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Nucleation for a long range magnetic model

by

Francis COMETS
UA 743, CNRS, Laboratoire de Statistique Appliquée,
Bâtiment 425, Univ. Paris-Sud, 91405 Orsay Cedex

ABSTRACT. — We are interested in a local mean-field Ising model on the torus which exhibits two stable equilibria at low temperature and in the limit of infinite number of particles. Using large deviations techniques, we analyse the behaviour of the system during dynamical transitions from one equilibrium to the other: it is shown to be crucially dependent on the temperature and the interaction structure; symmetry breaking may occur, as in the asymptotic behaviour of the Gibbs measure.

Key-words: Mean-field, ising model, large deviations, nucleation.

RÉSUMÉ. — On considère un modèle d'Ising de champ moyen local sur le tore, qui présente deux états d'équilibre stable, dans l'asymptotique d'un nombre infini d'aimants et à température suffisamment basse. A l'aide de techniques de grandes déviations, on décrit le comportement du systeme lors des transitions dynamiques d'un de ces équilibres à l'autre : il dépend crucialement de la température ainsi que de la structure fine des interactions, et peut présenter une brisure de symétrie analogue à celle de la mesure de Gibbs.

I. INTRODUCTION

We are interested in long-time behaviour for a magnetic system, consisting in a large number N of Ising spins with fixed sites, and weak pair interaction (depending on distance between particles).

In the case of a ferromagnetic mean-field model without external influence, the Gibbs measure is concentrated on the neighbourhood of two stable steady states u^+, $-u^+$, at low temperature [*11*]. We consider a dynamic process, whose invariant probability is the Gibbs measure; on finite time intervals, it behaves—in first approximation—like the solution of an ordinary differential equation (the bigger N the better approximation) with u^+, $-u^+$ as stable equilibria. Because of ergodicity, the process starting near u^+ leaves the domain of attraction of u^+ in a finite time. Through this paper we study this type of dynamical phase transition and establish results conjectured by G. Ruget [*24*]. Such transitions can be studied using the theory of large deviations: one can refer to [*2*] [*15*] for finite dimensional processes. A quite recent reference to large deviations for distribution-valued processes is [*8*], with an application to the empirical distribution of a system of N weakly coupled diffusions; however, their model is quite different from the one studied in this paper.

Using large deviations estimates, we show under some conditions that the transition occurs at the neighbourhood of one of the « lowest saddle points » separating the two domains of attraction. We then give an example, where these saddle points can be found explicitely, and show how these results yield an explanation to nucleation [*23*]: at low temperature, the decisive step during a transition is the constitution of nuclei (of macroscopic size) in which local magnetization approaches that of the new equilibrium; these nuclei will later agregate as the whole system tends to the new equilibrium. The structure of the nuclei depends on the interaction function.

To make this more precise, we first define the *static model.*

For every integer n, we consider on $\mathbb{T} = (\mathbb{R}_{/\mathbb{Z}})^d$, the d-dimensional torus, $\mathrm{N} = n^d$ magnets located at each point x of a square lattice with mesh $\frac{1}{n}$; the magnetization at each point is represented by a spin $\eta^n(x) \in \{-1, +1\}$.

Let $\mathscr{S}^n = \left\{ x \in \mathbb{T};\ x = \left(\frac{r_1}{n_n}, \ldots, \frac{r_d}{n}\right),\ r_1, \ldots, r_d \in \{0, 1, \ldots, n-1\} \right\}$ be the set of N sites, and $\mathscr{E}^n = \{-1, +1\}^{\mathscr{S}^n}$ the set of configurations η^n, $\eta^n = (\eta^n(x))_{x \in \mathscr{S}^n}$.

These magnets undergo an external field, represented by an element h of $\mathrm{C}(\mathbb{T})$, the space of real continuous functions on $\mathbb{T}$, and interact according to a symmetric translation-invariant coupling represented by a symmetric function $\mathrm{J} \in \mathrm{C}(\mathbb{T})$. In statistical mechanics (cf. [*25*] [*26*]), one defines the internal energy of a configuration η^n as:

$$\mathrm{H}^n(\eta^n) = -\sum_{x \in \mathscr{S}^n} h(x)\eta^n(x) - \frac{1}{2\mathrm{N}} \sum_{x, y \in \mathscr{S}^n} \mathrm{J}(x - y)\eta^n(x)\eta^n(y) \qquad (1.1)$$

and the Gibbs measure on $\mathscr{E}^n$ as:

$$G^n(\eta^n) = \frac{1}{2^N Z_h^n} \exp - \beta H^n(\eta^n) \tag{1.2}$$

where β is proportional to the inverse of the temperature, and where the constant Z_h^n makes G^n a probability. The multiplicative coefficient $\frac{1}{N}$ in the interaction term in (1.1) ensures the existence of asymptotics when n goes to infinity. Notice that the interaction is long range, wherefore this model is qualitatively different from nearest neighbour ones (for example see [*26*]); but interaction intensity depends on the distance between particles, thus being more general than the *Curie-Weiss model*, in which h and J are constant [*12*] [*13*]. This is a *local mean field* model (or long range model).

Let us describe now the *dynamics*.

For each N-particles system, the configuration will evolve with time, according to a stationary and reversible Markov process, whose invariant measure is the Gibbs measure G^n; spins are allowed to flip, at most one at a time (Glauber's dynamics, see [*17*]).

For $x \in \mathscr{S}^n$, let $\tau_x : \mathscr{E}^n = \{-1, +1\}^{\mathscr{S}^n} \to \mathscr{E}^n$ the operator of flip at site x:

$$\tau_x \eta^n(y) = \begin{cases} \eta^n(y) & \text{if} \quad y \neq x \\ -\eta^n(x) & \text{if} \quad y = x \end{cases}$$

and Δ_x operating on functions $f : \mathscr{E}^n \to \mathbb{R}$,

$$\Delta_x f = f \circ \tau_x - f.$$

The configuration being η^n at time t, we imagine for each site x a clock delivering a random time t_x with exponential law with intensity parameter $c^n(x, \eta^n)$.

All these variables are supposed to be independent of one another, and of the past. Let x_0 be the site with shortest time t_{x_0}; at time $t + t_{x_0}$, one flips the spin in x_0, and the previous mechanism is restarted. The resulting random process of configurations is denoted by η_t^n; its infinitesimal generator is

$$L^n f(\eta^n) = \sum_{x \in \mathscr{S}^n} c^n(x, \eta^n) \Delta_x f(\eta^n) \tag{1.3}$$

In order to obtain the previous properties together with asymptotics as n goes to infinity, we will restrict to jump parameters c^n of a suitable form given below in (1.9 to 1.11). Our purpose is to establish large deviation results for the configuration process: these being closely related to the large deviations results for Gibbs measure, we recall now the latter ones.

As the set $\mathscr{E}^n$ of configurations depends on n, we will represent the state of the system by a measure σ^n

$$\sigma^n = \frac{1}{N} \sum_{x \in \mathscr{S}^n} \eta^n(x) \delta_x = \eta^n \lambda^n \tag{1.4}$$

where δ_x is the Dirac mass at point x, and $\lambda^n = \frac{1}{N} \sum_{x \in \mathscr{S}^n} \delta_x$. As in [*11*], we could as well consider the density of magnetization

$$\xi^n = \sum_{x \in \mathscr{S}^n} \eta^n(x) \mathbb{1}_{x + \left[0, \frac{1}{n}\right[^d} \tag{1.5}$$

which is constant on the cubes $x + \left[0, \frac{1}{n}\right[^d, x \in \mathscr{S}^n$.

It's easy to transfer properties obtained for one of the representations to the other. We will use (1.4) for calculations, which can be written formally in a simpler way: for instance, $H^n(\eta^n)$ is equal to $-N \left\langle h + \frac{1}{2} J^* \sigma^n, \sigma^n \right\rangle$ where $*$ denotes the convolution and $\langle , \rangle$ duality brackets. Nevertheless, in § 8, 9, we will consider ξ^n which is more suggestive.

Then σ^n belongs to the set $M_1(\mathbb{T})$ of all bounded measures μ on the Borel field of $\mathbb{T}$ with total variation norm $\| \mu \| \leqq 1$. $M_1(\mathbb{T})$ will be furnished with the weak-* topology τ^* (weakened by $C(\mathbb{T})$); since $\lambda^n \xrightarrow[n \to \infty]{\tau^*} \lambda$ the Haar probability measure on $\mathbb{T}$, the states of the system will be represented in the limit $n \to \infty$ by measures $u\lambda$, with density $u \in B$ the closed unit ball of $\mathbb{L}^\infty(\mathbb{T}) = \mathbb{L}^\infty(\mathbb{T}; \lambda)$.

The following results are due to Eisele and Ellis [*11*], for general spin distribution; see [*5*] for the lower bound; the techniques of [*16*] also extend to this situation.

THEOREM 1.1.

1) $$\lim_{n \to \infty} \frac{-1}{N\beta} \log Z_h^n = F_h$$

where the specific free energy F_h *is given by the variational problem*

$$F_h = \inf \{ V_h(u) ; u \in B \} \tag{1.6}$$

The potential V_h *is the* τ^**-lower-semi-continuous (l. s. c.) functional*

$$V_h(\mu)=-\left\langle u, h+\frac{1}{2}J*u\right\rangle+\frac{1}{\beta}\int_{\mathbb{T}}\phi(u(x))dx \text{ if } \mu=u\lambda, \text{ for some } u\in B \quad (1.7)$$

$V_h(\mu)=\infty$ *otherwise*

and ϕ *denotes the Cramer transform of the single spin distribution* $\frac{1}{2}(\delta_1+\delta_{-1})$:

$$\phi(W)=\frac{1+W}{2}\log(1+W)+\frac{1-W}{2}\log(1-W), \qquad W\in[-1,1].$$

2) *For all* $A\subset M_1(\mathbb{T})$,

$$F_h-\inf_{\mu\in\mathring{A}}V_h(\mu)\leq \underline{\lim}_{n\to\infty}\frac{1}{N\beta}\log G^n(A)\leq \overline{\lim}_{n\to\infty}\frac{1}{N\beta}\log G^n(A)\leq-\inf_{\mu\in\bar{A}}V_h(\mu)+F_h$$

(here, and in the following, we identify G^n *and its image by the application* $\eta^n\to\sigma^n$*).*

Therefore the support of any accumulation point of the sequence of probabilities G^n (on $M_1(\mathbb{T})$) is contained in the set of all the solutions of the variational problem (1.6). We will call stable equilibrium (or phase) any global minimum of V_h, metastable equilibrium any τ^*-local minimum of V_h, and more generally equilibrium any zero for the gradient (1)

$$dV_h(u) = -h-J*u+\frac{1}{\beta}\tanh^{-1}u \qquad (1.8)$$

Notice that an equilibrium is λ-equivalent to some element of $\mathscr{C}(\mathbb{T};\,]-1,1[)$.

If $h=0$ and $J\geq 0$, the model shows a phase transition (see previous references); for β greater the critical value $\beta_c=(\langle 1,J\rangle)^{-1}$, there are two stable equilibria, with constant densities u^+, and $-u^+$, where u^+ is the unique positive solution of the real equation associated to (1.7):

$$\tanh\frac{\beta}{\beta_c}u^+=u^+.$$

Now we define the jump parameters

$$c^n(x,\eta^n)=c(x,\sigma^n)\exp\{-\eta^n(x)\beta(h+J*\sigma^n)(x)\} \qquad (1.9)$$

(1) V_h is differentiable on $\{u;\|u\|_\infty<1\}$ with respect to uniform norm with differential $v\to\langle dV_h(u),v\rangle$. In (1.8), $\tanh^{-1}$ denotes the inverse function of tanh.

with c a *continuous* function on $\mathbb{T} \times \mathrm{M}(\mathbb{T})$ (set of all bounded measures on $\mathbb{T}$, furnished with topology τ^*) to $]0, +\infty[$. We furthermore assume that

$$\forall x \in \mathbb{T}, \quad \forall \mu \in \mathrm{M}_1(\mathbb{T}), \qquad c(x, \mu - \mu\{x\}\delta_x) = c(x, \mu) \tag{1.10}$$

and that there exists some C_0 (capital C will denote constants) such that

$$\| c(u_1) - c(u_2) \|_1 \leqq \mathrm{C}_0 \| u_1 - u_2 \|_1 \qquad \forall u_1, u_2 \in \mathbb{L}^1(\mathbb{T}) \tag{1.11}$$

Relations (1.9, 1.10) imply that « detailed balanced conditions » are fulfilled with respect to G^n (see [*25*]); the form of the multiplicative factor c of the exponential in (1.9) ensures us with the existence of asymptotics and (1.11) with the uniqueness of the limit process.

The simplest case is $c(x, \mu) = 1$, which is the situation considered in [*5*]. Other examples are given by $c(x, \mu) = f(\theta_1 * \mu(x), \ldots, \theta_\mathrm{K} * \mu(x))$ with $\theta_k \in \mathscr{C}(\mathbb{T})$ and $\theta_k(0) = 0$ for $k = 1, \ldots, \mathrm{K}$, and f a Lipschitz continuous function on $\mathbb{R}^\mathrm{K}$.

For any sign $\eta \in \{-1, +1\}$ let

$$c_\eta(x, \mu) = c(x, \mu) \exp\{ -\eta\beta(h + \mathrm{J} * \mu)(x) \} \tag{1.11 b}$$

Then
$$c^n(x, \eta^n) = \sum_{\eta \in \{-1,+1\}} \frac{1 + \eta\eta^n(x)}{2} c_\eta(x, \sigma^n) \tag{1.12}$$

Let g be a bounded measurable function on $\mathbb{T}$, $\mathrm{F}_g : \mu \to \langle g, \mu \rangle$; applying (1.3) to $f(\eta^n) = \frac{1}{\mathrm{N}} \sum_{x \in \mathscr{S}^n} g(x)\eta^n(x)$, we derive the infinitesimal generator ([2]) of the measure-value process σ_t^n, restricted to such linear functional F_g:

$$\mathrm{L}^n \mathrm{F}_g(\mu) = - \sum_{\eta \in \{-1,+1\}} \langle \mu + \eta\lambda^n, g c_\eta(\mu) \rangle \tag{1.13}$$

Because the particles are weakly interacting, it turns out that this process converges uniformly on finite time intervals to the solution $u_t \in \mathrm{B}$ of the ordinary differential equation

$$\frac{d}{dt} u_t = - \sum_{\eta \in \{-1,+1\}} (u_t + \eta) c_\eta(u_t) \tag{1.14}$$

$$= -2c(u_t)\sqrt{1 - u_t^2} \sinh \beta d \mathrm{V}_h(u_t) \tag{M. E.}$$

([2]) Still denoted by L^n.

the mean evolution equation [3]; the right hand side of (1.14) is obtained in taking the limit $n \to \infty$ in (1.13). In the simpler case of Curie-Weiss model, this law of large numbers may be found in physical literature (see [*18*]), and in [*22*] for a global mean field on $\mathbb{Z}^d$.

Notice that the equilibrium are the stationary points for equation (1.14). Furthermore, one can show that V_h is a Lyapunov function [4] for the dynamical system (1.14), in the sense that V_h is decreasing along its trajectories.

Hence, the transitions from the neighbourhood of a stable equilibrium to another are large deviations from the law of large numbers: we need estimates for the probability of such an event. We will obtain the following result:

let $T > 0$, $u_0 \in \mathscr{C}(\mathbb{T};]-1, 1[)$, σ_0^n a sequence of initial magnetization measures such that $\tau^* - \lim_{n\to\infty} \sigma_0^n = u_0$, and $A \subset \mathscr{D}\{ [0, T]; M_1(\mathbb{T}) \}$ the space of all right-continuous left-limited functions on $[0, T]$, with values in $(M_1(\mathbb{T}); \tau^*)$.

Let (A) be the set of interior points of A with respect to the uniform convergence topology, [A] its closure.

THEOREM I.2. — *There exists a functional* I_{0T} *such that the inequalities*

$$- \inf \{ I_{0T}(\varphi); \varphi \in (A), \varphi_0 = u_0 \} \le \liminf_{n\to\infty} \frac{1}{N} \operatorname{Log} P^n_{\sigma_0^n} \{ \sigma^n \in A \} \le$$

$$\le \limsup_{n\to\infty} \frac{1}{N} \operatorname{Log} P^n_{\sigma_0^n} \{ \sigma^n \in A \} \le - \inf \{ I_{0T}(\varphi); \varphi \in [A], \varphi_0 = u_0 \}$$

hold whenever $\{ \sigma^n \in A \}$ *is measurable for all* n($P^n_{\sigma_n^0}$ *denotes the law of the magnetisation process starting at* σ_0^n).

The action functional I_{0T}, or « energy », will be defined in section 3. It is such that $I_{0T}(\varphi) \ge 0$, with equality if and only if φ satisfies (1.14); furthermore, the least energy trajectories which leave a potential wells are time-reversed solutions of M. E., this least energy being related to the potential V_h.

In section 3, we also give some properties of I_{0T}, which are proved in appendix. We establish the Vent'sel-Freidlin estimates for large deviations in § 4.5. Technical difficulties essentially arise from the lack of regularity

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(3) From (1.11), (1.14) has a unique solution in $\mathbb{L}^1(\mathbb{T})$; a precise study on the of B shows that $\| u_t \|_\infty < 1$ for all $t > 0$.

(4) Use inequality $z \sinh z \ge z^2$, z real (notice that the vector field is not a gradient field).

of various functionals at the boundary (local magnetization equal to $+1$ or -1), this boundary not being rare enough (in the sense of large deviations probability) to be negligible. The lower bound for the large deviations probability is obtained in a manner slightly different from [*27*] in the finite dimensional case (another problem being the structure of neighbourhood of 0 in the weak topology); as for the upper bound, we first show a local estimate, then extend it similarly to the proof of Sanov's theorem [*3*]. The law of large numbers is a by-product of theorem V.1: it justifies intuitively some further choices, but will not be used in the proofs: therefore we do not give a more precise statement of it. Theorem I.2 is a straight consequence of theorems IV.1 and V.1 (see 7.6 in [*2*] for the proof). In § 7, we solve the problem of exit points from a basin of attraction; the result extends the well known one in [*15*]. The quasi-potential $W(u_e, u)$, which represents the minimal energy to go from an equilibrium u_e to u, is a lower semi-continuous function of u; but this doesn't change anything compared to the classical situation, as we can guess from the result of [*14*]. As an application, we study nucleation in a simple model.

