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On the joint distribution of the maximum and its location for a linear diffusion

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by

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ABSTRACT. — For a linear diffusion X let $M_t = \sup_{0 \le s \le t} X_s$ and $T = \inf \{ s : X_s = M_t \}$. In this note we compute the joint distribution of M_t, X_t and T. As an application of our formula we rederive a result of Borodin which gives the distribution for the supremum of the Brownian local time. Further, some examples are presented.

Key-words: Linear diffusion, maximum, location of the maximum, joint distribution, Brownian local time.

AMS classification: 60 J 60, 60 J 55.

RÉSUMÉ. — Soient X une diffusion linéaire, $M_t = \sup_{0 \le s \le t} X_s$ et $T = \inf \{ s : X_s = M_t \}$. Nous déterminons la distribution simultanée de M_t, X_t et T. Notre formule est employée pour redériver un résultat de Borodin donnant la distribution du maximum du temps local Brownien. Nous donnons aussi quelques exemples.

1. INTRODUCTION

Let X be a regular one-dimensional diffusion (in the sense of Itô-McKean [8]) taking values on the interval $I \in \mathcal{R}$. Let for a fixed $t > 0$

$M_t = \sup_{0 \le s \le t} X_s$ and $T = \inf \{ s : X_s = M_t \}$. The main aim of this note is to derive the joint distribution of X_t, M_t and T. In the literature one can find expressions for this in some special cases. Lévy [10], Vincze [18], and Louchard [11] treated Brownian motion, Shepp [16] Brownian motion with drift, Imhof [6] [7], and Louchard [12] killed Brownian motion and three-dimensional Bessel process. Proofs in these works are based on special analytical properties of the particular processes.

The formula presented here with two proofs covers all linear diffusions. The first proof, which uses excursion theory, duality, and properties of processes with independent increments, shows the probabilistic structure of the formula. The second proof is short, but purely analytical and completely unmotivated. As an application of our formula we rederive a result of Borodin [2], which gives the distribution for the supremum of the Brownian local time. Further we work out some specific examples to give more flavor to our formula.

2. DISTRIBUTION OF (X_t, M_t, T)

Let X be as above, and denote with $\mathbf{P}_x$ the probability measure associated with X when started from x. Further denote with S and m the scale function and the speed measure of X (see [8]).

THEOREM A. — For a fixed $t > 0$ let $M_t = \sup_{0 \le s \le t} X_s$ and $T = \inf \{ s: X_s = M_t \}$. Then we have for $x \le y, z \le y$

$$(2.1) \quad \mathbf{P}_x(M_t \in dy, X_t \in dz, T \in ds) = n_x(s; y)n_z(t-s; y)S(dy)m(dz)ds,$$

where $n_x(.; y)$ is the $\mathbf{P}_x$-density of $\tau_y = \inf \{ s: X_s = y \}$.

(2.2) *Remarks.* — *i*) The timepoint when X attains its maximum value before the time t is $\mathbf{P}_x$-a. s. unique. We refer to Williams [19]; although our case with a fixed time t is not treated there, it is quite obvious that a similar argument applies.

ii) The densities $n_x(.; y)$ exist for all x and y. This must be well known, but the only reference we have is Getoor [5] (10.10). This is not however directly formulated for diffusions. In the course of our proof of (2.1) it is seen how this result applies.

We give below two proofs of Theorem A. The first one might be called probabilistic, and the second one computational. In our probabilistic proof Theorem A is derived from a more general result—Theorem B—,

which we now formulate. Let $Z = \{ Z_t ; t \geq 0, Z_0 = 0 \}$ be a continuous non-decreasing stochastic process such that its right-continuous inverse $H = \{ H_t ; t \geq 0, H_0 = 0 \}$ is a stochastically continuous, increasing process with independent increments. In this case the Lévy-Khintchine representation takes the form

$$(2.3) \quad \mathbf{E} \left(\exp \left(- \lambda (H_b - H_a) \right) \right)$$
$$= \exp \left(- \int_0^{+\infty} (1 - e^{-\lambda u}) \Pi((a, b) \times du) - \Pi((a, b) \times +\infty) \right).$$

Here for all $a > 0$ $\Pi((0, a) \times .)$ is a measure on $\mathbf{B}_\varepsilon$ $(=$ Borel sets on $[\varepsilon, +\infty)$, where $\varepsilon > 0$ is given), and for all $U \in \mathbf{B}_\varepsilon$ the function $a \mapsto \Pi((0, a) \times U)$ is non-decreasing and continuous. The second term on the right hand side of (2.3) gives the probability for the explosions, i. e.

$$\mathbf{P}(H_b < \infty) = \exp \left(- \Pi((0, b) \times +\infty) \right).$$

For typographical reasons we introduce the following convention

$$\int_{(0, +\infty]} (1 - e^{-\lambda u}) \Pi((a, b) \times du)$$
$$:= \int_0^{+\infty} (1 - e^{-\lambda u}) \Pi((a, b) \times du) + \Pi((a, b) \times \infty).$$

