui est non linéaire du premier ordre.

Dans cet article nous montrons que la même méthode s'applique au processus de zéro-range asymétrique, l'équation limite étant du même type. Une amélioration de [3] nous permet de traiter le cas de transitions quelconques.

Nous en déduisons facilement l'équilibre local.

Classification A.M.S. : Primary 6 0 F; Secondary 3 5 F.

INTRODUCTION

In [3] we proved a strong law of large numbers and we deduced the local equilibrium for the one-dimensional asymmetric simple exclusion process by using the monotonicity and the existence of a family of equilibrium measures for this process.

In this article we show that the same method applies to a larger class of processes having the same properties.

In particular we are interested in the asymmetric one-dimensional zero-range process which preserves the stochastic order (monotonicity) and has a family $\{\nu^\rho\}$ of equilibrium measures indexed by a continuous parameter ranging from 0 to $+\infty$. We will suppose that the transition probabilities are translation invariant and have a first moment different from zero.

We consider this process starting from a product measure $\nu^{a,b}$ corresponding to two half-spaces in equilibrium at different levels a and b, and we study its asymptotic behavior.

After a suitable space and time rescaling, the distribution of particles at time t defines a random measure on the real line, each particle contributing an equal mass. We show that this measure converges weakly almost surely to a deterministic measure which has a density called the density profile (section II).

In section III we show that this density is a weak solution of a nonlinear hyperbolic P.D.E.; without the nearest neighbor assumption, as in [3], we cannot longer apply the interface argument but a coupling procedure and the particular initial distribution enable us to obtain the result. The identification of the density profile as the unique weak solution satisfying the entropy condition is again a consequence of the monotonicity of the process.

In section IV we deduce from the preceding result the local equilibrium at each point of continuity of the density profile.

This problem is solved by E. Andjel and C. Kipnis in [2] when the particles move in only one direction to the nearest neighbor, the exponential time between two jumps at a given site not depending on the number of particles at that site.

The symmetric case is treated in [5], the limiting equation being parabolic, the method is completely different.

I. THE ZERO-RANGE PROCESS

I.1. Generator, invariant measures and initial distribution

Consider on the space $E=\mathbb{N}^{\mathbb{Z}}$ with elements (also called configurations) $\eta=\{\eta(k), k\in\mathbb{Z}\}$ the markovian evolution which can be described intuitively in the following way: at each site k of $\mathbb{Z}$ we have a number of particles $X(k)$ and after an exponential time with parameter depending on $X(k)$, one of the particles (if any) will jump to the site l with probability $p(k, l)$.

We will suppose the transition probabilities translation invariant and set: $p(k, l)=p(0, l-k)=p(l-k)$ with $p(0)=0$. We also suppose that $\sum_{k\in\mathbb{Z}} |k|\, p(k)$ is finite and that $\sum_{k\in\mathbb{Z}} kp(k)$, denoted by β, is not zero.

Let the jump rates be described by a non decreasing function $\{g(n), n\in\mathbb{N}\}$ such that $g(0)=0$, $g(1)>0$ and $\sup_n (g(n+1)-g(n))$ is finite.

The existence and uniqueness of a Markov process corresponding to this description is given in [1]. If $g(n)$ is not bounded, in order to avoid infinitely many particles jumping to a given site, it is necessary to restrict the set of allowed configurations to a subset E_0 of E (*cf.* [1]).

In any case the generator L of this Markov process is defined on cylindrical functions on E by:

$$Lf(\eta)=\sum_{k, l\in\mathbb{Z}} p(l)\, g(\eta(k))\,[f(\eta^{k, k+l})-f(\eta)] \tag{1}$$

where

$$\eta^{k, k+l}(m)=\begin{cases}\eta(m) & \text{if} \quad m\neq k \quad \text{and} \quad m\neq k+l \text{ or } \eta(k)=0\\ \eta(m)-1 & \text{if} \quad m=k \quad \text{and} \quad \eta(k)\geqq 1\\ \eta(m)+1 & \text{if} \quad m=k+l \quad \text{and} \quad \eta(k)\geqq 1.\end{cases}$$

Denote by $(T_t)_{t\geqq 0}$ the semigroup generated by L and by $\mathscr{L}$ the set of real functions on E [E_0 if $g(n)$ is not bounded] on which this semigroup operates.

