SHINTARO NAKAO

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On weak convergence of sequences of continuous local martingales

by

Shintaro NAKAO
Department of Mathematics, Osaka University,
Toyonaka, Osaka 560, Japan

SUMMARY. — Let $\{ M^n(t) \}_{n=1,2,\dots}$ be a sequence of Hilbert space valued continuous local martingales. We give a necessary and sufficient condition for which $\{ M^n \}$ is tight in C in terms of $\{ \langle M^n \rangle \}$. Using this result we show the preservation of the local martingale property under the weak convergence in C of $\{ M^n \}$.

Mots-clés : Continuous local martingale. Weak convergence.

RÉSUMÉ. — Soit $\{ M^n(t) \}_{n=1,2,\dots}$ une suite de martingales locales continues à valeurs dans un espace de Hilbert. Nous donnons une condition nécessaire et suffisante pour que $\{ M^n \}$ soit tendue dans C à l'aide d'une condition de tension sur $\{ \langle M^n \rangle \}$. A l'aide de ce résultat, nous montrons que la propriété de martingale faible locale est conservée par la convergence faible dans C.

1. INTRODUCTION

Let H be a real separable Hilbert space with inner product (,) and $\{ e_j \}_{j \in J}$ be an orthonormal basis of H. Consider a sequence $\{ M^n(t) \}_{n=1,2,\dots}$

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of H-valued continuous local martingales starting at 0, where the time interval I is $[0, T]$ $(0 < T < +\infty)$ or $[0, +\infty)$. Denote by dia $\langle M^n \rangle (t)$ the vector of all diagonal components of $(\langle (M^n, e_k), (M^n, e_l) \rangle (t))_{k,l \in J}$, where $\langle (M^n, e_k), (M^n, e_l) \rangle$ is the quadratic variational process of the martingales (M^n, e_k) and (M^n, e_l) (cf. [3] [6] [9]). We then regard dia $\langle M^n \rangle$ as an l^1-valued continuous process in case that H is infinite-dimensional.

Rebolledo [11] proved, using Lenglart's inequality [8], that in case of $H = \mathbb{R}$ the tightness in C of $\{ M^n(t) \}_{n=1,2,\dots}$ and the tightness in C of $\{ \text{dia} \langle M^n \rangle (t) \}_{n=1,2,\dots} (= \{ \langle M^n \rangle (t) \}_{n=1,2,\dots})$ are equivalent to each other, where C is the space of continuous sample functions. A main purpose of this article is to show the above equivalence holds even if H is infinite-dimensional (Theorem A in Section 2).

By applying Theorem A we show that any accumulation point of $\{ M^n(t) \}_{n=1,2,\dots}$ under the weak convergence in C is also a continuous local martingale (Theorem B in Section 3). Moreover the continuity of the mapping dia $\langle \ \rangle$ under the weak convergence in C is stated in Theorem C in Section 3.

In Appendix we give an elementary proof of Theorem A in case of $H = \mathbb{R}$ which is best adapted to the situation where all the processes considered are continuous and which is based on change of time.

2. TIGHTNESS OF $\{ M^n \}$ AND $\{ \text{dia} \langle M^n \rangle \}$

For a real separable Banach space E, set

$$C_0(I, E) = \{ w; w: I \to E, w \text{ is continuous and } w(0) = 0 \}$$

and we define the usual metric on $C_0(I, E)$ (see [5]); that is, $\{ w_n \}$ converges to w if and only if $\{ w_n \}$ converges uniformly to w on each compact interval in I. Let $\mathscr{B}(C_0(I, E))$ be the topological σ-algebra of $C_0(I, E)$. We denote the space of all probability measures on $(C_0(I, E), \mathscr{B}(C_0(I, E)))$ by $\mathscr{P}(C_0(I, E))$ and endow $\mathscr{P}(C_0(I, E))$ with the weak convergence topology. Consider the following subspace of $\mathscr{P}(C_0(I, E))$;

$$\mathscr{P}_{lm}(C_0(I, E)) = \{ P \in \mathscr{P}(C_0(I, E)); \langle w(t), f \rangle$$

is continuous local martingale for every $f \in E' \}$,

where E′ is the dual space of E.