Assume further that there exist a measure S on $[0, +\infty)$ and a kernel $K(a, U)$, $a \geq 0$, $U \in \overline{\mathbf{B}}_\varepsilon$ $(=$ smallest σ-field which contains $\mathbf{B}_\varepsilon$ and $\{ +\infty \})$ such that

$$(2.4) \qquad \Pi(A \times U) = \int_A \int_U K(a, du) S(da), \qquad A \in \mathbf{B}[0, +\infty).$$

THEOREM B. — Let Z and H be as above. Then for $a < b$

$$\mathbf{P}(Z_t \in (a, b]) = \int_a^b \int_0^t \mathbf{P}(H_x \in ds) \Pi(dx \times [t - s, +\infty])$$
$$= \int_a^b S(dx) \int_0^t \mathbf{P}(H_x \in ds) K(x, [t - s, +\infty]).$$

Proof. — By the independence of the increments we have

$$\mathbf{P}(Z_t \in (a, b]) = \mathbf{P}(H_a < t, H_b \geq t)$$
$$= \int_0^t \mathbf{P}(H_a \in ds, H_b - H_a \geq t - s)$$
$$= \int_0^t \mathbf{P}(H_a \in ds) \mathbf{P}(H_b - H_a \geq t - s).$$

Next we compute

$$\int_0^{+\infty} e^{-\lambda u} \mathbf{P}(H_b - H_a \geq u) du = \frac{1}{\lambda} \mathbf{E}(1 - \exp(-\lambda(H_b - H_a)))$$

$$= \frac{1}{\lambda}\left(1 - \exp\left(-\int_{(0, +\infty]} (1 - e^{-\lambda u}) \Pi((a, b) \times du)\right)\right).$$

Using (2.4) this gives

$$\lim_{b \to a} \int_0^{+\infty} \lambda e^{-\lambda u} \frac{\mathbf{P}(H_b - H_a \geq u)}{S(a, b)} du = \int_{(0, +\infty]} (1 - e^{-\lambda u}) K(a \times du),$$

and, by the uniqueness of the Laplace transforms,

$$(2.5) \qquad \lim_{b \to a} \frac{\mathbf{P}(H_b - H_a \geq u)}{S(a, b)} = K(a \times [u, +\infty]).$$

The function $u \mapsto \mathbf{P}(H_b - H_a \geq u)$ being non-increasing it is seen that (2.5) holds uniformly in u on compact sub-intervals of $(0, +\infty)$. Further it is obvious by the regularity properties of the functions $b \mapsto \mathbf{P}(H_b - H_a \geq u)$ and $b \mapsto S(a, b)$ that there exists a division $a = a_0 < a_1 < \ldots < a_n = b$ such that for a given $\varepsilon > 0$ and for all $v < t$

$$(1 - \varepsilon) \sum_{i=0}^{n-1} S(a_i, a_{i+1}) \int_0^v \mathbf{P}(H_{a_i} \in ds) K(a_i \times [t-s, +\infty]) \leq$$

$$\leq \int_0^v \mathbf{P}(H_a \in ds) \mathbf{P}(H_b - H_a \geq t - s) \leq$$

$$\leq (1 + \varepsilon) \sum_{i=0}^{n-1} S(a_i, a_{i+1}) \int_0^v \mathbf{P}(H_{a_i} \in ds) K(a_i \times [t-s, +\infty]).$$

Letting $n \to +\infty$ and $v \uparrow t$ concludes the proof.

Proof of Theorem A. — Here the process $t \mapsto M_t$ and its right continuous inverse $\tau_a = \inf\{t : X_t > a\}$ play the roles of Z and H in Theorem B. It is well known (see Itô and McKean [8] (4.10)) that $a \mapsto \tau_a$ is a stochastically continuous process with independent increments having the Lévy-Khintchine representation $(a < b)$

$$\mathbf{E}(\exp(-\beta(\tau_b - \tau_a))) = \exp\left(-\int_a^b S(dx) \int_{(0, +\infty]} (1 - e^{-\beta l}) K_-(x \times dl)\right),$$

where S is the scale measure and the kernel K_- is given by

$$(2.6)\,i) \qquad\qquad K_-(x \times dl) = \lim_{y\uparrow x} \frac{\mathbf{P}_y(\tau_x \in dl)}{S(x) - S(y)}$$

$$K_-(x \times +\infty) = \lim_{y\uparrow x} \frac{\mathbf{P}_y(\tau_x = +\infty)}{S(x) - S(y)}.$$

We introduce also

$$(2.6)\,ii) \qquad\qquad K_+(x \times dl) = \lim_{y\downarrow x} \frac{\mathbf{P}_y(\tau_x \in dl)}{S(y) - S(x)}$$

$$K_+(x \times +\infty) = \lim_{y\downarrow x} \frac{\mathbf{P}_y(\tau_x = +\infty)}{S(y) - S(x)},$$

and set $K := K_+ + K_-$. Now Theorem B gives

$$(2.7) \qquad \mathbf{P}_x(M_t \in (a, b)) = \int_a^b S(dy) \int_0^t ds\, n_x(s; y) K_-(y \times (t - s, +\infty]).$$