Let $\{X_t=\{X_t(k), k\in\mathbb{Z}\}, t\geqq 0\}$ be the right continuous version with left limits of the Markov process with semigroup (T_t) [$X_t(k)\in\mathbb{N}$ for every $t\geqq 0$ and $k\in\mathbb{Z}$].

The extremal invariant and translation invariant measures (*see* [1]) are the product measures $\nu(\gamma)$ given by:

$$\nu(\gamma)\{\eta \in E/\eta(0)=n\}=Z^{-1}\frac{\gamma^n}{g(n)!} \tag{2a}$$

$$Z=\sum_{n \geqq 0} \frac{\gamma^n}{g(n)!} \tag{2b}$$

where $g(n)!=g(1)\ldots g(n)$, $g(0)!=1$ and $\gamma \in [0, \sup_n g(n))$. If $g(n)$ is not bounded we have $\nu(\gamma)(E_0)=1$ (*cf.* [1]).

We have $\gamma=\int g(\eta(0))\,d\nu(\gamma)$ and if $\rho=\int \eta(0)\,d\nu(\gamma)$, it is not difficult to see that $\rho=Z^{-1}\sum_{n=1}^{+\infty}\frac{n\gamma^n}{g(n)!}$ is a strictly increasing C^∞-function of γ such that $\rho(0)=0$ and $\lim_{\gamma \to \sup_n g(n)} \rho(\gamma)=+\infty$.

The well-defined inverse function $\gamma=G(\rho)$, continuous and strictly increasing from 0 to $\sup_n g(n)$, will be important in the sequel: if $\gamma=G(\rho)$ we also write $\nu(\gamma)=\nu^\rho$.

Our *initial distribution* will be a *product measure* on E such that its restriction to $\mathbb{Z}_+=\{k>0\}$ is ν^b and its restriction to $\mathbb{Z}_-=\{k \leqq 0\}$ is ν^a for given a and b such that $0 \leqq b \leqq a$; we will denote this initial distribution by $\nu^{a,b}$.

We may see $\nu^{a,b}$ as the juxtaposition of two half-spaces in equilibrium and then observe its evolution according to the zero-range process.

Note that $p(1)=1$ or $p(-1)=1$ (i. e. particles move in only one direction to the nearest neighbor), $g(n)=1$ for every $n \geqq 1$ is the case studied in [2] where $G(\rho)=\frac{\rho}{1+\rho}$.

Note also that in the case $g(n)=n$ for every n, particles move independently and $G(\rho)=\rho$.

I.2. Rescaling and heuristic

For every $\varepsilon>0$, x in $\mathbb{R}$ and $t \geqq 0$ we define:

$$X_t^\varepsilon(x)=\sum_{k \in \mathbb{Z}} X_{t/\varepsilon}(k)\,I_{[\varepsilon k,\, \varepsilon(k+1))}(x). \tag{3}$$

We want to prove the almost sure weak convergence of the random measure $X_t^\varepsilon(x)\,dx$, as ε goes to 0, to a deterministic measure $u(x, t)\,dx$.

Observe that for $t=0$, the strong law of large numbers holds: $X_0^\varepsilon(x)\,dx$ converges weakly almost surely to the deterministic measure: $u_0(x)\,dx=(a\,I_{(-\infty, 0)}+b\,I_{[0, +\infty)})(x)\,dx$; u_0 is the *initial density profile.*

In order to guess the limit for $t>0$, we need an equation for $\mathbb{E}(X_t^\varepsilon)$ also denoted by $u_t^\varepsilon(x)$.

For $k\in\mathbb{Z}$ we compute $d\mathbb{E}\{X_t(k)\}$ using the formula: $d\mathbb{E}\{f(X_t)\}=\mathbb{E}\{L f(X_t)\}\,dt$ with $f(\eta)=\eta(k)$.

$$d\mathbb{E}\{X_t(k)\}=-\sum_{l\in\mathbb{Z}^*} p(l)\,[\mathbb{E}\{g(X_t(k))\}-\mathbb{E}\{g(X_t(k-l))\}]\,dt. \tag{4}$$

On the set of functions $u: \mathbb{R}\to\mathbb{R}$ we define the following functionals: $D_{-\varepsilon l}u(x)=(\varepsilon l)^{-1}[u(x)-u(x-l)]$.