II. BASIC PROPERTIES AND PRELIMINARIES

Since $\mathcal{S}^n$ is finite, there exist a probability space $(\Omega^n, \mathbb{F}, P^n)$ and a process η^n on Ω^n with generator L^n given by (1.3). For $\eta_0^n \in \mathcal{E}^n$, $P^n_{\eta_0^n}$ will denote the law of the configuration process $(\eta_t^n)_{t \in \mathbb{R}^+}$ starting at η_0^n, or, equivalently, of the measure value process $(\sigma_t^n)_{t \in \mathbb{R}^+}$. Let $\mathbb{F}_t$ be the σ-field generated by the variables η_s^n, $s \leq t$.

Let $g(t, x)$ be a bounded measurable function on $\mathbb{R}^+ \times \mathbb{T}$, such that the set $\{ t \in \mathbb{R}^+ ; \exists x \in \mathbb{T}, s \to g(s, x) \text{ is discontinuous at point } t \}$ is discrete. The process η_t^n is of bounded variation on every finite interval of $\mathbb{R}^+$ with probability 1, so we can define as Stieltjes integrals the quantities

$$\int_0^t \langle g_s, d\sigma_s^n \rangle = \frac{1}{N} \sum_{x \in \mathcal{S}^n} \int_0^t g(s^-, x) d\eta_s^n(x).$$

In the following, we shall use the following probabilistic results (see [*19*] or [*20*]), and use (1.12):

$i)$ $$M_t^n(g) = \int_0^t \langle g_s, d\sigma_s^n \rangle - \int_0^t L^n F_{g_s}(\sigma_s^n) ds$$

is a $(\mathrm{P}^n, \mathbb{F}_t)$-martingale with increasing process

$$\langle \mathrm{M}^n(g) \rangle_t = \frac{2}{\mathrm{N}} \int_0^t \sum_{\eta \in \{-1,+1\}} \langle \lambda^n + \eta\sigma_s^n, g_s^2 c_\eta(\sigma_s^n) \rangle \, ds. \tag{2.1}$$

ii) For $\mu \in \mathrm{M}_1(\mathbb{T})$ and h' bounded measurable function on $\mathbb{T}$ (so-called because it is formally an external field) let's define

$$\Gamma_n^*(\mu, h') = \sum_{\eta \in \{-1,+1\}} \left\langle \frac{\lambda^n + \eta\mu}{2}, (e^{-\eta\beta h'} - 1) c_\eta(\mu) \right\rangle \tag{2.2}$$

Then
$$\mathrm{R}_t^n(g) = \exp\left\{ \mathrm{N} \int_0^t \frac{\beta}{2} \langle g_s, d\sigma_s^n \rangle - \int_0^t \Gamma_n^*(\sigma_s^n, g_s) ds \right\} \tag{2.3}$$

is a $(\mathrm{P}^n, \mathbb{F}_t)$-martingale.

Let's define the probability $\tilde{\mathrm{P}}^n$, by its restriction to $\mathbb{F}_\mathrm{T}$

$$d\tilde{\mathrm{P}}^n / \mathbb{F}_\mathrm{T}, = \mathrm{R}_\mathrm{T}^n(g) \, d\mathrm{P}^n / \mathbb{F}_\mathrm{T}.$$

Denoting by $\tilde{c}_t^n$, $\tilde{\mathrm{L}}_t^n$ for $t \leq \mathrm{T}$ the analogues to (1.3, 1.12) with

$$\tilde{c}_{\eta,t} = c_\eta . \exp - \eta\beta g_t \tag{2.4}$$

instead of c_η, $\tilde{\mathrm{L}}_t^\eta$ is the infinitesimal generator of the process $\tilde{\mathrm{P}}^n$. In particular, the analogue of property *i)* is valid for this last process.

Because of (1.12, 2.4), $\tilde{\mathrm{P}}^n$ is the law of the magnetization process evolving under external field $h + g_t$. This fact is the counterpart of the duality relationship (1.6), in which F_h is written like the Legendre transform of V_0 (i. e. V_h for $h = 0$): magnetization u and external field h are conjugate variables. We will prove that the law of large numbers remains valid —with the coefficients $\tilde{c}$— for a large class of such (non stationary) processes $\tilde{\mathrm{P}}^n$ (see (4.7)).

We need some topological properties of the space $\mathrm{M}_1(\mathbb{T})$, that we state here for convenience:

PROPOSITION II.1. — *$(\mathrm{M}_1(\mathbb{T}), \tau^*)$ is a metrizable compact space.*

$\mathrm{M}_1(\mathbb{T})$ is the closed unit ball of $\mathrm{M}(\mathbb{T})$, so it is compact for weak-$*$ topology. $\mathscr{C}(\mathbb{T})$ is a separable space according to Stone-Weierstrass theorem, and $\mathrm{M}_1(\mathbb{T})$ is strongly bounded; so *[21]* τ^* is metrizable on $\mathrm{M}_1(\mathbb{T})$, and defined by the metric ρ

$$\rho(\mu, \nu) = \sup_{m \in \mathbb{Z}^d} \{ (1 + |m|)^{-1} | \langle \mu - \nu, e^{2i\pi m.x} \rangle | \}.$$

Notice that $\mu \in \mathrm{B}$ is equivalent to $0 \leq \dfrac{\mu + \lambda}{2} \leq \lambda$, and therefore B is τ^*-compact too.

Let $\rho_{0\mathrm{T}}(\mu, \nu) = \sup \{ \rho(\mu_t, \nu_t) ; t \in [0, \mathrm{T}] \}$ be the uniform metric on the finite time interval $[0, \mathrm{T}]$. By computations similar to those of the end of § 4, we can show that $u_0 \rightarrow u$ the solution of (1.4) starting at u_0, is continuous on (B, τ^*) to $\mathscr{C}([0, \mathrm{T}] ; \mathrm{B})$.

Through this paper, $\mathscr{A} = \{ \mathrm{A}_k ; k = 1, 2 \ldots \mathrm{K} \}$ will denote a partition of $\mathbb{T}$ in rectangles (i. e.: product of connected sets of $\mathbb{R}/\mathbb{Z}$) with non-empty interior.

Let $\pi^{\mathscr{A}}$ the projection operator associating to a measure μ the Radon-Nikodym derivative of its restriction $\mu/\mathscr{A}$ to the algebra generated by $\mathscr{A}$ with respect to $\lambda/\mathscr{A}$:

$$\pi^{\mathscr{A}} \mu = \frac{d\mu/\mathscr{A}}{d\lambda/\mathscr{A}} = \sum_{k=1}^{\mathrm{K}} \frac{\mu(\mathrm{A}_k)}{\lambda(\mathrm{A}_k)} \mathbb{1}_{\mathrm{A}_k}. \tag{2.5}$$

For $\mathscr{A}_n$ the algebra generated by the cubes $x + \left[0, \dfrac{1}{n}\right[^d$, $x \in \mathscr{S}^n$, one sees that $\xi^n = \pi^{\mathscr{A}_n} \sigma^n$. In § IV, V, we will use operator $\pi^{\mathscr{A}}$ to define sets that are approximately neighbourhoods of 0:

Proposition II.2. — i) *Given such a partition* $\mathscr{A}_0$, *and a* τ^*-*neighbourhood* $\mathscr{V}$ *of* 0 *in* $\mathrm{M}(\mathbb{T})$, *there exist a finer partition* $\mathscr{A}$ *and* $\varepsilon > 0$ *such that*

$$\forall \mu, \nu \in \mathrm{M}_1(\mathbb{T}), \qquad \| \pi^{\mathscr{A}}(\mu - \nu) \|_1 < \varepsilon \Rightarrow \mu - \nu \in \mathscr{V}.$$

ii) *Given* $\mathscr{A}$ *and* $\varepsilon > 0$, *there exist and integer* n_0 *and a weak neighbourhood* $\mathscr{V}$ *of* 0 *in* $\mathrm{M}(\mathbb{T})$ *such that for all* $u \in \mathrm{B}$, $n \geq n_0$ *and* $\sigma^n \in \mathscr{E}^n$.

$$\sigma^n - u \in \mathscr{V} \Rightarrow \| \pi^{\mathscr{A}}(\sigma^n - u) \|_1 < \varepsilon .$$

To prove *i*) use uniform approximation of continuous functions by step functions on $\mathscr{A}$, then recall the inequality $\| \mu \| \leq 1$; for *ii*) notice that a strip of width α on the torus contains at most $\left(\alpha + \dfrac{1}{n}\right)\mathrm{N}$ points of $\mathscr{S}^n$ lattice.

III. THE ACTION FUNCTIONAL $\mathrm{I}_{0\mathrm{T}}$

In this section we state some standard properties of the action functional $\mathrm{I}_{0\mathrm{T}}$. The proofs of the results III.3, 4 and 6, somewhat technical, are carried out in the appendix.

First of all, we anticipate the demonstration of theorem IV.1 in order to introduce the action functional in a heuristic manner. Let's fix some time T, and consider a smooth enough trajectory φ defined on $[0, T]$ with values in B; let's try and estimate the probability for the process σ^n to be uniformly close to φ on $[0, T]$, following the idea of [27].

We look for some exponential change of probability making φ the central path; since magnetization and external field are conjugate variables (see § 2), it will consist in an adequate choice of some extra external field $\tilde{h}_t$, under which φ *satisfies the mean evolution equation* M. E.: let $\tilde{P}^n$ be the probability law on $(\Omega^n, \mathbb{F}_T)$ of the magnetization process with external field $h + \tilde{h}_t$

$$\frac{d\tilde{P}^n}{dP^n} = R^n_T = \exp N \left\{ \int_0^T \frac{\beta}{2} \langle \tilde{h}_t, d\sigma^n_t \rangle - \int_0^T \Gamma^*_n(\sigma^n_t, \tilde{h}_t) dt \right\} \tag{3.1}$$

We then require the analogue of (1.14) for $\tilde{P}^n$

$$\dot{\varphi}_t = - \sum_{\eta\{-1,1\}} (\varphi_t + \eta)\tilde{c}_{\eta,t}(\varphi_t) \tag{3.2}$$

with $\dot{\varphi}_t$ the time derivative of φ_t and $\tilde{c}_{\eta,t}$ given by relation (2.4). Using (1.11 *b*), we derive the following expression for $\tilde{h}_t$:

$$\tilde{h}_t = - h - J * \varphi_t + \beta^{-1} \tan h^{-1} \varphi_t + \beta^{-1} \sin h^{-1} \frac{\dot{\varphi}_t}{2c(\varphi_t)\sqrt{1-\varphi_t^2}} \tag{3.3}$$

where $\tan h^{-1}$, $\sin h^{-1}$ denote the inverse functions of tan h, sin h.

Formally, the computation will consist in writing $P^n(\sigma^n \sim \varphi)$ as $\tilde{\mathbb{E}}^n \{ \mathbb{1}_{\{\sigma^n \sim \varphi\}} (R^n_T)^{-1} \}$, with $\tilde{\mathbb{E}}^n$ the expectation for $\tilde{P}^n$. For trajectories σ^n, close to φ, we replace approximately $\Gamma^*_n(\sigma^n_t, \tilde{h}_t)$ with $\Gamma^*_n(\varphi_t, \tilde{h}_t)$ and $\int_0^T \langle \tilde{h}_t, d\sigma^n_t \rangle$ with $\int_0^T \langle \tilde{h}_t, \dot{\varphi}_t \rangle dt$ using the law of large numbers for $\tilde{P}^n$. We now recall that φ is the central path for the process $\tilde{P}^n$, and obtain the estimate $\exp - N \int_0^T \{ \langle \tilde{h}_t, \dot{\varphi}_t \rangle - \Gamma^*_n(\varphi_t, \tilde{h}_t) \} dt$ for the previous probability. This justifies the

III.1. **Definition of the action functional I_{OT}.**

Because of (1.11 *b*, 2.2), we define for $u \in B$, $a \in \mathbb{R}$ and $h' \in \mathbb{L}^\infty(\mathbb{T})$

$$\Gamma(u, a, x) = c(\varphi) \sum_{\eta \in \{-1,+1\}} \frac{1 + \eta u}{2} e^{-\eta\beta(h + J*u)}(e^{-\eta\beta a} - 1)(x)$$

and

$$\Gamma^*(u, h') = \int_{\mathbb{T}} \Gamma(u, h'(x), x)dx. \tag{3.4}$$

For an evolution speed $v \in \mathbb{L}^1(\mathbb{T})$ of the magnetization, its Legendre transform is

$$\mathcal{H}^*(u, v) = \sup_{h' \in \mathbb{L}^\infty(\mathbb{T})} \left\{ \frac{\beta}{2} \langle v, h' \rangle - \Gamma^*(u, h') \right\} \tag{3.5}$$

$\Gamma^*(u, .)$ is a convex differentiable function on $\mathbb{L}^\infty(\mathbb{T})$. If $\| u \|_\infty < 1$, the supremum (3.5) is achieved for h' given by the right-hand side of formula (3.3) with u instead of φ_t, and is equal to

$$\mathcal{H}^*(u, v) = \int_{\mathbb{T}} \left[\frac{v}{2} \operatorname{Log} \frac{\dfrac{v}{2c(u)} + \sqrt{1 - u^2 + (v/2c(u))^2}}{1 - u} \right.$$
$$- \beta \frac{v}{2}(h + \mathrm{J} * u) + c(u) \left\{ -\sqrt{1 - u^2 + (v/2c(u))^2} \right.$$
$$\left. \left. + \cosh \beta(h + \mathrm{J} * u) - u \sinh \beta(h + \mathrm{J} * u) \right\} \right](x)dx \tag{3.6}$$

Troughout this paper, we furnish $\mathscr{C}([0, \mathrm{T}]; \mathrm{B})$ with metric $\rho_{0\mathrm{T}}$ defined in § 2; for an element φ of this space, we denote by (D) the following differentiability condition:

$\exists \dot{\varphi} \in \mathbb{L}^1([0, \mathrm{T}] \times \mathbb{T})$ *such that for all* $t \leq \mathrm{T}$,

$$\varphi_t(x) - \varphi_0(x) = \int_0^t \dot{\varphi}(s, x)ds \quad \lambda\text{-a. s.}$$

We will then denote $\dot{\varphi}(s, x) = \dot{\varphi}_s(x)$.

Definition III.1. — *The action functional* $\mathrm{I}_{0\mathrm{T}}$ *is*

$$\mathrm{I}_{0\mathrm{T}}(\varphi) = \begin{cases} \displaystyle\int_{[0,\mathrm{T}]} \mathcal{H}^*(\varphi_t, \dot{\varphi}_t)dt & \textit{if } \varphi \textit{ satisfies to property } (\mathrm{D}) \\ \infty & \textit{otherwise} \end{cases}$$

We shall say that an element φ of $\mathscr{C}([0, \mathrm{T}]; \mathrm{B})$ is absolutely continuous if for all $\varepsilon > 0$, there exists some $\Delta > 0$ such that for all integer i_0 and all rectangles $\mathrm{A}_1, \ldots, \mathrm{A}_{i_0}$ of $\mathbb{T}$, and all real numbers $s_1, t_1, \ldots, s_{i_0}, t_{i_0}$ satisfying to $0 \leq s_i < t_i \leq \mathrm{T}$, the inequality $\sum_{i \leq i_0} | t_i - s_i | \lambda(\mathrm{A}_i) < \Delta$ implies $\sum_{i \leq i_0} | \langle \varphi_{t_i} - \varphi_{s_i}, \mathbb{1}_{\mathrm{A}_i} \rangle | < \varepsilon$.

PROPOSITION III.2. — *$\varphi \in \mathscr{C}([0, \mathrm{T}]; \mathrm{B})$ satisfies* (D) *if and only if φ is absolutely continuous.*

The proof of the proposition is standard (see [*10*]), and is not carried out here.

III.2. **Some properties of the action functional.**

We first notice that if φ satisfies to (D), we can find a modification of $\dot{\varphi}$ such that $\varphi\dot{\varphi} \leq 0$ at all points (t, x) such that $| \varphi | = 1$. We will then suppose this condition fulfilled by functions u, v in the following of this section. We need some technical results for obtaining usual properties of $\mathrm{I}_{0\mathrm{T}}$:

PROPERTIES III.3.

a) $$\mathrm{I}_{0\mathrm{T}}(\varphi) = \sup_{f \in \mathbb{L}^\infty([0,\mathrm{T}] \times \mathbb{T})} \left\{ \int_0^{\mathrm{T}} \left[\frac{\beta}{2} \langle f(t,.), \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, f(t,.)) \right] dt \right\}$$
$$= \int_{[0,\mathrm{T}] \times \mathbb{T}} \mathscr{H}(\varphi_t, \dot{\varphi}_t(x), x) dt dx$$

with

$$\mathscr{H}(u, v(x), x) = \sup_{a \in \mathbb{R}} \left\{ \frac{\beta}{2} v(x) a - \Gamma(u, a, x) \right\} \tag{3.7}$$

b) $\mathrm{I}_{0\mathrm{T}}(\varphi) < \infty$ if and only if $\dot{\varphi} \operatorname{Log} | \dot{\varphi} |$, $\dot{\varphi} \operatorname{Log} \dfrac{1}{1 - \varphi} \mathbb{1}_{\{\dot{\varphi} > 0\}}$ and $\dot{\varphi} \operatorname{Log} \dfrac{1}{1 + \varphi} \mathbb{1}_{\{\dot{\varphi} < 0\}}$ are elements of $\mathbb{L}^1([0, \mathrm{T}] \times \mathbb{T})$.

c) There exists some constant K such that

$$\mathscr{H}(u, v, x) \leq \frac{|v|}{2} \left[\operatorname{Log} |v| + \mathbb{1}_{v>0} \operatorname{Log} \frac{1}{1-u} + \mathbb{1}_{v<0} \operatorname{Log} \frac{1}{1+u} + \mathrm{K} \right](x) + \mathrm{K}$$

(here, and up to property *e*) we write v for $v(x)$, no confusion being possible).

d) There exists some constant $\mathrm{K} > 0$ such that

$$\mathscr{H}(u, v, x) \geq \frac{1}{2} | v | \, [\operatorname{Log} | v | - \mathrm{K}] - \mathrm{K}$$

e) For $\gamma > 0$ we have

$$| \mathcal{H}(u, v, x) - \mathcal{H}(u_1, v, y) |$$
$$= (1 + | v |) \{ \mathcal{O}_\gamma [| u(x) - u_1(y) |] + \varepsilon_\gamma [| x - y | + \rho(u, u_1)] \}$$

for all u, u_1 such that $\| u \|_\infty, \| u_1 \|_\infty \leq 1 - \gamma$, all $x, y \in \mathbb{T}$ and $v \in \mathbb{R}$.