Next we show that

$$(2.8) \qquad\qquad K_-(y \times (t, +\infty]) = \int_l^y n_z(t; y) m(dz),$$

where l is the left endpoint of the state space I. To see this consider the local time (with the Itô-McKean normalization) at the point y. Let A^y be its right continuous inverse. The process A^y is a Lévy-process with

$$\mathbf{E}\left(\exp\left(-\beta A_a^y\right)\right) = \exp\left(-a \int_{(0, +\infty]} (1 - e^{-\beta l}) K(y \times dl)\right)$$

(see Itô-McKean [8] (6.1) (6.2)). Now consider excursions of X from the point y going *below* y. We use results of Getoor [5], and assume therefore that $\zeta = +\infty$ $\mathbf{P}_x$-a. s. for all x, where ζ is the lifetime of X. From [5] (7.29) it follows that for $z < y, s < t$

$$(2.9) \qquad \mathbf{P}_x(X_t \in dz \mid \lambda_y^t = s, X_t < y) = \frac{Q_{t-s}^y(dz)}{K_-(y \times (t - s, +\infty])},$$

where Q_t^y is the excursion entrance law and $\lambda_y^t = \sup\{s < t : X_s = y\}$. Now (2.8) clearly follows from (2.9), if we prove

$$(2.10) \qquad\qquad Q_t^y(dz) = n_z(t; y) m(dz).$$

To see this we use [5] (10.10). Because the transition density (w. r. t. the speed measure) of X is symmetric i. e. $p(t; x, y) = p(t; y, x)$ (see Itô-McKean [8], p. 149) it follows that X is self dual. Let $q_*(t; z, y)$ be the density of Q_t^y

w. r. t. m (which exists by [5] (10.7)). Consequently by [5] (10.10) and self duality of X we obtain

(2.11) $$\mathbf{P}_x(\tau_y \in dt) = q_*(t; x, y)dt.$$

By our previous notation $n_x(t; y) = q_*(t; x, y)$ and (2.10) results. Note that (2.11) says that the density of τ_y always exists (cf. (2.2) ii)). So we have proved

$$\mathbf{P}_x(M_t \in (a, b)) = \int_a^b S(dy) \int_0^t ds \, n_x(s; y) K_-(y \times (t - s, + \infty])$$

$$= \int_a^b S(dy) \int_0^t ds \, n_x(s; y) \int_l^y m(dz) n_z(t - s; y).$$

It is clear from the probabilistic structure of this formula that

$$\mathbf{P}_x(M_t \in da, T \in ds) = n_x(s; a) K_-(a \times (t - s, + \infty]) ds \, S(da).$$

Hence to deduce (2.1) we have to do the identification ($0 < s < t$, $x, z < a$)

(2.12) $$\mathbf{P}_x(X_t \in dz \mid M_t = a, T = s) = \frac{Q^a_{t-s}(dz)}{K_-(a \times (t - s, + \infty])}.$$

But because the time point of the maximum is unique almost surely (cf. (2.2) i)) (2.12) is equivalent with

$$\mathbf{P}_x(X_t \in dz \mid \tau_a = s, \lambda_a^t = \tau_a, X_t < a) = \frac{Q^a_{t-s}(dz)}{K_-(a \times (t - s, + \infty])}.$$

This follows, however, immediately by taking $v = s$ in the following generalization (based on the strong Markov property) of (2.9)

$$\mathbf{P}_x(X_t \in dz \mid \tau_a = s, \lambda_a^t - \tau_a = v - s, X_t < a) = \frac{Q^a_{t-v}(dz)}{K_-(a \times (t - v, + \infty])}.$$

The proof is now complete in the case $\zeta = + \infty$ almost surely. If $\zeta < + \infty$ with a positive probability we still have (2.7) but now we are interested in the quantity $\mathbf{P}_x(M_t \in (a, b), \zeta > t)$. Because the entrance law measures all the paths living at the time point t it is obvious that (2.1) holds also in this case.

Second proof of Theorem A. — Let A be the infinitesimal operator of X, and denote with $\phi^\uparrow$ and $\phi^\downarrow$ the increasing and decreasing solution, respectively, of the equation

$$Au = \alpha u, \qquad \alpha > 0$$

such that

i) $\quad g(x, y) = \displaystyle\int_0^{+\infty} e^{-\alpha t} p(t; x, y) dt = \begin{cases} \dfrac{1}{B} \phi^{\uparrow}(x) \phi^{\downarrow}(y), & x \leq y; \\[2ex] \dfrac{1}{B} \phi^{\downarrow}(x) \phi^{\uparrow}(y), & x \geq y, \end{cases}$

ii) $\quad\quad\quad \mathbf{E}_x \left(\exp\left(-\alpha \tau_y \right) \right) = \begin{cases} \dfrac{\phi^{\uparrow}(x)}{\phi^{\uparrow}(y)}, & x \leq y; \\[2ex] \dfrac{\phi^{\downarrow}(x)}{\phi^{\downarrow}(y)}, & x \geq y, \end{cases}$

where $p(t; x, y)$ is the transition density w. r. t. the speed measure and B is the Wronskian (a constant)

$$(2.13) \quad\quad\quad B = \phi^{\downarrow}(x) \phi^{\uparrow +}(x) - \phi^{\downarrow +}(x) \phi^{\uparrow}(x).$$