With this notation and (3), (4) becomes:

$$du_t^\varepsilon=-\sum_{l\in\mathbb{Z}^*} l p(l)\,D_{-\varepsilon l}\,\mathbb{E}\{g(X_t^\varepsilon)\}\,dt. \tag{5}$$

Intuitively, when the process evolves under the invariant measure $\nu^\rho=\nu(\gamma)$, we have $u_t^\varepsilon=\mathbb{E}_{\nu^\rho}\{X_t^\varepsilon\}=\rho$ and $\mathbb{E}_{\nu^\rho}\{g(X_t^\varepsilon)\}=\gamma=G(\rho)=G(u_t^\varepsilon)$. From this observation, which will be the basic idea in section III, we must have: if $X_t^\varepsilon(x)\,dx$ converges, as ε goes to 0, weakly almost surely to the deterministic measure $u(x, t)\,dx$, then u_t^ε converges to $u(x, t)$, $D_{-\varepsilon l}$ converges to $\dfrac{\partial}{\partial x}$ and $\mathbb{E}(g(X_t^\varepsilon))$ converges to $G(u(x, t))$.

Therefore u should be a solution to the following first order P.D.E.:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\beta\dfrac{\partial G(u)}{\partial x}=0\\ u(.,0)=u_0\end{cases} \tag{6}$$

where $\beta=\sum_{l\in\mathbb{Z}^*} l p(l)\neq 0$.

In general we do not have unicity of the weak solution for this type of equation.

In the appendix of [3] we summarized its properties, recalled the notion of entropy condition and the result of existence and uniqueness of the weak solution satisfying the entropy condition.

In this paper we will make the following hypothesis:

(H) G IS CONCAVE.

Then from the appendix of [3] we know that the entropy condition is equivalent to:

$$u^{-}(x, t)=\lim_{y \uparrow x} u(y, t) \geqq u^{+}(x, t)=\lim_{y \downarrow x} u(y, t) \qquad \text{if} \quad \beta<0$$

and

$$u^{-}(x, t) \leqq u^{+}(x, t) \qquad \text{if} \quad \beta>0.$$

II. CONVERGENCE

In [1] it is proved that the zero-range evolution preserves the stochastic order (*monotonicity*). One also can find in [1] the generator of the *coupled process* which allows us to construct simultaneously on the same space versions of the zero-range process starting from an arbitrary configuration at an arbitrary time; these properties are a consequence of the hypothesis $\sup_n (g(n+1)-g(n))<+\infty$ and enable us to apply the method described in [3], chapter II, based on a subadditive ergodic theorem due to Liggett ([5], Chapt. I, Thm. 2.6).

The proof of the follwoing convergence result can be found in [3], Chapt. II.

PROPOSITION 1. — *There exists a function $u(x, t)$, decreasing and right continuous in the x-variable such that: $b \leqq u(x, t) \leqq a$ and $X_t(x)\,dx$ converges weakly almost surely to $u(x, t)\,dx$.*

In order to identify u it will be enough to study $u_t^{\varepsilon}(x)=\mathbb{E}\{X_t^{\varepsilon}(x)\}=\mathbb{E}\{X_{t/\varepsilon}([x/\varepsilon])\}$ as ε goes to 0, since by dominated convergence

$$\lim_{\varepsilon \to 0} \int_x^y \mathbb{E}\{X_t^{\varepsilon}(z)\}\,dz=\int_x^y u(z, t)\,dz$$

and therefore $u(z, t)=\lim_{\varepsilon \to 0} \mathbb{E}\{X_t^{\varepsilon}(z)\}$ for almost every z in $\mathbb{R}$.

III. IDENTIFICATION OF THE DENSITY PROFILE

One can rewrite the equation (5) satisfied by u_t^ε in the weak form: for every smooth function $\varphi: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, with compact support, we have:

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} \left[u_t^\varepsilon \frac{\partial \varphi}{\partial t} \right] + \sum_{l \in \mathbb{Z}^*} l p(l) \mathbb{E}(g(\mathrm{X}_t^\varepsilon)) \mathrm{D}_{\varepsilon l}(\varphi) \, dx \, dt = - \int_{\mathbb{R}} u_0^\varepsilon(x) \varphi(x, 0) \, dx \quad (7)$$

where $u_0^\varepsilon = a \mathrm{I}_{(-\infty, \varepsilon)} + b \mathrm{I}_{[\varepsilon, +\infty)}$.