For an orthonormal basis $\{ e_j \}_{j\in J}$ of H, define the mapping ϕ: $C_0(I, H) \to C_0(I, l^2(J))$ by

(2.1) $$\phi(w)(t) = ((w(t), e_j))_{j\in J} \quad \text{for} \quad w \in C_0(I, H),\ t \in I,$$

where for $p = 1, 2$

$$l^p(J) = \begin{cases} l^p & \text{if} \quad \dim H = +\infty \\ \mathbb{R}^d & \text{if} \quad \dim H = d < +\infty . \end{cases}$$

Then it is obvious that

(2.2) $$\phi : C_0(I, H) \to C_0(I, l^2(J)) \quad \text{is a homeomorphism}.$$

According to the mapping ϕ, we introduce the mapping Φ:

$$\mathscr{P}(C_0(I, H)) \to \mathscr{P}(C_0(I, l^2(J)))$$

by

(2.3) $$\Phi(P) = P \circ \phi^{-1} \quad \text{for} \quad P \in \mathscr{P}(C_0(I, H)).$$

We have then by (2.2)

(2.4) $$\Phi : \mathscr{P}(C_0(I, H)) \to \mathscr{P}(C_0(I, l^2(J))) \quad \text{is a homeomorphism}$$

and

(2.5) $$\Phi : \mathscr{P}_{lm}(C_0(I, H)) \to \mathscr{P}_{lm}(C_0(I, l^2(J))) \quad \text{is a homeomorphism}.$$

We denote by $\mathring{w}(t) = (\mathring{w}_j(t))_{j\in J}$ the sample path of $C_0(I, l^2(J))$. Since the process $(\text{dia} \langle \mathring{w} \rangle, \mathring{P}) = ((\langle \mathring{w}_j \rangle)_{j\in J}, \mathring{P})$ is a $l^1(J)$-valued continuous process for $\mathring{P} \in \mathscr{P}_{lm}(C_0(I, l^2(J)))$, we can define the mapping dia $\langle\ \rangle$:

$$\mathscr{P}_{lm}(C_0(I, l^2(J))) \to \mathscr{P}(C_0(I, l^1(J)))$$

in the following manner; for $\mathring{P} \in \mathscr{P}_{lm}(C_0(I, l^2(J)))$

(2.6) $\text{dia} \langle \mathring{P} \rangle =$ the probability measure on $C_0(I, l^1(J))$ induced by $(\text{dia} \langle \mathring{w} \rangle, \mathring{P})$.

We get the following theorem about the tightness of a family of H-valued continuous local martingales.

THEOREM A. — Let $P_\alpha \in \mathscr{P}_{lm}(C_0(I, H))$ $(\alpha \in A)$. Then the following three conditions are equivalent to each other.

i) $\{ P_\alpha ; \alpha \in A \}$ is tight in $C_0(I, H)$.

ii) $\{ \Phi(P_\alpha) ; \alpha \in A \}$ is tight in $C_0(I, l^2(J))$.

iii) $\{ \text{dia} \langle \Phi(P_\alpha) \rangle ; \alpha \in A \}$ is tight in $C_0(J, l^1(J))$.

As stated in the introduction Rebolledo [11] proved the above theorem using Lenglart's inequality in case of $H = \mathbb{R}$. So we will prove the above theorem in case of $\dim H = +\infty$. For this purpose we estimate the tail behaviors of $\{ \Phi(P_\alpha); \alpha \in A \}$ and $\{ \text{dia} \langle \Phi(P_\alpha); \alpha \in A \}$ uniformly in $\alpha \in A$.

Let K be a subset of l^p $(p = 1, 2)$. It is well known ([4]) that K is relatively compact if and only if

$$(2.7) \quad \begin{cases} \sup_{c \in K} \| c \|_{l^p} < +\infty \\ \lim_{n \to \infty} \sup_{c \in K} \sum_{m=n}^{\infty} | c_m |^p = 0, \qquad \text{where} \qquad c = (c_1, c_2, \ldots). \end{cases}$$

Let $I = [0, T]$ and $Q_\alpha \in \mathscr{P}(C_0(I, l^p))$ $(\alpha \in A)$. Noting (2.7), it is easy to see that $\{ Q_\alpha ; \alpha \in A \}$ is tight in $C_0(I, l^p)$ if and only if for any $\varepsilon > 0$ and $i = 1, 2, \ldots$ there exist a compact set $K \subset l^p$ and $\delta(i) > 0$ such that

$$(2.8) \quad \sup_{\alpha \in A} Q_\alpha\Big(\sup_{\substack{0 \le s, t \le T \\ |s-t| \le \delta(i)}} | w_i(t) - w_i(s) | < \varepsilon \Big) > 1 - \varepsilon, \qquad i = 1, 2, \ldots$$

and

$$(2.9) \qquad \sup_{\alpha \in A} Q_\alpha(w(t) \in K \quad \text{for any} \quad t \in I) > 1 - \varepsilon,$$

where $w = (w_1, w_2, \ldots) \in C_0(I, l^p)$.