Here $\phi^{\uparrow +}$ and $\phi^{\downarrow +}$ are the right derivatives w. r. t. the scale function. For these facts see Itô-McKean [8]. Now the key observation is the following expression for the Green function

$$(2.14) \quad\quad g(x, y) = \phi^{\uparrow}(x) \phi^{\uparrow}(y) \left(\int_{x \vee y}^r \frac{S(du)}{(\phi^{\uparrow}(u))^2} + \frac{1}{B} \frac{\phi^{\downarrow}(r)}{\phi^{\uparrow}(r)} \right),$$

where r is the right hand endpoint of I. Using (2.13) it is easily seen that (2.14) holds. Next we compute

$$\int_0^{+\infty} e^{-\alpha t} \mathbf{P}_x(X_t \in dz, \tau_y > t) dt$$

$$= \int_0^{+\infty} e^{-\alpha t} \mathbf{P}_x(X_t \in dz) dt - \int_0^{+\infty} e^{-\alpha t} \mathbf{P}_x(X_t \in dz, \tau_y \leq t) dt$$

$$= \int_0^{+\infty} e^{-\alpha t} \mathbf{P}_x(X_t \in dz) dt - \int_0^{+\infty} e^{-\alpha t} \left(\int_0^t \mathbf{P}_x(\tau_y \in ds) \mathbf{P}_y(X_{t-s} \in dz) ds \right) dt$$

$$= g(x, z) m(dz) - \frac{\phi^{\uparrow}(x)}{\phi^{\uparrow}(y)} g(y, z) m(dz)$$

$$= \phi^{\uparrow}(x) \phi^{\uparrow}(z) \int_{x \vee z}^y \frac{S(du)}{(\phi^{\uparrow}(u))^2} m(dz),$$

where we used the strong Markov property and (2.14). Differentiate w. r. t. y to obtain

$$\int_0^{+\infty} e^{-\alpha t} \mathbf{P}_x(X_t \in dz, M_t \in dy) dt = \frac{\phi^{\uparrow}(x) \phi^{\uparrow}(z)}{(\phi^{\uparrow}(y))^2} S(dy) m(dz).$$

Inverting the Laplace transform gives

$$\mathbf{P}_x(X_t \in dz, M_t \in dy) = \left(\int_0^t n_x(s; y)n_z(t - s; y)ds \right) S(dy)m(dz).$$

Finally using again the strong Markov property we have

$$\mathbf{P}_x(X_t \in dz, M_t \in dy, T > s)$$

$$= \int_l^y \mathbf{P}_x(X_s \in du, M_s < y)\mathbf{P}_u(X_{t-s} \in dz, M_{t-s} \in dy)$$

$$= \int_l^y \mathbf{P}_x(X_s \in du, M_s < y)\left(\int_s^t n_u(v - s; y)n_z(t - v; y)dv \right) S(dy)m(dz)$$

$$= \left(\int_s^t n_x(v; y)n_z(t - v; y)dv \right) S(dy)m(dz),$$

which completes the proof.

3. EXAMPLES

3.1. **Brownian motion.**

The scale function and the speed measure for a Brownian motion are $S(x) = x$ and $m(dx) = 2dx$, respectively. Hence the formula (2.1) has the form

$$(3.1) \qquad \mathbf{P}_x(M_t \in dy, X_t \in dz, T \in ds) = 2n_x(s; y)n_z(t - s; y)dydzds.$$

I. For a standard Brownian motion we have

$$n_x(s; y) = \frac{y - x}{\sqrt{2\pi s^3}} \exp\left(-\frac{(y - x)^2}{2s} \right),$$

and (3.1) is due to Lévy [10], see also Louchard [11].

II. For a reflected (at zero) Brownian motion we have ($x < y$)

$$\mathbf{E}_x(\exp(-\beta\tau_y)) = \frac{\cosh(\sqrt{2\beta}x)}{\cosh(\sqrt{2\beta}y)}$$

(see Itô-McKean [8], p. 29), and inverting this (see Erdélyi [4] (37), p. 258) gives

$$(3.2) \quad n_x(t; y) = \frac{\pi}{y^2} \sum_{j=0}^{+\infty} (-1)^j \left(j + \frac{1}{2} \right) \cos\left(\left(j + \frac{1}{2} \right)\frac{\pi x}{y} \right) \exp\left(-\left(j + \frac{1}{2} \right)^2 \frac{\pi^2 t}{y^2} \right).$$

The formula (3.1) with (3.2) is probably known but we have no reference where this is explicitly stated.

III. For a killed (at zero) Brownian motion we have $x < y$

$$E_x \left(\exp\left(-\beta\tau_y \right) \right) = \frac{\sinh\left(\sqrt{2\beta}x\right)}{\sinh\left(\sqrt{2\beta}y\right)},$$

(see Itô-McKean [8], p. 29), and inverting this (see Erdélyi [4] (31), p. 258) gives

$$(3.3) \quad n_x(t; y) = \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-\infty}^{+\infty} (y - x - 2ky) \exp\left(-\frac{(y - x - 2ky)^2}{2t} \right).$$

The formula (3.1) with (3.3) is explicitly stated in Imhof [6].