The main difficulty to take a limit in (7) is to prove that $\mathbb{E}(g(\mathrm{X}_t^\varepsilon(x)))$ converges to $\mathrm{G}(u(x, t))$, where G is the function defined in section I and $u(x, t)$ the density profile obtained in section II.

In order to do that we will compare our process (X_t) with the same zero-range process (B_t^c) starting from the invariant measure ν^c defined in (2).

Initially we impose $\mathrm{X}_0(k) \geqq \mathrm{B}_0^c(k)$ for $k \leqq 0$ and $\mathrm{X}_0(k) \leqq \mathrm{B}_0^c(k)$ for $k > 0$; the initial distribution allows us to do that and we let the processes evolve according to the coupling procedure.

Under the nearest neighbor assumption the interface argument given in [3] would have given the result directly; without this assumption the proof is more complicated.

We denote by $\nu^{a, b; c}$ the initial coupled distribution previously described and by τ the shift on $\mathbb{Z}$; $(\tilde{\mathrm{T}}_t)$ being the semigroup of the coupled process associated with the generator $\tilde{\mathrm{L}}$, we have the following result:

LEMMA 1. — *For every (x, t) such that $u(x, t)$ is continuous in the x-variable, every cluster point of the precompact set* $\{\tau^{[x/\varepsilon]} \tilde{\mathrm{T}}_{t/\varepsilon} \nu^{a, b; c}, \varepsilon \searrow 0\}$ *is a measure on* $\mathrm{E} \times \mathrm{E}$ *supported by* $\{\eta \geqq \xi\} \cup \{\eta \leqq \xi\}$.

Proof. — Observe that the compactness is given by the monotonicity. Let $\tilde{\nu}$ be a weak limit along the sequence (ε_n). The second marginal of $\tilde{\nu}$ is obviously ν^c and its first marginal has a one point correlation equated to $u(x, t)$. By shifting the initial distribution it is not difficult to see that $\tau\tilde{\nu}$ is stochastically larger than $\tilde{\nu}$ and has the same one point correlation; it follows that $\tilde{\nu}$ is translation invariant.

The next step is to show that $\tilde{\nu}$ is invariant by $(\tilde{\mathrm{T}}_t)$. The method is very similar to what we did in [3], Chapt. II, to prove the nonrandomness of I. Let the system evolve up to time t_0 and couple it with the same one where particles have been added in such a way that restricted to $\mathbb{Z}_-$ we have our initial distribution $\nu^{a, b; c}$.

After t_0 let the system evolve along the sequence (ε_n). The particles added at time t_0 will not change the law of large numbers and then the one point correlation. This fact and the stochastic domination give again $\tilde{T}_{t_0}\tilde{\nu}=\tilde{\nu}$.

$\tilde{\nu}$ being translation invariant and invariant by the semigroup $(\tilde{T}_t)$ the conclusion is given by [1], § 4.

The next result shows that there is a limiting interface between particles of the X-system alone and particles of the B-system alone, in terms of densities.

LEMMA 2. — *For every real numbers x and y such that $x < y$, either* $\varepsilon \sum_{j=-\infty}^{j=[x/\varepsilon]} \{B_{t/\varepsilon}(j)-X_{t/\varepsilon}(j)\} \vee 0$ *or* $\varepsilon \sum_{j=[y/\varepsilon]}^{j=+\infty} \{X_{t/\varepsilon}(j)-B_{t/\varepsilon}(j)\} \vee 0$ *goes to zero in probability as* ε *goes to zero.*