From now on we denote by $\hat{w} = (\hat{w}_1, \hat{w}_2, \ldots)$ the sample path of $C_0(I, l^1)$. We prepare a lemma for the proof of Theorem A.

LEMMA 2.1. — Let $I = [0, T]$ and $\mathring{P}_\alpha \in \mathscr{P}_{lm}(C_0(I, l^2))$ $(\alpha \in A)$. Then the following two properties are equivalent to each other.

i) For any $\varepsilon > 0$ there exists a compact set $K \subset l^2$ such that

$$(2.10) \quad \mathring{P}_\alpha(\mathring{w}(t) \in K \quad \text{for any} \quad t \in I) \geqq 1 - \varepsilon \quad \text{for} \quad \alpha \in A.$$

ii) For any $\varepsilon > 0$ there exists a compact set $\hat{K} \subset l^1$ such that

$$(2.11) \quad \text{dia} \langle \mathring{P}_\alpha \rangle (\hat{w}(t) \in \hat{K} \quad \text{for any} \quad t \in I) \geqq 1 - \varepsilon \quad \text{for} \quad \alpha \in A.$$

Proof. — We show that *i*) implies *ii*). Assume that (2.10) holds. Set

$$\gamma = \begin{cases} \inf \{ t \in I; \mathring{w}(t) \notin K \} \\ T \quad \text{if} \quad \{ \ \} = \phi. \end{cases}$$

Since it holds that

$$\sup_{\alpha \in A} E^{\mathring{P}_\alpha}[\| \text{dia} \langle \mathring{w} \rangle (T \wedge \gamma) \|_{l^1}] = \sup_{\alpha \in A} E^{\mathring{P}_\alpha}[\| \mathring{w}(T \wedge \gamma) \|_{l^2}^2]$$
$$\leqq \sup_{c \in K} \| c \|_{l^2}^2 = \beta < +\infty,$$

we get by Chebyshev's inequality

$$(2.12)\qquad \sup_{\alpha\in A} \overset{\circ}{P}_\alpha(\| \operatorname{dia} \langle \overset{\circ}{w} \rangle (T \wedge \gamma) \|_{l^1} > G) \leqq \frac{\beta}{G} \quad \text{for} \quad G > 0 .$$

On the other hand it is easy to see that

$$(2.13)\qquad \lim_{n\to\infty} \sup_{\alpha\in A} E^{\overset{\circ}{P}_\alpha}\left[\sum_{m=n}^{\infty} \langle \overset{\circ}{w}_m \rangle (T \wedge \gamma) \right] \leqq \lim_{n\to\infty} \sup_{c\in K} \sum_{m=n}^{\infty} c_m^2 = 0 .$$

Let $\{ \varepsilon_r \}_{r=1,2,\ldots}$ be a sequence such that $\varepsilon_r \downarrow 0$ and $\sum_{r=1}^{\infty} \varepsilon_r \leqq \varepsilon/2$. Then in view of (2.13) for any $r = 1, 2, \ldots$ there exists n_r such that

$$(2.14)\qquad \sup_{\alpha\in A} E^{\overset{\circ}{P}_\alpha}\left[\sum_{m=n_r}^{\infty} \langle \overset{\circ}{w}_m \rangle (T \wedge \gamma) \right] \leqq \varepsilon_r^2 .$$

Putting

$$\Omega_r = \left\{ \overset{\circ}{w} \in C_0(I, l^2) ; \sum_{m=n_r}^{\infty} \langle \overset{\circ}{w}_m \rangle (T \wedge \gamma) > \varepsilon_r \right\},$$

we get from (2.14)

$$(2.15)\qquad \overset{\circ}{P}_\alpha\left(\bigcup_{r=1}^{\infty} \Omega_r \right) \leqq \sum_{r=1}^{\infty} \varepsilon_r \leqq \varepsilon/2 .$$