The property *a*) shows that one can reverse the order of the supremum and the integrals; *b*) is a characterisation of finite energy trajectories. With upper bound *c*) one can limit to consider magnetization densities avoiding the boundary points -1, $+1$. The continuity property *e*) is somewhat similar to condition (C) in [*27*]; « outer » speeds being forbidden at these boundary points, it only holds for non-zero γ. The regularity in the x variable is a (new) property that enables us to replace magnetization u with a smooth function on $\mathbb{T}$ in proposition III.6 *d*) shows how $\mathcal{H}$ increases at infinity; it is an usual property for Cramer transforms.

Furthermore one can notice that the condition required in [*2*] is not satisfied here, because the set of possible speeds is discontinuous at the boundary points -1, $+1$.

THEOREM III.4. — 1) $D_{I_0} = \{ \varphi ; I_{0T}(\varphi) \leq I_0 \}$ *is compact in* $\mathcal{C}([0, T]; B)$ *for all non negative* I_0.

2) *The functional* I_{0T} *is lower semi-continuous on* $\mathcal{C}([0, T]; B)$.

This result ensures us with existence of solution to variational problem $\min \{ I_{0T}(\varphi) ; \varphi \in A \}$ for closed subset A of $\mathcal{C}([0, T]; B)$.

Remark. — Whenever φ satisfies to (D), φ is continuous on $[0, T]$ with values in B furnished with $\| \,.\, \|_1$ norm; but this topology is too fine to make D_{I_0} compact.

In the proof of theorem IV.1, we will need a large enough class of smooth functions: piecewise $\mathcal{C}^{1,0}$ functions.

DEFINITIONS III.5. — *We define* $\mathcal{C}P_T^{1,0}$ *as the class of all* φ *of* $\mathcal{C}([0, T] \times \mathbb{T} ;]-1, 1[)$ *such that there exists a subdivision* $S = (t_k)_{k \leq k_0}$ *of* $[0, T]$ *with*:

$$\forall k \leq k_0 - 1, \frac{\partial \varphi}{\partial t} \text{ exists on } [t_k, t_{k+1}] \times \mathbb{T} \text{ and is continuous.}$$

Then φ satisfies to (D), and $\dot{\varphi} = \dfrac{\partial \varphi}{\partial t}$.

PROPOSITION III.6. — *Let* φ *with* $I_{0T}(\varphi) < \infty$, $\varphi_0 \in \mathcal{C}(\mathbb{T} ;]-1, 1[)$, γ, $\delta > 0$. *Then, there exists* $\tilde{\varphi} \in \mathcal{C}P^{1,0}$ *such that*

$$\varphi_0 = \tilde{\varphi}_0, \qquad \rho_{0T}(\varphi, \tilde{\varphi}) < \delta \qquad \textit{and} \qquad | I_{0T}(\varphi) - I_{0T}(\tilde{\varphi}) | < \gamma \quad (3.8)$$

IV. LARGE DEVIATIONS: LOWER BOUND FOR THE PROBABILITY OF PASSAGE IN A TUBELET

For $\varphi \in \mathscr{C}([0,T]; B)$ and $\delta > 0$, we define the $[0,T]$-tubelet with axis φ and radius δ as the set of all μ: $[0,T] \to M_1(\mathbb{T})$ such that $\rho_{0T}(\mu, \varphi) < \delta$. We shall denote it shortly by $\{ \varphi \}^\delta$.

THEOREM IV.1. — *Let $\delta>0$ and $\varphi \in \mathscr{C}([0,T]; B)$ with $\varphi_0 \in \mathscr{C}(\mathbb{T};]-1,1[)$. For all $\gamma > 0$, there exist an integer n_0 and $\delta_1 > 0$ such that $n \geq n_0$ implies*

$$P^n_{\sigma^n_0}\{ \rho_{0T}(\sigma^n, \varphi) < \delta \} \geq \exp - N\{ I_{0T}(\varphi) + \gamma \}$$

on the set $\{ \rho(\sigma^n_0, \varphi_0) < \delta_1 \}$.

□ *Proof.* — Suppose first $\varphi \in \mathscr{C}P^{1,0}_T$ (see def. III.5).

We can define the extra external field $\tilde{h}_t$ by (3.3) and the probability $\tilde{P}^n$ by (3.1); as written in the beginning of § III,

$$P^n_{\sigma^n_0}(\{ \varphi \}^\delta) = \tilde{\mathbb{E}}^n_{\sigma^n_0}\{ (R^n_t)^{-1} \mathbb{1}_{\{\varphi\}^\delta} \}. \tag{4.1}$$

φ being a smooth function, there exists a finite subset S of $[0,T]$ such that the family $\{ \tilde{h}_t; t \notin S \}$ be equicontinuous on $\mathbb{T}$; so is $\{ c_\eta(\mu); \eta \in \{ -1, +1 \}, \mu \in M_1(\mathbb{T}) \}$. Then, the Riemann sums in $\Gamma^*_n(\mu, \tilde{h}_t)$ converges to the λ-integral, uniformly for $t \notin S$ and $\mu \in M_1(\mathbb{T})$, and this last quantity converges uniformly to

$$\Gamma^*(\mu, \tilde{h}_t) = \sum_{\eta \in \{-1,+1\}} \left\langle \frac{\lambda + \eta\mu}{2}, (e^{-\eta\beta\tilde{h}_t} - 1)c_\eta(\mu) \right\rangle \tag{4.2}$$

which is an extension of (3.4). Recall that the generator $\tilde{L}^n_t$ of the process $\tilde{P}^n$ is given by (1.13) with $\tilde{c}_{\eta,t} = c_\eta . \exp - \eta\beta\tilde{h}_t$ instead of c_η: in particular, $M^n(\tilde{h}) = \int_0^T \langle \tilde{h}_t, d\sigma^n_t \rangle - \int_0^T \tilde{L}^n_t(F_{\tilde{h}_t})(\sigma^n_t)dt$ (notation F_g being defined just before (1.13) is a random variable with mean 0 for $\tilde{P}^n$ and variance less than $C_1 N^{-1}$, relation (2.1) showing that the constant C_1 depends only on φ. Using Chebicheff's inequality, we choose some integer n_1 such that

$$\forall n \geq n_1, \quad \forall \sigma^n_0 \in \mathscr{E}^n, \quad \tilde{P}^n_{\sigma^n_0}\{ | M^n_T(\tilde{h}) | \leq \gamma/6 \} \geq 3/4 \tag{4.3}$$

As above, we notice that $\tilde{L}^n_t(F_{\tilde{h}_t})(\mu)$ converges informly to $\tilde{L}_t(F_{\tilde{h}_t})(\mu)$, with

$$\tilde{L}_t(F_g)(\mu) = - \sum_{\eta \in \{-1,+1\}} \langle \mu + \eta\lambda, g\tilde{c}_{\eta,t}(\mu) \rangle \tag{4.4}$$

we then choose n_2 such that, for $n \geq n_2$, we can replace on $\{ M_T^n(\tilde{h}) \leq \gamma/6 \}$, up to an error of magnitude $\gamma/6$ for each operation, $\int_0^T \langle \tilde{h}_t, d\sigma_t^n \rangle$ with $\int_0^T \tilde{L}_t^n(F_{\tilde{h}_t})(\sigma_t^n)dt$, this last term with $\int_0^T \tilde{L}_t(F_{\tilde{h}_t})(\sigma_t^n)dt$, and Γ_n^* with Γ^*; we obtain:

$$\mathbb{1}_{\left\{M_t^n(\tilde{h}) \leq \frac{\gamma}{6}\right\}} R_T^n \leq \exp N \left\{ \int_0^T \left[\frac{\beta}{2} \tilde{L}_t(F_{\tilde{h}_t})(\sigma_t^n) - \Gamma^*(\sigma_t^n, \tilde{h}_t) \right] dt + \frac{\gamma}{2} \right\}. \quad (4.5)$$

We need the following result, where $L^{h'}$ denotes the operator given by (4.4) with h' instead of $\tilde{h}_t$ (i. e.: $c_\eta \exp - \eta\beta h'$ instead of $\tilde{c}_{\eta,t}$):

LEMMA IV.2. — *Let* Q *be a compact subset of* $\mathscr{C}(\mathbb{T})$. *The family* $\{ \mu \to \Gamma^*(\mu, h'), \mu \to L^{h'}(F_{h'})(\mu) ; h' \in Q \}$ *is equicontinuous on* $(M_1(\mathbb{T}), \rho)$.

We go on the proof of the theorem: since φ is smooth, Ascoli's theorem shows that $\{ \tilde{h}_t ; t \notin S \}$ is relatively compact; the lemma yields some $\delta' < \delta$ such that:

$$\left\{ \begin{array}{l} \mu_1, \mu_2 \in M_1(\mathbb{T}) \\ \rho(\mu_1, \mu_2) < \delta' \end{array} \right. \Rightarrow \left\{ \begin{array}{l} | \tilde{L}_t(F_{\tilde{h}_t})(\mu_1) - \tilde{L}_t(F_{\tilde{h}_t})(\mu_2) | < (3\beta T)^{-1}\gamma \\ | \Gamma^*(\mu_1, \tilde{h}_t) - \Gamma^*(\mu_2, \tilde{h}_t) | \leq (6T)^{-1}\gamma \end{array} \right. \forall t \in [0, T] - S \quad (4.6)$$

Now, we claim it's enough to find $n_3 \in \mathbb{N}$, $\delta_1 > 0$ with:

$$\rho(\sigma_0^n, \varphi_0) < \delta_1, \qquad n \geq n_3 \Rightarrow \tilde{P}_{\sigma_0^n}^n(\{ \varphi \}^\delta) \geq \frac{3}{4}. \quad (4.7)$$

Indeed, for $n \geq n_1 \vee n_2 \vee n_3$, relations (4.4, 8, 9) imply:

$$P_{\sigma_0^n}^n(\{ \varphi \}^\delta) \geq \tilde{\mathbb{E}}_{\sigma_0^n}^n \{ (R_T^n)^{-1} \mathbb{1}_{\{\varphi\}^{\delta'}} \mathbb{1}_{\left\{ |M_T^n(\tilde{h})| \leq \frac{\gamma}{6} \right\}} \}$$

$$\geq P_{\sigma_0^n}^n\left(\{ \varphi \}^{\delta'} \cap \left\{ | M_T^n(h) | \leq \frac{\gamma}{6} \right\} \right)$$

$$\exp - N\left(\int_0^T \left[\frac{\beta}{2} \tilde{L}_t(F_{\tilde{h}_t})(\varphi_t) - \Gamma^*(\varphi_t, h_t) \right] dt + 5\frac{\gamma}{6} \right).$$

Combining (4.3 and 7), we see that the last probability is not less than 0.5. φ being the central path for $\tilde{P}^n$, $\tilde{L}_t(F_{\tilde{h}_t})(\varphi_t) = \langle \tilde{h}_t, \dot{\varphi}_t \rangle$ holds for all $t \notin S$ (one can compute it from (3.2) and (4.4)): recalling then that $\tilde{h}_t$ is the solution to variational problem (3.5), we see that the term between brackets in the last exponential is equal to $\mathscr{H}^*(\varphi_t, \dot{\varphi}_t)$, this yields the desired result.

We now prove (4.7). From proposition I.4.*i*) we first fix some partition $\mathscr{A}$ of $\mathbb{T}$ in rectangles with non empty interior and positive ε such that

$$\forall \mu, \quad \nu \in \mathrm{M}_1(\mathbb{T}), \quad \| \pi^{\mathscr{A}}(\mu - \nu) \|_1 \leq \varepsilon \Rightarrow \rho(\mu, \nu) < \delta' \tag{4.8}$$

Let's consider a finer partition $\mathscr{A}_0 = \{ \mathrm{A}_k ; k \leq \mathrm{K}_0 \}$; for $\eta \in \{ -1, +1 \}^{\mathrm{K}_0}$, we set $h'_{\underline{\eta}} = \sum_{k=1}^{\mathrm{K}_0} \eta_k \mathbb{1}_{\mathrm{A}_k}$. We recall property *i)* in section II:

$$\mathrm{M}_t^n(\underline{\eta}) = \langle \sigma_t^n - \sigma_0^n, h'_{\underline{\eta}} \rangle - \int_0^t \tilde{\mathrm{L}}_s^n(\mathrm{F}_{h'_{\underline{\eta}}})(\sigma_s^n) ds$$

are $(\tilde{\mathrm{P}}^n - \mathrm{F}_t)$ martingales, which increasing process is uniformly bounded over $[0, \mathrm{T}]$ with C_2/N for some constant C_2 depending on φ. Since equality $\tilde{\mathrm{L}}_s(\mathrm{F}_g)(\varphi_s) = \langle g, \dot{\varphi}_s \rangle$ holds for all bounded measurable function g on $\mathbb{T}$ and all $s \notin \mathrm{S}$, and since $\langle \mu, \pi^{\mathscr{A}_0} f \rangle = \langle \pi^{\mathscr{A}_0} \mu, \pi^{\mathscr{A}_0} f \rangle$ for all $f \in \mathscr{C}(\mathbb{T})$, we derive:

$$\mathrm{M}_t^n(\underline{\eta}) = \langle \pi^{\mathscr{A}_0}(\sigma_t^n - \varphi_t), h'_{\underline{\eta}} \rangle - \langle \pi^{\mathscr{A}_0}(\sigma_0^n - \varphi_0), h'_{\underline{\eta}} \rangle - \int_0^t \mathrm{X}_s ds - \int_0^t [\tilde{\mathrm{L}}_s^n(\mathrm{F}_{h'_{\underline{\eta}}})(\sigma_s^n) - \tilde{\mathrm{L}}_s(\mathrm{F}_{h'_{\underline{\eta}}})(\sigma_s^n)] ds \tag{4.9}$$

with $\mathrm{X}_s = \tilde{\mathrm{L}}_s(\mathrm{F}_{h'_{\underline{\eta}}})(\sigma_s^n) - \tilde{\mathrm{L}}_s(\mathrm{F}_{h'_{\underline{\eta}}})(\varphi_s)$.

We state it's enough to show:

$$\forall s \notin \mathrm{S} \qquad | \mathrm{X}_s | \leq \mathrm{C}_3 \| \pi^{\mathscr{A}_0}(\sigma_s^n - \varphi_s) \|_1 + \varepsilon_0 (\operatorname{diam} \mathscr{A}_0) \tag{4.10}$$

where diam $\mathscr{A}_0$ denotes the diameter $\sup \{ | x - y | ; x, y \in \mathrm{A}_k, k = 1, \ldots, \mathrm{K}_0 \}$ of partition $\mathscr{A}_0$, ε_0 a function with limit zero, and C_3 some positive constant.

Indeed, we then fix partition $\mathscr{A}_0$ finer than $\mathscr{A}$ such that last term in (4.10) be less than $(\varepsilon/4\mathrm{T}) \exp - \mathrm{C}_3\mathrm{T}$. As above, we can suppose the last integral in (4.9) to be bounded with $(\varepsilon/4) \exp - \mathrm{C}_3\mathrm{T}$ for all n superior to some n_4: this time, the functions to be integrated with λ^n are equicontinuous on the rectangles A_k. At last, using property (2.1, *ii*) we can choose $\delta_1 > 0$ and n_5 such that $n \geq n_5$, $\rho(\sigma_0^n, \varphi_0) < \delta_1$ imply $\| \pi^{\mathscr{A}_0}(\sigma_0^n - \varphi_0) \|_1 < (\varepsilon/4) \exp - \mathrm{C}_3\mathrm{T}$. Then, (4.9) yields

$$| \langle \pi^{\mathscr{A}_0}(\sigma_t^n - \varphi_t), h'_{\underline{\eta}} \rangle | \leq | \mathrm{M}_t^n(\underline{\eta}) | + (3\varepsilon/4) \exp - \mathrm{C}_3\mathrm{T} + \mathrm{C}_3 \int_0^t \| \pi^{\mathscr{A}_0}(\sigma_s^n - \varphi_s) \|_1 ds \tag{4.11}$$

Using Doob's inequality for each martingale $\mathrm{M}_t^n(\underline{\eta})$, $\underline{\eta} \in \{ -1, +1 \}^{\mathrm{K}_0}$, we can control the probability of

$$\mathscr{X}^n = \{ \max_{\underline{\eta}} \max_{t \leq \mathrm{T}} | \mathrm{M}_t^n(\underline{\eta}) | \leq (\varepsilon/4) \exp - \mathrm{C}_3\mathrm{T} \}$$

with

$$\tilde{\mathrm{P}}^n_{\sigma^n_0}(\mathscr{X}^n) \geq 1 - 2^{\mathrm{K}_0}(25\mathrm{C}_2/\varepsilon^2\mathrm{N}) \exp 2\mathrm{C}_3\mathrm{T},$$
$$\geq 3/4 \text{ whenever } n \text{ is more than some } n_6.$$

Notice that $\| \pi^{\mathscr{A}_0}(\sigma^n_t - \varphi_t) \|_1 = \max_{\underline{\eta}} \langle \pi^{\mathscr{A}_0}(\sigma^n_t - \varphi_t), h'_{\underline{\eta}} \rangle$: for $n \geq n_3 = n_4 \vee n_5 \vee n_6$, relation (4.11) shows that

$$\| \pi^{\mathscr{A}_0}(\sigma^n_t - \varphi_t) \|_1 \leq \mathrm{C}_3 \int_0^{\mathrm{T}} \| \pi^{\mathscr{A}_0}(\sigma^n_s - \varphi_s) \|_1 ds + \varepsilon \exp - \mathrm{C}_3\mathrm{T}$$

holds on the set $\mathscr{X}^n \cap \{ \rho(\sigma^n_0, \varphi_0) < \delta_1 \}$. Using Gromwall's lemma, we derive $\sup_{t \leq \mathrm{T}} \| \pi^{\mathscr{A}_0}(\sigma^n_t - \varphi_t) \|_1 \leq \varepsilon$; since $\mathscr{A}_0$ is finer than $\mathscr{A}$, Jensen's inequality and (4.8) imply (4.7).