IV. Next consider a Brownian bridge with duration l. This is a Brownian motion conditioned by $\{ X_t = 0 \}$. We denote its law by $\mathbf{P}^l_{00}$. Then formula (3.1) gives

$$(3.4) \qquad \mathbf{P}^l_{00}(M_l \in dy, \, T \in ds) = \frac{n_0(s; y)n_0(l - s; y)}{p(l, 0, 0)} \, dyds$$

$$= 2\sqrt{2\pi l}n_0(s; y)n_0(l - s; y)dyds,$$

where

$$p(l; x, y) = \frac{1}{2\sqrt{2\pi l}} \exp\left(-\frac{(x - y)^2}{2l} \right)$$

is the transition density w. r. t. the speed measure of a standard Brownian motion. The formula (3.4) is due to Vincze [18].

V. Finally we consider a Brownian motion with a drift μ. This is a diffusion with $S(dx) = e^{-2\mu x}dx$, $m(dx) = 2e^{2\mu x}dx$ and the first passage time density

$$n_x(t; y) = \exp\left(-\mu(x - y) - \frac{\mu^2 t}{2} \right) \frac{y - x}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y - x)^2}{2t} \right), \quad x < y.$$

So we have the formula

$$(3.5) \quad \mathbf{P}_x(M_t \in dy, \, X_t \in dz, \, T \in ds) =$$

$$= 2\frac{y - x}{\sqrt{2\pi s^3}} \exp\left(-\frac{(y - x)^2}{2s} \right) \frac{y - z}{\sqrt{2\pi (t - s)^3}} \exp\left(-\frac{(y - z)^2}{2(t - s)} \right)$$

$$\exp\left(-\mu(x - z) - \frac{\mu^2 t}{2} \right) dydzds.$$

Shepp [16] derives (3.5) from the corresponding formula for a standard Brownian motion using absolute continuity.

3.2. **Three-dimensional Bessel process.**

A Bessel process is a diffusion on $[0, + \infty)$ with the generator

$$R = \frac{1}{2} \frac{d^2}{dx^2} + \frac{r-1}{2x} \frac{d}{dx}, \qquad r \in \mathscr{R}.$$

For positive integer values of r the corresponding Bessel process is identical in law with the radial part of an r-dimensional Brownian motion. We consider the three-dimensional case, i. e. $r = 3$, and denote the process with R. The scale and speed measures for R are $S(dx) = x^{-2}dx$ and $m(dx) = 2x^2dx$, respectively. Further it is well known that R may be described also as a Brownian motion killed when it hits 0 conditioned never to hit 0. In other words R is an excessive transform of a killed Brownian motion and the excessive function used is $h(x) = x$—the scale function of the killed Brownian motion. Therefore

$$(3.6) \qquad \mathbf{E}_x \left(\exp\left(-\beta \tau_y \right) \right) = \frac{y \sinh\left(\sqrt{2\beta}x \right)}{x \sinh\left(\sqrt{2\beta}y \right)}$$

and so (cf. (3.3))

$$n_x(t; y) = \frac{y}{x} \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-\infty}^{+\infty} (y - x - 2ky) \exp\left(-\frac{(y - x - 2ky)^2}{2t} \right).$$

The boundary point 0 is entrance for R and we have

$$\mathbf{E}_0 \left(\exp\left(-\beta \tau_y \right) \right) = \lim_{x \downarrow 0} \mathbf{E}_x \left(\exp\left(-\beta \tau_y \right) \right) = \frac{\sqrt{2\beta}y}{\sinh\left(\sqrt{2\beta}y \right)}.$$

By Chung [3] this is the Laplace transform of the function

$$n_0(t; y) = \left(\frac{\pi}{y} \right)^2 f_1\left(\frac{\pi^2 t}{2y^2} \right),$$

where

$$f_1(x) = 2 \sum_{n=1}^{+\infty} (-1)^{n+1} n^2 \exp\left(-n^2 x \right), \qquad 0 < x < +\infty,$$

is the probability density function having the distribution function

$$F_1(x) = \sum_{n=-\infty}^{+\infty} (-1)^n \exp\left(-n^2 x \right)$$

(see also Knight [9]). Using the observations above and (2.1) we may write down the joint distribution of M_t, T and X_t. This is a result of Imhof [6].

Next we compute (cf. (2.7), (2.8))

$$K_-(y \times [t, +\infty]) = \int_0^y n_x(t; y)m(dx).$$

This may also be found in an implicit form in Imhof [6], Theorem 4 i).
By Itô-McKean [8], p. 214 (see also [15]) we have

$$\int_0^{+\infty} \alpha e^{-\alpha t} K_-(y \times [t, +\infty])dt = \lim_{x \uparrow y} \frac{1 - E_x(\exp(-\alpha \tau_y))}{S(y) - S(x)}.$$

Using (3.6) we obtain

$$\lim_{x \uparrow y} \frac{1 - E_x(\exp(-\alpha \tau_y))}{S(y) - S(x)} = \sqrt{2\alpha} y^2 \frac{\cosh(\sqrt{2\alpha} y)}{\sinh(\sqrt{2\alpha} y)} - y$$

and, consequently,

$$\int_0^{+\infty} e^{-\alpha t} K_-(y \times [t, +\infty])dt = \sqrt{2} \frac{y^2}{\sqrt{\alpha}} \frac{\cosh(\sqrt{2\alpha} y)}{\sinh(\sqrt{2\alpha} y)} - \frac{y}{\alpha}.$$