We then set $\hat{K} = \left\{ c = (c_1, c_2, \ldots) \in l^1 ; \| c \|_{l^1} \leqq 2\beta/\varepsilon, \sum_{m=n_r}^{\prime} | c_m | \leqq \varepsilon_r \right.$ for $r = 1, 2, \ldots \}$. Obviously $\hat{K}$ is compact and satisfies from (2.12) and (2.15)

$$\sup_{\alpha\in A} \overset{\circ}{P}_\alpha(\{ \overset{\circ}{w} ; \operatorname{dia} \langle \overset{\circ}{w} \rangle (t \wedge \gamma) \in \hat{K} \quad \text{for} \quad t \in I \}) \geqq 1 - \varepsilon .$$

Consequently we get from (2.10)

$$\sup_{\alpha\in A} \overset{\circ}{P}_\alpha(\{ \overset{\circ}{w} ; \operatorname{dia} \langle \overset{\circ}{w} \rangle (t) \in \hat{K} \quad \text{for} \quad t \in I \}) \geqq 1 - 2\varepsilon .$$

We can prove the implication *ii*) → *i*) in the same way as above. □

We now return to the proof of Theorem A.

Proof of Theorem A. — It is obvious that *i*) and *ii*) are equivalent to each other. We show the equivalence between *ii*) and *iii*). We may assume that $I = [0, T]$ and $\dim H = +\infty$. The Rebolledo's result [Cor. II, 3.14 in

[11]) implies that the condition (2.8) with $\{ \Phi(P_\alpha) \}$ instead of $\{ Q_\alpha \}$ is equivalent to the condition (2.8) with $\{ \text{dia} \langle \Phi(P_\alpha) \rangle \}$ instead of $\{ Q_\alpha \}$. On the other hand we get by Lemma 2.1 that the condition (2.9) with $\{ \Phi(P_\alpha) \}$ instead of $\{ Q_\alpha \}$ is equivalent to the condition (2.9) with $\{ \text{dia} \langle \Phi(P_\alpha) \rangle \}$ instead of $\{ Q_\alpha \}$. Consequently *ii*) and *iii*) are equivalent to each other. The proof is completed. □

3. PRESERVATION OF LOCAL MARTINGALE PROPERTY

First we show that any accumulation point of a sequence of H-valued continuous local martingales under the weak convergence is also a H-valued continuous local martingale.

THEOREM B. — $\mathscr{P}_{lm}(C_0(I, H))$ is closed in $\mathscr{P}(C_0(I, H))$.

Proof. — We may assume $H = l^2$. Let $\{ \mathring{P}_n \}$ converges weakly to $\mathring{P}(\mathring{P}_n \in \mathscr{P}_{lm}(C_0(I, l^2)), \mathring{P} \in \mathscr{P}(C_0(I, l^2)))$. Denote by R_n the probability measure on $C_0(I, l^2) \times C_0(I, l^1)$ induced by $((\mathring{w}, \text{dia} \langle \mathring{w} \rangle), \mathring{P}_n)$. Theorem A implies $\{ R_n \}$ is tight. Therefore there exists a subsequence $\{ n' \}$ such that $\{ R_{n'} \}$ converges weakly to R. For $N = 1, 2, \ldots$ set

$$\sigma_N(\hat{w}) = \inf \{ t; t + \| \hat{w}(t) \|_{l^1} > N \}.$$

For each $i = 1, 2, \ldots, s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_q \leqq s < t$ and bounded continuous function f on $(l^2)^p \times (l^1)^q$, we have

$$\begin{aligned}(3.1)\quad & E^{R_{n'}}[\mathring{w}_i(t \wedge \sigma_N) f(\mathring{w}(s_1), \ldots, \mathring{w}(s_p), \hat{w}(t_1), \ldots, \hat{w}(t_q))] \\ & = E^{R_{n'}}[\mathring{w}_i(s \wedge \sigma_N) f(\mathring{w}(s_1), \ldots, \mathring{w}(s_p), \hat{w}(t_1), \ldots, \hat{w}(t_q))].\end{aligned}$$

Since σ_N is R-a. s. continuous on

$$C_0(I, l^2) \times C_0(I, l^1) \quad \text{and} \quad \sup_{n'} E^{R_{n'}}[\mathring{w}_i(t \wedge \sigma_N)^2] < +\infty,$$

the equality (3.1) holds with R instead of $R_{n'}$. Therefore we have

$$\mathring{P} \in \mathscr{P}_{lm}(C_0(I, l^2)). \quad \square$$

Next we state a continuity property of the mapping dia $\langle \quad \rangle$ defined by (2.6).