Now, let's prove (4.10): denoting the random function

$$2c(\sigma^n_s) \cosh \beta(h + \tilde{h}_s + \mathrm{J}^*\sigma^n_s) \quad \text{by} \quad \psi_s,$$

we have:

$$\begin{aligned} | \mathrm{X}_s | \leq\ & | \langle \varphi_s - \sigma^n_s, h'_{\underline{\eta}} \pi^{\mathscr{A}_0}\psi_s \rangle | + | \langle \varphi_s - \sigma^n_s, h'_{\underline{\eta}}(\psi_s - \pi^{\mathscr{A}_0}\psi_s) \rangle | \\ & + \sum_{\eta \in \{-1,+1\}} \langle | \varphi_s + \eta | e^{-\eta\beta(h + \tilde{h}_s + \mathrm{J}^*\sigma^n_s)}, c(\varphi_s) | e^{-\eta\beta \mathrm{J}^*(\varphi_s - \sigma^n_s)} - 1 | \\ & + | c(\varphi_s) - c(\pi^{\mathscr{A}_0}\sigma^n_s) | + | c(\pi^{\mathscr{A}_0}\sigma^n_s) - c(\sigma^n_s) | \rangle . \end{aligned}$$

The first bound is not more than $\| \psi_s \|_\infty \| \pi^{\mathscr{A}_0}(\sigma^n_s - \varphi_s) \|_1$; the second one can be controled with the continuity modulus of the (equicontinuous) family $\{ c(\mu);\ \mu \in \mathrm{M}_1(\mathbb{T}) \} \cup \{ \tilde{h}_t; t \notin \mathrm{S} \} \cup \{ \mathrm{J}, h \}$. For the last one, we use mean-value theorem for the derivative and inequality

$$| \mathrm{J}^*\mu | (x) \leq \| \mathrm{J} \|_\infty \| \pi^{\mathscr{A}_0}\mu \|_1 + \| \mu \| \ \| \mathrm{J}_x - \pi^{\mathscr{A}_0}\mathrm{J}_x \|_\infty$$

(denoting by $\mathrm{J}_x : y \to \mathrm{J}(x - y)$): $| e^{-\eta\beta\mathrm{J}^*(\varphi_s - \sigma^n_s)} - 1 |$ is bounded with $\mathrm{C}_4 \| \pi^{\mathscr{A}_0}(\sigma^n_s - \varphi_s) \|_1 + \varepsilon_1 (\operatorname{diam} \mathscr{A}_0)$ for some function ε_1 with limit zero.

Next we use relation (1.11) to get

$$\| c(\varphi_s) - c(\pi^{\mathscr{A}_0}\sigma^n_s) \|_1 \leq \mathrm{C}_0 \{ \| \varphi_s - \pi^{\mathscr{A}_0}\varphi_s \|_1 + \| \pi^{\mathscr{A}_0}(\sigma^n_s - \varphi_s) \|_1 \} .$$

At last, $\| c(\mu) - c(\nu) \|_\infty$ goes to zero with $\rho(\mu, \nu)$; but for all continuous function f on $\mathbb{T}$ and all measure $\mu \in \mathrm{M}_1(\mathbb{T})$,

$$\begin{aligned} | \langle \mu - \pi^{\mathscr{A}_0}\mu, f \rangle | &= | \langle \mu, f - \pi^{\mathscr{A}_0} f \rangle | \\ &\leq \langle | \mu |, | f - \pi^{\mathscr{A}_0} f | \rangle \\ &\leq \| f - \pi^{\mathscr{A}_0} f \|_\infty \end{aligned}$$

then $\sup \{ \rho(\mu, \pi^{\mathscr{A}_0}\mu); \mu \in \mathrm{M}_1(\mathbb{T}) \}$ goes to zero with $\operatorname{diam} \mathscr{A}_0$. All the considered functions being bounded, these estimates prove the statement (4.10).

In the general case, we suppose $I_{0T}(\varphi) < \infty$, the opposite case being trivial. Then proposition III.6 for φ, γ, $\delta/2$ yields a smooth trajectory $\tilde{\varphi}$, for which the previous computation apply. $\square$

$\square$ To end, we prove lemma IV.2: the functionals Γ^* and $L^{h'}$ are composed of two kinds of terms,

$$\mathscr{C}_1(\mu) = \langle \lambda, \theta \exp \eta\beta J^*\mu \rangle \qquad \text{and} \qquad \mathscr{C}_2(\mu) = \langle \mu, \theta c(\mu) \exp \eta\beta J * \mu \rangle$$

with

$$\theta \exp - \eta\beta h = 1, \exp \eta\beta h', \qquad \text{or} \qquad h' \exp \eta\beta h'.$$

Since $\mu \in M_1(\mathbb{T}) \to c(\mu) \in \mathscr{C}(\mathbb{T})$ is equicontinuous and bounded, equicontinuity for the first kind term will result from that of $\mu \to J^*\mu$. According to Stone-Weierstrass theorem one can uniformly approximate J with some trigonometric polynomial $f(x) = \sum_{\substack{q \in \mathbb{Z}^d \\ |q| \le m}} a_q \exp 2i\pi q.x$. Furthermore

$$\| J^*(\mu - \nu) \|_\infty \le 2 \| J - f \|_\infty + \sum_{|q| \le m} | a_q | \, | \langle \mu - \nu, \exp 2i\pi q.x \rangle |$$

for $\mu, \nu \in M_1(\mathbb{T})$, where the last duality brackets are linear continuous forms: one easily derive that $| \mathscr{C}_1(\mu) - \mathscr{C}_1(\nu) | = \varepsilon(\rho(\mu, \nu))$ for some function ε independent of μ, ν, with limit zero.

Using this to control $| \mathscr{C}_2(\mu) - \mathscr{C}_2(\nu) |$, one sees that the only extra work necessary is to bound $| \langle \mu - \nu, \theta c(\mu) \exp \eta\beta J^*\mu \rangle |$. Because of Ascoli's theorem, the family $\theta c(\mu) \exp \eta\beta J^*\mu$ is equicontinuous and bounded on $\mathbb{T}$, then totally bounded: taking a finite covering of this set with $\| - \|_\infty$-balls centered at points $g_k \in \mathscr{C}(\mathbb{T})$, $k \le K$, and radius $\delta > 0$, one can see that the previous quantity is some $\mathcal{O}\left(\delta + \sum_{k \le K} | \langle \mu - \nu, g_k \rangle | \right)$, which ends the proof. $\square$

V. UPPER BOUND FOR LARGE DEVIATIONS PROBABILITIES

Recall that $D_{I_0} = \{ \varphi \in \mathscr{C}([0, T]; B); I_{0T}(\varphi) \le I_0 \}$.

THEOREM V.1. — *Let $\gamma > 0$, $\varepsilon > 0$, $I_0 > 0$. There exists an integer n_0 such that for all $n \ge n_0$ and all σ_0^n,*

$$P^n_{\sigma_0^n} \{ \rho_{0T}(\sigma^n, D_{I_0}) \ge \varepsilon \} \le \exp \{ - N(I_0 - \gamma) \}.$$

In order to prove theorem, we need the two following results; the first one is a local upper bound, and will be obtained from Markov exponential inequality; the second one is a (very) rough global estimate.

LEMMA V.2. — *For all* $\varphi \in \mathscr{C}([0,T]; B)$, *and all* $I < I_{0T}(\varphi)$ *there exist* $\delta' > 0$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\forall n \geq n_1, \qquad P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} \leq \exp(-NI).$$

LEMMA V.3. — *For all* $a \geq 0$, *there exist a compact subset* Λ *of* $\mathscr{C}([0,T]; B)$ *with following property*: $\forall \delta > 0$, $\exists n_2$ *such that* $\forall n \geq n_2$,

$$P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \Lambda) \geq \delta \} \leq \exp(-Na).$$

□ We first prove theorem V.1:

Choose a compact set Λ from lemma V.3 with $a = I_0$. Then $\Lambda \cap \{ \varphi; \rho_{0T}(\varphi, D_{I_0}) \geq \varepsilon/2 \}$ is compact; for each element φ of this set, apply Lemma V 2 with $I = I_0$, and obtain some integer $n_1(\varphi)$ and some $\delta'(\varphi)$, that can be supposed less than ε without loss of generality. Then make a covering of the previous compact with a finite number K of open neighbourhoods $\left\{ \varphi; \rho_{0T}(\varphi, \varphi_k) < \frac{1}{2}\delta'(\varphi_k) \right\}$ where the φ_k belong to this compact.

Let $\delta = \frac{1}{2}\min\{ \delta'(\varphi_k); k \leq K \}$; $\rho_{0T}(\sigma^n, \Lambda) \leq \delta$ and $\rho_{0T}(\sigma^n, D_{I_0}) \geq \varepsilon$ imply $\rho_{0T}(\sigma^n, \varphi_k) < \delta'(\varphi_k)$ for some $k \leq K$; hence

$$P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, D_{I_0}) \geq \varepsilon \} \leq P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \Lambda) > \delta \} + \sum_{k \leq K} P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \varphi_k) < \delta'(\varphi_k) \},$$

which is less than $(K+1)e^{-NI_0}$, when $n \geq n_2 \vee \max_{k \leq K} n_1(\varphi_k)$; finally, for large n the last bound is less than $e^{-N(I_0-\gamma)}$. □

□ We now prove lemma V.2:

a) If φ is absolutely continuous, let $I < I_{0T}(\varphi)$ and $\gamma > 0$ with $I + 3\gamma < I_{0T}(\varphi)$; according to property III.3 *a*, there exists some $f \in \mathbb{L}^\infty([0,T] \times \mathbb{T})$ such that

$$\int_0^T \left[\frac{\beta}{2} \langle \dot{\varphi}_t, f_t \rangle - \Gamma^*(\varphi_t, f_t) \right] dt \geq I + 3\gamma.$$

The functions h, J and f being bounded, Lusin's theorem shows that we can suppose f to be continuous with respect to (t, x), and even $f \in \mathscr{C}^{1,0}$, using a density argument.

Let $\tilde{P}^n$ be the probability on $(\Omega^n, \mathbb{F}_T)$ defined by its Radon-Nikodym derivative with respect to the restriction of P^n to $\mathbb{F}_T$:

$$\frac{d\tilde{P}^n}{dP^n/\mathbb{F}_T} = R_T^n = \exp N\left\{ \int_0^T \frac{\beta}{2}\langle f_t, d\sigma_t^n \rangle - \int_0^T \Gamma_n^*(\sigma_t^n, f_t)dt \right\} \tag{5.1}$$

Then, we have

$$P_{\sigma_0^n}^n\{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} = \tilde{\mathbb{E}}_{\sigma_0^n}^n([R_T^n]^{-1}.\mathbb{1}_{\{\rho_{0T}(\sigma^n,\varphi)<\delta'\}}). \tag{5.2}$$

As $\eta_t^n(x)$ is $\tilde{P}^n$-almost surely of bounded variation on $[0, T]$ for every $x \in \mathscr{S}^n$, we integrate by parts:

$$\int_0^T \langle f_t, d\sigma_t^n \rangle = \int_0^T \langle f_t, \dot{\varphi}_t \rangle dt - \int_0^T \langle \sigma_t^n - \varphi_t, \dot{f}_t \rangle dt + [\langle \sigma_t^n - \varphi_t, f_t \rangle]_0^T \tag{5.3}$$

Like in the proof of theorem IV.1, we have for large n:

$$\left| \int_0^T [\Gamma_n^*(\sigma_t^n, f_t) - \Gamma^*(\sigma_t^n, f_t)]dt \right| \leq \gamma \quad \text{for all path } \sigma^n.$$

According to lemma IV.2, the family $\{ \mu \to \Gamma^*(\mu, f_t); t \leq T \}$ is equicontinuous on the compact $M_1(\mathbb{T})$; furthermore, $\{ \dot{f}_t, t \leq T \}$ is totaly bounded in $\mathscr{C}(\mathbb{T})$. Therefore we can choose $\delta' > 0$ such that $\rho_{0T}(\sigma^n, \ ) < \delta'$ implies the inequalities

$$\left| \int_0^T [\Gamma^*(\sigma_t^n, f_t) - \Gamma^*(\varphi_t, f_t)]dt \right| \leq \gamma$$

and

$$\left| \int_0^T \langle \sigma_t^n - \varphi_t, \dot{f}_t \rangle dt + [\langle \sigma_t^n - \varphi_t, f_t \rangle]_0^T \right| \leq \gamma .$$

Then, the last three inequalities, together with relations (5.1 to 3) yield for large n:

$$\begin{aligned} P_{\sigma_0^n}^n\{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} &\leq \exp\left\{ -N\int_0^T \left[\frac{\beta}{2}\langle f_t, \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, f_t)\right]dt + 3\gamma N \right\} \\ &\leq \exp(-NI). \end{aligned}$$

b) In the case of a non absolutely continuous function φ, let's fix $\gamma > 0$ such that for all $\Delta > 0$ there exist $s_i, t_i \in [0, T]$, $i = 1, \ldots, i_0$, $s_i < t_i$, and i_0

rectangles A_i of $\mathbb{T}$ with positive λ-measure, satisfying both inequalities

$$\sum_{i=1}^{i_0}(t_i - s_i)\lambda(A_i) < \Delta \text{ and } \sum_{i=1}^{i_0}|\langle \varphi_{t_i} - \varphi_{s_i}, \mathbb{1}_{A_i}\rangle| \geq \gamma.$$

By parting some A_i', s, then modifying them at the boundary, and increasing i_0, we can suppose without loss of generality that $\{A_i ; i \leq i_0\}$ is included in some partition $\mathscr{A}$ of $\mathbb{T}$ in rectangles with non-empty interior. Let b a positive real number, η_i the sign of

$$\langle \varphi_{t_i} - \varphi_{s_i}, \mathbb{1}_{A_i}\rangle \qquad \text{and} \qquad f = b\sum_{i=1}^{i_0}\eta_i \mathbb{1}_{]s_i,t_i]\times A_i}.$$

Let's define probability $\tilde{P}^n$ by (5.1) and this function f; we have:

$$-\frac{1}{N}\operatorname{Log} R_T^n = -\frac{\beta}{2}b\sum_{i=1}^{i_0}\eta_i\langle \mathbb{1}_{A_i}, \varphi_{t_i} - \varphi_{s_i}\rangle$$

$$-\frac{\beta b}{2}\sum_{i=1}^{i_0}\eta_i\langle \mathbb{1}_{A_i}, (\sigma_{t_i}^n - \varphi_{t_i}) - (\sigma_{s_i}^n - \varphi_{s_i})\rangle + \int_0^T \Gamma_n^*(\sigma_t^n, f_t)dt \quad (5.3b)$$

In the right-hand side member of this equality, the first term is not more than $-\frac{\beta b}{2}\gamma$, the second one not more than $\frac{\beta b}{2} i_0 \sup_{t\leq T}\|\pi^{\mathscr{A}}(\sigma_t^n - \varphi_t)\|_1$.

For the measure $\lambda^n + \eta\sigma_t^n$ is positive,

$$\Gamma_n^*(\sigma_t^n, b\mathbb{1}_A) \leq C_1 \sum_{\eta\in\{-1,+1\}}\left\langle \frac{\lambda^n + \eta\sigma_t^n}{2}, e^{\beta b \mathbb{1}_A} - 1\right\rangle = C_1\lambda^n(A)(e^{\beta b} - 1)$$

with constant $C_1 = \max\{c_\eta(x, \mu); \eta \in \{-1, +1\}, x \in \mathbb{T}, \mu \in M_1(\mathbb{T})\}$; so last term of (5.3 b) is less than $C_1(e^{\beta b} - 1)(\Delta + i_0 T \|\pi^{\mathscr{A}}\lambda^n - 1\|_1)$.

We now choose $b = 8I(\beta\gamma)^{-1}$ and $\Delta = I[C_1(e^{\beta b} - 1)]^{-1}$ .Proposition I.4.*ii*) for partition $\mathscr{A}$ and $\varepsilon = 2I(i_0\beta b)^{-1}$ yields $\delta' > 0$ such that, for large n, $\rho_{0T}(\sigma^n, \varphi) < \delta'$ implies $\|\pi^{\mathscr{A}}(\sigma_t^n - \varphi_t)\|_1 < \varepsilon$ for all $t \leq T$. $\|\pi^{\mathscr{A}}\lambda^n - 1\|_1 \leq \Delta(i_0 T)^{-1}$ for large enough n, so that (5.2) leads to

$$P_{\sigma_0^n}^n\{\rho_{0T}(\sigma^n, \varphi) < \delta'\} \leq \sup\{[R_T^n]^{-1}; \rho_{0T}(\sigma^n, \varphi) < \delta'\}$$
$$\leq \exp - NI \qquad \square$$

□ We next prove lemma V.3: suppose $a > 0$ (the case $a = 0$ is trivial). First of all, fix a sequence $\Delta_j, j = 1, 2, \ldots$ such that $T/\Delta_1 \in \mathbb{N}$,

$$\Delta_j \leq (a+1)/C_1 (\exp [2\beta(a+1)j] - 1), \qquad \Delta_j/\Delta_{j+1} \in \mathbb{N} - \{ 0, 1 \},$$

for $j = 1, 2, \ldots$; these conditions imply in particular that $T/\Delta_j \in \mathbb{N}$ and $j\Delta_j$ is decreasing.
For φ: $[0, T] \to$ B, let $\Delta^\varphi(j) = \sup \left\{ \Delta \geq 0; \sup_{\substack{|t-t'| \leq \Delta \\ t,t' \in [0,T]}} \| \varphi_t - \varphi_{t'} \|_1 < \frac{1}{j} \right\}$ be its modulus of uniform continuity for $\| . \|_1$ norm. Let's define $\Lambda' = \{ \varphi \in \mathscr{C}([0, T]; B); \Delta^\varphi(j) \geq \Delta_j, \forall j \geq 1 \}$; because $\| . \|_1$ norm dominates metric ρ Ascoli's theorem proves that Λ' is relatively compact in space $(\mathscr{C}([0, T]; B), \rho_{0T})$; so its closure Λ is compact.