Erdélyi [4] (36), p. 258 gives

$$K_-(y \times [t, +\infty]) = \sqrt{\frac{2}{\pi t}} y^2 \sum_{n=-\infty}^{+\infty} \exp\left(-\frac{2n^2 y^2}{t}\right) - y.$$

It is obvious by (2.6 i)) that

$$\lim_{t \to +\infty} K_-(y \times [t, +\infty]) = K_-(y \times +\infty) = 0.$$

For $x > y$ we obtain

$$E_x(\exp(-\beta \tau_y)) = \frac{y}{x} \exp(-\sqrt{2\beta}(x - y))$$

and, consequently,

$$K_+(y \times [t, +\infty]) = \sqrt{\frac{2}{\pi t}} y^2 + y.$$

Note that

$$\lim_{t \to +\infty} K_+(y \times [t, +\infty]) = K_+(y \times +\infty) = \lim_{x \downarrow y} \frac{P_x(\tau_y = +\infty)}{S(x) - S(y)}$$

$$= \frac{1}{S(+\infty) - S(y)} = y.$$

Finally we consider a three-dimensional Bessel bridge of length l from 0

to 0. This is a three-dimensional Bessel process started from 0 and conditioned to be at 0 at time l. We refer to [15] for a discussion of diffusion bridges. The law of our Bessel bridge is denoted with P_{00}^l, and we compute the distribution of the maximum. To obtain this, note that (cf. (3.4))

$$\frac{P_0(M_t \in da, T \in ds, X_l \in dx)}{P_0(X_l \in dx)} = \frac{n_0(s; a)n_x(l - s; a)}{p(l; 0, x)} S(da)ds,$$

where

$$p(l; 0, x) = \frac{1}{\sqrt{2\pi l^3}} \exp\left(-\frac{x^2}{2l}\right)$$

is the transition density of R w. r. t. the speed measure. Consequently

$$(3.10) \qquad P_{00}^l(M_l \in da, T \in ds) = \frac{\sqrt{2\pi l^3}}{a^2} n_0(s; a)n_0(l - s; a)dads.$$

This is (4.5.f) in [15], where it is explained how (3.10) is connected with the excursion theory of Brownian motion.

4. DISTRIBUTION OF THE MAXIMUM
OF THE BROWNIAN LOCAL TIME

Let X, $X_0 = 0$, be a standard Brownian motion and $L(x, \cdot)$ its local time at x i. e. the jointly continuous version of the density (w. r. t. the Lebesgue measure) of the occupation measure. Borodin [2] shows that

$$(4.1) \qquad P_0\left(\sup_x L(x, \hat{T}) < y\right) = 1 - \frac{4y\sqrt{2\lambda} \exp(y\sqrt{2\lambda})I_1\left(\frac{y\sqrt{\lambda}}{\sqrt{2}}\right)}{(\exp(y\sqrt{2\lambda}) - 1)^2 I_0\left(\frac{y\sqrt{\lambda}}{\sqrt{2}}\right)},$$

where $\hat{T} \sim \exp(\lambda)$ independently of X, and I_0, I_1 are modified Bessel functions.

As an application of our formula (2.1) we rederive (4.1). Our proof —as well as Borodin's—is based on the following result due to Ray [14] (see also McGill [13]): Let $L(0, \hat{T}) = x$, $X(\hat{T}) = y > 0$, and $L(X(\hat{T}), \hat{T}) = z$ be given. Then

i) $\qquad \{L(-a, \hat{T}); a \geq 0\} \sim \{R_a^0; a \geq 0; R_0^0 = x\}$

ii) $\qquad \{L(a, \hat{T}); 0 \leq a \leq y\} \sim \{R_a^2; 0 \leq a \leq y; R_0^2 = x\}$

iii) $\qquad \{L(a, \hat{T}); y \leq a\} \sim \{R_a^0; y \leq a; R_y^0 = z\},$

where « $\sim$ » means that the processes are identical in law and R^r, $r = 0, 2$, are diffusions having the generator

$$R^r = 2x \frac{d^2}{dx^2} - (2\beta x - r)\frac{d}{dx}, \qquad \beta = \sqrt{2\lambda},$$

respectively. Now consider differential equations $(r = 0, 2)$

$$R^r y = \alpha y, \qquad \alpha > 0.$$

These equations can be transformed to Kummer's equation (see [1], p. 504), and we find the unique strictly increasing and decreasing solutions $\phi_r^\uparrow$ and $\phi_r^\downarrow$, $r = 0, 2$, respectively; these are

$$\phi_0^\uparrow(x) = xM\left(1 + \frac{\alpha}{2\beta}; 2; \beta x\right), \qquad \phi_0^\downarrow(x) = xU\left(1 + \frac{\alpha}{2\beta}; 2; \beta x\right)$$

(4.2) $$\phi_2^\uparrow(x) = M\left(\frac{\alpha}{2\beta}; 1; \beta x\right), \qquad \phi_2^\downarrow(x) = U\left(\frac{\alpha}{2\beta}; 1; \beta x\right),$$

where M and U are the first and second Kummer's function, respectively.