THEOREM C. — $\text{dia} \langle \quad \rangle : \mathscr{P}_{lm}(C_0(I, l^2)) \to \mathscr{P}(C_0(I, l^1))$ is continuous.

Proof. — Within the situation of the above proof, we can show that

$(\mathring{w}_i^2 - \hat{w}_i, \mathrm{R})$ is a continuous martingale for any $i = 1, 2, \ldots$. Therefore R is the probability measure on $\mathrm{C}_0(\mathrm{I}, l^2) \times \mathrm{C}_0(\mathrm{I}, l^1)$ induced by $((\mathring{w}, \mathrm{dia} \langle \mathring{w} \rangle), \mathring{\mathrm{P}})$. This implies that $\{ \mathrm{R}_n \}$ converges weakly to R. Hence $\{ \mathrm{dia} \langle \mathring{\mathrm{P}}_n \rangle \}$ converges weakly to $\mathrm{dia} \langle \mathring{\mathrm{P}} \rangle$ and the proof is completed. □

Applying the above theorems, we can easily get the following remark about the central limit theorem for a sequence of Hilbert space valued continuous local martingales.

Remark. — Let $\mathrm{V} = (v_{ij})_{i,j=1,2,\ldots}$ be a nonnegative definite, symmetric real matrix such that $\sum_{i=1}^{\infty} v_{ii} < +\infty$ and $\mathring{\mathrm{P}}_w \in \mathscr{P}_{lm}(\mathrm{C}_0(\mathrm{I}, l^2))$ be a Wiener measure such that the mean vector is zero and the covariance function is $((t \wedge s)v_{ij})$. Consider a sequence $\{ \mathring{\mathrm{P}}_n \}$ $(\mathring{\mathrm{P}}_n \in \mathscr{P}_{lm}(\mathrm{C}_0(\mathrm{I}, l^2)))$. Then the following two conditions are equivalent to each other.

i) $\{ \mathring{\mathrm{P}}_n \}$ converges weakly to $\mathring{\mathrm{P}}_w$ in $\mathrm{C}_0(\mathrm{I}, l^2)$.

ii) $\{ \mathrm{dia} \langle \mathring{\mathrm{P}}_n \rangle \}$ converges weakly to the distribution of $(v_{11}t, v_{22}t, \ldots)$ in $\mathrm{C}_0(\mathrm{I}, l^1)$ and for any $i, j = 1, 2, \ldots$ and $t > 0$ $\{$ the distribution of $(\langle \mathring{w}_i, \mathring{w}_j \rangle (t), \mathring{\mathrm{P}}_n)$ on $\mathbb{R}$ $\}$ converges weakly to the δ-distribution at $v_{ij}t$ on $\mathbb{R}$.

Finally we note that such infinite-dimensional central limit theorem for current valued stochastic processes are investigated by Ochi [*10*] and Ikeda and Ochi [*5*].

APPENDIX

As stated in the introduction, in the case of $H = \mathbb{R}$, Theorem A is proved by Rebolledo as a consequence of a general Aldous theorem for right continuous processes and using Lenglart's inequality (cf. [7] [*11*]). The considered processes M^n may be discontinuous and only the limits of $\{ M^n \}$ and $\{ \langle M^n \rangle \}$ are assumed continuous. The proof is quite heavy. We give here a direct simple proof adapted to the special case of continuous processes.

LEMMA A.1. — Let $M(t)$ $(t \in [0, T])$ be an $\mathbb{R}$-valued continuous martingale defined on an usual filtered probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$ starting at 0 with $E[M(T)^4] < +\infty$. For each finite partition $\Delta = \{ 0 = s_0 < s_1 < \ldots < s_l = T \}$ of $[0, T]$, set

$$\text{(A.1)} \qquad I_\Delta(M, t) = \sum_{j=0}^{l-1} M(s_j)\{ M(s_{j+1} \wedge t) - M(s_j \wedge t) \} \qquad (0 \leqq t \leqq T).$$