Given $\delta > 0$, from proposition I.4.*i*) we can choose a partition $\mathscr{A} = \{ A_k; k \leq K \}$ of $\mathbb{T}$ in rectangles with non-empty interior and $\varepsilon > 0$ such that for every $\mu \in M_1(\mathbb{T})$, the δ-neighbourhood of μ in metric ρ contains all $\nu \in M_1(\mathbb{T})$ satisfying to $\| \pi^{\mathscr{A}}(\mu - \nu) \|_1 < \varepsilon$. Let's fix now $j_0 = [2/\varepsilon] + 1$, $t_m = m\Delta_{j_0}, m = 1, 2 \ldots m_0 = T/\Delta_{j_0}$. For $b > 0$ and $\underline{\eta} \in \{ -1, 1 \}^K$ we define $h'_\eta = \sum_{k=1}^{K} \eta_k \mathbb{1}_{A_k}$. Then,

$$M_t^{n,b}(m, \underline{\eta}) = \exp N \left\{ \langle bh'_{\underline{\eta}}, \sigma_t^n - \sigma_{t_m}^n \rangle - \int_{t_m}^t \Gamma_n^*(\sigma_s^n, bh'_{\underline{n}}) ds \right\} \tag{5.4}$$

is a P^n-martingale for $t \geq t_m$. Recall inequality $\Gamma_n^*(\sigma_s^n, bh'_{\underline{\eta}}) \leq C_1(e^{\beta b} - 1)$ from the proof of lemma V.2 (part *b*); since $\langle h'_{\underline{\eta}}, \sigma_t^n - \sigma_{t_m}^n \rangle = \langle h'_{\underline{\eta}}, \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \rangle$ is equal to $\| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1$ for at least one choice of $\underline{\eta}$,

$$\left\{ \sup_{t_m \leq t \leq t_{m+1}} \| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1 \geq \frac{1}{j_0} \right\}$$
$$\subset \bigcup_{\eta} \{ \sup_{t_m \leq t \leq t_{m+1}} M_t^{n,b}(m, \underline{\eta}) \geq \exp N \{ (b/j_0) - \Delta_{j_0} C_1(e^{\beta b} - 1) \} \}.$$

For each $\underline{\eta}$, we bound from above the conditional probability of this event using Kolmogorov's maximal inequality [7] by

$$\exp - N \{ (b/j_0) - \Delta_{j_0} C_1(e^{\beta b} - 1) \}.$$

Take $b = 2(a + 1)j_0$; this term is not more than $\exp - N(a + 1)$, because of the properties of the sequence Δ_j.

As a conclusion,

$$P_{\sigma_0^n}^n \{ R(j_0)^c \} \leq 2^K m_0 \exp - N(a + 1) \tag{5.5}$$

where $R(j_0)$ denotes the event $\{ \sup_{m < m_0} \sup_{t_m \leq t \leq t_{m+1}} \| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1 < 1/j \}$.

Let l^n be the random polygon on $[0, T]$, $\mathscr{A}$-measurable valued, with vertices at the points $(t_m, \pi^{\mathscr{A}}\sigma^n_{t_m})$. Of course, on $R(j_0)$, $\rho_{0T}(\sigma^n, l^n) < \delta$, so

$$P^n_{\sigma_0^n}\{\rho_{0T}(\sigma^n, \Lambda) \geq \delta\} \leq P^n_{\sigma_0^n}(R(j_0)^c) + P^n_{\sigma_0^n}(R(j_0) \cap \{l^n \notin \Lambda'\}) \quad (5.6)$$

In the following, we bound the last term of (5.6).

On $R(j_0)$, the slope of l^n satisfies

$$\| l^n_t - l^n_{t'} \|_1 / | t - t' | < (j_0 \Delta_{j_0})^{-1} \leq (j \Delta_j)^{-1} \quad \text{if} \quad j \geq j_0$$

(see remark at the beginning of the proof).

We derive from this $\Delta^{l^n}(j) \geq \Delta_j$ for $j \geq j_0$. For $j < j_0$, we show first that:

$$\max_{|t-t'| \leq \Delta_j} \| l^n_t - l^n_{t'} \|_1 = \max_{\substack{m,r \\ |t_m - t_r| \leq \Delta_j}} \| l^n_{t_m} - l^n_{t_r} \|_1 . \quad (5.7)$$

t [resp. t'] belongs to some interval $[t_m, t_{m+1}]$ [resp. $[t_r, t_{r+1}]$].

For $s \in [t_m \vee (t + t_r - t'), t_{m+1} \wedge (t + t_{r+1} - t')]$, $s \to l^n_s - l^n_{s+t'-t}$ is an affine function; $u \to \| u \|_1$ being a convex function, so is the product function: this one achieves its maximum value on the boundary of the interval. Thus, it's enough to show that $\| l^n_t - l^n_{t_m} \|_1$ is not more than the right-hand side of (5.7) when $t \in]t_r, t_{r+1}[$ and $| t - t_m | \leq \Delta_j$. In this situation, Δ_j/Δ_{j_0} being integer implies that $| t_r - t_m | \vee | t_{r+1} - t_m | \leq \Delta_j$. Combining this with the convexity of $s \to \| l^n_s - l^n_{t_m} \|_1$ yields the desired result.

From (5.7), we derive the inclusion

$$\{\Delta^{l^n}(j) < \Delta_j\} \subset \bigcup_{(m,r)} \{ \| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r}) \|_1 > 1/j \},$$

where the union extends to all couples (m, r) such that

$$0 \leq m < r \leq m_0 \wedge (m + \Delta_{j_0}/\Delta_j) .$$

For such a couple and $b_j > 0$, $\| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r} \|_1 \geq 1/j$ implies for at least one $\underline{\eta}$:

$$M^{n,b_j}_{t_r}(m, \underline{\eta}) \geq \exp N \{ (b_j/j) - \Delta_j C_1(e^{\beta b_j} - 1) \} .$$

We now choose $b_j = 2(a + 1)j$, we apply Bienaymé inequality to the positive variables $M^{n,b_j}_{t_r}(m, \underline{\eta})$ with expected value 1:

$$P^n_{\sigma_0^n}\{ \| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r}) \|_1 \geq 1/j \} \leq 2^K \exp - N(a + 1) .$$

Next, using the rough upper bound $m_0\Delta_j/\Delta_{j_0}$ of the number of couples (m, r), we get

$$\mathrm{P}^n_{\sigma^n_0}\left(\bigcup_{j<j_0}\{\Delta^{l^n}(j) < \Delta_j\}\right) \leq m_0\left(\sum_{j=1}^{j_0-1}\Delta_j/\Delta_{j_0}\right)2^{\mathrm{K}}\exp - \mathrm{N}(a+1).$$

Combining this with (5.5 and 6), we find

$$\mathrm{P}^n_{\sigma^n_0}\{\rho_{0\mathrm{T}}(\sigma^n, \Lambda) \geq \delta\} \leq 2^{\mathrm{K}}m_0\left(\sum_{j=1}^{j_0}\Delta_j/\Delta_{j_0}\right)\exp - \mathrm{N}(a+1),$$

so we can choose n_2 (depending on δ) such that the last bound is less than $\exp - \mathrm{N}a$ for all $n \geq n_2$. □

VI. PROPERTIES OF THE QUASIPOTENTIAL $\mathrm{W}(u_e, u)$

The quasipotential $\mathrm{W}(u_e, u)$ is the least energy necessary to join an equilibrium u_e to some point $u \in \mathrm{B}$:

$$\mathrm{W}(u_e, u) = \inf\{\mathrm{I}_{0\mathrm{T}}(\varphi);\ \varphi \in \mathscr{C}([0, \mathrm{T}]; \mathrm{B}),\ \varphi_0 = u_e,\ \varphi_\mathrm{T} = u,\ \mathrm{T} \in \mathbb{R}^+\} \quad (6.1)$$

Before studying the exit points of an attracting domain, we show some properties of the quasipotential. We say that u is attracted by u_e (or u is in the basin of attraction of u_e) if the solution u_t of (M. E.) starting at u goes to u_e as t tends to ∞ (τ^*-convergence implying here convergence in norm $\|\,.\,\|_\infty$, see [5]).

Hamilton-Jacobi equation corresponding to the free-time variational problem (6.1) is $\Gamma^*\left(u, \frac{2}{\beta}d\mathrm{W}\right) = 0$ where $d\mathrm{W}$ denotes the gradient of $\mathrm{W}(u_e, u)$ with respect to u. Combining (1.6), (3.4) one computes that $\Gamma^*(u, 2d\mathrm{V}_h(u)) = 0$; this shows the relation between large deviations results for the magnetization process, and the ones for the Gibbs measure we recalled in § 1.

PROPOSITION VI.1. — *a)* $\forall u \in \mathrm{B}$, $\mathrm{W}(u_e, u) \geq \beta\{\mathrm{V}_h(u) - \mathrm{V}_h(u_e)\}$.

b) *If u is attracted by u_e, the equality holds in* a).

c) *If φ is the line segment $[u, u']$ covered in the time $\mathrm{T} = \|u - u'\|_2$ with constant speed, $\mathrm{I}_{0\mathrm{T}}(\varphi) = \mathcal{O}(\|u - u'\|_2^{1/2-\varepsilon})$ for all $\varepsilon > 0$ and $u, u' \in \mathrm{B}$.*

We just give the sketch of the proof; refer to [5] for more details. Using the above remark, we have

$$\mathcal{H}^*(\varphi_t, \dot{\varphi}_t) \geq \frac{\beta}{2} \langle \dot{\varphi}_t, 2\,d\mathrm{V}_h(\varphi_t) \rangle - \Gamma^*(\varphi_t, 2\,d\mathrm{V}_h(\varphi_t)) \rangle \geq \beta \langle \dot{\varphi}_t, d\mathrm{V}_h(\varphi_t) \rangle .$$

Integrating over $[0, \mathrm{T}]$, we obtain the first inequality.

The path φ described in *c)* is $\varphi_t = u + \frac{t}{\mathrm{T}}(u' - u)$. Let's fix $x \in \mathbb{T}$, and suppose that $u'(x) > u(x)$, the other case being treated similarly; we shortly denote $u(x)$ by u, $u'(x)$ by u'. From property III.3.*c)*, we derive

$$\int_0^t \mathcal{H}(\varphi_t, \dot{\varphi}_t, s)dx \leq \frac{u' - u}{2} [\log(u' - u) - \log \mathrm{T}] + [\theta(\log\theta - 1)]_{1-u}^{1-u'}$$
$$+ \frac{\mathrm{K}}{2}(u' - u) + \mathrm{KT} \tag{6.2}$$

$f(\theta) = \theta \log \theta$ being a convex function on $[0, 2], 0 \leq \theta' \leq \theta \leq 2$ implies $f(\theta') - f(\theta) \leq f(0) - f(\theta - \theta') = -f(\theta - \theta')$; the second term of (6.2) is then bounded from above by $-(u' - u)\log(u' - u) + (u' - u)$. Using Hölder's inequality together with the boundedness of $\theta^\varepsilon \log \theta$ on $[0, 2]$, we can easily prove *c)*.

In order to show *b)* let's notice that there exists a unique function on $]-\infty, 0]$ in B such that $\varphi_0 = u$ and the field h' maximizing (3.5) along the trajectory be equal to $2\,d\mathrm{V}_h(\varphi_t)$: because of (3.2), it is the solution starting at $\varphi_0 = u$ of the mean evolution equation time being reversed

$$\frac{d}{dt}\varphi_t = 2c(\varphi_t)\sqrt{1 - \varphi_t^2}\ \mathrm{sh}\ \beta d\mathrm{V}_h(\varphi_t). \tag{6.3}$$

Such a trajectory φ will be called and extremal. Since φ_t converges to u_e in $\mathbb{L}^2(\mathbb{T})$ as t goes to $-\infty$, *b)* is a consequence of *c)*.

The previous results are valid in general finite dimensional situations [*15*]. But, in our case, the potential V_h (and $d\mathrm{V}_h$ too) is not continuous in the weak topology. We then need some extra results:

PROPOSITION VI.2. — *There exist positive constants* K, K′ *such that for all trajectories* φ *on* $[0, \infty[$ *with values in* B, *and all* $\mathrm{T} > 0$

$$\mathrm{I}_{0\mathrm{T}}(\varphi) \geq -\mathrm{K} + \mathrm{K}' \int_0^{\mathrm{T}} \|\, 1 \wedge |\, d\mathrm{V}_h(\varphi_t)\,|\,\|_2^2 dt .$$

In particular, whenever $\mathrm{I}_{0\infty}(\varphi) < \infty$, *there exists a sequence* $t_m \to \infty$ *such that* φ_{t_m} *converges to an equilibrium in* $\mathbb{L}^2(\mathbb{T})$.

By an easy calculation, one sees that

$$\Gamma(u, d\mathrm{V}_h(u), x) = c(u)\left[- \sum_{\eta \in \{+1,-1\}} \frac{1+u}{2} e^{-\eta\beta(h+\mathrm{J}*u)} + \sqrt{1-u^2} \right],$$

the last quantity being evaluated at point x.

On one hand we see that

$$\Gamma(u, d\mathrm{V}_h(u), x) \leq c(u)[-\mathrm{K}'' + \sqrt{1-u^2}] \tag{6.4}$$

with constant $\mathrm{K}'' = \exp - \beta(\| h \|_\infty + \| \mathrm{J} \|_\infty) < 1$. On the other hand,

$$\begin{aligned} \Gamma(u, d\mathrm{V}_h(u), x) &= -2c(u)\sqrt{1-u^2}\, \mathrm{sh}^2 \frac{\beta}{2} d\mathrm{V}_h(u) \\ &\leq -2c(u)\sqrt{1-u^2}\left[\frac{\beta}{2} d\mathrm{V}_h(u)\right]^2 \end{aligned} \tag{6.5}$$

because of the inequality $|\mathrm{sh}\, z| \geq |z|$ for $z \in \mathbb{R}$.

Combining (6.4 and 5), we deduce that $\Gamma^*(u, d\mathrm{V}_h(u)) \leq -\mathrm{K}' \| 1 \wedge | d\mathrm{V}_h(u) | \|_2^2$ for some positive constant K' depending on K'' and $\min_{x,u} c(x, u)$. We have

$$\begin{aligned} \mathrm{I}_{0\mathrm{T}}(\varphi) &\geq \int_0^{\mathrm{T}} \left[\frac{\beta}{2} \langle d\mathrm{V}_h(\varphi_t), \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, d\mathrm{V}_h(\varphi_t)) \right] dt \\ &\geq \frac{\beta}{2} [\mathrm{V}_h(\varphi_\mathrm{T}) - \mathrm{V}_h(\varphi_0)] + \mathrm{C}' \int_0^{\mathrm{T}} \| 1 \wedge | d\mathrm{V}_h(\varphi_t) | \|_2^2 dt ; \end{aligned}$$

since V_h is bounded on B, this yields the desired inequality.

Suppose now that $\mathrm{I}_{0\infty}(\varphi) < \infty$. We can find some sequence $t_m \to \infty$ such that $\| 1 \wedge | d\mathrm{V}_h(\varphi_{t_m}) | \|_2^2 = \lambda(\mathrm{A}_m^c) + \| d\mathrm{V}_h(\varphi_{t_m}) . \mathbb{1}_{\mathrm{A}_m} \|_2^2$ goes to 0, A_m denoting the subset $\{ | d\mathrm{V}_h(\varphi_{t_m}) | < 1 \}$ of $\mathbb{T}$.

Then, write φ_{t_m} as $\tanh \{ \beta(h + \mathrm{J}*\varphi_{t_m}) + d\mathrm{V}_h(\varphi_{t_m}) . \mathbb{1}_{\mathrm{A}_m} \} . \mathbb{1}_{\mathrm{A}_m} + \varphi_{t_m} . \mathbb{1}_{\mathrm{A}_m^c}$; $\| \varphi_{t_m} \|_\infty \leq 1$ implies that the second term tends to 0 in $\mathbb{L}^2(\mathbb{T})$; considering a subsequence of $(t_m)_m$, we may suppose that φ_{t_m} converges in the weak topology to some $u \in \mathrm{B}$; then, $\mathrm{J} * \varphi_{t_m}$ goes to $\mathrm{J} * u$ uniformly, and we easily deduce that the first term converge in $\mathbb{L}^2(\mathbb{T})$ to $\tanh \beta(h + \mathrm{J} * u)$. To see that u is an equilibrium, write

$$\tanh \beta(h + \mathrm{J} * u) = \| \cdot \|_2 - \lim_{m \to \infty} \varphi_{t_m} = \tau^* - \lim_{m \to \infty} \varphi_{t_m} = u.$$

VII. THE EXIT POINTS FROM THE ATTRACTING DOMAIN OF A METASTABLE STATE

In the next two sections, we will consider the magnetization density ξ^n given by (1.5). Obviously there exists some function ε with $\lim_{n\to\infty} \varepsilon(n) = 0$ and: for all configuration

$$\rho(\xi^n, \sigma^n) \leq \varepsilon(n). \tag{7.1}$$

Hence theorems IV.1 and V.1 are still valid for the process ξ_t^n. For a subset Z of B, we will denote by $\partial Z, \overline{Z}, \ldots$ [resp. $\partial^2 Z, \overline{Z}^2, \ldots$] the boundary, the closure,... of Z in τ^* topology [resp. in the $\| \cdot \|_2$ norm topology on B]; for positive δ, $\mathscr{V}_\delta(Z)$ will denote the (closed) δ-neighbourhood of Z in metric ρ: $\mathscr{V}_\delta(Z) = \{ u \in B; \rho(u, Z) \leq \delta \}$.

In this section, we make somewhat general assumptions, which are satisfied in the example of § VIII: we consider an equilibrium u_e, τ^*-asymptotically stable in the Lyapunov sense for the mean evolution equation (M. E.). Because of the continuity of $u_0 \to u$ (see § 2), its basin of attraction $\mathscr{B}_e$ is a weakly open subset of B. We are interested in the situation where there exist at least two locally stable equilibria, so we suppose $\overline{\mathscr{B}_e} \neq B$. Let Ex be the set of the « lowest saddle points » on the boundary:

$$\mathrm{E}x = \{ u \in \partial\mathscr{B}_e; V_h(u) = \min \{ V_h(w); w \in \partial\mathscr{B}_e \} \} \tag{7.2}$$

V_h being l. s. c., Ex is weakly compact; since $u_0 \to u$ is continuous, and since V_h is a Lyapunov function for M. E., its elements are equilibria.

Throughout this section, $V_h(\mathrm{E}x)$ will denote the value of V_h on Ex, and $\Delta = \beta \{ V_h(\mathrm{E}x) - V_h(u_e) \}$ the height of the potential barrier.