Proof of (4.1). — Recall the joint distribution of $L(0, \hat{T})$ and $X(\hat{T})$:

$$\mathbf{P}_0(L(0, \hat{T}) \in dl, \ X(\hat{T}) \in dx) = \left(\int_0^{+\infty} \lambda e^{-\lambda t} \frac{l + |x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{(l + |x|)^2}{2t}\right) dt\right) dl \, dx$$

$$= \frac{1}{2} \beta^2 e^{-\beta(l + |x|)} dx \, dl,$$

where $\beta = \sqrt{2\lambda}$. For $X(\hat{T}) > 0$ let

$$M(\hat{T}) = \sup_{0 \le a \le X(\hat{T})} L(a, \hat{T}), \qquad H(\hat{T}) = \inf\{a : L(a, \hat{T}) = M(\hat{T})\},$$

and denote by $f(x, l, k, m, s)$, $x > 0$, $m > \max(l, k)$, $s > 0$, the joint $\mathbf{P}_0$-density of the variables $X(\hat{T})$, $L(0, \hat{T})$, $L(X(\hat{T}), \hat{T})$, $M(\hat{T})$, and $H(\hat{T})$. Conditioning on the values of $L(0, \hat{T})$ and $X(\hat{T})$, using *ii*) above, and (2.1) we obtain

$$f(x, l, m, k, s) = n_l(s; k)n_m(x - s; k)\frac{1}{k} e^{\beta k}\frac{1}{2} e^{-\beta m}\frac{1}{2} \beta^2 e^{-\beta(l + x)}.$$

Notice that

$$m(dx) = \frac{1}{2} e^{-\beta x} dx, \qquad S(dx) = \frac{1}{x} e^{\beta x} dx$$

are the speed and the scale measure, respectively, for the R^2 diffusion. Given $L(0, \hat{T})$ and $L(X(\hat{T}), \hat{T})$ the random variables $\sup_{a \le 0} L(a, \hat{T})$,

$\sup_{0 \le a \le X(\hat{T})} L(a, \hat{T})$ and $\sup_{a \ge X(\hat{T})} L(a, \hat{T})$ are independent. Further by i) and ii) above

$$\mathbf{P}_0 \left(\sup_{a \le 0} L(a, \hat{T}) < y \mid L(0, \hat{T}) = l \right) = \mathbf{R}_l^0(\tau_0 < \tau_y) = \frac{e^{\beta y} - e^{\beta l}}{e^{\beta y} - 1},$$

$$\mathbf{P}_0 \left(\sup_{a \ge X(\hat{T})} L(a, \hat{T}) < y \mid L(X(\hat{T}), \hat{T}) = m \right) = \frac{e^{\beta y} - e^{\beta m}}{e^{\beta y} - 1},$$

where $\mathbf{R}^0$ is the probability measure associated with the $\mathbf{R}^0$-diffusion. Notice that the scale function of $\mathbf{R}^0$ is $S(x) = e^{\beta x}$. Using these facts we obtain the desired distribution

$$\mathbf{P}_0 \left(\sup_a L(a, \hat{T}) < y \right) = 2\mathbf{P}_0 \left(\sup_a L(a, \hat{T}) < y, X(\hat{T}) > 0 \right)$$

$$= 2 \int \frac{e^{\beta y} - e^{\beta l}}{e^{\beta y} - 1} f(x, l, m, k, s) \frac{e^{\beta y} - e^{\beta m}}{e^{\beta y} - 1} \, dx \, dl \, dm \, dk \, ds,$$

where the integration is over the domain

$$D = \left\{ (x, l, m, k, s): 0 \le x < +\infty, 0 \le l \le k, 0 \le m \le k, 0 \le k \le y, 0 \le s \le x \right\}.$$

To check (4.1) we have to perform a tedious integration. To do this integrate first over x and s using (4.2)

$$(4.3) \quad \mathbf{P}_0 \left(\sup_a L(a, \hat{T}) < y \right) = \frac{1}{2} \frac{1}{(e^{\beta y} - 1)^2} \int_0^y \left(\frac{G_1(\beta, y, k)}{M\left(\frac{1}{2}, 1, \beta k\right)} \right)^2 \frac{1}{k} e^{\beta k} dk,$$

where

$$(4.4) \qquad G_1(\beta, y, k) = \int_0^k (e^{\beta y} - e^{\beta l})\beta e^{-\beta l} M\left(\frac{1}{2}, 1, \beta l\right) dl$$

$$= e^{\beta(y-k)}\beta k M\left(\frac{3}{2}, 2, \beta k\right) - \beta k M\left(\frac{1}{2}, 2, \beta k\right)$$

by [17] 2.1.12 and 2.1.5. Denoting the integral in (4.3) by S, using the integration (4.4) once, and changing the order of integration in the remaining double integral we obtain

$$S = \int_0^y (e^{\beta y} - e^{\beta m})\beta e^{-\beta m} M\left(\frac{1}{2}, 1, \beta m\right) G_2(\beta, y, m) dm,$$