Then there exists a universal positive constant C such that

$$\text{(A.2)} \quad E\left[\sup_{0 \le t \le T} \left| \int_0^t M(s)dM(s) - I_\Delta(M, t) \right|^2 \right] \leqq 4\varepsilon^2 E[\langle M \rangle(T)] + CE[\langle M \rangle(T)^2]^{1/2} E[\langle M \rangle(T)^2; \Omega_\varepsilon^c]^{1/2} \quad \text{for} \quad \varepsilon > 0,$$

where

$$\Omega_\varepsilon = \{ \omega \in \Omega : \sup_{\substack{|s-t| \le |\Delta| \\ s,t \le T}} | M(t, \omega) - M(s, \omega) | \leqq \varepsilon \}$$

and

$$| \Delta | = \max_{0 \le j \le l-1} (s_{j+1} - s_j).$$

Proof. — It is easy to see that the left hand side of (A.2)

$$\leqq 4E\left[\int_0^T (M(s) - M([s]))^2 d\langle M \rangle(s) \right] \leqq 4\varepsilon^2 E[\langle M \rangle(T)] + 16E[\sup_{0 \le s \le T} M(s)^2 \langle M \rangle(T); \Omega_\varepsilon^c],$$

where $[s] = s_j$ for $s_j \leqq s < s_{j+1}$ $(j = 0, 1, \ldots, l-1)$. Applying Schwarz's inequality to the second part of the right hand side of the above inequality, we get (A.2). □

LEMMA A.2. — Let $M^n(t)$ $(n = 1, 2, \ldots)$ be an $\mathbb{R}$-valued continuous martingale defined on a filtered probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t^n)$ with $M^n(0) = 0$. Suppose that there exists a stochastic process M on $(\Omega, \mathscr{F}, P)$ such that with probability one $\{ M^n \}$ converges to M uniformly on each compact interval. Further suppose that there exists a constant $\beta > 0$ such that $\sup_n E[| M^n(t) |^{4+\beta}] < \infty$ $(t > 0)$. Then M is a continuous martingale with $E[| M(t) |^{4+\beta}] < \infty$ $(t > 0)$ and satisfies

$$\text{(A.3)} \qquad \lim_{n \to \infty} E[\sup_{0 \le t \le T} | \langle M^n \rangle(t) - \langle M \rangle(t) |^2] = 0 \quad \text{for} \quad T > 0.$$

Proof. — It is obvious that M is a continuous martingale with $E[| M(t) |^{4+\beta}] < \infty$ $(t > 0)$. Itô formula implies

$$M^n(t)^2 = 2\int_0^t M^n(s)dM^n(s) + \langle M^n \rangle(t).$$

We then get from the assumption

$$\text{(A.4)} \qquad \lim_{n\to\infty} \mathrm{E}[\sup_{0\leq t\leq \mathrm{T}} |\mathrm{M}^n(t)^2 - \mathrm{M}(t)^2|^2] = 0.$$

For any finite partition $\Delta = \{0 = s_0 < s_1 < \ldots < s_l = \mathrm{T}\}$ of $[0, \mathrm{T}]$ we have

$$\mathrm{E}\left[\sup_{0\leq t\leq \mathrm{T}} \left| \int_0^t \mathrm{M}^n(s)d\mathrm{M}^n(s) - \int_0^t \mathrm{M}(s)d\mathrm{M}(s)\right|^2\right]$$

$$\leq 3\mathrm{E}\left[\sup_{0\leq t\leq \mathrm{T}} \left| \int_0^t \mathrm{M}^n(s)d\mathrm{M}^n(s) - \mathrm{I}_\Delta(\mathrm{M}^n, t)\right|^2\right] + 3\mathrm{E}\,[\sup_{0\leq t\leq \mathrm{T}} |\mathrm{I}_\Delta(\mathrm{M}^n, t) - \mathrm{I}_\Delta(\mathrm{M}, t)|^2]$$

$$+ 3\mathrm{E}\left[\sup_{0\leq t\leq \mathrm{T}} \left| \int_0^t \mathrm{M}(s)d\mathrm{M}(s) - \mathrm{I}_\Delta(\mathrm{M}, t)\right|^2\right] = a_n(\Delta, 1) + a_n(\Delta, 2) + a_n(\Delta, 3) \quad \text{say},$$

where $\mathrm{I}_\Delta(\mathrm{M}^n, t)$ and $\mathrm{I}_\Delta(\mathrm{M}, t)$ are defined in the same way as (A.1). Since $\{\langle \mathrm{M}^n \rangle(\mathrm{T})^2\}$ is uniformly integrable, Lemma A.1 implies that for any $\varepsilon > 0$ there exists a positive constant δ such that

$$a_n(\Delta, 1) + a_n(\Delta, 3) < \varepsilon \qquad \text{for} \qquad |\Delta| < \delta, \qquad n = 1, 2, \ldots,$$