We require the following hypothesis (H):

i) $\mathrm{E}x \cap \partial^2\mathscr{B}_e \neq \emptyset$; let $u_{\mathrm{E}x}$ be one of its elements.

ii) $\mathrm{E}x \subset \partial^2 [(\overline{\mathscr{B}_e})^c]$.

There exists positive δ_0 such that

iii) $\mathscr{V}_{\delta_0}(\mathrm{E}x) \cap \{ u \in \partial\mathscr{B}_e - \mathrm{E}x; dV_h(u) = 0 \} = \emptyset$

iv) $\mathscr{V}_{\delta_0}(\partial\mathscr{B}_e) \cap \{ u \in B; dV_h(u) = 0 \} \subset \partial\mathscr{B}_e$.

THEOREM VII.1. — *Let τ be the exit time for ξ_t^n from the basin of* *attraction $\mathscr{B}_e$. Under assumptions* (H), *we have for all weakly closed subset* F *of $\mathscr{B}_e$ and all $\delta > 0$*

$$\lim_{n\to\infty} \inf_{\xi_0^n \in \mathscr{F}} \mathrm{P}^n_{\xi_0^n}\{ \xi^n_\tau \in \mathscr{V}_\delta(\mathrm{E}x) \} = 1$$

where $\mathrm{P}^n_{\xi_0^n}$ denotes the law of process ξ^n_t starting at ξ^n_0.

The theorem states that, for large enough n, the process leaves the basin of attraction of such an equilibrium at the neighbourhood of one of the « lowest saddle points » on the boundary. It extends Vent'sel-Freidlin result ([15] [*1*]) about the exit point of a compact set strictly contained in an attracting domain, under the assumption that the vector field at the boundary be transverse and pointed inwards.

The technical assumptions (H) *i*) *ii*) cannot reduce to only local conditions holding at exit points; they are satisfied whenever the frontier $\partial\mathscr{B}_e$ is smooth, for example a one dimensional Banach $\mathscr{C}^1$ manifold. The hypothesis (H) *iii*) *iv*) concern the accumulation points of equilibria at the neighbourhood of $\partial\mathscr{B}_e$.

As in the previous references, we will study long time behaviour using finite time estimates of theorems IV.1 and V.1 together with the fact that the magnetization process restarts afresh from Markov stopping times. The structure of the stopping times we use is quite different from the one of [*15*] [*1*], because the quasipotential W is not continuous in the weak topology, but only in a strong one; we must furthermore take into account the equilibria located in $\partial\mathscr{B}_e$.

We will outline the proof after relation (7.5); we first reduce the analysis of the random path to its final part.

□ It's enough to prove the theorem for $\delta < \delta_0 \wedge \rho(u_e, \mathrm{E}x)$. Recall definition (7.2); since V_h is lower semicontinuous, and $\partial\mathscr{B}_e - \mathscr{V}_\delta(\mathrm{E}x)$ is τ^*-compact, one can find positive numbers α and $\delta_1 < \delta/2$ such that (5)

$$\forall u \in \mathscr{V}_{2\delta_1}(\mathrm{NE}x), \qquad \beta\mathrm{V}_h(u) \geq \beta\mathrm{V}_h(\mathrm{E}x) + \alpha \tag{7.3}$$

where NEx stands for $\partial\mathscr{B}_e - \mathscr{V}_\delta(\mathrm{E}x)$.

We consider (small) neighbourhoods $\mathscr{V}_{\gamma_e}(u_e)$, $\mathscr{V}_{\gamma_3}(\mathrm{E}x)$, and (large) time T. We first carry out the proof with initial condition ξ_0^n in $\mathscr{V}_{\gamma_e}(u_e)$ instead of $\mathscr{F}$.

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(5) Subscripts 1 will be used for NEx, subscripts 3 for Ex, e for u_e and 2 for points outside of $\mathscr{B}_e$. (NEx is defined after next relation (7.3)).

Let's define the stopping times:

$\tau_e^0 = 0$

$\tau_3^0 =$ the entrance time of ξ_t^n in $\mathscr{V}_{\gamma_3}(\mathrm{E}x)$, and for $k = 0, 1, \dots$

$\tau_e^{k+1} = \min \{ t \geq \tau_3^k \wedge (\tau_e^k + \mathrm{T}) ; \xi_t^n \in \mathscr{V}_{\gamma_e}(u_e) \}$

$\tau_3^{k+1} = \min \{ t \geq \tau_e^{k+1} ; \xi_t^n \in \mathscr{V}_{\gamma_3}(\mathrm{E}x) \}$

$\tau_1 =$ the entrance time in $\mathscr{V}_{\delta_1}(\mathrm{NE}x)$.

Let ν_e be the last integer k such that $\tau_e^k < \tau$, and $\mathrm{R}_k = \{ \nu_e = k \}$.

It is enough to show that, for $\xi_0^n \in \mathscr{V}_{\gamma_e}(u_e)$ and sufficiently large n:

$$\forall k, \quad \mathrm{P}^n_{\xi_0^n} \{ \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x) ; \mathrm{R}_k \} \geq q^2 \mathrm{P}^n_{\xi_0^n} \{ \mathrm{R}_k \} \tag{7.4}$$

with $q = 1 - 2 \exp - \mathrm{N}\alpha/6$: indeed, summing this relation over all k provides us with the theorem.

Using strong Markov property on the set $\{ \tau_e^k < \tau \}$, we obtain:

$$\begin{aligned}\mathrm{P}^n_{\xi_0^n} \{ \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x) ; \mathrm{R}_k \} &= \mathbb{E}^n_{\xi_0^n} \{ \mathbb{1}_{\tau_e^k < \tau} . \mathrm{P}^n_{\xi_0^n} \{ \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x) ; \tau \leq \tau_e^{k+1} / \mathbb{F}_{\tau_e^k} \} \} \\ &= \mathbb{E}^n_{\xi_0^n} \{ \mathbb{1}_{\tau_e^k < \tau} . \mathrm{P}^n_{\xi^n_{\tau_e^k}} \{ \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x) ; \tau \leq \tau_e^1 \} \} .\end{aligned}$$

The same computation for the right hand side probability in (7.4) shows we only need to prove this inequality for $k = 0$. We now decompose R_0 according to

$$\begin{aligned}\Omega^n = \{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2\mathrm{T} \} \cup \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2\mathrm{T} \} \\ \cup \{ \tau \leq \tau_3^0 \wedge 2\mathrm{T}, \tau < \tau_1 \} \cup \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2\mathrm{T} \}\end{aligned} \tag{7.5}$$

The main contribution in decomposition (7.5) to the probability of R_0 is given by the two last terms. The contribution of first term will be negligible, because the process cannot spend too much time far away from the equilibria (lemma 4). The trajectories close to the second set hit

$$\mathscr{V}_{2\delta_1}(\partial \mathscr{B}_e - \mathscr{V}_\delta(\mathrm{E}x)),$$

so they have a large action functional value (lemma 3); this set will be negligible too. On the third set, we have $\xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x)$ for large enough n. To bound from below the contribution to $\mathrm{P}^n_{\xi_0^n}(\mathrm{R}_0)$ of the last set in (7.5), we shall look for some tubelet in it; but we also need to study the random paths starting close to $\mathrm{E}x$ which leave $\mathscr{B}_e$ before returning near u_e:

LEMMA VII.2. — *There exists γ_3 such that for all $\gamma_e < \gamma_e' = \rho(u_e, \mathscr{V}_\delta(\mathrm{E}x))$, the inequality $\mathrm{P}^n_{\xi_0^n} \{ \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x), \tau_e > \tau \} \geq q \mathrm{P}^n_{\xi_0^n} \{ \tau_e > \tau \}$ holds on the set $\{ \xi_0^n \in \mathscr{V}_{\gamma_3}(\mathrm{E}x) \}$ for all sufficiently large n, where τ_e denotes the entrance time in $\mathscr{V}_{\gamma_e}(u_e)$, and q is the same as in (7.4).*

From now on, the radius γ_3 is fixed as above. As for γ_e, we use it for controling the energy value of some trajectories.

LEMMA VII.3. — *There exists $\gamma_e'' > 0$ such that for all* $T > 0$, $\varphi \in \mathscr{C}([0, T]; B)$, *the conditions*

$$\varphi_0 \in \mathscr{V}_{2\gamma_e''}(u_e), \quad \varphi_T \in \mathscr{V}_{2\delta_1}(\mathrm{NE}x), \quad \varphi[0, T] \subset \mathscr{V}_{\delta_0}(\mathscr{B}_e) - \mathscr{V}_{\gamma_3/2}(\mathrm{E}x)$$

imply $I_{0T}(\varphi) \geq 3\alpha/4$.

These two lemmas will be proved further. In order to fix γ_e, we now look for a tubelet included in the last set of (7.5). According to the assumption (H) *i*), and to the $\|.\|_2$-continuity (6) of V_h, we can pick some $u_3 \in \mathscr{B}_e \cap \mathscr{V}_{\gamma_3/2}(u_{\mathrm{E}x})$ with $V_h(u_3) \leq V_h(\mathrm{E}x) + \alpha/5$; using proposition VI.1.*b*) we can find some trajectory $\overline{\varphi}$ on $[0, T_3]$ joining u_e to u_3 with energy $I_{0,T_3}(\overline{\varphi}) \leq \Delta + \alpha/4$; in particular we derive from (7.3) that $\overline{\varphi}$ does not enter $\mathscr{V}_{2\delta_1}(\mathrm{NE}x)$. Furthermore, we can assume that $\overline{\varphi}$ does not return to u_e on $]0, T_3]$. We can therefore choose $\gamma < \delta_1 \wedge (\gamma_3/2)$ such that the random paths ξ^n with $\rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma$ don't return in $\mathscr{V}_\gamma(u_e)$ after reaching $\mathscr{V}_{\gamma_3}(\mathrm{E}x)$, and enter $\mathscr{V}_{\gamma_3}(\mathrm{E}x)$ before time T_3 and before hitting $\mathscr{V}_{\delta_1}(\mathrm{NE}x)$. Applying theorem IV.1, we obtain (7), for some $\gamma_e < \gamma$,

$$P^n_{\xi_0^n}\{\rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma\} \geq \exp - N(\Delta + \alpha/3) \tag{7.6}$$

for sufficiently large n and $\xi_0^n \in \mathscr{V}_{\gamma e}(u_e)$. Of course, we can impose the condition $\gamma_e < \gamma_e' \wedge \gamma_e''$.

At last, we need the following

LEMMA VII.4. — *Let $\mathscr{F}_1$, $\mathscr{F}_1'$ be weakly closed subsets of* B *such that* $\mathscr{F}_1 \subset \overset{\circ}{\mathscr{F}_1'}$, *and no equilibrium lies in $\mathscr{F}_1'$. Then for all positive* I *there exists* T_0 *with* $P^n_{\xi_0^n}\{\xi_t^n \in \mathscr{F}_1; \forall t \leq T_0\} \leq \exp - NI$ *for all sufficiently large n, and all ξ_0^n.*

Because of (H) *iii*) and *iv*), we can apply this result to

$$\mathscr{F}_1' = \mathscr{V}_{\delta_0}(\mathscr{B}_e) - \overset{\circ}{\mathscr{V}}_{\gamma_e/2}(u_e) - \overset{\circ}{\mathscr{V}}_{\gamma_3/2}(\mathrm{E}x) - \overset{\circ}{\mathscr{V}}_{\delta_1/2}(\mathrm{NE}x),$$
$$\mathscr{F}_1 = \mathscr{B}_e - \overset{\circ}{\mathscr{V}}_{\gamma_e}(u_e) - \overset{\circ}{\mathscr{V}}_{\gamma_3}(\mathrm{E}x) - \overset{\circ}{\mathscr{V}}_{\delta_1}(\mathrm{NE}x),$$

and $I = \Delta + \alpha/2$. We now fix $T = T_0 \vee T_3$, and come back to decomposition (7.5).

(6) See [5] 1.2 lemme 1. Or, for this particular point, use proposition VI.1 together with τ^*-lower semicontinuity of V_h.

(7) Recall that every equilibrium belongs to $\mathscr{C}(\mathbb{T};]-1, 1[)$.

Since $\{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2T \} \cap R_0$ is contained in $\{ \xi_t^n \in \mathscr{F}_1 ; \forall t \in [T, 2T] \}$, we bound its probability using the Markov property and the previous lemma.

Because of lemma VII.3, trajectories φ such that

$$\rho_{0,2T}(\varphi, \xi^n) \leq (\delta_0/2) \wedge \delta_1 \wedge \gamma_e \wedge (\gamma_3/2)$$

for some

$$\xi^n \in \{\tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T \} \cap \{ \xi_0^n \in \mathscr{V}_{\gamma_e}(u_e) \}$$

satisfy $I_{0,2T}(\varphi) \geq \Delta + 3\gamma/4$; theorem V.I provides:

$$P_{\xi_0^n}^n \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T; R_0 \} \leq \exp - N(\Delta + \alpha/2) \quad \text{on} \quad \xi_0^n \in \mathscr{V}_{\gamma_e}(u_e).$$

At last, $\{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T, R_0 \}$ contains the tubelet $\{ \rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma \}$. Combining (7.6) and the two last inequalities, we obtain for large enough n

$$P_{\xi_0^n}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T; R_0 \} \geq 2^{-1} \exp(N\alpha/6)$$
$$P_{\xi_0^n}^n([\{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2T \} \cup \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T \}] \cap R_0).$$

Recalling that $\{ \tau \leq \tau_3^0, \tau < \tau_1 \} \subset \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex) \}$, we then derive from (7.5)

$$q^{-1} P_{\xi_0^n}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T; R_0 \} + P_{\xi_0^n}^n \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex), \tau \leq \tau_3^0; R_0 \} \geq P_{\xi_0^n}^n(R_0). \quad (7.7)$$

Furthermore,

$$P_{\xi_0^n}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T; R_0 \} = \mathbb{E}_{\xi_0^n}^n \{ \mathbb{1}_{\tau_3^0 \leq \tau \wedge \tau_1 \wedge 2T} . P_{\xi_0^n}^n(\tau_e^1 \geq \tau/\mathbb{F}_{\tau_3^0}) \quad (7.8)$$

Applying the strong Markov property on the set $\{ \tau_3^0 \leq \tau \}$, we see that $P_{\xi_0^n}^n(\tau_e^1 \geq \tau/\mathbb{F}_{\tau_3^0}) = P_{\xi_{\tau_3^0}^n}^n(\tau_e \geq \tau)$ with τ_e as in lemma VII.2; from this lemma we deduce that (7.8) is not more than

$$q^{-1} \mathbb{E}_{\xi_0^n}^n \{ \mathbb{1}_{\tau_3^0 \leq \tau \wedge \tau_1 \wedge 2T} . P_{\xi_{\tau_3^0}^n}(\tau_e^1 \geq \tau, \xi_\tau^n \in \mathscr{V}_\delta(Ex)) \}$$
$$\leq q^{-1} P_{\xi_0^n}^n \{ \tau_3^0 \leq \tau, \quad \xi_\tau^n \in \mathscr{V}_\delta(Ex); R_0 \}.$$

Then, (7.7) yields

$$q^{-2} P_{\xi_0^n}^n \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex); R_0 \} \geq P_{\xi_0^n}^n \{ R_0 \}$$

which is the desired result.

We end with the case $\xi_0^n \in \mathscr{F}$: denoting by τ_e^0 the entrance time in $\mathscr{V}_{\gamma_e}(u_e)$, we must show $\lim_{n \to \infty} \inf_{\xi_0^n \in \mathscr{F}} P_{\xi_0^n}^n(\tau_e^0 < \tau) = 1$. This can be carried out in the same way as in [*15*]. (Lyapunov stability implies that $\rho(\mathscr{B}_e^c, \hat{\mathscr{F}}) > 0$, where $\hat{\mathscr{F}}$ denotes the set of all points visited by the solutions of (M. E.) starting from $\mathscr{F}$) $\square$

□ *Proof of lemma* VII.2. — Let $\gamma_3, \mathrm{T}' > 0$ and $\gamma_e < \gamma'_e$. We define stopping times τ_1 as above, and

$$\tau_3^0 = 0, \qquad \tau_3^{k+1} = \min\{\, t > \tau_3^k + \mathrm{T}' ; \xi_t^n \in \mathscr{V}_{\gamma_3}(\mathrm{E}x) \,\}.$$

Let $\nu_3 = \max\{\, k > 0 ; \tau_3^k < \tau \,\}$, and $\mathrm{R}'_k = \{\, \tau < \tau_e, \nu_3 = k \,\}$.

Using the same argument as above, we see it is enough to show that

$$\mathrm{P}^n_{\xi_0^n}\{\, \xi_\tau^n \in \mathscr{V}_\delta(\mathrm{E}x), \mathrm{R}'_0 \,\} \geq q\mathrm{P}^n_{\xi_0^n}(\mathrm{R}'_0) \tag{7.9}$$

holds for $\xi_0^n \in \mathscr{V}_{\gamma_3}(\mathrm{E}x)$ and sufficient large n.