Proof. — If it is not the case we can find a sequence (ε_n), $\delta_1 > 0$, $\delta_2 > 0$ and $\alpha > 0$ such that:

$$\lim_n \mathbb{P}\left\{ \sum_{j=-\infty}^{j=[x/\varepsilon_n]} \{B_{t/\varepsilon_n}(j)-X_{t/\varepsilon_n}(j)\} \vee 0 \geqq \delta_1/\varepsilon_n, \right.$$
$$\left. \sum_{j=[y/\varepsilon_n]}^{j=+\infty} \{X_{t/\varepsilon_n}(j)-B_{t/\varepsilon_n}(j)\} \vee 0 \geqq \delta_2/\varepsilon_n \right\} \geqq \alpha.$$

Intuitively from time 0 to time t/ε_n we must have a number of jumps of particles of one type over particles of different type at least of order $\delta_1 \delta_2/\varepsilon_n^2$ since at time 0 particles of different types are well ordered. This will not be possible since by Lemma 1, locally, particles alone are of the same type.

The particular shape of the initial distribution of the X-particles alone and B-particles alone shows that the number of jumps of size greater than N from time 0 to time t/ε_n involving X-particles alone over B-particles alone (or vice versa) is at most of order $(\sum_{|l| > N} p(l))(t/\varepsilon_n)^2$. This enables us to suppose that $p(l)=0$ for every l such that $|l| > N$.

Let $W(t, N)$ be the number of pairs $(B_t(k), X_t(j))$ with $|j-k| \leqq N$. Lemma 1 shows that $W(t/\varepsilon_n, N)$ converges to 0 in probability. Then the number of jumps of size less than N from time 0 to time t/ε_n involving X-particles alone over B-particles alone (or vice versa) is of order less than $(t/\varepsilon_n)^2$ which is not compatible with the order $\delta_1 \delta_2/\varepsilon_n^2$.

COROLLARY 1. — *For almost every x in $\mathbb{R}$,*

$$\limsup_{\varepsilon \to 0} \left| \mathbb{E}\{g(X_t^\varepsilon(x))\} - G(u_t^\varepsilon(x)) \right| = 0. \tag{8}$$

Proof. — Let C be the set of nonnegative real numbers which are the values taken at most one time by $u(x, t)$ as a function of x, t being fixed; u is decreasing in the x-variable then C is dense in $\mathbb{R}_+$. For c in C we define $\alpha(c, t)$ as the only real number, $-\infty$ and $+\infty$ included, such that $u(x, t) > c$ for every $x < \alpha(c, t)$ and $u(x, t) < c$ for every $x > \alpha(c, t)$. Lemma 2 tells us that the density of B^c-particles alone on the left of $\alpha(c, t)$ is zero and that the density of X-particles alone on the right of $\alpha(c, t)$ is zero. Therefore

$$\lim_{\varepsilon \to 0} \mathbb{E}\{[B_t^\varepsilon(x) - X_t^\varepsilon(x)] \vee 0\} = 0$$

and

$$\liminf_{\varepsilon \to 0} \mathbb{E}\{g(X_t^\varepsilon(x))\} > \mathbb{E}\{g(B_t^\varepsilon(x))\} = G(c)$$

for almost every $x < \alpha(c, t)$. Similarly

$$\lim_{\varepsilon \to 0} \mathbb{E}\{[X_t^\varepsilon(x) - B_t^\varepsilon(x)] \vee 0\} = 0$$

and

$$\limsup_{\varepsilon \to 0} \mathbb{E}\{g(X_t^\varepsilon(x)\} < \mathbb{E}\{g(B_t^\varepsilon(x))\} = G(c)$$

for almost every $x > \alpha(c, t)$.

For $\delta > 0$ such that $c+\delta$ is also in C we deduce that for almost every x such that $\alpha(c+\delta, t) < x < \alpha(c, t)$ we have:

$$c \leqq \lim_{\varepsilon \to 0} \mathbb{E}\{X_t^\varepsilon(x)\} = u(x, t) \leqq c+\delta$$

$$G(c) = \mathbb{E}\{g(B_t^{c, \varepsilon}(x))\} \leqq \liminf_{\varepsilon \to 0}(\sup)\, \mathbb{E}\{g(X_t^\varepsilon(x))\}$$
$$\leqq \mathbb{E}\{g(B_t^{c+\delta}(x))\} = G(c+\delta).$$

(8) is obtained by taking a limit as δ goes to zero and using the density of C in $\mathbb{R}_+$.

THEOREM 1. — *u is a weak solution to* (6).