where

$$G_2(\beta, y, m) = \int_m^y \beta e^{\beta y} \frac{M\left(\frac{3}{2}, 2, \beta k\right)}{\left(M\left(\frac{1}{2}, 1, \beta k\right)\right)^2} - \beta e^{\beta k} \frac{M\left(\frac{1}{2}, 2, \beta k\right)}{\left(M\left(\frac{1}{2}, 1, \beta k\right)\right)^2} \, dk$$

$$= -\frac{4 e^{\beta y}}{M\left(\frac{1}{2}, 1, \beta y\right)} + \frac{2(e^{\beta y} + e^{\beta m})}{M\left(\frac{1}{2}, 1, \beta m\right)}$$

by [17] 2.1.1 and 2.1.8. Therefore

$$S = -\frac{4 e^{\beta y}}{M\left(\frac{1}{2}, 1, \beta y\right)} G_3(\beta, y) + 2 G_4(\beta, y),$$

where

$$G_3(\beta, y) = G_1(\beta, y, y) = \beta y M\left(\frac{3}{2}, 2, \beta y\right) - \beta y M\left(\frac{1}{2}, 2, \beta y\right)$$

$$G_4(\beta, y) = \int_0^y (e^{\beta y} - e^{\beta m}) \beta e^{-\beta m} (e^{\beta y} + e^{\beta m}) \, dm = (e^{\beta y} - 1)^2.$$

Finally using [17] 2.2.4 and 1.8.3 we obtain

$$\mathbf{P}_0 \left(\sup_a L(a, \hat{T}) < y\right) = 1 - \frac{\beta y e^{\beta y}}{(e^{\beta y} - 1)^2} \frac{\left(2 M\left(\frac{3}{2}, 2, \beta y\right) - 2 M\left(\frac{1}{2}, 2, \beta y\right)\right)}{M\left(\frac{1}{2}, 1, \beta y\right)}$$

$$= 1 - \frac{e^{\beta y} (\beta y)^2 M\left(\frac{3}{2}, 3, \beta y\right)}{(e^{\beta y} - 1)^2 M\left(\frac{1}{2}, 1, \beta y\right)}$$

$$= 1 - \frac{4 \beta y e^{\beta y} I_1\left(\frac{\beta y}{2}\right)}{(e^{\beta y} - 1)^2 I_0\left(\frac{\beta y}{2}\right)},$$

since $\beta = \sqrt{2\lambda}$ we have (4.1).

REFERENCES

[1] M. ABRAMOWITZ, I. A. STEGUN, *Handbook of Mathematical Functions*. Dover, New York, 1965.

[2] A. N. BORODIN, Distribution of integral functionals of Brownian motion (in Russian). *Zap. Naučn. Sem. LOMI*, t. **119**, 1982, p. 19-38.

[3] K. L. CHUNG, Excursions in Brownian motion. *Ark. Mat.*, t. **14**, 1976, p. 155-177.

[4] A. ERDÉLYI (ed.). *Tables of Integral Transforms*, vol. I. McGraw-Hill, New York, Toronto, London, 1954.

[5] R. K. GETOOR, Excursions of a Markov process. *Ann. Prob.*, t. **7** (2), 1979, p. 244-266.

[6] J.-P. IMHOF, Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Prob.*, t. **21**, 1984, p. 500-510.

[7] J.-P. IMHOF, On Brownian bridge and excursion. *Studia Sci. Math. Hungar* (to appear).

[8] K. ITÔ, H. McKEAN, *Diffusion Processes and Their Sample Paths*. Springer, Berlin, 1965.

[9] F. B. KNIGHT, On the excursion process of Brownian motion. *Trans. Amer. Math. Soc.*, t. **258**, 1980, p. 77-86.

[10] P. LÉVY, *Processus Stochastiques et Mouvement Brownien*. Gauthier-Villars, Paris, 1948.

[11] G. LOUCHARD, Mouvement Brownien et valeurs propres du Laplacien. *Ann. Inst. H. Poincaré*, IV (4), 1968, p. 331-342.

[12] G. LOUCHARD, Kac's formula, Lévy's local time and Brownian excursion. *J. Appl. Prob.*, t. **21**, 1984, p. 479-499,

[13] P. McGILL, Markov properties of diffusion local time: a martingale approach. *Adv. Appl. Prob.*, t. **14**, 1982, p. 789-810.

[14] D. RAY, Sojourn times of diffusion processes. *Illinois J. Math.*, t. **7**, 1963, p. 615-630.

[15] P. SALMINEN, Brownian excursions, revisited. *Sem. Stochastic Processes*, 1983, eds E. Çinlar, K. L. Chung, R. K. Getoor, Birkhäuser, Boston, 1984.

[16] L. A. SHEPP, The joint density of the maximum and its location for a Wiener process with drift. *J. Appl. Prob.*, t. **16**, 1979, p. 423-427.

[17] L. J. SLATER, *Confluent Hypergeometric Functions*. University Press, Cambridge, 1960.

[18] I. VINCZE, Einige zweidimensionale Verteilungs- und Grenzverteilungssätze in der Theorie der geordneten Stichproben. *Publ. Math. Inst. Hungar. Acad. Sci.*, t. **11**, 1957, p. 183-203.

[19] D. WILLIAMS, Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.*, t. **28** (3), 1974, p. 738-768.

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