This time, we will decompose R'_0 according to

$$\Omega^n = \{\, \tau \wedge \tau_1 > 2\mathrm{T}' \,\} \cup \{\, \tau_1 \leq 2\mathrm{T}' \wedge \tau \,\} \cup \{\, \tau \leq 2\mathrm{T}', \tau < \tau_1 \,\} \tag{7.10}$$

To show that the most important contribution to $\mathrm{P}^n_{\xi_0^n}(\mathrm{R}'_0)$ comes from the last set in (7.10), we look for a family of tubelets included in it. Using hypothesis (H) *ii*) and proposition VI.1.*c*, we find for each $u \in \mathrm{E}x$ some $u_2[u] \in (\overline{\mathscr{B}_e})^c$ with $\rho(u_2[u], u) < \delta/4$ and some line segment $\varphi[u]$ with endpoints $u, u_2[u]$ on the time interval $[0, \mathrm{T}_2[u]]$ such that $\mathrm{I}_{0,\mathrm{T}_2[u]}(\varphi[u]) \leq \alpha/6$. Let $\delta_2[u] \leq \delta_1/2$ with $\mathscr{V}_{\delta_2[u]}(u_2[u]) \subset (\overline{\mathscr{B}_e})^c$; then,

$$\mathscr{V}_{\delta_2[u]}(\varphi[u][0, \mathrm{T}_2[u]]) \subset \mathscr{V}_{\delta/2}(\mathrm{E}x). \tag{7.11}$$

Since u is an equilibrium, we apply theorem IV.1 and find some $\delta_3[u] < \delta_2[u]$ such that for large enough n

$$\mathrm{P}^n_{\xi_0^n}\{\, \rho_{0,\mathrm{T}_2[u]}(\xi^n, \varphi[u]) < \delta_2[u] \,\} \geq \exp - \mathrm{N}\alpha/3$$

on
$$\{\, \rho(\xi_0^n, u) < \delta_3[u] \,\}. \tag{7.12}$$

Ex being compact, there exist $\delta_3 > 0$ and a finite number K of elements u_k of Ex with $\mathscr{V}_{\delta_3}(\mathrm{E}x) \subset \bigcup_{k \leq \mathrm{K}} \mathring{\mathscr{V}}_{\delta_3[u_k]}(u_k)$. We now claim the analogue to lemma VII.3:

LEMMA VII.5. — *There exists* $\gamma_3 > 0$ *such that for all* $\mathrm{T} \geq 0$ *and* $\varphi \in \mathscr{C}([0, \mathrm{T}]; \mathrm{B})$ *the conditions* $\varphi_0 \in \mathscr{V}_{2\gamma_3}(\mathrm{E}x)$,

$$\varphi_\mathrm{T} \in \mathscr{V}_{2\delta_1}(\mathrm{NE}x) \qquad \textit{and} \qquad \varphi[0, \mathrm{T}] \subset \mathscr{V}_{\delta_0}(\mathscr{B}_e)$$

imply $\mathrm{I}_{0\mathrm{T}}(\varphi) \geq 3\alpha/4$.

We fix γ_3; of course we may suppose $\gamma_3 \leq \delta_3$. Recall time T_0 we obtained from lemma VII.4: from now, we set $\mathrm{T}' = \mathrm{T}_0 \vee \max_{k \leq \mathrm{K}} \mathrm{T}_2[u_k]$.

We now come back to the decomposition (7.10). The set $\{\tau \wedge \tau_1 > 2\mathrm{T}'\} \cap \mathrm{R}'_0$ being contained in $\{\, \xi_t^n \in \mathscr{F}_1, \forall t \in [\mathrm{T}, 2\mathrm{T}] \,\}$, we derive an uniform upper bound for its probability from lemma VII.4.

The trajectories φ on $[0, 2T']$ uniformly close to the set $\{ \tau_1 \leq \tau \wedge 2T', \xi_0^n \in \mathscr{V}_{\gamma_3}(Ex) \}$ up to a distance $(\delta_0/2) \wedge \delta_1 \wedge \gamma_3$ have action functional value not less than $3\alpha/4$, according to lemma VII.5: thus, theorem V.1 for $D_{2\alpha/3}$ yields for large enough n

$$P^n_{\xi_0^n} \{ \tau_1 \geq 2T' \wedge \tau, R'_0 \} \leq \exp - N\alpha/2 \quad \text{on} \quad \{ \xi_0^n \in \mathscr{V}_{\gamma_3}(Ex) \}.$$

At last, whenever $\xi_0^n \in \mathscr{V}_{\gamma_3}(Ex)$, ξ_0^n lies in some $\mathscr{V}_{\delta_3[u_k]}(u_k)$; but (7.11), and the conditions on δ_1, $\delta_2[u_k]$, T' show that the tubelet with axis $\varphi[u_k]$ and radius $\delta_2[u_k]$ is included in $\{ \tau \leq 2T', \tau < \tau_1 \} \cap R'_0$: thus, the $P^n_{\xi_0^n}$-probability of this set is not less than $\exp - N\alpha/3$ because of (7.12), and combining the three last estimates:

$$P^n_{\xi_0^n} \{ \tau \leq 2T', \tau < \tau_1 ; R'_0 \} \geq 2^{-1} \exp - N\alpha/6$$
$$. [P^n_{\xi_0^n} \{ \tau \wedge \tau_1 > 2T'; R'_0 \} + P^n_{\xi_0^n} \{ \tau_1 \leq 2T' \wedge \tau; R'_0 \}].$$

Because of (7.10), the term between brackets is equal to

$$P^n_{\xi_0^n}(R'_0) - P^n_{\xi_0^n} \{ \tau \leq 2T', \tau < \tau_1 ; R'_0 \} ;$$

then, relation (7.9) easily follows from $\{ \tau < \tau_1 \} \subset \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex) \}$. □

□ *Proof of lemma* VIII.3. — Suppose the results is false: then, there exist time T^k, trajectories φ^k with $\tau^*-\lim\limits_{k\to\infty} \varphi_0^k = u_e$, $\varphi^k_{T_k} \in \mathscr{V}_{2\delta_1}(NEx)$,

$$\varphi^k[0, T^k] \subset \mathscr{V}_{\delta_0}(\mathscr{B}_e) - \mathring{\mathscr{V}}_{\gamma_3}(Ex), \qquad I_{0T^k}(\varphi^k) \leq \Delta + 3\alpha/4 \quad (7.11)$$

We may suppose—shortening T^k if necessary—that T^k is the entrance time of φ^k in $\mathscr{V}_{2\delta_1}(NEx)$. If $(T^k)_k$ was bounded, say with $T^\infty \in \mathbb{R}^+$, we would extend φ^k on $[T^k, T^\infty]$ as being the solution of (M. E.) starting at $\varphi^k_{T^k}$ (without changing action value); according to the theorem III.4, there should exist some accumulation point φ^∞ with $\varphi_0^\infty = u_e$, $I_{0T^\infty}(\varphi^\infty) \leq \Delta + 3\alpha/4$ and $\varphi_t^\infty \in \mathscr{V}_{2\delta_1}(NEx)$ for some accumulation point t of $(T^k)_k$, which would contradict (7.3).

So we may suppose that the times T^k increase to infinity. Let's shift φ^k in ψ^k: $\psi_t^k = \varphi^k_{t+T^k}$, $t \in [-T^k, 0]$. Using the same argument as before for all $K \in \mathbb{N}^*$, one can find a subsequence—still denoted by ψ^k—uniformly converging on $[-K, 0]$ to some $\tilde{\psi}^k$ such that

$$I_{-K,0}(\tilde{\psi}^k) \leq \Delta + 3\alpha/4, \qquad \tilde{\psi}_0^K \in \mathscr{V}_{2\delta_1}(NEx),$$
$$\tilde{\psi}^K[-K, 0] \subset \mathscr{V}_{\delta_0}(\mathscr{B}_e) - \mathring{\mathscr{V}}_{\gamma_3}(Ex) - \mathring{\mathscr{V}}_{\delta_1}(NEx) \qquad (7.12)$$

By a classical argument, we then find a subsequence—still denoted by ψ^k—uniformly converging on $[-\mathrm{K}, 0]$ to $\tilde{\psi}^{\mathrm{K}}$ for all K; of course, the $\tilde{\psi}^{\mathrm{K}}$ are the restrictions of some $\tilde{\psi}$ defined on $]-\infty, 0]$ with $\mathrm{I}_{-\infty,0}(\tilde{\psi}) \leq \Delta + 3\alpha/4$; thus, using proposition VII.2, there exist times $t^l < 0$ such that $\tilde{\psi}_{t^l}$ converges in $\| \, . \, \|_2$ norm to an equilibrium which must be u_e because of (7.12). For large enough l, $\tilde{\psi}_{t^l} \in \mathscr{B}_e$: hence we derive from proposition VI.1 that $\mathrm{W}(u_e, \tilde{\psi}_{t^l}) = \beta \{ \mathrm{V}_h(\tilde{\psi}_{t^l}) - \mathrm{V}_h(u_e) \}$ goes to zero; for large l, we can find a function $\tilde{\varphi}$ on $[0, s]$ with $\tilde{\varphi}_0 = u_e$, $\tilde{\varphi}_s = \tilde{\psi}_{t^l}$, $\mathrm{I}_{0s}(\tilde{\varphi}) < \alpha/4$; making a trajectory φ from pieces $\tilde{\varphi}$ on $[0, s]$ and $\tilde{\psi}$ on $[s, s - t^l]$, we would obtain $\varphi_{s-t^l} \in \mathscr{V}_{2\delta_1}(\mathrm{NE}x)$ and

$$\Delta + \alpha > \mathrm{I}_{0,s-t^l}(\varphi) > \beta \{ \mathrm{V}_h(\varphi_{s-t^l}) - \mathrm{V}_h(u_e) \},$$

which contradicts (7.3). □

□ The proof of lemma VII.5 is carried out in the same way as the previous one: if the result was false, we could find an accumulation point $\tilde{\psi}$ of some sequence satisfying to $\mathrm{I}_{-\infty,0}(\tilde{\psi}) \leq 3\alpha/4$, $\tilde{\psi}_0 \in \mathscr{V}_{2\delta_1}(\mathrm{NE}x)$; this time, there would exist a sequence of times t^l such that $\tilde{\psi}_{t^l}$ converges in the $\| \, . \, \|_2$ norm to u_e or some element of Ex. In both cases, we are lead to a contradiction. □

The proof of lemma VII.4 is much simpler here than in general frameworks; ω-limit sets (8) consist in equilibria. First, notice that $\min \{ \| 1 \wedge | d\mathrm{V}_h(u) | \|_2^2 ; u \in \mathscr{F}_1' \} > 0$: otherwise, an argument we used in the end of the proof of proposition IV.2 would conclude to the existence of some equilibrium in $\mathscr{F}_1'$.

This proposition therefore shows there exist constants C, C′ such that $\mathrm{I}_{0\mathrm{T}}(\varphi) \geq \mathrm{CT} - \mathrm{C}'$ for all $\mathrm{T} > 0$ and trajectory φ on $[0, \mathrm{T}]$ with values in $\mathscr{F}_1'$. So the lemma is an easy consequence of the theorem V.I. □

VIII. AN EXAMPLE. NUCLEATION PHENOMENON

Studying the equilibrium equation $d\mathrm{V}_h(u) = 0$ is difficult in the general situation; it requires techniques of bifurcation (parameter β varying in $\mathbb{R}^+$). In the case $h \neq 0$ one can hardly derive a few quantitative results [*5*]. If $h = 0$ the energy landscape defined by V_0 only depends on β and the Fourier structure of interaction J: somewhat general results about bifurcation branches in the set of equilibria are shown in [*6*].

(8) For the mean equation (M. E.).

We consider here the simplest example exhibiting nucleation phenomenon, which is the (ferromagnetic) case

$$J(x) = 1 + 2b \cos 2\pi p.x, \quad h = 0 \tag{8.1}$$

with $p \in \mathbb{Z}^d - \{ 0 \}$, $0 < b \leq 1/2$.

Then, all the equilibria are (see [6]) $u = 0$, constants u^+, $-u^+$ if $\beta > \beta_c = 1$ (given in the end of section I.1), and u_{p,x_0}, $x_0 \in \mathbb{T}$ if $\beta > \beta_p = [\hat{J}(p)]^{-1} = b^{-1}$, where u_{p,x_0} is given by

$$u_{p,x_0}(x) = \tanh \{ 2\beta ba \cos 2\pi p.(x - x_0) \} \tag{8.2}$$

and a is the unique positive root of $a = \langle \tanh \{ 2\beta ba \cos 2\pi p.x \}, \cos 2\pi p.x \rangle$; a depends on β, and is equal to $\hat{u}_{p,0}(p)$.

At critical value β_c, the branch of constant solutions $\pm u^+$ bifurcates from the branch of null solution, with some stability transfer: $\pm u^+$ are stable equilibria for $\beta > \beta_c$, while 0, being stable up to β_c, becomes a saddle point and $\mathrm{E}x = \{ 0 \}$ for $\beta \in]\beta_c, \beta_p]$; this is symmetry breaking. At value β_p, the branch $\{ u_{p,x_0}; x_0 \in \mathbb{T} \}$ bifurcates from zero solution branch with stability transfer: as in [5] we can compute that the relation

$$V_0(u) = (2\beta)^{-1} \langle \theta(u), 1 \rangle,$$

θ being the concave function $u \tanh^{-1} u + \log(1 - u^2)$, holds for all equilibrium u, and therefore $\mathrm{E}x = \{ u_{p,x_0}; x_0 \in \mathbb{T} \}$ as soon as $\beta > \beta_p$. It's easy to see that the assumptions of section VII are satisfied with the stable equilibria $\pm u^+$.

For the sake of simplicity let's assume $d = 1$. If $\beta > \beta_p$, theorem VIII.1 shows that the magnetization process when leaving the attracting domain of u^+ must pass the potential barrier close to one of lowest saddle points u_{p,x_0}; these states exhibit p areas on the torus—« nuclei »—where local magnetization approaches the new phase $- u^+$.

In this simple example, it seems to be difficult to study the extremal trajectories from u^+ to u_{p,x_0} (recall these are the solutions to (6.2) with $\lim_{t \to -\infty} \varphi_t = u^+$, $\lim_{t \to +\infty} \varphi_t = u_{p,x_0}$), which are the exit paths from the attracting domain of u^+ for the process (see [1] [15]). Nevertheless one may conjecture, with a slight act of faith, that, during such a dynamic phase transition and for $\beta > \beta_p$, small clusters initially appear, among which some, very small, are due to stochastic fluctuations; they next order in p main nuclei, and grow untill they attain approximately the structure of some u_{p,x_0}. At last, the process is attracted by $- u^+$, the nuclei go on spreading till they occupy the entire space.

For $\beta \in]\beta_c, \beta_p]$ the only one exit point is 0, so nucleation phenomenon does not occur. In the Curie-Weiss model (J = 1), every equilibrium is a constant function, and so nucleation never occurs. For more general ferromagnetic interaction function J, a bifurcation temperature is given by $\beta_p = [\hat{J}(p)]^{-1}$ with $\hat{J}(p) = \max_{q \in \mathbb{Z}^d} \{ \hat{J}(q); q \neq 0, \pm p \}$, and, under additional assumptions, (8.2) still defines saddle points when $\beta > \beta_p$ (see [5]).

IX. APPENDIX: PROOFS OF III.3, 4 AND 6

$\square$ We begin with properties III.3:

a) We show the different formulas for I_{0T}. Let's denote by I_1, I_2 the second and last expression in the desired equality. We have clearly $I_1 \leq I_{0T}(\varphi) \leq I_2$. Let's define for all t, x, $a_0(t, x) \in \overline{\mathbb{R}}$ maximizing (3.7): a_0 is the (measurable) function given by (3.3) if $| \varphi_t(x) | < 1$, and, if $\varphi_t(x) = \eta \in \{ -1, +1 \}$, by $-h(x) - J * \varphi_t(x) - \eta\beta^{-1} \log \dfrac{-\eta\dot{\varphi}_t(x)}{2c(\varphi_t, x)}$ with the convention $\log y = -\infty$ for $y \leq 0$. Then, for fixed (t, x), $a_m = \operatorname{sign}(a_0) \times [\, | a_0 | \wedge m]$ converges to a_0 in $\overline{\mathbb{R}}$ as $m \to \infty$, and $b_m(t, x) = \dfrac{\beta}{2} \dot{\varphi}_t(x) . a_m(t, x) - \Gamma(\varphi_t, a_m(t, x), x)$ converges to $\mathcal{H}(\varphi_t, \dot{\varphi}_t(x), x)$ in $\overline{\mathbb{R}}^+$. As $a \to \Gamma(\varphi_t, a, x)$ is convex and $a_m(t, x)$ is between 0 and $a_0(t, x)$, b_m is non negative. Fatou's lemma then shows that $I_2 \leq \underline{\lim}_{m \to \infty} \int_{[0,T] \times \mathbb{T}} a_m$; this last term being less than I_1, we obtain property *a*).

Proof of the lower bound *d*) of $\mathcal{H}$: using the inequality $e^{\beta a} - 1 \leq e^{\beta|a|} - 1$, we obtain for $\Gamma(u, a, x)$ an uniform upper bound $A(e^{\beta|a|} - 1)$ with constant A, whose Legendre transform in the sense of (3.7) is

$$\max \left\{ \frac{|v|}{2} \left[\log \frac{|v|}{2A} - 1 \right] + A, 0 \right\}.$$

To show the upper bound *c*) for $\mathcal{H}$, we notice that

$$\theta_c(w, v) = \frac{v}{2} \log \frac{\dfrac{v}{2c} + \sqrt{1 - w^2 + (v/2c)^2}}{1 - w}$$

(with parameter $c \in \mathbb{R}^+$) is an even function on $\mathbb{R}^2$; we restrain to $v > 0$, and see that $|\theta_c| \leq \frac{v}{2}\left\{1{,}5 \log 2 + e^{-1} + \left(\log \frac{v}{c}\right)^+ + \log \frac{1}{1-w}\right\}$. Then, the result can be easily derived.

The conditions mentioned in *b*) are sufficient because of *c*); the first one is necessary because of *d*). Inequalities

$$\theta_c(w, v) - \frac{v}{2} \log \frac{v}{2c} > \frac{v}{2} \log \frac{1}{1-u} \geq -\frac{v}{2} \log 2$$

hold for positive v, so the last two ones are necessary too.

In the proof of the regularity property *e*) of $\mathscr{H}$, the most difficult bound to get is for

$$\begin{aligned} &|\theta_{c_1}(w_1, v) - \theta_c(w, v)| \\ &\quad \leq |v| \left\{ \left| \log \frac{1-w}{1-w_1} \right| + \left| \log \frac{c}{c_1} \right| + \left| \log \frac{v + \sqrt{4c_1(1-w_1^2) + v^2}}{v + \sqrt{4c(1-w^2) + v^2}} \right| \right\}; \end{aligned}$$

we use inequality $\log 1 + z \leq z$ to bound the first two terms, and control the derivative of $a \to \log v + \sqrt{a + v^2}$ for the last one.

□ *Proof of the theorem III.4.* — We show first that D_{I_0} is relatively compact. Let φ^m be a sequence in D_{I_0}; because of property III.3.*d*) $(\dot{\varphi}^m)_{m \in \mathbb{N}}$ is uniformly integrable on $[0, T] \times \mathbb{T}$. According to Dunford-Pettis' theorem [9], there exists a subsequence that we still denote by φ^m, such that $\dot{\varphi}^m$ converges to some $\dot{\varphi}^\infty \in \mathbb{L}^1([0, T] \times \mathbb{T})$ in the weak topology $\sigma(\mathbb{L}^1([0, T] \times \mathbb{T}); \mathbb{L}^\infty([0, T] \times \mathbb{T}))$.