Proof. – We define $A(\varepsilon, \varphi)$ for $\varepsilon > 0$ and φ as in (7) by:

$$A(\varepsilon, \varphi)=\iint_{\mathbb{R}_+\times\mathbb{R}}\left\{u_t^\varepsilon\frac{\partial\varphi}{\partial t}+\beta\,G(u_t^\varepsilon)\frac{\partial\varphi}{\partial x}\right\}dx\,dt+\int_{\mathbb{R}}u_0^\varepsilon(x)\,\varphi(x, 0)\,dx$$

where $\beta=\sum_{l\in\mathbb{Z}} lp(l)$.

In order to prove that

$$\iint_{\mathbb{R}_+\times\mathbb{R}}\left\{u\frac{\partial\varphi}{\partial t}+\beta\,G(u)\frac{\partial\varphi}{\partial x}\right\}dx\,dt=\int_{\mathbb{R}}u_0(x)\,\varphi(x, 0)\,dx,$$

by the continuity of G and the convergence of u_0^ε to u_0, it is enough to prove that $A(\varepsilon, \varphi)$ goes to zero as ε goes to zero.

From (7) we deduce:

$$\begin{aligned}A(\varepsilon, \varphi)&=\iint_{\mathbb{R}_+\times\mathbb{R}}\left\{\beta\,G(u_t^\varepsilon)\frac{\partial\varphi}{\partial x}-\mathbb{E}(g(X_t^\varepsilon))\sum_{l\in\mathbb{Z}} lp(l)\,D_{\varepsilon l}(\varphi)\right\}dx\,dt\\
&=\iint_{\mathbb{R}_+\times\mathbb{R}}\beta\frac{\partial\varphi}{\partial x}\{G(u_t^\varepsilon)-\mathbb{E}(g(X_t^\varepsilon))\}\,dx\,dt\\
&\quad+\iint_{\mathbb{R}_+\times\mathbb{R}}\mathbb{E}(g(X_t^\varepsilon))\sum_{l\in\mathbb{Z}} lp(l)\left\{\frac{\partial\varphi}{\partial x}-D_{\varepsilon l}(\varphi)\right\}dx\,dt\\
&\leqq\iint_{\mathbb{R}_+\times\mathbb{R}}|\beta|\times\left|\frac{\partial\varphi}{\partial x}\right|\times|G(u_t^\varepsilon)-\mathbb{E}(g(X_t^\varepsilon))|\,dx\,dt\\
&\quad+\iint_{\mathbb{R}_+\times\mathbb{R}}|\mathbb{E}(g(X_t^\varepsilon))|\times\left(\sum_{l\in\mathbb{Z}}|l|\,p(l)\left|\frac{\partial\varphi}{\partial x}-D_{\varepsilon l}(\varphi)\right|\right)dx\,dt.\end{aligned}$$

The first integral going to zero by the Corollary 2 and the second integral going to zero since

$$|\mathbb{E}(g(X_t^\varepsilon))|\leqq G(a)<+\infty\qquad\text{and}\qquad\sum_{l\in\mathbb{Z}}|l|\,p(l)\left|\frac{\partial\varphi}{\partial x}-D_{\varepsilon l}(\varphi)\right|$$

converges to zero; $\lim_{\varepsilon\to 0} A(\varepsilon, \varphi)=0$.

THEOREM 2. – *u is the weak solution to* (6) *satisfying the entropy condition.*

Proof. – We first look at the case $g(n)=\lambda n$ for every $n\in\mathbb{N}$ with λ a strictly positive constant; in that case G is linear, $G(\rho)=\lambda\rho$, (6) is linear and u is its unique solution in the following sense: $u(x, t)$ is almost everywhere equal to $\tilde{u}$ given by:

$$\tilde{u}(x, t)=a\qquad\text{if}\quad x<\beta\lambda t\quad\text{and}\quad b\text{ if } x>\beta\lambda t.$$

This a. e. equality, the monotonicity (in x) of $u(x, t)$ and the fact that $u(x, t)=u(x/t, 1)$ imply that the equality holds for all (x, t) such that $x \neq \beta\lambda t$.