On the other hand it is easy to see that $\lim_{n\to\infty} a_n(\Delta, 2) = 0$ for any Δ. Therefore we have

$$\text{(A.5)} \qquad \lim_{n\to\infty} \mathrm{E}\left[\sup_{0\leq t\leq \mathrm{T}} \left| \int_0^t \mathrm{M}^n(s)d\mathrm{M}^n(s) - \int_0^t \mathrm{M}(s)d\mathrm{M}(s)\right|^2\right] = 0.$$

Combining (A.4) with (A.5), we then get (A.3). □

Proof of Theorem A *in case of* $\mathrm{H} = \mathbb{R}$. — First we prove that *iii*) implies *ii*). Let $\mathrm{M}^n(t)$ $(n = 1, 2, \ldots)$ be an $\mathbb{R}$-valued continuous local martingale with $\mathrm{M}^n(0) = 0$ defined on a filtered probability space $(\Omega_n, \mathscr{F}_n, \mathrm{P}_n, \mathscr{F}_t^n)$. We assume that $\{\langle \mathrm{M}^n \rangle\}$ is tight in $\mathrm{C}_0([0, +\infty), \mathbb{R})$. Then for any $n = 1, 2, \ldots$ there exist an extension $(\Omega_n', \mathscr{F}_n', \mathrm{P}_n', \mathscr{F}_t^{n'})$ of $(\Omega_n, \mathscr{F}_n, \mathrm{P}_n, \mathscr{F}_t^n)$ and a one-dimensional Brownian motion $\mathrm{B}^n(t)$ on $(\Omega_n', \mathscr{F}_n', \mathrm{P}_n', \mathscr{F}_t^{n'})$ with $\mathrm{B}^n(0) = 0$ such that $\mathrm{M}^n(t) = \mathrm{B}^n(\langle \mathrm{M}^n \rangle(t))$. The tightness of $\{(\mathrm{B}^n, \langle \mathrm{M}^n \rangle)\}$ implies the tightness of $\{\mathrm{M}^n\}$.

Next we prove that *ii*) implies *iii*). Let $\mathrm{M}^n(t)$ $(n = 1, 2, \ldots)$ be an $\mathbb{R}$-valued continuous local martingale defined on a filtered probability space $(\Omega, \mathscr{F}, \mathrm{P}, \mathscr{F}_t^n)$ with $\mathrm{M}^n(0) = 0$. We assume that with probability one $\{\mathrm{M}^n(t)\}$ converges uniformly on each compact interval. It is sufficient to prove that $\{\langle \mathrm{M}^n \rangle\}$ is tight in $\mathrm{C}_0([0, \mathrm{T}], \mathbb{R})$ $(\mathrm{T} > 0)$. For $\mathrm{N} = 1, 2, \ldots$ and $n = 1, 2, \ldots$ put

$$\mathrm{S}_n(\mathrm{N}) = \begin{cases} \inf\{t \in [0, \mathrm{T}]; |\mathrm{M}^n(t)| > \mathrm{N}\} \\ \mathrm{T} \quad \text{if} \quad \{\ \} = \phi. \end{cases}$$

It is easy to see that for any $\varepsilon > 0$ there exists N_0 such that

$$\text{(A.6)} \qquad \mathrm{P}(\mathrm{S}_n(\mathrm{N}_0) = \mathrm{T}) > 1 - \varepsilon \qquad \text{for} \qquad n = 1, 2, \ldots.$$

Since $\{\mathrm{M}^n(t \wedge \mathrm{S}_n(\mathrm{N}))\}$ is tight in $\mathrm{C}_0([0, \mathrm{T}], \mathbb{R})$ for $\mathrm{N} = 1, 2, \ldots$, we have by Lemma A.2 that $\{\langle \mathrm{M}^n \rangle(t \wedge \mathrm{S}_n(\mathrm{N}))\}$ is tight in $\mathrm{C}_0([0, \mathrm{T}], \mathbb{R})$ for $\mathrm{N} = 1, 2, \ldots$. Therefore the tightness in $\mathrm{C}_0([0, \mathrm{T}], \mathbb{R})$ of $\{\langle \mathrm{M}^n \rangle\}$ follows from (A.6). □

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