Since $\| \varphi_{t'}^m - \varphi_t^m \|_1 \leq \int_{[t,t'] \times \mathbb{T}} | \dot{\varphi}_s^m(x) | \, ds dx$, uniform integrability shows that (φ^m) is equicontinuous on $[0, T]$ in the $\| \, . \, \|_1$ norm, and then so it is in the metric ρ; B being τ^*-compact, Ascoli's theorem in the space $(\mathscr{C}([0, T]; B), \rho_{0T})$ yields the relative compactness of D_{I_0}.

Let φ be an accumulation point of (φ^m): we now show $\varphi \in D_{I_0}$. Without loss of generality, we may suppose that φ^m goes to φ in metric ρ_{0T}, and that $\dot{\varphi}^m$ goes to some $\dot{\varphi}^\infty$ in weak topology $\sigma(\mathbb{L}^1; \mathbb{L}^\infty)$.

For $t \leq T$ and $g \in \mathscr{C}(\mathbb{T})$, we have

$$\begin{aligned} \langle g, \varphi_t - \varphi_0 \rangle &= \lim_{m \to \infty} \langle g, \varphi_t^m - \varphi_0^m \rangle \\ &= \lim_{m \to \infty} \int_{[0,t] \times \mathbb{T}} \dot{\varphi}_s^m(x) g(x) ds dx \\ &= \int_{[0,t] \times \mathbb{T}} \dot{\varphi}_s^\infty(x) g(x) ds dx, \end{aligned}$$

since $\mathbb{1}_{[0,t]} \times g \in \mathbb{L}^{\infty}([0,T] \times \mathbb{T})$. So φ satisfies to the differentiability condition (D) with $\dot{\varphi} = \dot{\varphi}^{\infty}$.

Let's suppose $I_{0T}(\varphi) = I < \infty$. For $\varepsilon > 0$, property III.3.*a*) yields some $f \in \mathbb{L}^{\infty}([0,T] \times \mathbb{T})$ such that

$$\int_{[0,T]\times\mathbb{T}} \left[\frac{\beta}{2}\dot{\varphi}f - \Gamma(\varphi, f, x)\right] dtdx \geq I - \varepsilon. \tag{9.1}$$

Because of the convergence of $\dot{\varphi}^m$ to $\dot{\varphi} = \dot{\varphi}^{\infty}$, we have

$$\lim_{m\to\infty} \int_{[0,T]\times\mathbb{T}} \dot{\varphi}^m f = \int_{[0,T]\times\mathbb{T}} \dot{\varphi} f .$$

We then study the convergence of $\int_{[0,T]\times\mathbb{T}} \Gamma(\varphi^m, f, x) dtdx$. The difficult point concerns terms of the type

$$a(\varphi^m) - a(\varphi) \quad \text{with} \quad a(\psi) = c(\psi)\psi \exp \beta(h + f + J * \psi);$$

using Lebesgue's theorem, it's enough to show that $\int_{[0,T]\times\mathbb{T}} b^m \underset{m\to\infty}{\longrightarrow} 0$ where $b^m = (\varphi - \varphi^m)c(\varphi)e^{\beta(h+f+J*\varphi)}$. φ_t^m being uniformly bounded, we may suppose $f(t,.)$ to be a continuous function according to Lusin's theorem; then we derive $\int_{\mathbb{T}} b^m \to 0$, and, together with Lebesgue's theorem, $\int_{[0,T]\times\mathbb{T}} b^m \to 0$.

We have showed

$$\underline{\lim}_{m\to\infty} \int_{[0,T]\times\mathbb{T}} \left[\frac{\beta}{2}\dot{\varphi}^m f - \Gamma(\varphi^m, f, x)\right] dtdx \geq I - \varepsilon,$$

so

$$\underline{\lim}_{m\to\infty} I_{0T}(\varphi^m) \geq I - \varepsilon .$$

The case $I_{0T}(\varphi) = \infty$ is impossible, because (9.1) would otherwise be true for all I, and the previous demonstration would conclude to $I_0 \geq I$. So we showed the first part of the result. Since $I_{0T}^{-1}([0, I_0])$ is closed, I_{0T} is a l. s. c. function. □

□ At last, we prove the result III.6 of approximation by smooth trajectories: we first show that (3.8) is satisfied by some trajectory $\tilde{\varphi}$ staying far away from the boundary points $-1, +1$; then by a polygon in t variable, with vertices on the previous trajectory; to end, by a last one which is furthermore continuous on $\mathbb{T}$.

Let $\varphi_t^m = \varphi_0 + \left(1 - \frac{1}{m}\right)(\varphi_t - \varphi_0)$. For $\delta = 1 - \| \varphi_0 \|_\infty$, we have $\| \varphi_t^m \|_\infty \leq 1 - \frac{\delta}{m}$; furthermore $\varphi^m \to \varphi$ and $\dot{\varphi}^m = \left(1 - \frac{1}{m}\right)\dot{\varphi} \to \dot{\varphi}$ for all (t, x), one can notice that, for all (t, x) such that $\varphi_t(x) = \eta \in \{ -1, +1 \}$,

$$\frac{1}{2}\dot{\varphi}^m \log \frac{[\dot{\varphi}^m/2c(\varphi^m)] + \sqrt{1-(\varphi^m)^2 + [\dot{\varphi}^m/2c(\varphi^m)]^2}}{1-\varphi^m} \underset{m\to\infty}{\longrightarrow} \frac{|\dot{\varphi}^m|}{2}\log - \eta \frac{\dot{\varphi}^m}{2c(\varphi^m)}$$

with the previous notation $\log a = -\infty$ for $a \in \mathbb{R}_*^-$, so that

$$\mathscr{H}(\varphi_t^m, \dot{\varphi}_t^m(x), x) \to \mathscr{H}(\varphi_t, \dot{\varphi}_t(x), x).$$

In order to apply Lebesgue's theorem, we look for an upper bound of $\mathscr{H}(\varphi_t^m, \dot{\varphi}_t^m(x), x)$ using property III.3.*c*): remark that, on $\{ \varphi^m \geq 1 - \delta \}$, $\varphi - \varphi_0 \geq 0$ and $1 - \varphi^m \geq 1 - \varphi$ which is $(t - x)$ a. s. non zero on the set $\{ \dot{\varphi} > 0 \} = \{ \dot{\varphi}^m > 0 \}$. We then obtain the following bound, independent of m:

$$\mathscr{H}(\varphi_t^m, \dot{\varphi}_t^m, x)$$
$$\leq \mathrm{K}(\delta)\left[|\dot{\varphi}| \{ \log |\dot{\varphi}| + 1 \} + 1 + \sum_{\eta \in \{-1,+1\}} \mathbb{1}_{\{\eta\dot{\varphi} > 0 \text{ and } 1-\delta < \eta\varphi < 1\}} |\dot{\varphi}| \log \frac{1}{1 - \eta\varphi}\right].$$

Property III.3.*b*) and hypothesis $\mathrm{I}_{0\mathrm{T}}(\varphi) < \infty$ imply that the bound is integrable; then $\lim_{m\to\infty} \mathrm{I}_{0\mathrm{T}}(\varphi^m) = \mathrm{I}_{0\mathrm{T}}(\varphi)$. As φ^m clearly goes to φ in metric $\rho_{0\mathrm{T}}$, we can fix some m such that φ^m satisfies to (3.8).

We will prove further on the following

LEMMA A.1. — *Let ψ satisfying to* (D) *and $\gamma > 0$ with $\mathrm{I}_{0\mathrm{T}}(\psi) < \infty$, $\sup_{t \leq \mathrm{T}} \| \psi_t \|_\infty \leq 1 - \gamma$. For all subdivision $\mathrm{S} = \{ t_0 = 0 < t_1 < \ldots < t_{k_0} = \mathrm{T} \}$, we define the polygon l^{S} with vertices at points (t_k, ψ_{t_k}). As* S *becomes finer, $\mathrm{I}_{0\mathrm{T}}(l^{\mathrm{S}})$ goes to $\mathrm{I}_{0\mathrm{T}}(\psi)$ and l^{S} goes to ψ uniformly on $[0, \mathrm{T}]$ in $\| . \|_1$ norm.*

Applying this to $\psi = \varphi^m$, we find a polygon l satisfying to (3.8).

To end, we make l smoother in the x variable, using a kernel $\alpha^m \in \mathscr{C}(\mathbb{T}; \mathbb{R}^+)$, with support contained in $\left[-\frac{1}{m}, \frac{1}{m} \right]$ and integral equal to 1:

LEMMA A.2. — *Let ψ satisfying to* (D) *and $\gamma > 0$ with $\mathrm{I}_{0\mathrm{T}}(\psi) < \infty$, $\sup_{t \leq \mathrm{T}} \| \psi_t \|_\infty \leq 1 - \gamma$ and $\psi_0 \in \mathscr{C}(\mathbb{T})$. Denote by ψ^m the function*

$$\psi_t^m(x) = \psi_0(x) + \alpha^m * (\psi_t - \psi_0)(x).$$

Then, as $m \to \infty, \psi^m \to \psi$ *uniformly on* $[0, T]$ *in* $\|\,.\,\|_1$ *norm*, $I_{0T}(\psi^m) \to I_{0T}(\psi)$ *and* $\varlimsup_{m\to\infty} (\sup_{t\leq T} \| \psi_t^m \|_\infty) \leq 1 - \gamma$.

We apply this lemma to $\psi = l$, and find some m such that (3.8) is satisfied, and $\sup_{t\leq T} \| l_t^m \|_\infty \leq 1 - \gamma/2$; we show in the proof of the lemma that $\dot{l}_t^m = a^m * l_t$, which is a stepwise function on $[0, T]$, with values in $\mathscr{C}(\mathbb{T})$; so $l^m \in \mathscr{C}P_T^{1,0}$, which ends the proof. □

□ We now prove lemma A.1; we will forget the index S in notation l^S. l satisfies to (D), with

$$\dot{l}_t = \frac{\psi_{t_{k+1}} - \psi_{t_k}}{t_{k+1} - t_k}, \quad \text{for} \quad t \in I_k =]t_k, t_{k+1}[, \qquad \text{and} \qquad \sup_{t\leq T} \| l_t \|_\infty \leq 1 - \gamma .$$

Applying Jensen's inequality to the convex function $a \to \mathscr{H}(l_{t_k}, a, x)$ and to $\dot{\psi}_t(x)$, we derive

$$(t_{k+1} - t_k)\mathscr{H}(l_{t_k}, \dot{l}_{t_k^+}(x), x) \leq \int_{I_k} \mathscr{H}(l_{t_k}, \dot{\psi}_t(x), x)dt - \lambda \quad \text{a. s.}$$

Next, we integrate this relation on $\mathbb{T}$, and use property III.3.*e*) with $y = x$: we obtain:

$$\int_{I_k \times \mathbb{T}} \mathscr{H}(l_t, \dot{l}_t(x), x)dx \leq \int_{I_k \times \mathbb{T}} \mathscr{H}(\psi_t, \dot{\psi}_t(x), x)dtdx + A_k \tag{9.2}$$

where

$$A_k = \int_{I_k \times \mathbb{T}} \{ (1 + | \dot{\psi}_t(x) |)(\mathscr{O}_\gamma[| \psi_t - l_{t_k} | (x)] + \varepsilon_\gamma[\rho(\psi_t, l_{t_k})]) \\ + (1 + | \dot{l}_{t_k^+}(x) |)(\mathscr{O}_\gamma[| \psi_t - l_{t_k} | (x)] + \varepsilon_\gamma[\rho(\psi_t, l_{t_k})] \} \, dtdx .$$

Let's denote $r = \sup \{ \| \psi_t - \psi_{t_{k'}} \|_1 ; \; t \in I_{k'}, \; k' = 0, \ldots, k_0 - 1 \}$; then $\| l_t - l_{t_k} \|_1 \leq r$ if $t \in I_k$. Metric ρ being dominated by $\|\,.\,\|_1$ norm and $| \dot{l}_{t_k^+}(x) |$ being bounded from above by $| t_{k+1} - t_k |^{-1} \int_{I_k} | \dot{\psi}_s(x) | \, dsdx$, we see that

$$A_k \leq \varepsilon'_\gamma(r) \int_{I_k \times \mathbb{T}} (1 + | \dot{\psi}_t |)dtdx + K_\gamma A'_k \tag{9.3}$$

for some constant K_γ depending on γ and

$$A'_k = \int_{I_k \times \mathbb{T}} | \psi_t - l_{t_k} | \, | \dot{\psi}_t | \, dtdx + (t_{k+1} - t_k)^{-1} \int_{I_k \times I_k \times \mathbb{T}} | l_t - l_{t_k} | \, | \dot{\psi}_s | \, dtdsdx .$$

Recall that $|l_t - l_{t_k}|, |\psi_t - l_{t_k}| \le 2$. For any $C > 0$ we have

$$A'_k \le 4\left[Cr(t_{k+1} - t_k) + \int_{I_k \times \mathbb{T}} |\dot{\psi}_t| \mathbb{1}_{\{|\dot{\psi}_t| > C\}} dtdx\right].$$

Combining this with (9.2 and 3) we finally derive

$$I_{0T}(l) \le I_{0T}(\psi) + \varepsilon''_{\gamma,C}(r)\left[1 + \int_{[0,T]\times\mathbb{T}} |\dot{\psi}| dtdx\right] + 4K_\gamma \int_{\{|\dot{\psi}| > C\}} |\dot{\psi}| dtdx \quad (9.4)$$

where $\varepsilon''_{\gamma,C}$ depends on γ, C and goes to 0 with r. According to the remark after theorem III.4, $t \to \psi_t$ is continuous in $\|\cdot\|_1$ norm; so l goes to ψ in $\mathscr{C}([0,T], \mathbb{L}^1(\mathbb{T}))$ as the subdivision S becomes finer, and then theorem III.4 implies $\underline{\lim}\ I_{0T}(l) \ge I_{0T}(\psi)$. On the other hand, the integrability of $\dot{\psi}$ and (9.4) shows that $\overline{\lim}\ I_{0T}(l) \le I_{0T}(\psi)$; so the lemma is proved. □

□ At last we prove lemma A.2:

1) Note that $\|\psi_t^m\|_\infty \le \|\alpha^m * \psi_t\|_\infty + \|\psi_0 - \alpha^m * \psi_0\|_\infty$. ψ_0 being continuous, the last term converges to 0; the first one being less than $1-\gamma$, we derive the last part of the result.

In the following, we will suppose m large enough so that

$$\sup_{t\le T} \|\psi_t^m\|_\infty \le 1 - \frac{\gamma}{2}.$$

Notice that $\psi_t^m = \psi_0^m + \int_0^t \alpha^m * \dot{\psi}_s ds$. As $\alpha^m * \dot{\psi}_s$ goes to $\dot{\psi}_s$ in $\|\cdot\|_1$ norm for a. e. $s \in [0,T]$, the inequality $\|\psi_t^m - \psi_t\|_1 \le \int_{[0,T]\times\mathbb{T}} |\alpha^m * \dot{\psi}_s - \dot{\psi}_s| dsdx$ shows that $\lim_{m\to\infty} \psi^m = \psi$ in $\mathscr{C}([0,T]; \mathbb{L}^1(\mathbb{T}))$; in particular, $\underline{\lim}_{m\to\infty} I_{0T}(\psi^m) \ge I_{0T}(\psi)$.

2) First apply Jensen's inequality to the probability $\alpha^m(x-y)dy$:

$$\mathscr{H}(\psi_t, \alpha^m * \dot{\psi}_t(x), x) \le \int_{\mathbb{T}} \alpha^m(y)\mathscr{H}(\psi_t, \dot{\psi}_t(x-y), x)dy \quad \text{for a. e. } (t,x).$$

Combining the relation

$$\int_{\mathbb{T}\times\mathbb{T}} \alpha^m(y)\mathscr{H}(\psi_t, \dot{\psi}_t(x-y), x-y)dydx = \int_{\mathbb{T}} \mathscr{H}(\psi_t, \dot{\psi}_t(x), x)dx,$$

and the property III.3.f) we obtain for a. e. t:

$$\mathscr{H}^*(\psi_t^m, \dot{\psi}_t^m) - \mathscr{H}^*(\psi_t, \dot{\psi}_t)$$
$$\leq \int_{\mathbb{T}} \{ \mathcal{O}_{\frac{\gamma}{2}}[\,|\,\psi_t - \psi_t^m\,|\,(x)] + \varepsilon_{\frac{\gamma}{2}}(\,\|\,\psi_t - \psi_t^m\,\|_1) \} (1 + |\,\dot{\psi}_t^m\,|\,(x))dx$$
$$+ \int_{\mathbb{T}\times\mathbb{T}} \alpha^m(y) \{ \mathcal{O}_{\frac{\gamma}{2}}[\,|\,\psi_t(x) - \psi_t(x-y)\,|\,] + \varepsilon_{\frac{\gamma}{2}}(\,|\,y\,|\,) \} (1 + |\,\dot{\psi}_t(x-y)\,|\,)dxdy \qquad (9.5)$$

the first bound in (9.5) can be studied as we did in the proof of Lemma A.1: $(\dot{\psi}^m)_m$, being convergent in $\mathbb{L}^1([0, T] \times \mathbb{T})$, is uniformly integrable on $[0, T] \times \mathbb{T}$; as for $|\,\psi_t - \psi_t^m\,|$, it is less than 2 and goes to 0 in space $\mathbb{L}^1(\mathbb{T})$.

In order to use the same arguments for the last term of (9.5), we only need to show that $z \to \int_{\mathbb{T}} \alpha^m(x-z)\,|\,\psi_t(x) - \psi_t(z)\,|\,dx$ goes to 0 in $\mathbb{L}^1(\mathbb{T})$ (we set $z = x - y$). Denoting by $\mathscr{C}_{-y}\psi_t : x \to \psi_t(x - y)$, we have

$$\int_{\mathbb{T}\times\mathbb{T}} \alpha^m(x - z)\,|\,\psi_t(x) - \psi_t(z)\,|\,dxdz = \int_{\mathbb{T}} \alpha^m(y)\,\|\,\psi_t - \mathscr{C}_{-y}\psi_t\,\|_1 dy\,;$$

but translation operator is continuous in space $\mathbb{L}^1(\mathbb{T})$, so this last term goes to 0. We then showed $\overline{\lim}_{m\to\infty} \mathrm{I}_{0T}(\psi^m) \leq \mathrm{I}_{0T}(\psi)$, which ends the proof. □

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