In the case $\beta < 0$, as recalled in section I, the entropy condition reduces to $u^+ \leqq u^-$ which is clearly satisfied since u is decreasing in x. The previous argument shows that:

$$u(x, t)=\begin{cases} a & \text{if} \quad x < \beta \dfrac{G(b)-G(a)}{b-a} t \\ b & \text{if} \quad x > \beta \dfrac{G(b)-G(a)}{b-a} t. \end{cases}$$

In the case $\beta > 0$, g not linear, G' is (continuous) strictly decreasing and we have: $\beta G'(a) \leqq \beta G'(u) \leqq \beta G'(b)$.

As in [3] the method of characteristics gives:

$$x > \beta G'(b) t \quad \Rightarrow \quad u(x, t)=u_0(x-\beta G'(u) t)=b$$

$$x < \beta G'(a) t \quad \Rightarrow \quad u(x, t)=u_0(x-\beta G'(u) t)=a.$$

By comparison with $u^{a, c}$ and $u^{c, b}$ obtained as a profile for the process starting from $\nu^{a, c}$ and $\nu^{c, b}$ and monotonicity we get that $u(., t)$ is *continuous* and therefore satisfies the entropy condition. Moreover this method gives u explicitly:

$$u(x, t)=\begin{cases} a & \text{if} \quad x \leqq \beta G'(a) t \\ (G')^{-1}\left(\dfrac{x}{\beta t}\right) & \text{if} \quad \beta G'(a) t \leqq x \leqq \beta G'(b) t \\ b & \text{if} \quad x \geqq \beta G'(b) t. \end{cases}$$

IV. LOCAL EQUILIBRIUM

We deduce from Theorem 2 the limiting behavior of the particle process seen by a travelling observer (i. e. the weak limit of the distribution of $\{X_t([xt]+k), k \in \mathbb{Z}\}$ as t goes to $+\infty$ for fixed x in $\mathbb{R}$).

THEOREM 3. — *For all points of continuity of* $u(x, 1)$, *for all values of* $\beta \neq 0$) *we have:*

$$w.\lim_{t \to +\infty} \text{Law}\{X_t([xt]+k), k \in \mathbb{Z}\} = \nu^{u(x, 1)}.$$

Remark. — Except for one case $\left(\beta < 0,\ x = \beta \dfrac{G(b) - G(a)}{b - a} t\right)$, the limiting distribution is a product measure invariant under the action of the semigroup (T_t). We say that *propagation of chaos* holds and that the system is in *local equilibrium*.

Proof. — As in [3], it is easy from the proof of Theorem 1 to get that for any finite set $\{f_1, \ldots, f_n\}$ of increasing functions from $\mathbb{N}$ to $\mathbb{R}_+$:

$$\lim_{t \to +\infty} \mathbb{E}\left\{\prod_{i=1}^{i=n} f_i(X_t([xt] + k_i))\right\} = \prod_{i=1}^{i=n} \int f_i(\eta(k_i))\, d\nu^{u(x,1)}$$

where the k_i's are all distinct.

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REFERENCES

[1] E. D. ANDJEL, Invariant Measures for the Zero-Range Process, *The Annals of Probability*, Vol. **10**, No. 3, 1982, pp. 525-547.

[2] E. D. ANDJEL and C. KIPNIS, Derivation of the Hydrodynamical Equations for the Zero-Range Interaction Process: a Non-Linear Euler Equation, *The Annals of Probability*, Vol. **12**, No. 2, 1984, pp. 325-334.

[3] A. BENASSI and J. P. FOUQUE, Hydrodynamical Limit for the Asymmetric Simple Exclusion Process, *The Annals of Probability*, Vol. **15**, No. 2, 1987, pp. 546-560.

[4] A. DE MASI, N. IANIRO, A. PELLEGRINOTTI and E. PRESUTTI, *A Survey of the Hydrodynamical Behavior of Many Particle Systems. Studies of Statistical Mechanics*, Vol. II, North-Holland, Amsterdam, 1984.

[5] A. DE MASI and P. FERRARI, A Remark on the Hydrodynamics of the Zero-Range Processes, *Journal of Statistical Physics*, Vol. **36**, Nos. 1/2, 1984, pp. 81-87.

[6] T. M. LIGGETT, *Interacting Particle Systems*, Berlin, Springer-Verlag, 1985.

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